

Article

Risk Control for Synchronizing a New Economic Model

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Abstract: Risk analysis in control problems is a critical but often overlooked issue in this research area. The main goal of this analysis is to assess the reliability of designed controllers and their impact on applied systems. The chaotic behavior of fractional-order economical systems has been extensively investigated in previous studies, leading to advancements in such systems. However, this chaotic behavior poses unpredictable risks to the economic system. This paper specifically investigates the reliability and risk analysis of chaotic fractional-order systems synchronization. Furthermore, we present a technique as a new mechanism to evaluate controller performance in the presence of obvious effects. Through a series of simulation studies, the reliability and risk associated with the proposed controllers are illustrated. Ultimately, we show that the suggested technique effectively reduces the risks associated with designed controllers.

Keywords: reliability; risk; synchronization; fractional-order; economical system



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1. Introduction

Reliability is defined as the ability of an item to start and continue a predefined process under certain operating conditions, whereas risk refers to the potential loss that arises from exposure to a hazard. As it is shown in [1–4], the risk value can be computed based on the reliability value.

In engineering applications, specifically in control engineering, conducting a reliability analysis of the controller is crucial for designing a reliable controller. For instance, in [5], the second-order reliability method is proposed, which computes probabilistic reliability measures to assess controlled structures. Additionally, a novel stability analysis based on reliability is introduced for the controlled structures in [6]. The reliability analysis is conducted for various system dynamics, including fractional-order systems [7]. It is worth noting that the reliability analysis can also be performed for controller systems [8,9].

In addition to reliability analysis, risk analysis and management are two important subjects with numerous applications in systems engineering [10–12]. The operational risk of a controller for a system can be defined in various ways depending on its specific application. In this paper, the risk of the controller is presented as a time-based function, which is defined as the likelihood of failure in each time sequence, meaning that the controlled system does not converge to the origin [13]. The risk analysis of the controller is a crucial aspect in control systems design. However, it has not received as much attention in engineering compared to other fields such as economics and management.

To address the reliability and risk analysis of economic systems, the first step is to model such systems. Fractional order models have been successfully employed to accurately represent economic systems, as demonstrated in [14–16]. Moreover, it has been well-studied that certain economic systems exhibit chaotic behavior [17–21]. While these

behaviors can be captured using integer-order models, empirical studies have indicated that the observed dynamics in economic systems can be more accurately represented by fractional-order models [22,23].

Economic models have traditionally employed classical systematic analyses. The significance of business cycle synchronization between countries is now commonly utilized [24].

The results of synchronization are influenced by various conditions. Therefore, in order to mitigate the negative effects in this field, it is necessary to find a way to identify these influences. The objective of synchronizing two economic systems is to have the second system, known as the slave system, behave similarly to the first system, referred to as the master system. The synchronization of fractional-order chaotic systems is extensively investigated in [25–29].

However, the main challenge of such studies is to handle the external disturbance. To address this challenge, multiple methods are presented to synchronize fractional-order chaotic systems in the presence of external disturbances [30,31]. Recently, these methods gained attention in real-world applications. For instance, ref. [32] has demonstrated the utilization of synchronization of fractional-order chaotic systems with disturbance for speech-secure communication. Designing a robust controller that can perform well in the presence of external disturbance is a complex problem. Additionally, these methods are employed for system controllers to endure external disturbances [33,34]. All of these methods rely on the state space model of the system, and identifying the accurate model of the system is an important task that may not always be feasible.

In this paper, a new analytical method is proposed based on the risk analysis of the controller system to assess its performance in the presence of external disturbances with addressing the drawbacks of the existing methods. Additionally, we present a method to reduce the effects of external disturbances and reduce the chance of controller failure, resulting in a more reliable controller. Furthermore, we aim to propose a strategy to minimize this risk. In order to reduce the risk of controllers in the presence of external perturbations, controllers are optimized under stability conditions for fractional order systems. Risk analysis has emerged as a new area of discussion in assessing the performance of controllers. Various common factors can significantly influence the performance of controllers in real-world applications [35,36]. The analysis of these controllers is primarily grounded in the theoretical stability of the system. However, it is crucial to acknowledge that in real-world applications, there are various additional factors that demand attention. Mitigating the risk associated with a controller is a pivotal aspect to consider. To mitigate the risk associated with controllers, various optimization algorithms can be employed, such as those mentioned in [37,38]. Many of these algorithms draw inspiration from nature. When selecting an optimization method for this particular problem, it is essential to consider different parameters. Additionally, certain parts of the algorithm may need to be revised to align with the specific optimization problem at hand. In our case, we utilize a novel type of evolutionary algorithm known as the Biography-based Optimization (BBO) algorithm [39]. This algorithm demonstrates superior performance compared to traditional approaches. Furthermore, we have made modifications to this algorithm to achieve even better results in our optimization process.

The main contributions of this paper can be listed as follows.

- A new method is proposed to compare the performance of different controllers.
- A new method is introduced to synchronize the fractional-order economic systems in the presence of external disturbances without traditional mathematical analysis of a system.
- The reliability and risk of the proposed method are analyzed.
- A new method is used to reduce the risk of the designed controller.

The structure of this paper is organized as follows: Section 2 provides an introduction to the basic concept of fractional calculation. Section 3 presents the fractional-order modeling of economic problems from a system engineering perspective. In Section 4, a synchronization method for two economic systems is defined, and an appropriate controller is designed. This section also takes into account the system's response to external distur-

bances. Section 5 offers a quantitative formulation of the suggested controller's reliability and risk analysis. Section 6 outlines two different optimization methods aimed at reducing risks in the proposed controllers. Simulation results are presented in Section 7. Section 8 consists of a discussion about the obtained results. Finally, in Section 9, the paper concludes by analyzing the simulation results.

