Published for SISSA by Springer

Received: October 29, 2021
Accepted: December 28, 2021
Published: January 13, 2022

Higher order RG flow on the Wilson line in $\mathcal{N} = 4$ SYM

M. Beccaria,[a] S. Giombi[b] and A.A. Tseytlin[c,1]

[a] *Università del Salento, Dipartimento di Matematica e Fisica "Ennio De Giorgi" and I.N.F.N., Sezione di Lecce, Via Arnesano, Lecce I-73100, Italy*

[b] *Department of Physics, Princeton University, Princeton, NJ 08544, U.S.A.*

[c] *Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2AZ, U.K.*

E-mail: matteo.beccaria@le.infn.it, sgiombi@princeton.edu, tseytlin@imperial.ac.uk

Abstract: Extending earlier work, we find the two-loop term in the beta-function for the scalar coupling ζ in a generalized Wilson loop operator of the $\mathcal{N} = 4$ SYM theory, working in the planar weak-coupling expansion. The beta-function for ζ has fixed points at $\zeta = \pm 1$ and $\zeta = 0$, corresponding respectively to the supersymmetric Wilson-Maldacena loop and to the standard Wilson loop without scalar coupling. As a consequence of our result for the beta-function, we obtain a prediction for the two-loop term in the anomalous dimension of the scalar field inserted on the standard Wilson loop. We also find a subset of higher-loop contributions (with highest powers of ζ at each order in 't Hooft coupling λ) coming from the scalar ladder graphs determining the corresponding terms in the five-loop beta-function. We discuss the related structure of the circular Wilson loop expectation value commenting, in particular, on consistency with a 1d defect version of the F-theorem. We also compute (to two loops in the planar ladder model approximation) the two-point correlators of scalars inserted on the Wilson line.

Keywords: AdS-CFT Correspondence, Supersymmetry and Duality

ArXiv ePrint: 2110.04212

[1] Also at the Institute for Theoretical and Mathematical Physics (ITMP) of Moscow University and Lebedev Institute.

https://doi.org/10.1007/JHEP01(2022)056

JHEP01(2022)056

Contents

1 Introduction and summary

The expectation value of the Wilson loop (WL) operator $\langle \mathrm{Tr}\,\mathrm{P}e^{i\oint A}\rangle$ is a key observable in gauge theories. In supersymmetric theories, it has a "locally supersymmetric" analog obtained by adding a particular coupling to extra fields in the gauge field multiplet. The prototypical example is the Wilson-Maldacena loop (WML) [1, 2] in the $\mathcal{N}=4$ Super Yang-Mills (SYM) theory which contains an extra scalar coupling, and as a result has a particularly simple structure controlled by the underlying supersymmetry. When the Wilson loop contour is a circle or a straight line, both the WL and WML preserve a one-dimensional conformal symmetry $SL(2,\mathbb{R})$ and may be regarded as 1d conformal defects of the $\mathcal{N}=4$ SYM theory. A circular or straight WML also globally preserves half of the superconformal symmetry.

For a generic smooth contour, the standard WL in YM theory is known to be renormalizable: power divergences exponentiate and factorize (or simply absent in dimensional regularization) while the logarithmic divergences disappear after the renormalization of gauge coupling [3–8]. The latter are absent in $\mathcal{N}=4$ SYM so both WL and WML have finite expectation values which in the planar limit are given by the functions of the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$.

It is of interest to study a family of more general Wilson loop operators that interpolate between the WML and the standard WL [9]. This one-parameter family was introduced

JHEP01(2022)056

in [10] and further studied in [11, 12]. It is of interest, in particular, in the context of the 1d defect QFT interpretation (see, in particular, [13–19]). Explicitly, this generalized Wilson loop operator depends on an arbitrary coefficient ζ in front of the coupling to the 6 scalars ϕ_m and so it interpolates between the standard the WL ($\zeta = 0$) and the WML ($\zeta = 1$) cases [10]

$$W^{(\zeta)}(C) = \frac{1}{N} \operatorname{Tr} \mathrm{P} \exp \oint_C d\tau \left[i\, A_\mu(x)\, \dot{x}^\mu + \zeta \phi_m(x)\, \theta^m\, |\dot{x}| \right], \qquad \theta_m^2 = 1\,. \tag{1.1}$$

We may choose the unit vector θ_m to be along 6-th direction, i.e. $\phi_m\theta^m = \phi_6 \equiv \phi$. The expectation value $\langle W^{(\zeta)}\rangle$ for a smooth contour C will have logarithmic divergences that can be absorbed into a renormalization of the coupling ζ. Then the renormalized value of $\langle W^{(\zeta)}\rangle$ will be given by (in the planar limit)

$$\langle W^{(\zeta)}\rangle \equiv \mathrm{W}(\lambda; \zeta(\mu), \mu)\,, \qquad \mu\frac{\partial}{\partial\mu}\mathrm{W} + \beta_\zeta \frac{\partial}{\partial\zeta}\mathrm{W} = 0\,, \qquad \beta_\zeta = \mu\frac{d\zeta}{d\mu}\,, \tag{1.2}$$

where μ is a renormalization scale.[1] The running of ζ then defines a 1d RG flow between the WL and WML operators. Since $\phi_m \to -\phi_m$ is a symmetry of the SYM path integral, W should be invariant under $\zeta \to -\zeta$. In what follows we shall assume that $\zeta \geq 0$. In the planar weak coupling expansion the leading order term in the beta-function was found in [10] to be

$$\beta_\zeta = -\frac{\lambda}{8\pi^2}\zeta(1-\zeta^2) + \mathcal{O}(\lambda^2)\,. \tag{1.3}$$

The WL ($\zeta = 0$) and WML ($\zeta = 1$) cases in (1.1) are expected to be the only two fixed points also at higher orders in λ.[2]

One may view the running of ζ as an RG flow in the effective 1d defect theory coupled to the bulk SYM theory. This interpretation may be made more explicit by representing the path ordering in (1.1) using the auxiliary 1d fermion path integral as in [5, 6, 20, 21] and thus getting an interacting 1d defect action. Considering a circular contour, $F = -\log \mathrm{W}$ may be interpreted as a 1d defect theory free energy on S^1 (normalized by the partition function of the bulk theory). It is then natural to expect that this quantity provides a defect analog of the F-theorem. Specifically, the $d = 1$ version of the generalized F-theorem [22–24] adapted to defects [25] requires that $\widetilde{F} \equiv \sin\frac{\pi d}{2} \log Z(S^d)\Big|_{d=1} = \log Z(S^1) = -F = \log \mathrm{W}$ decreases under RG flow: $\tilde{F}_{\rm UV} > \tilde{F}_{\rm IR}$. This is analogous to the g-theorem [26, 27] that applies to a 1d boundary of a 2d theory. The beta-function (1.3) implies that $\zeta = 0$ is the UV fixed point and $\zeta = 1$ the IR one, and so one shall find that

$$\log\langle W^{(\zeta=0)}\rangle > \log\langle W^{(\zeta=1)}\rangle\,. \tag{1.4}$$

[1] For a specific contour W will depend on μ (that has dimension of a mass) in combination with some effective length characterizing the loop geometry, like radius for a circular loop.

[2] While we shall mostly discuss only planar contributions let us mention that in general the coefficient N in λ (1.3) for a general simple group is the value of quadratic Casimir in the adjoint representation (and thus does not depend on the representation of the Wilson loop). For $\mathrm{U}(N)$ the coefficient in (1.3) is to be multiplied by $(1 - 1/N^2)$ and thus vanishes in the abelian $\mathrm{U}(1)$ case.

This was indeed verified in [11] to hold both in perturbation theory and in the strong coupling expansion. A proof of the $d = 1$ defect version of the F-theorem was recently proposed in [19], where a quantity that monotonically decreases along the flow (and coincides with $\tilde{F}$ at fixed points) was also given.

On general grounds, consistent with the interpretation of $\log \mathrm{W}$ as a defect free enenergygy, we should have

$$\frac{\partial}{\partial\zeta}\log \mathrm{W} = \mathcal{C}\,\beta_\zeta\,, \tag{1.5}$$

where the function $\mathcal{C} = \mathcal{C}(\lambda, \zeta)$ has the weak coupling expansion $\mathcal{C} = \frac{\lambda}{4} + \mathcal{O}(\lambda^2)$ [11].

In the case of a circle or straight line the flow of $\zeta(\mu)$ is driven by the scalar operator $\phi_6 \equiv \phi(x(\tau))$ in (1.1) restricted to the line. Then $\frac{\partial}{\partial\zeta}\langle W^{(\zeta)}\rangle\Big|_{\zeta=0,1} = 0$ implies that its one-point function vanishes at the fixed points, as required by the 1d conformal invariance on the defect line. From (1.5) we also have

$$\frac{\partial^2}{\partial\zeta^2}\log \mathrm{W}\Big|_{\zeta=0,1} = \mathcal{C}\,\frac{\partial\beta_\zeta}{\partial\zeta}\Big|_{\zeta=0,1}\,. \tag{1.6}$$

According to (1.1) this second derivative is given by the integrated 2-point function of ϕ restricted to the line [11]. The latter is determined by the corresponding anomalous dimensions at $\zeta = 0$ and $\zeta = 1$. Indeed, from (1.3) one finds that $\frac{\partial\beta_\zeta}{\partial\zeta}$ reproduces [10] the leading weak-coupling terms in the anomalous dimensions [9] of ϕ at the conformal points $\zeta = 1$ and $\zeta = 0$

$$\Delta^{(\zeta)} - 1 = \frac{\partial\beta_\zeta}{\partial\zeta}\,, \tag{1.7}$$

$$\Delta^{(1)} = 1 + \frac{\lambda}{4\pi^2} + \mathcal{O}(\lambda^2)\,, \qquad\qquad \Delta^{(0)} = 1 - \frac{\lambda}{8\pi^2} + \mathcal{O}(\lambda^2)\,. \tag{1.8}$$

Our main aim here will be to find the λ^2 term in the beta-function (1.3) which should also exhibit the factor $\zeta(1-\zeta^2)$. Let us start with writing down the general structure of β_ζ

$$\beta_\zeta = b_1\,\lambda\,\zeta(1-\zeta^2) + \lambda^2\,\zeta\,(1-\zeta^2)\,(b_2 + b_3\,\zeta^2) + \lambda^3\,\zeta\,(1-\zeta^2)\,(b_4 + b_5\,\zeta^2 + b_6\,\zeta^4) + \mathcal{O}(\lambda^4)\,, \tag{1.9}$$

where $b_1 = -\frac{1}{8\pi^2}$ (cf. (1.3)) and $b_2, b_3, \ldots$ are to be determined. The dependence on powers of ζ at each order in λ follows from the structure of the relevant perturbation-theory diagrams.

An important observation is that the coefficients of the highest ζ^{2n+1} powers at each λ^n order in (1.9), i.e. $b_1, b_3, b_6, \ldots$, are determined by diagrams with maximal number of scalar propagators attached to the line. Thus they should not have internal vertices, i.e. should be given just by the scalar ladders. We shall compute them using the vertex renormalization method of [4] generalizing the one-loop computation in [10]. The resulting terms in β_ζ may be written as

$$\beta_\zeta^{\rm ladder} = q_1\,\frac{\lambda}{4\pi^2}\,\zeta^3 + q_2\,\left(\frac{\lambda}{4\pi^2}\right)^2\,\zeta^5 + q_3\,\left(\frac{\lambda}{4\pi^2}\right)^3\,\zeta^7 + q_4\,\left(\frac{\lambda}{4\pi^2}\right)^4\,\zeta^9 + q_5\,\left(\frac{\lambda}{4\pi^2}\right)^5\,\zeta^{11} + \cdots\,,$$
$$q_1 = \frac{1}{2}, \qquad q_2 = -\frac{1}{4}, \qquad q_3 = \frac{1}{4} - \frac{\zeta_2}{8}, \qquad q_4 = -\frac{17}{48} + \frac{\zeta_2}{3} - \frac{\zeta_3}{12}, \tag{1.10}$$
$$q_5 = \frac{29}{48} - \frac{37\,\zeta_2}{48} + \frac{29\,\zeta_3}{96} + \frac{25\,\zeta_4}{128}, \qquad\qquad b_{\frac{n(n+1)}{2}} = -\frac{1}{(4\pi^2)^n}\,q_n\,,$$

where $\zeta_n \equiv \zeta(n)$ are the Riemann zeta-function values. In particular, at the two-loop order, we get from q_2

$$b_3 = \frac{1}{4}\,\frac{1}{(4\pi^2)^2}\,. \tag{1.11}$$

To find the five-loop expression for the $\beta_\zeta^{\rm ladder}$ in (1.10) we used planar loop equation (2.5) and dimensional regularization. As the higher coefficients q_n are transcendental, the exact expression for this planar ladder-theory beta-function should be complicated. Note that as this is essentially a one-coupling ($\xi \equiv \lambda\zeta^2$) model, the three and higher loop coefficients are scheme-dependent (see below).

An indirect way to fix some combinations of other coefficients in the full beta-function (1.9) is to use the relation (1.7) between β_ζ and the anomalous dimension of the scalar operator and the value of $\Delta^{(1)}$ that was already found earlier from diagrammatic computation for a cusp line in [28] (at the two-loop level) and from the quantum spectral curve in [29] (at several higher loop orders). Explicitly, according to [29]

$$\Delta^{(1)} - 1 = d_1\,\frac{\lambda}{4\pi^2} + d_2\,\left(\frac{\lambda}{4\pi^2}\right)^2 + d_3\,\left(\frac{\lambda}{4\pi^2}\right)^3 + d_4\,\left(\frac{\lambda}{4\pi^2}\right)^4 + \cdots\,, \tag{1.12}$$

$$d_1 = 1, \quad d_2 = -1, \quad d_3 = 2 - \frac{7\,\zeta_4}{4}, \quad d_4 = -5 + \zeta_2 + \frac{\zeta_3}{2} - \frac{\zeta_3\zeta_2}{2} - \frac{5\,\zeta_5}{8} + \frac{119\,\zeta_6}{16}, \quad \ldots\,. \tag{1.13}$$

Comparing this with (1.7), (1.9) gives

$$b_1 = -\frac{d_1}{2(4\pi^2)}\,, \qquad\qquad b_2 + b_3 = -\frac{d_2}{2(4\pi^2)^2}\,, \tag{1.14}$$

$$b_4 + b_5 + b_6 = -\frac{d_3}{2(4\pi^2)^3}\,, \qquad b_7 + b_8 + b_9 + b_{10} = -\frac{d_4}{2(4\pi^2)^4}\ \ldots\,. \tag{1.15}$$

Using (1.11) then (1.14) implies that

$$b_2 = \frac{1}{4}\,\frac{1}{(4\pi^2)^2}\,. \tag{1.16}$$

Thus the explicit form of the 2-loop β_ζ is given by

$$\beta_\zeta = -\frac{\lambda}{8\pi^2}\,\zeta\,(1-\zeta^2) + \frac{\lambda^2}{64\pi^4}\,\zeta(1-\zeta^4) + \mathcal{O}(\lambda^3)\,. \tag{1.17}$$

This in turn implies that the two-loop terms in the anomalous dimensions (1.8) are given by

$$\Delta^{(1)} = 1 + \frac{\lambda}{4\pi^2} - \frac{\lambda^2}{16\pi^4} + \mathcal{O}(\lambda^3), \qquad\qquad \Delta^{(0)} = 1 - \frac{\lambda}{8\pi^2} + \frac{\lambda^2}{64\pi^4} + \mathcal{O}(\lambda^3)\,, \tag{1.18}$$

JHEP01(2022)056

where $\Delta^{(1)}$ is of course the same as in (1.13) while the two-loop term in $\Delta^{(0)}$ is a new non-trivial result.[3] Note that the sign-alternating structure of λ-expansion in (1.17) and (1.18) is consistent with expectation that the planar weak coupling expansion should have a finite radius of convergence $\left|\frac{\lambda}{4\pi^2}\right| = 1$ with the strong-coupling $\lambda \gg 1$ asymptotics [9, 11, 14] $\Delta^{(1)} = 2 - \frac{5}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right)$, $\Delta^{(0)} = \frac{5}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right)$.[4]

It is important to stress that the two-loop coefficients b_2, b_3 in (1.9), (1.17), are scheme independent: they are invariant under redefinitions of ζ that do not change the positions of the fixed points

$$\zeta' = \zeta + \zeta\,(1-\zeta^2)\left[\lambda\, z_1 + \lambda^2\,(z_2 + z_3\,\zeta^2) + \cdots\right]. \tag{1.19}$$

Since the β_ζ transforms as a vector, we find that the coefficients in (1.9) change as

$$b_i' = b_i + \delta b_i\,, \qquad \delta b_1 = \delta b_2 = \delta b_3 = \delta b_4 = 0, \qquad \delta b_5 = -\delta b_6 = b_1\,(3z_1^2 + 2z_3) - 2b_3 z_1. \tag{1.20}$$

This means that the beta-function is scheme independent at two loops, while the invariant combinations of the three-loop coefficients are b_4 and $b_5 + b_6$. This is consistent with the fact that the dimensions $\Delta^{(0)} = 1 + b_1\lambda + b_2\lambda^2 + b_4\lambda^3 + \ldots$ and $\Delta^{(1)}$ in (1.13) (and thus d_n in (1.14), (1.15)) should be scheme independent.

Let us now comment on the implications of the above discussion for the structure of higher order terms the expectation value of $\langle W^{(\zeta)}\rangle$ in (1.2) on a circle (see [11, 12]). Since the first derivative of $\mathrm{W} = \langle W^{(\zeta)}\rangle$ at the conformal points $\zeta = 0, 1$ should vanish (cf. (1.5)), we should have

$$\mathrm{W} = \langle W^{(1)}\rangle\left[1 + w_1\,\lambda^2\,(1-\zeta^2)^2 + \lambda^3\,(1-\zeta^2)^2(w_2 + w_3\,\zeta^2) + \cdots\right], \tag{1.21}$$

where [31–33]

$$\langle W^{(1)}\rangle = \frac{2}{\sqrt{\lambda}}\,I_1\left(\sqrt{\lambda}\right) = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \mathcal{O}(\lambda^3)\,. \tag{1.22}$$

The coefficients w_1 and w_2 are scheme-independent, while w_3 is finite after renormalization of ζ and in general contains $\log\mu$ dependence on the renormalization scale [12]. Using the general relation (1.5) to the beta-function (1.9) where on general grounds we should have

$$\mathcal{C} = \lambda\, c_1 + \lambda^2(c_2 + c_3\zeta^2) + \mathcal{O}(\lambda^3)\,, \tag{1.23}$$

we find

$$w_1 = -\frac{1}{4}b_1 c_1, \qquad w_2 = -\frac{1}{12}[b_1\,(3c_2 + c_3) + (3b_2 + b_3)c_1], \qquad w_3 = -\frac{1}{6}(b_1 c_3 + b_3 c_1)\,. \tag{1.24}$$

[3] It would be interesting to reproduce it by a direct diagrammatic approach similar to the one in [28].

[4] Ref. [29] found also several higher-order terms in the strong-coupling expansion, correcting the leading terms [14] in $\Delta^{(1)} = 2 - \frac{5}{\sqrt{\lambda}} + \ldots$. These corrections were also obtained analytically from the bootstrap approach in [30].

As b_1, b_2, b_3, w_2 are scheme independent while w_3 is not this implies that c_3 in (1.23) is scheme dependent but $3c_2 + c_3$ is scheme independent.[5] Here [11]

$$w_1 = \frac{1}{128\pi^2}\,, \qquad c_1 = \frac{1}{4}\,. \tag{1.25}$$

w_3 is determined by the contribution of scalar ladder diagrams (which is not UV finite at order λ^3). Its renormalized value was found (using a particular regularization scheme) in [12] to be $w_3 = -\frac{1}{96\,(4\pi^2)^2}\big[5+6\log(\mu R)\big]$. Here R is the radius of the circle and the coefficient 6 of $\log\mu$ in the bracket is related to the coefficient in the one-loop beta-function (1.3) (cf. (1.2)) while 5 is scheme-dependent. In view of (1.24) this then also fixes the value of c_3 in the same scheme. The value of w_2 remains currently unknown. If we choose a scheme in which the value of w_3 is equal to $-w_2$ then the λ^3 term in (1.21) will be proportional to $(1-\zeta^2)^3$. One can then conjecture that there exists a scheme in which similar simplification happens to all orders in λ [11], i.e. one gets

$$\langle W^{(\zeta)}\rangle = \langle W^{(1)}\rangle \Big[1 + \mathcal{F}(\lambda\,(1-\zeta^2))\Big], \tag{1.26}$$

where the function $\mathcal{F}(x)$ has a regular power series expansion $\mathcal{F}(x) = w_1\,x + w_2\,x^2 + \cdots$.

The rest of this paper is organized as follows. In section 2 we shall explain how to compute the coefficients in the beta-function (1.10) in ladder approximation using dimensional regularization. We shall mention that defining the effective ladder theory in $d = 4-\epsilon$ dimensions the corresponding beta-function has a Wilson-Fisher-like zero and the value of the Wilson loop at the corresponding IR fixed point is consistent with the 1d defect version of F-theorem [11, 19].

In section 3 we shall consider the computation of two-point functions of scalars inserted on the Wilson loop at two loops in ladder approximation. We shall consider separately the cases of a "transverse" scalar not coupled to the loop and the scalar coupled to the loop. Using the Callan-Symanzik equation we shall find the corresponding anomalous dimensions of the scalar operators relating them to the beta-function β_ζ and its derivative.

There are also a few technical appendices. In particular, in appendix D we shall present some details of the computation of the linear in ζ term in β_ζ in (1.3) which were not spelled out in [10].

2 Scalar ladder contributions to the beta-function

In this section we shall present the derivation of the coefficients of the $\lambda^n\zeta^{2n+1}$ terms in the beta-function (1.10).

[5] This of course follows also directly from the transformation law $\mathcal{C}'(\zeta') = \left(\frac{d\zeta'}{dz}\right)^{-2} \mathcal{C}(\zeta)$ with ζ' in (1.19): $c_2' = c_2 - 2c_1 z_1$ and $c_3' = c_3 + 6c_1 z_1$ so that $3c_2' + c_3' = 3c_2 + c_3$. In general, for a set of couplings ζ^i we have $\beta^i = \mu\, d\zeta^i/d\mu$ transforming as a vector under their redefinitions and the relation $\frac{\partial}{\partial\zeta^i}\log \mathrm{W} = \mathcal{C}_{ij}\beta^j$ means that $\mathcal{C}_{ij}$ transforms as a tensor. At a conformal point $\beta^i(\zeta^*) = 0$, one has $\frac{\partial^2}{\partial\zeta^i\partial\zeta^j}\log \mathrm{W}\big|_{\zeta=\zeta^*} = \mathcal{C}_{ij}\,\frac{\partial\beta^j}{\partial\zeta^i}\big|_{\zeta=\zeta^*}$. Here $\frac{\partial\beta^j}{\partial\zeta^i}\big|_{\zeta=\zeta^*}$ does not transform under $\zeta' = \zeta + X(\zeta)$ that does not change the position of the conformal point, i.e. if $X(\zeta^*) = 0$.

2.1 Vertex renormalization method

To compute the highest in ζ terms in each order of the small λ expansion in the beta-function we need to consider ladder graphs with only scalar propagators attached to Wilson loop. The perturbative expansion of the corresponding $\langle W^{(\zeta)}\rangle^{\rm ladder}$ is then given in the planar limit by a power series in a single effective coupling ξ

$$\langle W^{(\zeta)}\rangle^{\rm ladder} = W(\xi), \qquad\qquad \xi \equiv \lambda\,\zeta^2\,, \tag{2.1}$$

where (for a closed contour parameterized by $\tau \in (0, 2\pi)$)

$$W = \mathcal{W}(2\pi), \qquad\qquad \mathcal{W}(\tau) = \lim_{N\to\infty}\frac{1}{N}\Big\langle {\rm Tr\,P}\,\exp\int_0^\tau d\tau'\,\phi(\tau')\Big\rangle\,. \tag{2.2}$$

Compared to (1.1) we have redefined the scalar $\phi \to \zeta^{-1}\phi$, and set $\phi(\tau) \equiv \phi(x(\tau))$. The averaging is done in the free adjoint scalar theory with the action

$$S = \frac{N}{\xi}\int d^4x\ {\rm Tr}(\partial\phi\partial\phi)\,. \tag{2.3}$$

In what follows we shall consider the case of a circular or straight line contour when the (unregularized) propagator $D(\tau-\tau') = \langle\phi(\tau)\,\phi(\tau')\rangle$ has the following form[6]

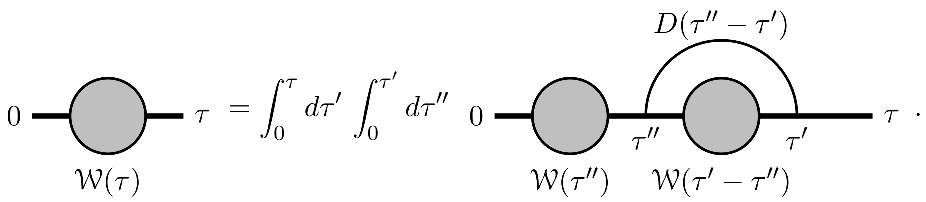
$$\text{circle:}\quad D(\tau) = \frac{\xi}{8\pi^2}\frac{1}{4\,\sin^2\frac{\tau}{2}}, \qquad\qquad \text{line:}\quad D(\tau) = \frac{\xi}{8\pi^2}\frac{1}{\tau^2}. \tag{2.4}$$

The problem of computing (2.2) with (2.4) is well defined and we expect W to admit a renormalizable perturbative expansion in ξ such that all UV divergences can be absorbed into a redefinition of ξ (any possible multiplicative renormalization of the loop operator should be absent in dimensional regularization).