2. Fractional Calculation

Fractional-order transformer functions describe systems with fractional orders. In previous studies, the fractional-order operator has been defined in various ways. The Caputo definition stands out as one of the most popular and useful definitions for fractional-order operators [40]:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \quad (1)$$

where, m is an integer number such that $n-1 < \alpha < m$, and $\Gamma(\cdot)$ is the Gamma function. For the case that $\alpha = 1$, this operator behaves like an ordinary first-order differential operator. In the whole of this paper, we use this definition to describe the fractional-order operators.

Stability of Fractional Order Systems

Like any other system, stability analysis of fractional-order systems is an important issue. The distinct properties of fractional-order systems, in comparison to integer systems, indicate that their stability regions differ. Consider the following fractional-order system with state variables $x \in R^n$, and orders $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_i, \dots, \alpha_n]^T$, ($0 < \alpha_i \leq 1$)

$$D^\alpha x = f(x) \quad (2)$$

We have the following stability conditions [41]:

(1) For a commensurate system, the stability region is defined as

$$|\arg(\lambda)| > \alpha\pi/2$$

(2) For an in-commensurate system, the stability region is

$$|\arg(\lambda)| > \pi/2M$$

where, $\alpha_i \in [0, 1]$. Also, we consider

$$\alpha_i = \frac{v_i}{u_i}, \quad \text{rem}(u_i, v_i) = 1$$

where $u_i, v_i \in N$, for $i = 1, 2, \dots, n$ and M_i is the minimum of common multiple of the denominators u_i of α_i s. Also, λ_i s are the roots of

$$\det(\text{diag}(M1, M2, \dots, Mn)A) = 0.$$

3. System Description

Economic systems involve interactions between enterprise units and markets, contributing to economic growth through commercial investments and demands. A dynamic model of the economic system has been recently introduced, utilizing the following set of ordinary differential equations [42,43].

$$\begin{cases} \dot{x} = z + (y-a)x \\ \dot{y} = 1 - by - x^2 \\ \dot{z} = -x - cz \end{cases} \quad (3)$$

Three system states are used in this model to explain the fundamental components of an economic system. Interest rates are shown by x , investor demand is shown by y , and prices are shown by z .

Typically, changes in the interest rate can be attributed to two primary factors:

- 1 Deviations in the investment market, such as an imbalance where investments exceed savings.
- 2 Structural modification on beneficial conditions.

The rate of investment exhibits a direct correlation with the interest rate, while an inverse relationship exists between the rate of investment and the interest rate. Additionally, changes in the price index can be influenced by the imbalance between supply and demand in commercial marketplaces, as well as fluctuations in inflation rates.

In the given dynamical model, the parameters a , b , and c are positive constants. Specifically: The parameter a represents the amount of savings. The parameter b denotes the per-investment cost. The parameter c represents the elasticity of demand for commercials. These constants play crucial roles in determining the dynamics and behavior of the economic system.

3.1. Fractional-Order Economic System

The fractional version of the system can be expressed as follows [22]:

$$\begin{cases} D^{\alpha_1}x = z + (y - a)x \\ D^{\alpha_2}y = 1 - by - x^2 \\ D^{\alpha_3}z = -x - cz \end{cases} \quad (4)$$

where $\alpha_1, \alpha_2, \alpha_3$ are fractional orders of systems. The system can be classified as commensurate or incommensurate based on these definitions: When $\alpha_1 = \alpha_2 = \alpha_3$, the system is considered commensurate. If the fractional orders differ ($\alpha_1 \neq \alpha_2 \neq \alpha_3$), the system is classified as incommensurate. It has been observed that the commensurate version of the fractional-order economic system exhibits chaotic behavior when the fractional orders satisfy $\alpha_i > 0.85$ ($i = 1, 2, 3$) [22]. Figure 1 shows the chaotic behavior of system (4) for $a = 1, b = 0.1, c = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0.9$.

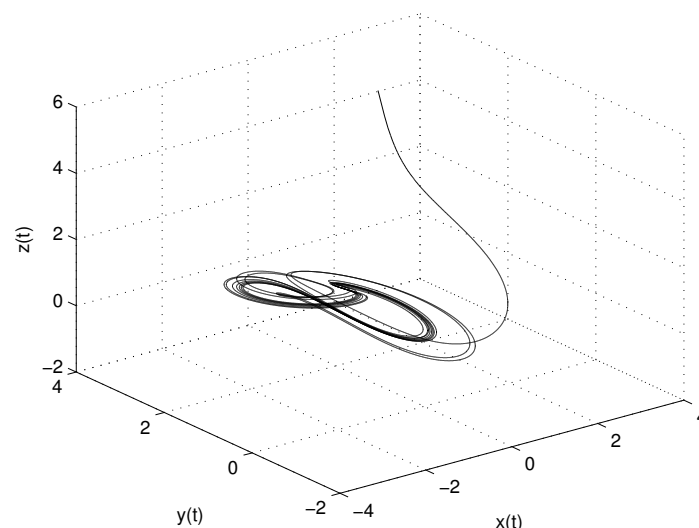


Figure 1. Chaotic behavior of fractional-order economical system based on Equation (4).