In the planar limit the function $\mathcal{W}(\tau)$ in (2.2) obeys the following integral equation[7]

$$\frac{\partial\mathcal{W}(\tau)}{\partial\tau} = \int_0^\tau d\tau'\,\mathcal{W}(\tau')\,\mathcal{W}(\tau-\tau')\,D(\tau-\tau'). \tag{2.5}$$

Note that by definition in (2.2) we have $\mathcal{W}(0) = 1$ and in (2.5) one has also $\mathcal{W}'(0) = 0$. Eq. (2.5) follows upon differentiation of the integral equation that is implied by the structure of the planar expansion of (2.2) in the free scalar theory (here thick line denotes the contour parameterised by τ)



(2.6)

[6]The original $\mathcal{N}=4$ SYM action is schematically $S = \frac{1}{g^2_{\rm YM}}\int d^4x\ {\rm Tr}(F^2 + D\phi D\phi + \phi^4 + \dots)$, and $\lambda = g^2_{\rm YM}N$. Eq. (2.4) takes into account a factor 1/2 from $T^aT^a = \frac{N}{2}\mathbf{1}$, valid for the generators T^a of SU(N) in the fundamental representation.

[7]K. Zarembo, private communication.

JHEP01(2022)056

Eq. (2.5) is valid for any propagator D. In particular, it can be applied to the case of the ladder contributions to the expectation value of WML (i.e. $\zeta = 1$ case of (1.1)) for a circle when the effective propagator is constant [31]

$$\Big\langle [iA^a(\tau) + \phi^a(\tau)]\,[iA^b(\tau') + \phi^b(\tau')]\Big\rangle = \delta^{ab}\,\frac{\lambda}{8\pi^2 N}, \qquad\qquad D(\tau) = D_0 = \frac{\lambda}{16\,\pi^2}, \tag{2.7}$$

where we used that $T^aT^a = \frac{N}{2}\mathbf{1}$. Taking the Laplace transform of (2.5) we have $-1 + s\,\widetilde{\mathcal{W}}(s) = D_0\,\widetilde{\mathcal{W}}^2(s)$ and thus finally

$$\widetilde{\mathcal{W}}(s) = \frac{s}{2D_0}\left(1 - \sqrt{1 - \frac{4D_0}{s^2}}\right) \quad \to \quad \mathcal{W}(\tau) = \frac{1}{\tau\sqrt{D_0}}\,I_1\left(2\,\tau\,\sqrt{D_0}\right), \tag{2.8}$$

reproducing (1.22) after setting $\tau = 2\pi$ as in (2.2). Similar Dyson integral equations also appear for the correlation function of two Wilson loops [34].

For a non-constant propagator $D(\tau)$, the solution of the loop equation (2.5) appears to be highly non-trivial, in particular, due to divergences starting at three loops. In appendix B we demonstrate how to use (2.5) to reproduce the two-loop result for (1.21), i.e. find the value of w_1 in (1.25).

Following [4] and adapting their discussion to the present case, to compute the renormalization of ζ it is useful to consider the "one-point" scalar correlator on a straight Wilson line segment. Before performing the rescaling of ζ into ϕ introduced above, one starts with the quantity

$$\frac{\langle \mathrm{Tr}\left(\phi(\tau_0)\,\mathrm{P}\exp\left[\int_{\tau_1}^{\tau_2} d\tau\,\zeta\phi(\tau)\right]\right)\rangle}{\langle \mathrm{Tr}\left(\mathrm{P}\exp\left[\int_{\tau_1}^{\tau_2} d\tau\,\zeta\phi(\tau)\right]\right)\rangle}\,. \tag{2.9}$$

Note that this is not exactly the standard one-point function of the operator ϕ on the Wilson line, because we are integrating only over a segment $\tau_1 < \tau < \tau_2$ (with far-separated points τ_1, τ_2) and τ_0 is also assumed to be far from that interval (hence $\phi(\tau_0)$ does not participate in the path-ordering). Following [4], all UV divergences should come from "internal" coinciding points not involving τ_0, τ_1, τ_2, so it is sufficient to look at this object in order to find the renormalization of ζ (or ξ). The normalization in (2.9) by the expectation value of the Wilson line segment without insertion is needed in order to remove some spurious divergences associated with the finite endpoints [4]. After performing the rescaling $\phi \to \zeta^{-1}\phi$, eq. (2.9) becomes

$$\frac{\langle \mathrm{Tr}\left(\phi(\tau_0)\,\mathrm{P}\exp\left[\int_{\tau_1}^{\tau_2} d\tau\,\phi(\tau)\right]\right)\rangle}{\zeta\,\langle \mathrm{Tr}\left(\mathrm{P}\exp\left[\int_{\tau_1}^{\tau_2} d\tau\,\phi(\tau)\right]\right)\rangle}. \tag{2.10}$$

In the planar limit, the numerator in this expression satisfies a relation analogous to (2.6) and can be written as

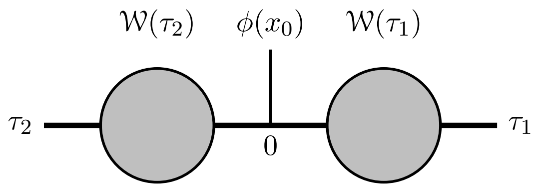
$$\lim_{N\to\infty}\frac{1}{N}\langle\,\mathrm{Tr}\left[\phi(\tau_0)\,\mathrm{P}\exp\int_{\tau_1}^{\tau_2} d\tau\,\phi(\tau)\right]\rangle = \int_{\tau_1}^{\tau_2} d\tau\,\mathcal{W}(\tau-\tau_1)\,D(\tau_0-\tau)\,\mathcal{W}(\tau_2-\tau). \tag{2.11}$$

On the other hand, the denominator factor in (2.10) is simply equal to $\zeta \mathcal{W}(\tau_2 - \tau_1)$. When performing the renormalization of (2.10), we may effectively factor out the τ integration on the right-hand-side of (2.11), since the integral over τ cannot bring in new UV divergences as the point τ_0 is supposed to be far away from τ_1, τ_2. Therefore, one can see that in the planar limit the calculation reduces to study the renormalization of the "vertex function"

$$V = \xi\, \mathcal{W}(\tau_1)\, \mathcal{W}(\tau_2)\,, \tag{2.12}$$

where τ_1, τ_2 are arbitrary (far separated) fixed points on the line. Note that at planar level the normalization by the denominator factor in (2.10) does not play an important role since, being equal to $\zeta \mathcal{W}(\tau_2 - \tau_1)$, it is essentially the same as the "square root" of the V function defined above (recall that $\xi = \lambda \zeta^2$), and hence it is finite once V is made finite by renormalization (or vice-versa). At the non-planar level, however, the denominator factor is expected to play a non-trivial role.

To summarize, in the planar limit it will be sufficient to consider the renormalization of V as given in (2.12). The relevant collection of diagrams may be represented symbolically as



. (2.13)

Here we have chosen the "middle" point τ in (2.11) as 0 (using translational invariance) and replaced the argument τ_0 of ϕ by a generic point x_0 that may or may not lie on the line. The vertical line represents the propagator $D(\tau_0 - \tau)$ or $D(x_0 - x(0))$ which will play only a spectator role.

The strategy is to find the divergent part of V in (2.12) and then absorb the divergences into the renormalization of ξ. This requires computing $\mathcal{W}$ from the corresponding sum of planar ladder diagrams (or using the loop equation (2.5)).

Let us note that the reason why this "scalar ladder" model is effectively an interacting one (despite the bulk theory being free) is due to the path ordering in (2.2). As was mentioned in the Introduction, one can cast the problem of renormalization of ζ or ξ in a more standard form by representing the path ordering using a functional integral over 1d fermions $\chi^i(\tau)$ in the fundamental representation. Integrating first over the free adjoint bulk scalar field we then get an effective 1d action of the following schematic form

$$I \sim \int d\tau\, \bar{\chi}_i \dot{\chi}^i + \xi \int d\tau\, d\tau'\, \bar{\chi}_j(\tau) \chi^i(\tau)\, \frac{1}{|\tau - \tau'|^2}\, \bar{\chi}_i(\tau') \chi^j(\tau')\,. \tag{2.14}$$

Introducing a cutoff into the propagator and expanding in ξ one should be able then to compute the corresponding β_ξ in the usual way.

2.2 Five-loop beta-function in ladder approximation

Let us consider the loop equation (2.5) on a line with a simple analytic regularization of the propagator

$$D(\tau - \tau') = \frac{\xi}{8\pi^2} \frac{1}{|\tau - \tau'|^{2-\epsilon}}. \tag{2.15}$$

JHEP01(2022)056

This is essentially the standard dimensional regularization with $d = 4 - \epsilon$, $\epsilon > 0$, where we did not include the usual ϵ-dependent normalization factor (this is equivalent to a redefinition of the renormalization scale). Solving the loop equation perturbatively, i.e. expanding in powers of ξ

$$\mathcal{W}(\tau) = 1 + \xi\, \mathcal{W}_1(\tau) + \xi^2\, \mathcal{W}_2(\tau) + \cdots\,, \tag{2.16}$$

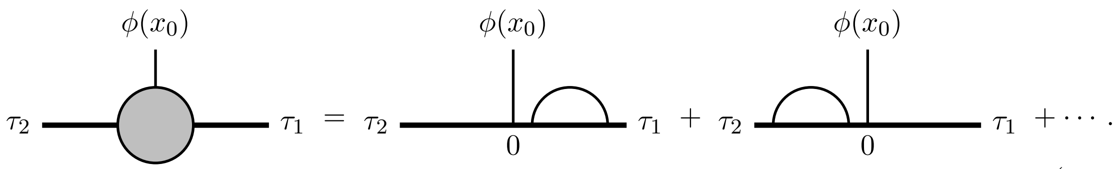
we find at the leading order

$$\frac{\partial}{\partial\tau}\mathcal{W}_1(\tau) = \frac{1}{\xi}\int_0^\tau d\tau'\, D(\tau-\tau') = \int_0^\tau d\tau' \frac{1}{8\pi^2(\tau-\tau')^{2-\epsilon}} = -\frac{\tau^{-1+\epsilon}}{8\pi^2(1-\epsilon)}. \tag{2.17}$$

Integrating this with the boundary condition $\mathcal{W}_1(0) = 0$ gives

$$\mathcal{W}_1(\tau) = -\frac{\tau^\epsilon}{8\pi^2(1-\epsilon)\epsilon}\,. \tag{2.18}$$

The corresponding one-loop correction to the vertex (2.13) is represented by $1 \times \mathcal{W}_1(\tau_1) + \mathcal{W}_1(\tau_2) \times 1$, i.e. by the following diagrams



(2.19)

Next, $\mathcal{W}_2(\tau)$ is given by

(2.20)

with the explicit expression being

$$\mathcal{W}_2(\tau) = \frac{\tau^{2\epsilon}\left[2\epsilon(-1+2\epsilon)\Gamma(-1+\epsilon)\Gamma(1+\epsilon) + \Gamma(1+2\epsilon)\right]}{128\pi^4(-1+\epsilon)\epsilon^2(-1+2\epsilon)\Gamma(1+2\epsilon)}. \tag{2.21}$$

All higher order functions $\mathcal{W}_n(\tau)$ may be expressed as

$$\mathcal{W}_n(\tau) = \frac{K_n(\epsilon)}{\Gamma(1+n\epsilon)}\,\tau^{n\epsilon}, \tag{2.22}$$

where the coefficients $K_n(\epsilon)$ are determined by the recurrence relation

$$K_n(\epsilon) = \frac{1}{8\pi^2}\sum_{p=0}^{n-1} K_p(\epsilon)\, K_{n-1-p}(\epsilon)\, \frac{\Gamma(-1+(p+1)\epsilon)}{\Gamma(1+p\epsilon)}, \qquad K_0(\epsilon) = 1. \tag{2.23}$$

This follows from the Laplace transform $\mathcal{L}$ of the loop equation on the line that reads[8]

$$s\,\widetilde{\mathcal{W}}(s) - 1 = \frac{\xi}{8\pi^2}\,\widetilde{\mathcal{W}}(s)\,\mathcal{L}\left[\frac{\mathcal{W}(\tau)}{\tau^{2-\epsilon}}\right] = \frac{\xi}{8\pi^2}\,\widetilde{\mathcal{W}}(s)\,(-\partial_s)^{\epsilon-2}\widetilde{\mathcal{W}}(s). \tag{2.24}$$

[8]The perturbative solution takes the form $\widetilde{\mathcal{W}}_n(s) = \frac{K_n(\epsilon)}{s^{1+n\epsilon}}$, cf. (2.22). Replacing this in the loop equation and using $(-\partial_s)^\alpha \frac{1}{s^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\frac{1}{s^{\alpha+\beta}}$ gives (2.23).

JHEP01(2022)056

In the following we shall present the explicit results to five loop order, i.e. including $\mathcal{W}_5(\tau)$ in (2.16).

We can then find the divergences in (2.12), i.e. coefficients of poles in $\epsilon \to 0$ in

$$V = \xi\, \mathcal{W}(\tau_1)\, \mathcal{W}(\tau_2) = \xi + V_2\, \xi^2 + V_3\, \xi^3 + \cdots \tag{2.25}$$

and cancel them by replacing the bare coupling ξ in terms of the renormalized one using the familiar general relation[9]

$$\begin{aligned}\xi = \mu^\epsilon \Bigg[\xi(\mu) + \frac{p_{11}}{\epsilon}\, [\xi(\mu)]^2 + \left(\frac{p_{21}}{\epsilon} + \frac{p_{22}}{\epsilon^2}\right) [\xi(\mu)]^3 + \left(\frac{p_{31}}{\epsilon} + \frac{p_{32}}{\epsilon^2} + \frac{p_{33}}{\epsilon^3}\right) [\xi(\mu)]^4 \\ + \left(\frac{p_{41}}{\epsilon} + \frac{p_{42}}{\epsilon^2} + \frac{p_{43}}{\epsilon^3} + \frac{p_{44}}{\epsilon^4}\right) [\xi(\mu)]^5 + \cdots \Bigg].\end{aligned} \tag{2.26}$$

The condition that the bare coupling does not depend on μ implies various relations between the coefficients in (2.26) and the resulting beta-function expressed in terms of the renormalized coupling is given by (see (A.15), (A.16))

$$\beta_\xi = \mu \frac{d}{d\mu}\xi = p_{11}\xi^2 + 2\, p_{21}\xi^3 + 3\, p_{31}\xi^4 + 4\, p_{41}\xi^5 + \cdots . \tag{2.27}$$

Requiring that the vertex (2.25) expressed in terms of $\xi(\mu)$ is finite gives

$$p_{11} = \frac{1}{4\pi^2}, \qquad p_{21} = -\frac{1}{64\pi^4}, \qquad p_{31} = \frac{12-\pi^2}{4608\pi^6}, \qquad p_{41} = \frac{-51+8\pi^2-12\zeta_3}{73728\pi^8}, \qquad \ldots \tag{2.28}$$

and thus

$$\begin{aligned}\beta_\xi = \xi \Bigg[\frac{\xi}{4\pi^2} - \frac{1}{2}\left(\frac{\xi}{4\pi^2}\right)^2 + \left(\frac{1}{2} - \frac{\zeta_2}{4}\right)\left(\frac{\xi}{4\pi^2}\right)^3 + \left(-\frac{17}{24} + \frac{2\zeta_2}{3} - \frac{\zeta_3}{6}\right)\left(\frac{\xi}{4\pi^2}\right)^4 \\ + \left(\frac{29}{24} - \frac{37\zeta_2}{24} + \frac{29\zeta_3}{48} + \frac{25\zeta_4}{64}\right)\left(\frac{\xi}{4\pi^2}\right)^5 + \cdots \Bigg],\end{aligned} \tag{2.29}$$

where we added also the five-loop contribution and ζ_n are the zeta-function values. Recalling that according to (2.1) $\xi = \lambda\zeta^2$ where λ is not running we may then read off the ladder contribution to the beta-function of ζ

$$\beta_\xi = \mu \frac{d}{d\mu}(\lambda\zeta^2) = 2\,\lambda\,\zeta\, \beta_\zeta^{\rm ladder}, \tag{2.30}$$

reproducing (1.10). Explicitly, at the two-loop order

$$\beta_\zeta^{\rm ladder} = \frac{\lambda}{8\pi^2}\zeta^3 - \frac{\lambda^2}{64\pi^4}\zeta^5 + \cdots . \tag{2.31}$$

Note that higher loop terms in $\beta_\zeta^{\rm ladder}$ obtained in this way have similar transcendental structure to higher loop terms in $\Delta^{(1)}$ in (1.12), (1.13) found in [29].

[9]Note that the factor μ^ϵ is required to match the dimensions: defining the theory in $d = 4-\epsilon$ dimensions we keep the dimension of the scalar field to be 1 (as it is coupled to the line which still has dimension 1) so that the bulk action $\frac{N}{\xi}\int d^{4-\epsilon}x\, \mathrm{Tr}(\partial\phi\partial\phi)$ should have ξ with the mass dimension ϵ. Equivalently, this follows from the fact that the propagator (2.15) should still have dimension 2.

2.3 Comment on Wilson-Fisher fixed point

Let us note that considering the theory in $d = 4 - \epsilon$ dimensions and thus keeping the order ϵ term in the beta-function (2.29), we get from (2.26)

$$\beta_\xi = -\epsilon\, \xi + \frac{\xi^2}{4\pi^2} - \frac{\xi^3}{32\pi^4} + \cdots . \tag{2.32}$$

This implies that the effective 1d theory corresponding to the scalar ladder approximation has, in addition to the trivial UV fixed point $\xi = 0$, a non-trivial IR fixed point

$$\xi^* = 4\pi^2\epsilon + 2\pi^2\epsilon^2 + \cdots . \tag{2.33}$$

The expectation value of $\langle W^{(\zeta)}\rangle$ for a circle found in $d = 4 - \epsilon$ in ladder approximation may be written as [11][10]

$$\langle W^{(\zeta)}\rangle^{\rm ladder} \equiv W(\xi) = 1 - \frac{\epsilon}{16}\,\xi + \frac{1}{128\pi^2}\,\xi^2 + \cdots . \tag{2.34}$$

Evaluated at the fixed point (2.33) this gives

$$W(\xi^*) = 1 - \frac{\pi^2}{8}\,\epsilon^2 + \cdots . \tag{2.35}$$

Thus $\log W(\xi^*) < \log W(0) = 0$ in agreement with the 1d defect version of F-theorem. On the other hand, note that directly in $d = 4$, the beta-function in the ladder approximation is positive in perturbation theory, which means that the coupling ξ grows in the UV. From the point of view of the F-theorem, this is consistent with the fact that $\log W(\xi) = \frac{1}{128\pi^2}\,\xi^2 + \cdots$ is positive in perturbation theory.

3 Two-point function of scalars on the Wilson line

3.1 Two-point function for "transverse" scalar

Using the scalar ladder approximation let us compute the defect two-point function for one of the "transverse" scalars $\phi_1, \dots, \phi_5$ (to be denoted by $\widetilde{\phi}$) which is not coupled to the scalar Wilson line operator defined in (2.2)

$$\widetilde{G}(\tau) = \frac{\langle \mathrm{Tr}\,[\mathrm{P}\,\widetilde{\phi}(0)\,\widetilde{\phi}(\tau)\,\exp\int_{-L}^{L} d\tau'\,\phi(\tau')]\rangle}{\langle \mathrm{Tr}\,[\mathrm{P}\,\exp\int_{-L}^{L} d\tau'\,\phi(\tau')]\rangle} . \tag{3.1}$$

Here we shall consider an infinite straight line with L as an IR cutoff. We shall treat $\widetilde{\phi}$ and ϕ on equal footing rescaling both by ζ, i.e. the averaging is done with the free scalar action (2.3) where now $S = \frac{N}{\xi}\int d^4x\,\mathrm{Tr}(\partial\phi\partial\phi + \partial\widetilde{\phi}\partial\widetilde{\phi})$. Then ξ will appear in the propagators of both ϕ and $\widetilde{\phi}$ as in (2.4). Alternative equivalent option is to rescale both fields in (3.1) by $\sqrt{\xi}$ getting the factor $\sqrt{\xi}$ in the exponents in (3.1), ξ-independent propagator and the overall factor of ξ in (3.1). It is then natural to remove it by rescaling

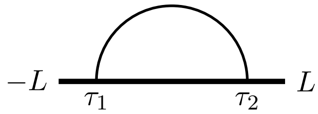
$$\widetilde{G} \;\to\; \xi^{-1}\widetilde{G} . \tag{3.2}$$

[10] Note that in [11] we used $d = 4 - 2\epsilon$ while here $d = 4 - \epsilon$.

The renormalization of $\widetilde{G}$ and $\xi^{-1}\widetilde{G}$ will differ just by a Z-factor corresponding to ξ, i.e. they will satisfy closely related Callan-Symanzik equations (see below).

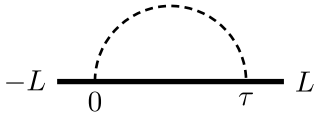
At the tree and one loop level we will then have from (3.1)

$$\langle \mathrm{Tr}\,[\mathrm{P}\,\exp \int_{-L}^{L} d\tau'\,\phi(\tau')]\rangle = 1 + \quad (3.3)$$

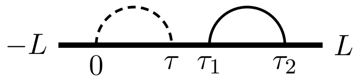


$+\cdots,$

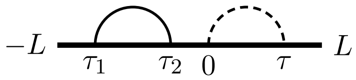
$$\langle \mathrm{Tr}\,[\mathrm{P}\,\widetilde{\phi}(0)\,\widetilde{\phi}(\tau)\,\exp \int_{-L}^{L} d\tau'\,\phi(\tau')]\rangle =$$



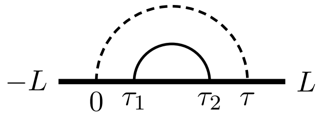
$+$



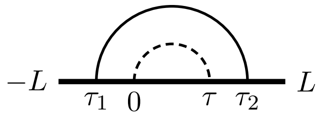
$+$



$+$



$+$



$+\cdots. \quad (3.4)$

Using the propagator (2.15) and explicit expressions in (C.1)–(C.6), we get

$$\widetilde{G}(\tau) = \frac{\tau^{-2+\epsilon}\xi}{8\pi^2} + \frac{\tau^{-2+\epsilon}((L-\tau)^\epsilon + 2\tau^\epsilon - (L+\tau)^\epsilon)\xi^2}{64\pi^4(-1+\epsilon)\epsilon} + \mathcal{O}(\xi^3). \quad (3.5)$$

Here ξ is the bare coupling. Applying the renormalization, i.e. the redefinition of ξ as in (2.26)–(2.28), we find a finite result

$$\widetilde{G}^{\mathrm{ren}}(\tau;\mu) = \frac{\xi}{8\pi^2}\frac{1}{\tau^2}\left[1 - \frac{\xi}{4\pi^2}\left(1 + \frac{1}{2}\log\frac{L-\tau}{L+\tau} + \log(\mu\tau)\right) + \mathcal{O}(\xi^2)\right], \quad (3.6)$$

where ξ is now the renormalized coupling $\xi(\mu)$ and the limit $L \to \infty$ is straightforward. Similarly, at two loops, from the results in (C.7)–(C.23) and using again the redefinition (2.26)–(2.28), we find a finite expression which in the limit $L \to \infty$ gives

$$\widetilde{G}^{\mathrm{ren}}(\tau;\mu) = \frac{\xi}{8\pi^2}\frac{1}{\tau^2}\left[1 - \frac{\xi}{4\pi^2}(1+\log(\mu\tau)) + \left(\frac{\xi}{4\pi^2}\right)^2\left(2 + \frac{\pi^2}{24} + \frac{5}{2}\log(\mu\tau) + \log^2(\mu\tau)\right) + \mathcal{O}(\xi^3)\right]. \quad (3.7)$$

Note that since $\widetilde{G}^{\mathrm{ren}}(\tau)$ requires only the renormalization of ξ for its finiteness (i.e. no extra Z-factor) it satisfies

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\xi\frac{\partial}{\partial\xi}\right)\widetilde{G}^{\mathrm{ren}}(\tau;\mu) = 0\,. \quad (3.8)$$

Since for generic ξ its beta-function is non-zero, i.e. the conformal invariance is broken in the ladder theory in $d = 4$, this correlator cannot be put into the standard conformal form.

A way to achieve conformal invariance is to consider the scalar ladder theory in $d = 4-\epsilon$ and specify to the Wilson-Fisher fixed point (2.33). Considering one loop order we have to keep the full ϵ dependence of the tree and one-loop terms. Then instead of (3.7) we get

$$\widetilde{G}^{\mathrm{ren}}(\tau) = \frac{\mu^\epsilon}{2\,\tau^{2-\epsilon}}\frac{\xi}{4\pi^2} + \frac{\mu^\epsilon(2 - 2\epsilon - \mu^\epsilon(L-\tau)^\epsilon - 2\mu^\epsilon\tau^\epsilon + \mu^\epsilon(L+\tau)^\epsilon)}{4\tau^{2-\epsilon}(1-\epsilon)\epsilon}\left(\frac{\xi}{4\pi^2}\right)^2 + \frac{1}{\tau^2}\left(1 + \frac{\pi^2}{48} + \frac{5}{4}\log(\mu\tau) + \frac{1}{2}\log^2(\mu\tau)\right)\left(\frac{\xi}{4\pi^2}\right)^3 + \cdots. \quad (3.9)$$

Evaluating this at $\xi = \xi^*$ in (2.33), expanding in ϵ, and taking the infinite line limit $L \to \infty$ gives simply

$$\widetilde{G}^{\rm ren}(\tau)\Big|_{\xi=\xi^*} = \frac{1}{2\tau^2}\left(\epsilon - \frac{1}{2}\,\epsilon + \frac{\pi^2}{24}\,\epsilon^3 + \cdots\right). \tag{3.10}$$

The fact that all $\log\tau$ terms cancel implies that the scaling with τ is non-anomalous at the conformal fixed point (i.e. the corresponding dimension is $\widetilde{\Delta}^* = 1$), at least to two loops.