3.2. Fractional-Order Economic System in the Presence of Disturbance

Models serve as approximations of real-world systems, and it is common for external disturbances to affect these systems. Therefore, in order to demonstrate the reliability and risk of a controller, it is necessary to introduce uncertainty or disturbances into the system. A modified version of the system (4), accounting for disturbances, can be expressed as follows:

$$\begin{cases} D^{\alpha_1}x = z + (y - a)x + w_1 \\ D^{\alpha_2}y = 1 - by - x^2 + w_2 \\ D^{\alpha_3}z = -x - cz + w_3 \end{cases} \tag{5}$$

where w_1, w_2 , and w_3 are disturbances.

4. Synchronization Method

Based on the characteristics of chaotic systems, even small deviations in initial conditions can result in significantly divergent behaviors, emphasizing the criticality of synchronization among these systems.

A synchronization model comprises two systems, known as the drive system and the response system, with control signals applied to the response system. The mathematical description of the synchronization error, which reflects the deviation between the synchronized states of the two systems, can be represented as follows:

$$E(t) = X_r(t) - X_d(t) \tag{6}$$

where the drive and response system states are represented by the variables X_d and X_r , respectively, and $E(t)$ is the synchronization error between them.

We consider a commensurate fractional-order economical System (4) as a set of drive and response systems. Then, we can formulate our model as

$$\text{Drivesystem} : \begin{cases} D^{\alpha}x_d = z_d + (y_d - a)x_d \\ D^{\alpha}y_d = 1 - by_d - x_d^2 \\ D^{\alpha}z_d = -x_d - cz_d \end{cases} \tag{7}$$

$$\text{Responsesystem} : \begin{cases} D^{\alpha}x_r = z_r + (y_r - a)x_r + u_1 \\ D^{\alpha}y_r = 1 - by_r - x_r^2 + u_2 \\ D^{\alpha}z_r = -x_r - cz_r + u_3 \end{cases} \tag{8}$$

where u_1, u_2 , and u_3 are control inputs that will be determined later to ensure the convergence of two responses and drive systems to each other to synchronize their states. By using the definition of synchronization error as Equation (6), we have

$$\begin{cases} D^{\alpha}e_1 = e_3 - ae_1 + y_r x_r - y_d x_d + u_1 \\ D^{\alpha}e_2 = -be_2 - x_r^2 + x_d^2 + u_2 \\ D^{\alpha}e_3 = -e_1 - ce_3 + u_3 \end{cases} \tag{9}$$

Using the active controller procedure [29], we can define controller inputs as

$$\begin{cases} u_1 = -y_r x_r + y_d x_d + k_{11}e_1 + k_{22}e_2 + k_{33}e_3 \\ u_2 = x_r^2 - x_d^2 + k_{21}e_1 + k_{22}e_2 + k_{23}e_3 \\ u_3 = e_3 = k_{31}e_1 + k_{32}e_2 + k_{33}e_3 \end{cases} \tag{10}$$

In Equation (10), k_{ij} s for $i, j = 1, 2, 3$ are the elements of matrix K , which are specified in accordance with the stability requirements of fractional-order systems and are added to the control signals. With these controllers, the errors of synchronization become

$$\begin{cases} D^{\alpha}e_1 = (-a + k_{11})e_1 + k_{22}e_2 + (k_{33} + 1)e_3 \\ D^{\alpha}e_2 = k_{21}e_1 + (-b + k_{22})e_2 + k_{23}e_3 \\ D^{\alpha}e_3 = (-1 + k_{31})e_1 + k_{32}e_2 + (-c + k_{33})e_3 \end{cases} \tag{11}$$

5. Utilizing Reliability and Risk in the Proposed Controller Analysis

5.1. Reliability

Here, we discuss the analysis of the reliability and risk of the designed controller. Reliability refers to the probability that a component or system will continue to function as intended and meet its operational requirements within a specified set of conditions for a given period of time. It quantifies the ability of the component or system to perform its intended function without failure or breakdown [2]. In the context of the synchronization problem, reliability can be defined as a function that is influenced by the synchronization error. The reliability function reaches its maximum value when synchronization is achieved, indicating that the controller's reliability is at its peak. In this case, the reliability of a controller in synchronized states is equal to 1.

According to the defined range of reliability in the interval $[0, 1]$, it is necessary to normalize synchronization errors within this range to express reliability in terms of those errors. The reliability of each controller can be written as:

$$R_i(t) = 1 - \hat{e}_i(t) \quad (12)$$

where \hat{e}_i denotes the normalized synchronization error of i th controller. Indeed, as indicated by Equation (12), there is an inverse relationship between the reliability of a controller and its synchronization error. When the synchronization errors are large, the reliability tends to approach 0, implying that the controller's performance is less reliable. Conversely, as the synchronization errors approach zero, the reliability tends to approach 1, indicating a higher degree of confidence in the controller's effectiveness.

5.2. Risk

In the introduction, the term "risk" is defined as the potential for a process to fail in delivering the desired outcomes [2]. Based on this definition, risk can be divided into two components: the causes of failure and the costs associated with these failures. A mathematical model of risk can be represented as follows:

$$Risk = FF * CF \quad (13)$$

where FF refers to the likelihood or probability of the process or system failing, and CF represents the potential negative consequences or losses incurred as a result of the failure. According to the given definition of risk in the context of a synchronization problem, the failure factor of a controller can be identified as the synchronization error between the systems being synchronized. The cost associated with this failure factor is related to the control signals used in the synchronization process.

$$Risk_i(t) = \hat{e}_i(t) * \int_0^t u_i^2(\tau) d\tau \quad (14)$$

Again \hat{e}_i denotes the normalized synchronization error of controller i . The risk of a controller shows the performance of the controller in the time domain. According to relation (13), the synchronized states lead to zero risk in a limited time.

5.3. Relation between Reliability and Risk

Equations (12) and (13) show that basically reliability and risk are related to each other as shown by Equation (15).

$$Risk_i(t) = (1 - R_i(t)) * \int_0^{t_f} u_i^2(\tau) d\tau \quad (15)$$

This means that by increasing reliability, the risk decreases. But it must be noted that in some problems, the cost part of the above equation ' (u_i^2) ' might have its own relation with reliability, which leads to a complex relationship between reliability and risk in controllers.

6. Risk Reduction in the Proposed Controllers

Biography Based Optimization Algorithm

The biography-based optimization (BBO) algorithm is a novel optimization technique inspired by the geographic distribution of biological organisms [39]. In the BBO algorithm, information sharing among different solutions, referred to as islands or habitats, is facilitated through migration. The algorithm employs various parameters to control its behavior and optimization process. These parameters include:

Suitability Index Variable (SIV): The SIV represents a variable used to evaluate the suitability of a habitat or island for a particular solution. It quantifies the fitness or quality of the solution within its respective habitat.

Habitat Suitability Index (HSI): The HSI indicates the overall suitability of a habitat or island for hosting solutions. It is a measure of the habitat's ability to support and promote good solutions.

Immigration Rate (λ): The immigration rate determines the frequency or probability at which solutions migrate from one habitat to another. It governs the movement of solutions across different islands in the optimization process.

Emigration Rate (μ): The emigration rate determines the likelihood or rate at which solutions leave a particular habitat. It controls the departure of solutions from one island to migrate to other habitats.

More details about this algorithm and optimizing approach can be found in [39].

For the given problem, the coefficients of the controllers in Equation (11) are chosen as habitats in the BBO algorithm. Furthermore, the cost function of the algorithm is selected as follows:

$$f = \text{mean}(\text{Risk}_1)^2 + \text{mean}(\text{Risk}_2)^2 + \text{mean}(\text{Risk}_3)^2 \quad (16)$$

7. Results of Simulation

In this section, two examples are simulated to demonstrate different scenarios. In the first case, a new fractional-order model for the financial crisis is introduced. The system and controller in this case are specifically designed to account for external disturbances. The aim is to analyze the behavior of the system under the influence of these disturbances and evaluate the effectiveness of the controller in mitigating their impact. In the second example, synchronization is achieved between two fractional-order economic systems. A suitable controller is developed to facilitate synchronization between these systems. One of the systems is subjected to an external disturbance, which further tests the robustness and performance of the synchronization controller in the presence of disturbances.

7.1. Example 1

In this example, a new fractional-order version of the financial crisis model is considered, based on the model introduced by Korobeinikov [44]. The chosen fractional-order financial crisis model can be expressed as follows:

$$\begin{cases} D^{q_1} X = -\beta XY^\alpha \\ D^{q_2} Y = \beta XY^\alpha - \frac{1}{\sigma} Y \end{cases} \quad (17)$$

In the considered model, the population is divided into two subpopulations: the healthy subpopulation and the activated subpopulation. The size of the healthy subpopulation is denoted by $X(t)$, while the size of the activated subpopulation, specifically focused on financial problems, is denoted by $Y(t)$.

Now we can consider external disturbances and control signals in this system as follows.

$$\begin{cases} D^{q_1} X = -\beta XY^\alpha + d_1(t) + u_1(t) \\ D^{q_2} Y = \beta XY^\alpha - \frac{1}{\sigma} Y + d_2(t) + u_2(t) \end{cases} \quad (18)$$

where d_1 and d_2 are external disturbances. u_1 and u_2 are control signals. Parameters of this model are selected as $\alpha = 1.92, \beta = 8.25, \sigma = 500$. Also, fractional orders are selected as $q_1 = q_2 = 1.55$. The results of controller risk for this case can be shown in Figure 2.

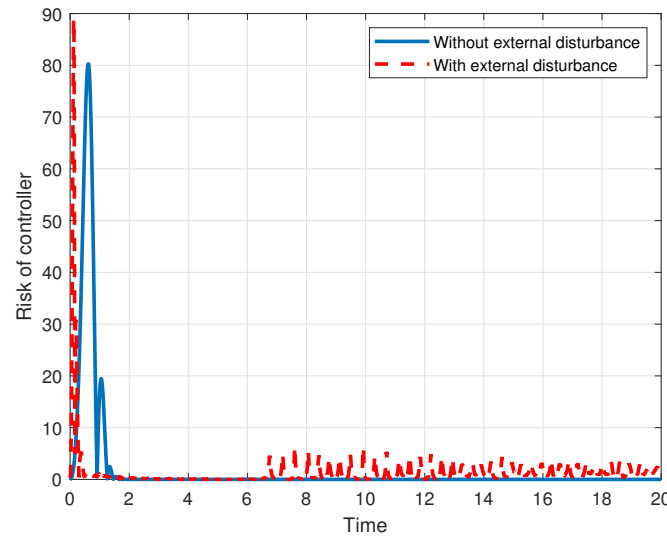


Figure 2. Risk of the second controller in the financial crisis model.

Based on the results obtained, it is evident that the mean risk of the designed controller is higher when external disturbances are considered. The calculated mean risk of 1.72 for the controller with external disturbances is greater than the mean risk of 1.38 for the controller without external disturbances. This finding suggests that the presence of external disturbances increases the overall risk associated with the performance of the designed controller.

7.2. Example 2

In the given scenario, a fractional-order economical system represented by Equation (7) is selected as the master system, while the fractional-order economical system represented by Equation (8) is chosen as the slave system. The parameters for both the master and slave systems are defined as $a = 1, b = 0.1,$ and $c = 1$. Additionally, both systems are characterized as chaotic systems with an order of 0.9.