This is analogous with what happens in full $\mathcal{N} = 4$ SYM theory case, where the insertion of the "transverse" scalars into the WML is BPS-protected. Here in the ladder model we do not have an argument based on supersymmetry, but we can explain why $\widetilde{\Delta}^* = 1$ as follows. If one starts with the WL coupled to two scalars as

$$\mathrm{Tr}\,\mathrm{P}\exp\int d\tau\,(\zeta_1\phi + \zeta_2\widetilde{\phi})\,, \tag{3.11}$$

then setting $\zeta_1 = \zeta\cos\alpha$, $\zeta_2 = \zeta\sin\alpha$ and redefining the scalars (using SO(6) symmetry of the bulk scalar action) we get back to the WL coupled to a single scalar with the coupling ζ. Thus the beta-function cannot be a function of α, but only of ζ, i.e. the angle α should be a parameter of an exactly marginal deformation. For infinitesimal α the integrand in the exponent in (3.11) is $\zeta\phi + \zeta\alpha\,\widetilde{\phi}$ so that the insertion of $\widetilde{\phi}$ into the WL coupled to ϕ should be exactly marginal with $\Delta = 1$ at a fixed point.

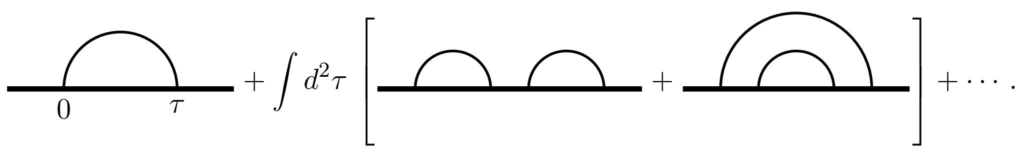
3.2 Two-point function for coupled scalar

The same calculation for the scalar coupled to the loop, i.e. for

$$G(\tau) = \frac{\langle \mathrm{Tr}\left[\mathrm{P}\,\phi(0)\,\phi(\tau)\,\exp\int_{-L}^{L}d\tau'\,\phi(\tau')\right]\rangle}{\langle \mathrm{Tr}\left[\mathrm{P}\,\exp\int_{-L}^{L}d\tau'\,\phi(\tau')\right]\rangle}, \tag{3.12}$$

requires us to consider all possible contractions, including those involving the fields at 0 and τ. At tree and one-loop orders we then get the following contributions

$$\langle \mathrm{Tr}\left[\mathrm{P}\,\phi(0)\,\phi(\tau)\,\exp\int_{-L}^{L}d\tau'\,\phi(\tau')\right]\rangle =$$



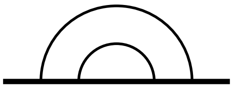
(3.13)

Each structure splits into 6 diagrams depending on the order of $\tau' < \tau''$ with respect to $0 < \tau$. The result is

$$\int d^2\tau$$ 

$$= \frac{\xi^2(L(L-\tau))^{-1+\epsilon}}{64\pi^4(-1+\epsilon)^2} + \frac{L^\epsilon\xi^2\tau^{-2+\epsilon}}{64\pi^4(-1+\epsilon)\epsilon} + \frac{\xi^2(L-\tau)^\epsilon\tau^{-2+\epsilon}}{64\pi^4(-1+\epsilon)\epsilon}$$
$$+ \frac{\xi^2(L\tau)^{-1+\epsilon}}{64\pi^4(-1+\epsilon)^2} + \frac{\xi^2((L-\tau)\tau)^{-1+\epsilon}}{64\pi^4(-1+\epsilon)^2} + \frac{\xi^2\tau^{-2+2\epsilon}\Gamma(-1+\epsilon)\Gamma(\epsilon)}{64\pi^4(-1+\epsilon)\Gamma(-1+2\epsilon)},$$

JHEP01(2022)056

$$\int d^2\tau$$  $$= \frac{\xi^2\tau^{-2+2\epsilon}}{64\pi^4(-1+\epsilon)\epsilon} + \frac{\xi^2\tau^{-2+\epsilon}(L^\epsilon\tau - L\tau^\epsilon)}{64L\pi^4(-1+\epsilon)^2}$$
$$+ \frac{\xi^2\tau^{-2+\epsilon}((-1+2^\epsilon)L^\epsilon + \tau^\epsilon - (L+\tau)^\epsilon)}{64\pi^4(-1+\epsilon)\epsilon} + \frac{\xi^2\tau^{-2+\epsilon}(-\tau^\epsilon(L+\tau) + \tau(L+\tau)^\epsilon)}{64\pi^4(-1+\epsilon)^2(L+\tau)}$$
$$- \frac{\xi^2\tau^{\,-2+2\epsilon}B_{\frac{\tau}{L}}(2-2\epsilon,\epsilon)}{64\pi^4(-1+\epsilon)} - \frac{2^{-5-2\epsilon}\xi^2\tau^{-2+2\epsilon}\Gamma\left(\frac{3}{2}-\epsilon\right)\Gamma(-1+\epsilon)}{\pi^{9/2}(-1+\epsilon)}$$
$$+ \frac{\xi^2\left((L(L+\tau))^{-1+\epsilon} - (L\tau)^{-1+\epsilon}\,{}_2F_1\left(1-\epsilon,-1+\epsilon;\epsilon;-\frac{L}{\tau}\right)\right)}{64\pi^4(-1+\epsilon)^2}. \tag{3.14}$$

Taking $L\to\infty$ limit in the B-function,[11] using

$${}_2F_1(1-\epsilon,-1+\epsilon,\epsilon;-x) = x^{1-\epsilon}\frac{\Gamma(2-2\epsilon)\Gamma(\epsilon)}{\Gamma(1-\epsilon)} + \mathcal{O}(x^{-1+\epsilon}), \qquad x\to+\infty, \tag{3.15}$$

and redefining ξ according to (2.26) we find that the renormalization of $G(\tau)$ in (3.12) requires also an additional Z-factor

$$G^{\text{ren}}(\tau;\mu) = \lim_{L\to\infty}\lim_{\epsilon\to 0} Z\,G(\tau), \qquad\qquad Z = 1 + \frac{1}{2\pi^2}\frac{\mu^{-\epsilon}\,\xi}{\epsilon} + \cdots. \tag{3.16}$$

As a result, we find

$$G^{\text{ren}}(\tau;\mu) = \mu^2\frac{\xi}{8\pi^2\,(\mu\tau)^{2-\epsilon}}\left[1 + (-2-3\log(\mu\tau))\frac{\xi}{4\pi^2} + \cdots\right], \tag{3.17}$$

where we restored the full ϵ-dependence of the tree-level term. Replacing now ξ by its Wilson-Fischer fixed point value $\xi^* = 4\pi^2\epsilon + 2\pi^2\epsilon^2 + \cdots$ in (2.33) we find

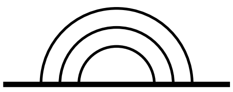
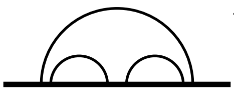
$$G^*(\tau;\mu) = \frac{\epsilon}{2\tau^2}\left[1 + (-2-2\log(\mu\tau))\right]\epsilon + \mathcal{O}(\epsilon^2). \tag{3.18}$$

This has the conformal form with a non-trivial dimension $\Delta^* = 1+\epsilon+\ldots$, in agreement with the expected relation between the anomalous dimension and the derivative of the beta-function in (2.32)

$$\Delta^* = 1 + \beta'_{\xi^*} = 1 + \epsilon - \frac{1}{2}\epsilon^2 + \cdots. \tag{3.19}$$

At the two-loop order we need the following contributions

$$\int d^4\tau \Bigg[$$  $+$  $+$ 

$+$  $+$  $$\Bigg]. \tag{3.20}$$

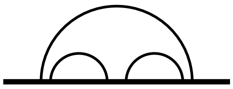
[11]Here B is the incomplete beta function $B_z(\alpha,\beta) = \int_0^z dw\,w^{\alpha-1}(1-w)^{\beta-1}$ with $B_0(\alpha,\beta)=0$.

JHEP01(2022)056

Each diagram correspond to 15 possible orderings of the positions on the line, two fixed at 0 and τ, and four at various possible positions consistent with the planarity constraint. To extract divergences from closed expressions for the diagrams here we use a simple cutoff regularization

$$D(\tau) \to D_a(\tau) = \frac{\xi}{8\pi^2} \frac{1}{(|\tau| + a)^2}\,, \qquad\qquad a \to 0\,. \tag{3.21}$$

Then one finds, for example, that

$$\int d^4\tau$$ 

$$= L\,\frac{(-2a+\tau)\xi^3 \log 2}{256 a \pi^6 \tau^3} - \frac{\xi^3\left(-1 + \log 256 + 16 \log \frac{\tau}{L}\right)}{1024 \pi^6 a^2}$$
$$+ \frac{\xi^3\left(-8 + \log 32 + 20 \log \frac{a}{L} - 12 \log \frac{\tau}{L}\right)}{512 \pi^6 \tau a} \tag{3.22}$$
$$+ \frac{\xi^3}{1024\ \pi^6 \tau^2}\Big[-19 + \pi^2 - 12 \log 2 - 16 \log \frac{a}{L} - 8 \log 2 \log \frac{a}{L}$$
$$+12 \log^2 \frac{a}{L} - 4 \log \frac{\tau}{L} + 2 \log 4 \log \frac{\tau}{L} - 32 \log \frac{a}{L} \log \frac{\tau}{L} + 18 \log^2 \frac{\tau}{L}\Big] + \cdots$$

where we already expanded in large L and small a (before the expansion the expression is unwieldy). Recomputing the one-loop diagrams with this regularization and adding the two-loop ones, one finds that one can define the renormalized correlator as[12]

$$G^{\rm ren}(\tau) = \lim_{L\to\infty} \lim_{a\to 0} Z\, G(\tau)\,. \tag{3.23}$$

The bare coupling $\xi(a)$ appearing in the expansion of G is related to the renormalized coupling $\xi = \xi(\mu)$ at the inverse length scale μ by

$$\xi(a) = \xi - \log(a\,\mu)\,\frac{\xi^2}{4\pi^2} + \left[\frac{1}{2}\log(a\,\mu) + \log^2(a\,\mu)\right]\frac{\xi^3}{(4\pi^2)^2} + \cdots\,, \tag{3.24}$$

corresponding to the beta-function in (2.29). The renormalization factor Z is given by

$$Z = 1 - 2\log(a\,\mu)\,\frac{\xi^2}{4\pi^2} + \left[2\,\log(a\,\mu) + \log^2(a\,\mu)\right]\frac{\xi^2}{(4\pi^2)^2} + \cdots\,. \tag{3.25}$$

The renormalized two-point function then reads

$$G^{\rm ren}(\tau;\mu) = \frac{1}{2\,\tau^2}\frac{\xi}{4\pi^2}\left[1 + (1 - 3\log(\mu\tau))\,\frac{\xi}{4\pi^2} + \left(-2 + \frac{5\pi^2}{24} - \frac{3}{2}\log(\mu\tau) + 6\log^2(\mu\tau)\right)\frac{\xi^2}{(4\pi^2)^2} + \cdots\right]. \tag{3.26}$$

Like (3.7) this expression cannot be put into the conformal form (with all τ dependence appearing only as a power $(\mu\tau)^{2\Delta}$) since the conformal invariance is broken for generic ξ.

The renormalized correlator satisfies the Callan-Symanzik equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\xi \frac{\partial}{\partial\xi} + 2(\Delta - 1)\right]\xi^{-1} G^{\rm ren}(\tau;\mu) = 0\,. \tag{3.27}$$

[12]The order of limits here is important.

We have written (3.27) for the rescaled (3.2) correlator corresponding to the picture where $\sqrt{\xi}$ coupling appears in the exponent of the WL operator and the two-point function is defined for the canonically normalized fields (i.e. without extra coupling factor in the bulk action compared to (2.3)). Thus $\Delta - 1$ in (3.27) is the anomalous dimension of the canonically normalized scalar ϕ.