Based on these system specifications, the synchronization errors can be calculated as follows:

$$\begin{cases} D^\alpha e_1 = e_3 - e_1 + y_r x_r - y_d x_d + u_1 \\ D^\alpha e_2 = -0.1e_2 - x_r^2 + x_d^2 + u_2 \\ D^\alpha e_3 = -e_1 - e_3 + u_3 \end{cases} \quad (19)$$

Also, the controller signals are

$$\begin{cases} u_1 = -y_r x_r + y_d x_d + k_{11}e_1 + k_{22}e_2 + k_{33}e_3 \\ u_2 = x_r^2 - x_d^2 + k_{21}e_1 + k_{22}e_2 + k_{23}e_3 \\ u_3 = e_3 = k_{31}e_1 + k_{32}e_2 + k_{33}e_3 \end{cases} \quad (20)$$

Initial conditions for the drive system are chosen as $(x_d(0), y_d(0), z_d(0)) = (2, 3, 5)$ and for the response system as $(x_r(0), y_r(0), z_r(0)) = (-9, -5, 14)$. The simulation time of the system is selected as 20 s. According to the proposed method, a control coefficient matrix

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

leads to the stable synchronization errors with roots on $(-1, -1.1, -2)$. The synchronization errors and control inputs are shown in Figure 3 and Figure 4, respectively.

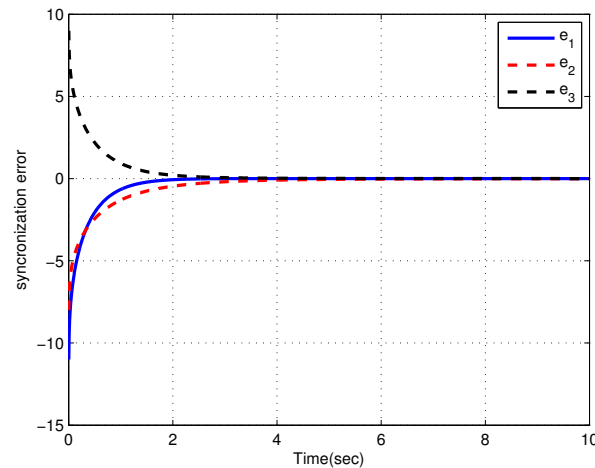


Figure 3. The synchronization errors of fractional-order economical systems.

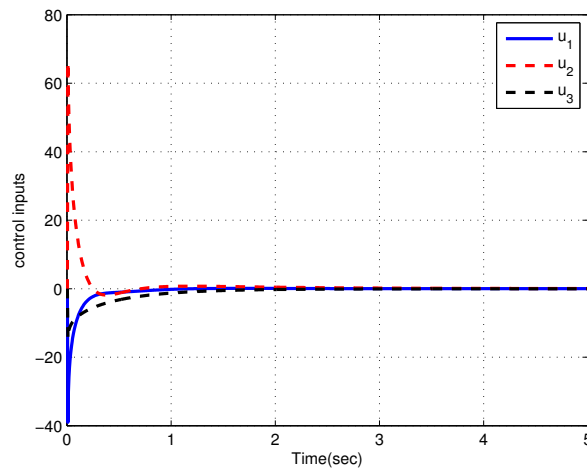


Figure 4. The control signals in synchronization of fractional-order economical systems.

Figure 5 also shows how reliable designed controllers can be. Finally, according to the mentioned relations, the risk of controllers is calculated, and the results are shown in Figure 6. For this case, we can calculate the relation between the reliability and risk of the controller with some simplifications, such as

$$\begin{cases} u_1 = (-y_r + k_{11})e_1 + (-x_d + k_{22})e_2 + k_{33}e_3 \\ u_2 = ((x_r + x_d) + k_{21})e_1 + k_{22}e_2 + k_{23}e_3 \\ u_3 = e_3 = k_{31}e_1 + k_{32}e_2 + k_{33}e_3 \end{cases} \quad (21)$$

According to the limited amplitude of each section of Equation (21), the convergence of controller input is guaranteed. Rewriting Equation (21) as

$$\begin{cases} u_1 = (-y_r + k_{11})e_1 + U_1 \\ u_2 = k_{22}e_2 + U_2 \\ u_3 = e_3 = k_{33}e_3 + U_3 \end{cases} \quad (22)$$

where $U_1 = (-x_d + k_{22})e_2 + k_{33}e_3$, $U_2 = ((x_r + x_d) + k_{21})e_1 + k_{23}e_3$ and $U_3 = k_{31}e_1 + k_{32}e_2$, and substituting Equation (22) in Equation (14) leads to

$$\begin{cases} Risk_1(t) = \hat{e}_1(t) * \int_0^t ((-y_r + k_{11})\hat{e}_1(\tau) + \hat{U}_1(\tau)^2) d\tau \\ Risk_2(t) = \hat{e}_2(t) * \int_0^t (k_{22}\hat{e}_2(\tau) + \hat{U}_2(\tau)^2) d\tau \\ Risk_3(t) = \hat{e}_3(t) * \int_0^t (k_{33}\hat{e}_3(\tau) + \hat{U}_3(\tau)^2) d\tau \end{cases} \quad (23)$$

where notation \hat{f} shows the normalized value of f . According to Equation (12), the relation between risk and reliability of each controller can be written as

$$\begin{cases} Risk_1(t) = (1 - R_1(t)) * \int_0^t ((-y_r + k_{11})(1 - R_1(\tau)) + \hat{U}_1(\tau)^2) d\tau \\ Risk_2(t) = (1 - R_2(t)) * \int_0^t (k_{22}(1 - R_2(\tau)) + \hat{U}_2(\tau)^2) d\tau \\ Risk_3(t) = (1 - R_3(t)) * \int_0^t (k_{33}(1 - R_3(\tau)) + \hat{U}_3(\tau)^2) d\tau \end{cases} \quad (24)$$

As mentioned in the previous section, the relationship between risk and reliability of each controller in the proposed controller coefficient matrix is not a simple reverse relation. While certain elements of the coefficient matrix, such as k_{11} , k_{22} , k_{33} , and the reference signal $y_r(t)$, have an impact on this relationship, it is not purely inverse.

In the implementation of the BBO algorithm for this specific scenario, the following parameter settings are used:

Maximum number of iterations: 50

Number of habitats: 30

Keep rate: 0.2

Lower bound of habitats: 0

Upper bound of habitats: 100

For emigration and immigration rates, we consider:

Emigration rate = linspace(1, 0, 30)

Immigration rate = 1 - Emigration rate

The results of risk reduction with this algorithm and the proposed controllers as Equation (20) are shown in Table 1.

Also, in these systems, disturbances are considered as random values drawn from a standard uniform distribution in $[0, 0.1]$. Here again, with the proposed method, controllers are designed to synchronize two systems. For this case, synchronization errors are shown in Figure 7. As seen in Figure 7 in comparison with Figure 3, there is not a mentionable difference between the two figures, which shows the effect of external disturbances. Control inputs are also illustrated in Figure 8. The results of compare method are shown in Figure 9. As seen in this figure, in the presence of external disturbances, the Ref. [45] method can not synchronize two systems. The reliability of the proposed controllers is figured in Figure 10. We calculate the risk of the proposed controllers. Figure 11 shows the risk of the system with and without disturbances. Figure 12 shows the result of Figure 11 with more details.

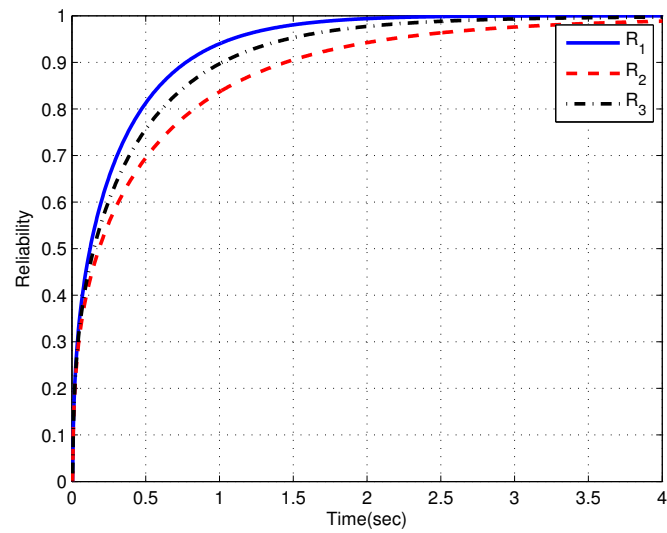


Figure 5. The reliability of designed controllers in synchronization of fractional-order economical systems.

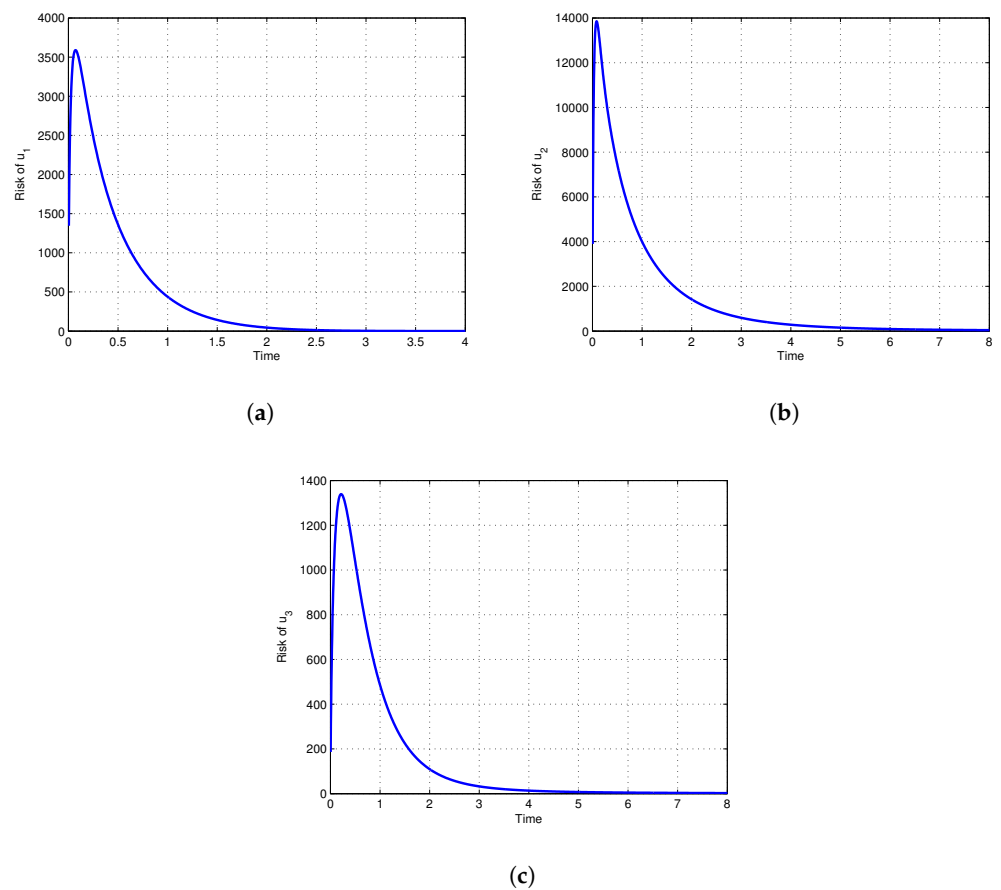


Figure 6. Controller risks in synchronization of fractional-order economical systems: (a) first controller, (b) second controller, (c) third controller.