Using (2.29), i.e. $\beta_\xi = \frac{1}{4\pi^2}\xi^2 - \frac{1}{32\pi^4}\xi^3 + \cdots$, we obtain

$$\Delta = 1 + \frac{3}{8\pi^2}\xi - \frac{5}{64\pi^4}\xi^2 + \cdots . \tag{3.28}$$

This expression is in agreement with the general relation for the anomalous dimension $\Delta - 1 = \frac{d}{dg}\beta_g$ where the coupling here is $g = \sqrt{\xi}$ appearing in front of the scalar ϕ in the exponent, i.e.

$$\beta_g = \mu\frac{\partial}{\partial\mu}g = \frac{1}{2\sqrt{\xi}}\beta_\xi = \frac{1}{8\pi^2}g^3 - \frac{g^5}{64\pi^4} + \cdots . \tag{3.29}$$

The dimension (3.28) in the effective ladder model corresponds to the terms with highest power of ζ in the dimension of the scalar $\phi = \phi_6$ in the full $\mathcal{N} = 4$ SYM theory (cf. (1.17))

$$\Delta^{(\zeta)} = 1 + \beta'(\zeta) = 1 + \frac{\lambda}{8\pi^2}(3\zeta^2 - 1) + \frac{\lambda^2}{64\pi^4}(1 - 5\zeta^4) + \mathcal{O}(\lambda^3). \tag{3.30}$$

Similarly, the Callan-Symanzik equation (3.8) for the two-point function $\widetilde{G}^{\rm ren}$ for the "transverse" scalar $\widetilde{\phi}$ in (3.7) rewritten for the rescaled (3.2) correlator, i.e. in the form (3.27), is

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\xi\frac{\partial}{\partial\xi}\right)\widetilde{G}^{\rm ren}(\tau;\mu) = \left[\mu\frac{\partial}{\partial\mu} + \beta_\xi\frac{\partial}{\partial\xi} + 2(\widetilde{\Delta} - 1)\right]\xi^{-1}\widetilde{G}^{\rm ren}(\tau;\mu) = 0\,. \tag{3.31}$$

It implies that the anomalous dimension $\widetilde{\Delta} - 1$ of the canonically normalized scalar field $\widetilde{\phi}$ found in the ladder approximation is directly proportional to the beta-function β_ξ in (2.29)

$$\widetilde{\Delta} = 1 + \frac{1}{2}\xi^{-1}\beta_\xi = 1 + \frac{\xi}{8\pi^2} - \frac{\xi^2}{64\pi^4} + \mathcal{O}(\xi^3). \tag{3.32}$$

To reconstruct the expression for the corresponding anomalous dimension $\widetilde{\Delta} - 1$ in the full SYM theory (i.e. the analog of (3.30)) which is a function of λ and ζ we may use that (i) it should reduce to (3.32) if one keeps only highest powers of ζ at each order in λ; (ii) it should vanish at $\zeta = \pm 1$; (iii) it should be equal to (3.30) at $\zeta = 0$ as then all 6 scalars are on equal footing (not coupled to the loop). This gives

$$\widetilde{\Delta}^{(\zeta)} = 1 + \frac{\lambda}{8\pi^2}(\zeta^2 - 1) - \frac{\lambda^2}{64\pi^4}(\zeta^4 - 1) + \mathcal{O}(\lambda^3). \tag{3.33}$$

This is simply related to the beta-function β_ζ in (1.17)[13]

$$\widetilde{\Delta}^{(\zeta)} = 1 + \zeta^{-1}\,\beta_\zeta\,. \tag{3.34}$$

Thus while the anomalous dimension of the coupled scalar (3.30) is given by the derivative of the beta-function of ζ, the anomalous dimension of the "transverse" scalar (3.34) is proportional to the beta-function itself.

[13]The relation between the prefactors in (3.32) and in (3.34) follows from $\frac{1}{2}\xi^{-1}\frac{d}{d\mu}\xi = \frac{1}{2}\zeta^{-2}\frac{d}{d\mu}\zeta^2 = \zeta^{-1}\beta_\zeta$.

JHEP01(2022)056

A way to understand why this relation of $\widetilde{\Delta}^{(\zeta)}$ to the beta-function should hold in general let us start again with the loop (3.11) coupled to the two scalars ($\phi = \phi_6$ and a "transverse" $\widetilde{\phi}$) with different coefficients, i.e.

$$\zeta_1\, \phi + \zeta_2\, \widetilde{\phi}\,, \qquad \zeta_1 = \zeta\cos\alpha, \qquad \zeta_2 = \zeta\sin\alpha\,. \tag{3.35}$$

In this case we may formally define two beta-functions $\beta_{\zeta_i} = \mu\frac{d}{d\mu}\zeta_i$ but since α is an exactly marginal parameter not running with μ (see discussion below (3.11)) we should have

$$\beta_{\zeta_1} = \cos\alpha\, \beta_\zeta\,, \qquad \beta_{\zeta_2} = \sin\alpha\, \beta_\zeta\,. \tag{3.36}$$

At the same time, the general relations for the anomalous dimensions of ϕ and $\widetilde{\phi}$ at the point $\zeta_2 = 0$ or $\alpha = 0$ (when $\widetilde{\phi}$ is not coupled to the loop) are given by

$$\Delta - 1 = \frac{\partial}{\partial\zeta_1}\beta_{\zeta_1}\Big|_{\zeta_2=0} = \beta'_\zeta\,, \qquad \widetilde{\Delta} - 1 = \frac{\partial}{\partial\zeta_2}\beta_{\zeta_2}\Big|_{\zeta_2=0} = \zeta^{-1}\beta_\zeta\,, \tag{3.37}$$

where in the last equality we used that $\frac{\partial}{\partial\zeta_2}\beta_{\zeta_2}\Big|_{\zeta_2=0} = \Big(\zeta^{-1}\cos\alpha\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\zeta}\Big)(\sin\alpha\beta_\zeta)\big|_{\alpha=0} = \zeta^{-1}\beta_\zeta$.

The above argument implies, in particular, that the fact that $\widetilde{\Delta} = 1$ at the fixed points when $\beta_\zeta = 0$ and $\zeta \neq 0$ follows essentially from the rotational symmetry in the scalar space. For the fixed point at $\zeta = 0$, where all scalars are not coupled to the loop and the full SO(6) rotational symmetry is restored, this does not apply as here all scalars have then the same dimension $\Delta^{(0)} = 1 + \beta'_\zeta|_{\zeta=0}$.

Acknowledgments

We are grateful to K. Zarembo for very useful discussions. We also thank N. Gromov and J. Henn for helpful communications. MB was supported by the INFN grant GSS (Gauge Theories, Strings and Supergravity). The work of SG is supported in part by the U.S. NSF under Grant No. PHY-1914860. AAT was supported by the STFC grant ST/T000791/1.

A Conventions and useful formulae

For SU(N) generators in the fundamental representation we have

$$[T^a, T^b] = i\, f^{abc}\, T^c, \qquad \mathrm{Tr}\, T^a = 0, \qquad \mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}, \quad (T^a T^a)_{ij} = \frac{N^2-1}{2N}\,\delta_{ij}. \tag{A.1}$$

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\Big(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\Big), \quad f^{acd} f^{bcd} = N\delta^{ab}. \tag{A.2}$$

In computing higher-order corrections one also needs

$$\mathrm{Tr}(T^aT^aT^bT^b) = \frac{1}{2}\,\mathrm{Tr}(1)\,\mathrm{Tr}(T^bT^b) - \frac{1}{2N}\,\mathrm{Tr}(T^bT^b) = \left(\frac{N}{2} - \frac{1}{2N}\right)\frac{N^2-1}{2} = \frac{(N^2-1)^2}{4N}, \tag{A.3}$$

$$\mathrm{Tr}(T^aT^bT^aT^b) = \frac{1}{2}\,\mathrm{Tr}(T^b)\,\mathrm{Tr}(T^b) - \frac{1}{2N}\,\mathrm{Tr}(T^bT^b) = -\frac{1}{2N}\frac{N^2-1}{2} = -\frac{N^2-1}{4N}, \tag{A.4}$$

$$\mathrm{Tr}(T^aT^bT^bT^a) = \mathrm{Tr}(T^aT^aT^bT^b) = \frac{(N^2-1)^2}{4N}. \tag{A.5}$$

Useful formulae for diagram computations are

$$\int_0^1 \prod_{i=1}^N d\alpha_i\, \alpha_i^{\nu_i-1}\, \delta\left(1-\sum_i \alpha_i\right) = \frac{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}{\Gamma(\nu_1+\cdots+\nu_N)}, \tag{A.6}$$

$$\int d^{2\omega}w \frac{1}{(w^2+M^2)^s} = \pi^\omega \frac{\Gamma(s-\omega)}{\Gamma(s)}(M^2)^{\omega-s}, \tag{A.7}$$

$$\begin{aligned}\int_{\tau>\tau_1>\tau_2>\tau_3>0} & d\tau_1\, d\tau_2\, d\tau_3\, (\tau_1-\tau_2)^{c_{12}}(\tau_2-\tau_3)^{c_{23}}(\tau_1-\tau_3)^{c_{13}} = \\ & \tau^{c_{12}+c_{23}+c_{13}+3}\, \frac{\Gamma(c_{12}+1)\Gamma(c_{23}+1)}{(c_{12}+c_{23}+c_{13}+2)(c_{12}+c_{23}+c_{13}+3)\Gamma(c_{12}+c_{23}+2)},\end{aligned} \tag{A.8}$$

$$\begin{aligned}\int_0^1 d\alpha \int_0^{1-\alpha} & d\beta\, (\alpha\beta)^p(\alpha+\beta)^q(1-\alpha-\beta)^r = \\ & \frac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)\Gamma(2+2p+q)\Gamma(1+r)}{\Gamma\left(\frac{3}{2}+p\right)\Gamma(3+2p+q+r)}.\end{aligned} \tag{A.9}$$

Let us also recall some general relations between pole coefficients and beta-function in dimensional regularization. If $\xi(\epsilon)$ is a bare coupling and $\xi(\mu)$ is a renormalized one we have

$$\xi(\epsilon) = \mu^\epsilon \left[\xi(\mu) + \frac{T_1(\xi(\mu))}{\epsilon} + \frac{T_2(\xi(\mu))}{\epsilon^2} + \cdots\right], \tag{A.10}$$

where $T_n(\xi)$ have perturbative expansions

$$\begin{aligned} &T_1(\xi) = p_{11}\xi^2 + p_{21}\xi^3 + p_{31}\xi^4 + p_{41}\xi^5 + \cdots, \qquad && T_2(\xi) = p_{22}\xi^3 + p_{32}\xi^4 + p_{42}\xi^5 + \cdots, \\ &T_3(\xi) = p_{33}\xi^4 + p_{43}\xi^5 + \cdots, && T_4(\xi) = p_{44}\xi^5 + \cdots, \ldots. \end{aligned} \tag{A.11}$$

Differentiating (A.10) over μ, using $\mu\frac{d}{d\mu}\xi(\epsilon) = 0$ and setting to zero the resulting coefficients of poles in ϵ gives differential constraints on T_n

$$-T_2 + \xi T_2' + T_1 T_1' - \xi T_1'^2 = 0, \tag{A.12}$$

$$-T_3 + T_2 T_1' - T_1 T_1'^2 + \xi T_1'^3 + T_1 T_2' - 2\xi T_1' T_2' + \xi T_3' = 0, \tag{A.13}$$

$$\begin{gathered}-T_4 + T_3 T_1' - T_2 T_1'^2 + T_1 T_1'^3 - \xi T_1'^4 \\ +T_2 T_2' - 2T_1 T_1' T_2' + 3\xi T_1'^2 T_2' - \xi T_2'^2 + T_1 T_3' - 2\xi T_1' T_3' + \xi T_4' = 0, \ldots.\end{gathered} \tag{A.14}$$

Plugging here the expansions (A.11) gives coefficients of all higher poles in terms of the coefficients of the simple one

$$p_{22} = p_{11}^2, \quad p_{32} = \frac{7}{3}\, p_{11}p_{21}, \quad p_{33} = p_{11}^3, \quad p_{42} = \frac{3}{2}p_{21}^2 + \frac{5}{2}p_{11}p_{31}, \quad p_{43} = \frac{23}{6}p_{11}^2 p_{21}, \quad p_{44} = p_{11}^4, \quad \ldots. \tag{A.15}$$

The beta-function can be expressed in terms of T_1 as

$$\beta(\xi) = \mu\frac{d}{d\mu}\xi(\mu) = -T_1 + \xi T_1' = p_{11}\xi^2 + 2\, p_{21}\xi^3 + 3\, p_{31}\xi^4 + 4\, p_{41}\xi^5 + \cdots. \tag{A.16}$$