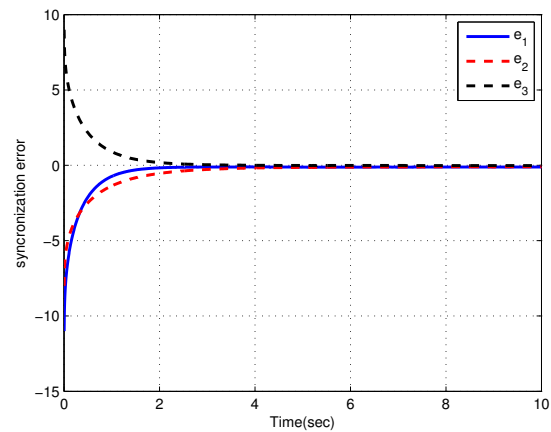


Figure 7. Synchronization of fractional-order economical systems in the presence of external disturbances errors.

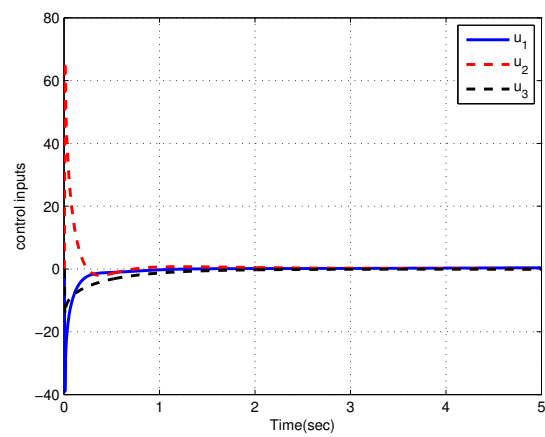


Figure 8. The control signals of in synchronization of fractional-order economical systems in the presence of external disturbances.

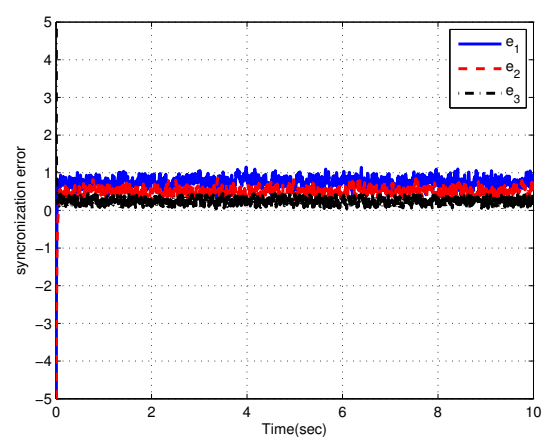


Figure 9. Synchronization of fractional-order economical systems in the presence of external disturbances errors using the ref. [45] method.

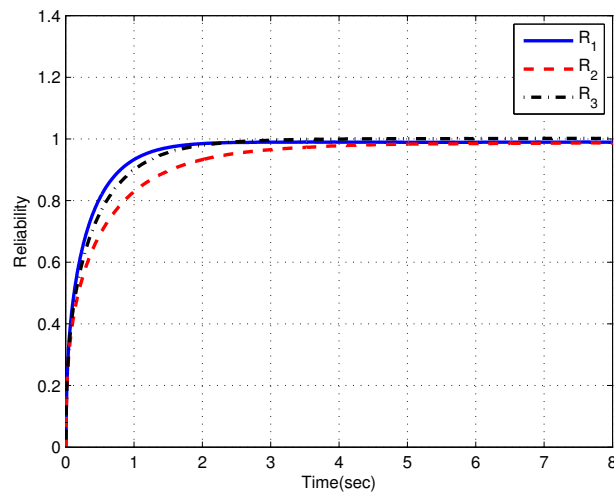
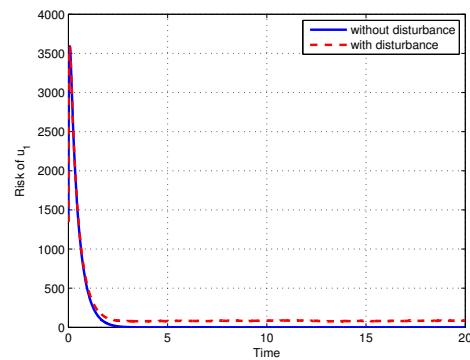
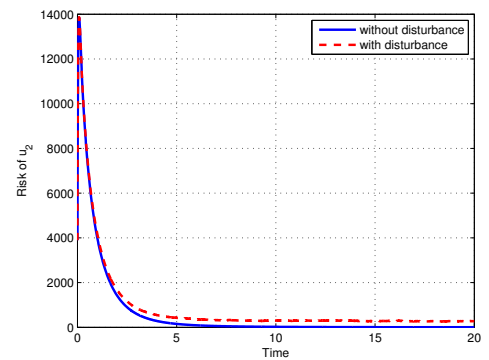


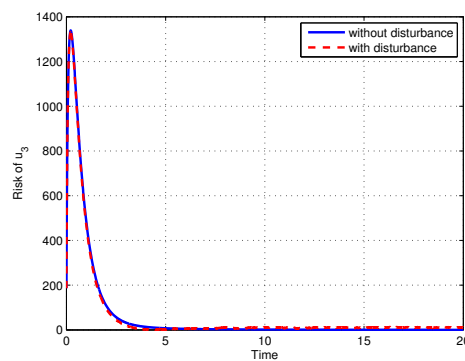
Figure 10. The reliability of control signal of in synchronization of fractional-order economical systems in the presence of external disturbances.