B Computation of Wilson loop on a circle in ladder approximation

Let us illustrate how to use the loop equation (2.5) to automatically generate relevant ladder diagrams in the planar expansion on the example of the two-loop calculation of the WL in (1.21). Following the mode regularization approach in [12], we replace the propagator on the circle in (2.4) by its regularized version

$$D_\epsilon(\tau) = \frac{\xi}{8\,\pi^2} \sum_{n=1}^{\infty} e^{-n\,\varepsilon}\,(-n)\,\cos(n\,\tau)\,, \tag{B.1}$$

where $\varepsilon = \frac{a}{R} \to 0$ and a is a cutoff of dimension of length. Using this in the loop equation (2.5), the first-order term in the expansion (2.16) is simply

$$\mathcal{W}_1(\tau) = \sum_{n=1}^{\infty} e^{-n\,\varepsilon} \frac{\cos(n\tau) - 1}{8\,\pi^2\,n}. \tag{B.2}$$

Setting $\tau \to 2\pi$ before summing over n, one gets

$$\mathcal{W}_1(2\pi) = 0. \tag{B.3}$$

Next, using (B.2) in the loop equation, we obtain

$$\begin{aligned}\mathcal{W}_2(\tau) = \sum_{n_1,n_2=1}^{\infty} e^{-\varepsilon(n_1+n_2)} \Bigg\{ &\frac{n_2^2 \sin(n_1\tau)\sin(n_2\tau)}{32\pi^4(n_1^2-n_2^2)^2} + \cos(n_1\tau)\left[\frac{(n_1^2+n_2^2)n_2\cos(n_2\tau)}{64\pi^4 n_1(n_1^2-n_2^2)^2}\right. \\ &\left.+ \frac{n_2}{64\pi^4(n_1^3-n_1n_2^2)}\right] + \frac{(n_2^2-2n_1^2)\cos(n_2\tau)}{64\pi^4 n_1(n_1^2-n_2^2)n_2} + \frac{(2n_1^4-5n_2^2n_1^2+n_2^4)}{64\pi^4 n_1(n_1^2-n_2^2)^2 n_2}\Bigg\}\,.\end{aligned} \tag{B.4}$$

Here the special case $n_1 = n_2$ should be treated separately and one gets

$$\mathcal{W}_2(\tau) = \sum_{n_1\neq n_2=1}^{\infty}(\cdots) + \sum_{n=1}^{\infty} e^{-2n\varepsilon}\left[\frac{15-2n^2\tau^2}{512\pi^4 n^2} - \frac{\cos(n\tau)}{32\pi^4 n^2} + \frac{\cos(2n\tau)}{512\pi^4 n^2} - \frac{\tau\sin(n\tau)}{128\pi^4 n}\right], \tag{B.5}$$

where the first term in the r.h.s. vanishes for $\tau = 2\pi$. As a result,

$$\mathcal{W}_2(2\pi) = -\frac{1}{64\pi^2}\sum_{n=1}^{\infty} e^{-2n\varepsilon} = -\frac{1}{128\pi^2\varepsilon} + \frac{1}{128\pi^2} + \mathcal{O}(\varepsilon). \tag{B.6}$$

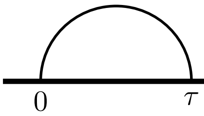
Dropping the singular term (linear divergence), we reproduce the ladder part of the expression in (1.21), (1.25) (cf. (2.1))

$$W = 1 + 0\cdot\xi + \frac{1}{128\pi^2}\,\xi^2 + \dots\,. \tag{B.7}$$

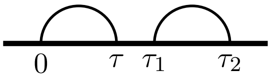
JHEP01(2022)056

C Contributions to two-point function for “transverse” scalar to two loops

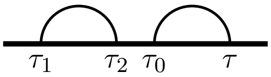
Using the propagator in (2.15), we have the following explicit expressions for the one loop diagrams in (3.4) and (3.3)

$-L$ 

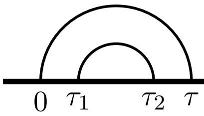
L $$= \frac{\xi}{8\pi^2}\frac{1}{\tau^{2-\epsilon}}, \tag{C.1}$$

$-L$ 

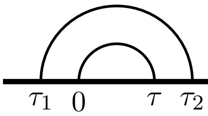
L $$= \frac{(L-\tau)^{\epsilon}\tau^{-2+\epsilon}}{64\pi^4(-1+\epsilon)\epsilon}\xi^2, \tag{C.2}$$

$-L$ 

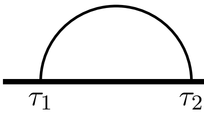
L $$= \frac{L^{\epsilon}\tau^{-2+\epsilon}}{64\pi^4(-1+\epsilon)\epsilon}\xi^2, \tag{C.3}$$

$-L$ 

L $$= \frac{\tau^{-2+2\epsilon}}{64\pi^4(-1+\epsilon)\epsilon}\xi^2, \tag{C.4}$$

$-L$ 

L $$= \frac{\tau^{-2+\epsilon}((-1+2^{\epsilon})L^{\epsilon}+\tau^{\epsilon}-(L+\tau)^{\epsilon})}{64\pi^4(-1+\epsilon)\epsilon}\xi^2, \tag{C.5}$$

$-L$ 

L $$= \frac{2^{-3+\epsilon}L^{\epsilon}}{\pi^2(-1+\epsilon)\epsilon}\xi. \tag{C.6}$$

At two loops, we have 15 diagrams N_i, $i=1,\ldots,15$, in the numerator of (3.1) and two diagrams D_1, D_2 in the denominator. Their expressions are[14]

$N_1 =$  $$= \frac{(L-\tau)^{2\epsilon}\tau^{-2+\epsilon}\Gamma(-1+\epsilon)^2}{512\pi^6\Gamma(1+2\epsilon)}\xi^3, \tag{C.7}$$

$N_2 =$  $$= \frac{(L(L-\tau)\tau)^{\epsilon}}{512\pi^6(-1+\epsilon)^2\epsilon^2\tau^2}\xi^3, \tag{C.8}$$

$N_3 =$  $$= \frac{L^{2\epsilon}\tau^{-2+\epsilon}\Gamma(-1+\epsilon)^2}{512\pi^6\Gamma(1+2\epsilon)}\xi^3, \tag{C.9}$$

$N_4 =$  $$= \frac{(L-\tau)^{2\epsilon}\tau^{-2+\epsilon}}{1024\pi^6(-1+\epsilon)\epsilon^2(-1+2\epsilon)}\xi^3, \tag{C.10}$$

$N_5 =$  $$= \frac{L^{\epsilon}\tau^{-2+2\epsilon}}{512\pi^6(-1+\epsilon)^2\epsilon^2}\xi^3, \tag{C.11}$$

$N_6 =$  $$\begin{aligned} &= \frac{\log\left(\frac{2\tau}{L+\tau}\right)}{512\pi^6\tau^2\epsilon}\xi^3 \\ &+\frac{1}{6144\pi^6\tau^2}\Big[\pi^2+12(\log^2 2+\log 4)+6\log 16\log L+6\log\tau(4+\log 4+3\log\tau) \\ &\qquad -6\log(L+\tau)(4+3\log(L+\tau))-12\mathrm{Li}_2\left(\frac{\tau}{L+\tau}\right)\Big]\xi^3+O(\epsilon), \end{aligned} \tag{C.12}$$

[14] Here we omit the labels $\pm L$ at the ends of the line.

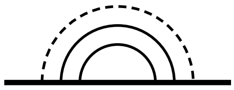
JHEP01(2022)056

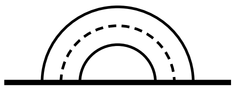
$$N_7 =$$  $$= \frac{L^{2\epsilon}\tau^{-2+\epsilon}}{1024\pi^6(-1+\epsilon)\epsilon^2(-1+2\epsilon)}\,\xi^3, \qquad \text{(C.13)}$$

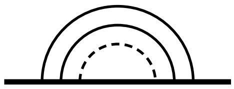
$$N_8 =$$  $$= \frac{(L-\tau)^{\epsilon}\tau^{-2+2\epsilon}}{512\pi^6(-1+\epsilon)^2\epsilon^2}\,\xi^3, \qquad \text{(C.14)}$$

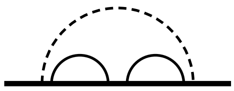
$$N_9 =$$  $$= -\frac{\log\left(\frac{L+\tau}{2\tau}\right)}{512(\pi^6\tau^2)\epsilon}\,\xi^3$$

$$+\frac{1}{1024\pi^6\tau^2}\Big[2\log^2 2+\log 16+\log 16\log L+\log 4\log(L-\tau)+4\log\tau+\log 4\log\tau$$
$$+2\log L\log\tau+3\log^2\tau-4\log(L+\tau)-\log 4\log(L+\tau)-2\log L\log(L+\tau)$$
$$-2\log\tau\log(L+\tau)-\log^2(L+\tau)+2\mathrm{Li}_2\left(-\frac{\tau}{L-\tau}\right)-2\mathrm{Li}_2\left(-\frac{L+\tau}{L-\tau}\right)\Big]\,\xi^3+O(\epsilon), \qquad \text{(C.15)}$$

$$N_{10} =$$  $$= \frac{\tau^{-2+3\epsilon}}{1024\pi^6(-1+\epsilon)\epsilon^2(-1+2\epsilon)}\,\xi^3, \qquad \text{(C.16)}$$

$$N_{11} =$$  $$= \frac{\tau^{-2+2\epsilon}((-1+2^{\epsilon})L^{\epsilon}+\tau^{\epsilon}-(L+\tau)^{\epsilon})}{512\pi^6(-1+\epsilon)^2\epsilon^2}\,\xi^3, \qquad \text{(C.17)}$$

$$N_{12} =$$  $$= \frac{1}{512\pi^6\tau^2}\left[-\frac{\pi^2}{12}+i\pi\log 2+\frac{\log^2 2}{2}-\log 2\log(L-\tau)+\log 2\log\tau\right.$$
$$\left.+\log\frac{2\tau}{L+\tau}-\mathrm{Li}_2\left(\frac{L}{L-\tau}\right)+\mathrm{Li}_2\left(\frac{2L}{L-\tau}\right)-\mathrm{Li}_2\left(-\frac{L}{\tau}\right)\right]\xi^3+\mathcal{O}(\epsilon), \qquad \text{(C.18)}$$

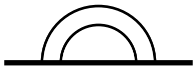
$$N_{13} =$$  $$= \frac{\tau^{-2+3\epsilon}\Gamma(-1+\epsilon)^2}{512\pi^6\Gamma(1+2\epsilon)}\,\xi^3, \qquad \text{(C.19)}$$

$$N_{14} =$$  $$= -\frac{\log\left(\frac{L+\tau}{2\tau}\right)}{512(\pi^6\tau^2)\epsilon}\,\xi^3$$

$$+\frac{1}{1024\pi^6\tau^2}\Big[\log^2 2+\log 16+\log 4\log L+\log 4\log(L-\tau)+\log 4\log\tau+\log^2\tau-\log^2(L+\tau)$$
$$-4\log\left(\frac{L+\tau}{\tau}\right)-2\log(L-\tau)\log\left(\frac{L+\tau}{\tau}\right)-2\log\tau\log\left(\frac{L+\tau}{\tau}\right)$$
$$-2\mathrm{Li}_2\left(1-\frac{L}{\tau}\right)+2\mathrm{Li}_2\left(\frac{-L+\tau}{L+\tau}\right)\Big]\,\xi^3+O(\epsilon), \qquad \text{(C.20)}$$

$$N_{15} =$$  $$= \frac{\log\left(\frac{2\tau}{L+\tau}\right)}{512\pi^6\tau^2\epsilon}\,\xi^3+\frac{1}{512\pi^6\tau^2}\left[-\frac{\pi^2}{12}+\frac{\log^2 2}{2}+\log 4+\frac{3\log^2\tau}{2}\right.$$
$$+\log L\log\left(\frac{4\tau}{L+\tau}\right)+\log\tau(2+\log 2-\log(L+\tau))$$
$$\left.-2\log(L+\tau)-\frac{1}{2}\log^2(L+\tau)-\mathrm{Li}_2\left(-\frac{L}{\tau}\right)\right]\xi^3+O(\epsilon). \qquad \text{(C.21)}$$

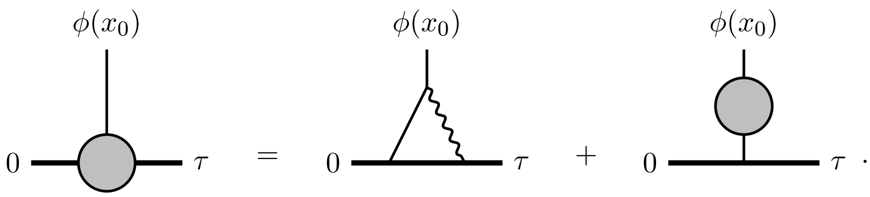
$$D_1 = \quad$$  $$= \frac{L^{2\epsilon}\Gamma(\epsilon)}{64\pi^{7/2}(-1+\epsilon)^2\epsilon\Gamma\left(\frac{1}{2}+\epsilon\right)}\,\xi^2, \tag{C.22}$$

$$D_2 = \quad$$  $$= \frac{2^{-7+2\epsilon}L^{2\epsilon}}{\pi^4(-1+\epsilon)\epsilon^2(-1+2\epsilon)}\,\xi^2. \tag{C.23}$$

D Computation of order ζ term in the one-loop beta-function

Here we shall present some details of the computation of the one-loop term in β_ζ in (1.3) that were not spelled out in [10]. Like the ζ^3 term in (1.3) that follows from the scalar ladder graphs (cf. (2.19)) the $-\zeta$ term comes from similar graphs with gluon propagator instead of the scalar one. For example, in the case of the Wilson line along Euclidean time $t = \tau$ direction the exponent in (1.1) is given by $\int d\tau (iA_t + \phi)$ and thus the gluon contribution is minus that of the scalar due to extra i factor. This already reproduces the expression (1.3) for the one-loop term in β_ζ.