(a)

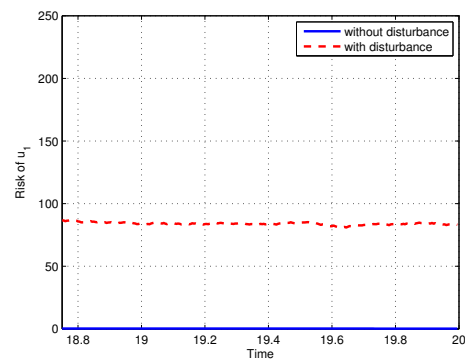


(b)

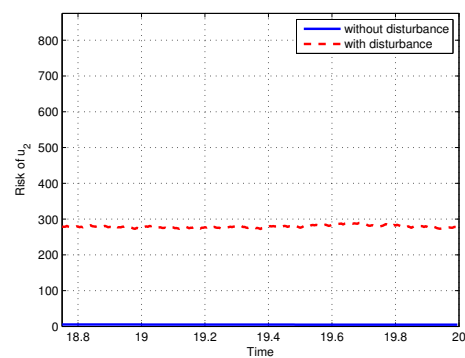


(c)

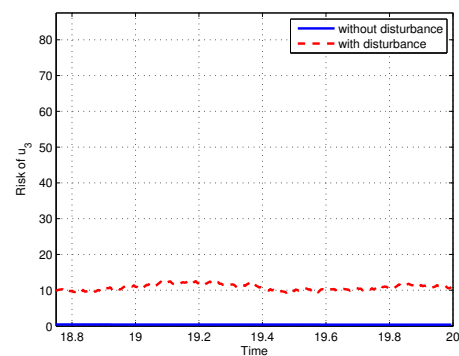
Figure 11. Designed controller risks in synchronization of fractional-order economical systems in the presence of external disturbances: (a) first controller, (b) second controller, (c) third controller.



(a)



(b)



(c)

Figure 12. Risk of designed controllers in synchronization of fractional-order economical systems with external disturbances: (a) first controller, (b) second controller, (c) third controller.

Table 1. Comparison of control signals risks.

Method	$Mean(Risk_1)$	$Mean(Risk_2)$	$Mean(Risk_3)$
Controllers as Equation (20) with matrix K	168.15	888.32	68.32
BBO optimized method	71.65	296.21	2.62

8. Discussion

In the presence of external disturbances, designing a controller has always been a challenging task, particularly for fractional-order systems. Simplifying the complex analysis of systems to facilitate controller design is the most important. In this paper, we propose a new

analysis for controller design that specifically addresses the presence of external disturbances. We introduce reliability and risk as new factors in the controller design process.

To thoroughly evaluate the risk associated with the proposed controller, it is necessary to understand the behavior of the controlled system when subjected to typical disturbances found in real systems. To achieve this, we incorporate disturbance input signals into both the master and slave systems. The findings indicate that the mean risk of the designed controller is higher in the presence of external disturbances compared to when they are absent. This demonstrates how external disturbances increase the risk of the proposed controller.

After introducing the concept of controller risk, we explore methods to minimize this risk, similar to the design of robust controllers in traditional control theory. However, the proposed method in this paper offers the advantage of simple calculation and analysis compared to traditional robust control methods. Additionally, our method exhibits better performance compared to previous approaches. To support this claim, we compare our method with the [45] approach through simulations, which reveal that the [45] approach is not effective in the presence of external disturbances. Furthermore, we quantify the risk associated with the controller in a new area of controller analysis and demonstrate that our approach carries less risk compared to others. This is a significant contribution of this paper.

Moreover, we provide a technique for reducing the risk associated with controllers in synchronization problems. We modify one of the best existing approaches to suit our specific problem. Statistical results demonstrate that the proposed method performs well in terms of risk reduction.

In summary, this paper presents a novel perspective on controller design in the presence of external disturbances. We introduce a new method for analyzing and designing controllers in such scenarios, specifically for the synchronization of fractional-order economic systems. Our proposed method outperforms existing approaches, making it a valuable contribution to the field.

9. Conclusions

The presence of disturbances in systems can lead to various issues, particularly in fractional-order systems with complex mathematical models. Consequently, studying these systems has posed a significant challenge. This study focused on the reliability and risk analysis of synchronization in fractional-order systems. The relationships between reliability and risk in each controller were explored. Specifically, a fractional-order economic system and a fractional-order model for financial crises were examined.

The study revealed that the relationship between a controller's reliability and risk in two identical systems can become highly complex in certain unusual situations. To further investigate, systems with external disturbances were introduced, and the reliability and risk of the developed controller were calculated. The simulation results indicated that small-amplitude disturbances have minimal impacts on synchronization errors and reliability, but significantly affected the controller risk. Consequently, it is concluded that considering the controller's risk can serve as a useful criterion for mitigating the impact of disturbances on the system.

Based on these findings, a new analysis based on risk was introduced to design a controller in the presence of external disturbances. Furthermore, an approach for reducing risk in the proposed controllers was described. Future works may include incorporating other factors such as uncertainty into the risk analysis. Additionally, different types of financial models can be utilized and their results can be compared. The proposed method is applicable to various financial models and can be extended to include new fractional order models for different economic data in future studies.

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