However, there are two extra types of diagrams that are potentially contributing at order ζ. Their total contribution should then be zero. Using the vertex renormalization method discussed in section 2.1 they are represented by[15]



(D.1)

Here the wavy line stands for the gauge field propagator and the blob in the second diagram is the scalar one-loop self-energy correction (given by the sum of the scalar-gluon loop and fermion loop). Below we shall demonstrate the mutual cancellation of these two contributions as claimed in [10] which is similar to the cancellation of the non-ladder diagram contributions to the expectation value of the circular WML observed in [31].

Explicitly, the contribution of the first diagram in (D.1) is[16]

$$V = \frac{i}{2!}\int d^2\tau\, \langle \mathrm{Tr}\,\{\phi(x_0)\,\mathrm{P}[2A(\tau_2)\phi(\tau_3)]\}\left(-\int d^4y\; f^{abc}\,\partial_\mu\phi_i^a(y)\,A_\mu^b(y)\,\phi_i^c(y)\right)\rangle, \tag{D.2}$$

where $A(\tau) = A_\mu^a \dot{x}^\mu T^a$ and $\phi(\tau) = \phi^a(x)|\dot{x}|T^a$ assuming we keep the contour general. If $\tau_2 > \tau_3$ this gives

$$\begin{aligned}
&-i\, f^{a'b'c'}\,\mathrm{Tr}(T^aT^bT^c)\int d^2\tau\,\langle\phi^a(x_0)A^b(\tau_2)\phi^c(\tau_3)\int d^4y\;\partial_\mu\phi_i^{a'}(y)\,A_\mu^{b'}(y)\,\phi_i^{c'}(y)\rangle\\
&\quad= -i\, f^{abc}\,\mathrm{Tr}(T^aT^bT^c)\int d^4y\int d^2\tau\;(\dot{x}^{(2)}\cdot\partial_y D(y-x_0))\,D(y-\tau_3)D(y-\tau_2)\\
&\qquad -i\, f^{cba}\,\mathrm{Tr}(T^aT^bT^c)\int d^4y\int d^2\tau\;(\dot{x}^{(2)}\cdot\partial_y D(y-\tau_3))\,D(y-x_0)D(y-\tau_2)
\end{aligned} \tag{D.3}$$

---

[15] There are, of course, two copies of the first diagram depending on ordering of the end-points of the propagators.

[16] Here $2iA(\tau_2)\phi(\tau_3)$ comes from the mixed term in the expansion of $(\phi+iA)(\phi+iA)$.

and for $\tau_2 < \tau_3$ we get

$$
\begin{aligned}
&-i\, f^{a'b'c'}\,\mathrm{Tr}(T^aT^bT^c)\int d^4y \int d^2\tau\ \langle \phi^a(x_0)A^c(\tau_2)\phi^b(\tau_3)\ \int d^4y\ \partial_\mu\phi_i^{a'}(y)\,A_\mu^{b'}(y)\,\phi_i^{c'}(y)\rangle \\
&\quad = -i\, f^{acb}\,\mathrm{Tr}(T^aT^bT^c)\int d^4y \int d^2\tau\ (\dot{x}^{(2)}\cdot\partial_y D(y-x_0))\,D(y-\tau_3)D(y-\tau_2) \\
&\qquad - i\, f^{bca}\,\mathrm{Tr}(T^aT^bT^c)\int d^4y \int d^2\tau\ (\dot{x}^{(2)}\cdot\partial_y D(y-\tau_3))\,D(y-x_0)D(y-\tau_2)\,.
\end{aligned}
\tag{D.4}
$$

Using that $f^{abc}\,\mathrm{Tr}(T^aT^bT^c) = \frac{i}{4}N^3 + \ldots$ where dots stand for subleading terms at large N we get

$$
V = N^3\int d^4y\,\mathcal{V}(y), \tag{D.5}
$$

$$
\mathcal{V} = -\frac{1}{4}\int d^2\tau\ \epsilon(\tau_2,\tau_3)\,(\dot{x}^{(2)}\cdot\partial_y D(y-\tau_3))\,D(y-x_0)D(y-\tau_2)\,, \tag{D.6}
$$

where $\epsilon(\tau_2,\tau_3)$ is the antisymmetric path ordering symbol. Specifying to the case of the contour being straight line we get[17]

$$
\mathcal{V} = \frac{1}{4}\int d^2\tau\ \epsilon(\tau_2,\tau_3)\,\frac{\partial}{\partial\tau_3}D(y-\tau_3)\,D(y-x_0)D(y-\tau_2). \tag{D.7}
$$

Integrating by parts (using that $\frac{\partial}{\partial\tau_3}\epsilon(\tau_2,\tau_3) = -2\delta(\tau_2-\tau_3)$) gives

$$
\begin{aligned}
\mathcal{V} &= \frac{1}{2}\int d^2\tau\ \delta(\tau_2-\tau_3)D(y-\tau_3)\,D(y-x_0)D(y-\tau_2) \\
&\quad + \frac{1}{4}\int d\tau_2\ \left[\epsilon(\tau_2,\tau_3)\,D(y-\tau_3)\,D(y-x_0)D(y-\tau_2)\right]_{\tau_3=0}^{\tau_3=\tau} \\
&= \frac{1}{2}\int d\tau'\ D(y-\tau')^2\,D(y-x_0) - \frac{1}{4}\int d\tau'\ D(y-\tau)\,D(y-x_0)D(y-\tau') \\
&\quad + \frac{1}{4}\int d\tau'\ D(y)D(y-x_0)D(y-\tau').
\end{aligned}
\tag{D.8}
$$

This may be written as

$$
\mathcal{V} = \frac{1}{4}\int d\tau'\,[2\,H(x_0,\tau',\tau') - H(x_0,\tau',\tau) + H(x_0,\tau',0)]. \tag{D.9}
$$

Here the function H is in general defined as (cf. [31])

$$
\begin{aligned}
&H(x^{(1)},x^{(2)},x^{(3)}) = \int d^{2\omega}y\,\Delta(x^{(1)}-y)\,\Delta(x^{(2)}-y)\,\Delta(x^{(3)}-y) \\
&= \frac{\Gamma(\omega-1)^3}{4^3\pi^{3\omega}}\int d^{2\omega}y\,\frac{1}{[(x^{(1)}-y)^2]^{\omega-1}}\,\frac{1}{[(x^{(2)}-y)^2]^{\omega-1}}\,\frac{1}{[(x^{(3)}-y)^2]^{\omega-1}} \\
&= \frac{\Gamma(\omega-1)^3}{4^3\pi^{3\omega}}\,\frac{\Gamma(3\omega-3)}{\Gamma(\omega-1)^3}\int d^{2\omega}y\int_0^1 d\alpha d\beta d\gamma\ \frac{\delta(1-\alpha-\beta-\gamma)\,(\alpha\beta\gamma)^{\omega-2}}{[\alpha(x^{(1)}-y)^2+\beta(x^{(2)}-y)^2+\gamma(x^{(3)}-y)^2]^{3(\omega-1)}}
\end{aligned}
\tag{D.10}
$$

[17] To simplify the analysis we may assume that the point x_0 also lies on the line but far away from other points and thus not participating in limits leading to short-distance singularities.

where Δ is the scalar propagator in $d = 2\omega = 4 - \epsilon$ dimensions (with canonical normalization). It can be put into the form

$$H(x^{(1)}, x^{(2)}, x^{(3)}) = \frac{\Gamma(2\omega - 3)}{4^3 \pi^{2\omega}} \int d\alpha d\beta d\gamma \, \frac{\delta(1 - \alpha - \beta - \gamma)\,(\alpha\beta\gamma)^{\omega-2}}{[M^2]^{2\omega-3}}, \tag{D.11}$$

$$\begin{aligned} M^2(x^{(1)}, x^{(2)}, x^{(3)}) &= \alpha(1-\alpha)\,(x^{(1)})^2 + \beta(1-\beta)\,(x^{(2)})^2 + \gamma(1-\gamma)\,(x^{(3)})^2 \\ &\quad - 2\alpha\beta\, x^{(1)} \cdot x^{(2)} - 2\alpha\gamma\, x^{(1)} \cdot x^{(3)} - 2\beta\gamma\, x^{(2)} \cdot x^{(3)}. \end{aligned} \tag{D.12}$$

The UV divergent contribution to (D.9) comes only from the limits when two points on the line approach each other, i.e. only from the first H-function term in (D.9). Focussing on this term and setting $x_0 = 0$ we have

$$\begin{aligned} \mathcal{V} &\to \frac{1}{2} \int d\tau' \, H(0, \tau', \tau') \\ &= \frac{N^3 \Gamma(2\omega - 3)}{2^7 \pi^{2\omega}} \int_0^\tau d\tau' \int d\alpha d\beta d\gamma \, \frac{\delta(1 - \alpha - \beta - \gamma)\,(\alpha\beta\gamma)^{\omega-2}}{((\alpha+\beta)\gamma)^{2\omega-3}} \tau'^{-2(2\omega-3)}. \end{aligned} \tag{D.13}$$

Using (A.9) then gives for the corresponding integral in (D.5)

$$V = N^3 \frac{2^{-3-2\omega} \pi^{\frac{3}{2}-2\omega} \tau^{7-4\omega} \csc(\pi\omega) \Gamma(-3 + 2\omega)}{(-7 + 4\omega)\Gamma(3 - \omega)\Gamma\left(-\frac{1}{2} + \omega\right)} = \frac{N^3}{64\pi^4 \tau} \, \frac{1}{\omega - 2} + \cdots. \tag{D.14}$$

Turning to the scalar self energy contribution, it can be written as (see [31])

$$S = -\frac{1}{2} \, \frac{\Gamma^2(\omega - 1)}{32\pi^{2\omega}(2 - \omega)(2\omega - 3)} \int_0^\tau d\tau' \, \frac{1}{(x_0 - \tau')^{2(2\omega-3)}}. \tag{D.15}$$

Setting as in (D.13) $x_0 = 0$ we get

$$S = -\frac{1}{2} \, \frac{\Gamma^2(\omega - 1)}{32\pi^{2\omega}(2 - \omega)(2\omega - 3)} \int_0^\tau d\tau' \, \frac{1}{\tau'^{2(2\omega-3)}} = -\frac{\pi^{-2\omega} \tau^{7-4\omega} \Gamma(-1 + \omega)^2}{64(7 - 4\omega)(2 - \omega)(-3 + 2\omega)}. \tag{D.16}$$

Then one can check that the contributions of the triangular graph and self-energy correction indeed cancel (even exactly in ω), i.e.

$$V + S = 0. \tag{D.17}$$

References

[1] J.M. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002] [INSPIRE].

[2] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J. C* **22** (2001) 379 [hep-th/9803001] [INSPIRE].

[3] A.M. Polyakov, *Gauge Fields as Rings of Glue*, *Nucl. Phys. B* **164** (1980) 171 [INSPIRE].

[4] V.S. Dotsenko and S.N. Vergeles, *Renormalizability of Phase Factors in the NonAbelian Gauge Theory*, *Nucl. Phys. B* **169** (1980) 527 [INSPIRE].

JHEP01(2022)056

[5] J.-L. Gervais and A. Neveu, *The Slope of the Leading Regge Trajectory in Quantum Chromodynamics*, Nucl. Phys. B **163** (1980) 189 [INSPIRE].

[6] I.Y. Arefeva, *Quantum contour field equations*, Phys. Lett. B **93** (1980) 347 [INSPIRE].

[7] H. Dorn, *Renormalization of Path Ordered Phase Factors and Related Hadron Operators in Gauge Field Theories*, Fortsch. Phys. **34** (1986) 11 [INSPIRE].

[8] R.M. Marinho and L. Boanerges Peixoto, *Charge renormalization of the Yang-Mills theory up to fourth order using dimensional regularization*, Nuovo Cim. A **97** (1987) 148 [INSPIRE].

[9] L.F. Alday and J.M. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT*, JHEP **11** (2007) 068 [arXiv:0710.1060] [INSPIRE].

[10] J. Polchinski and J. Sully, *Wilson Loop Renormalization Group Flows*, JHEP **10** (2011) 059 [arXiv:1104.5077] [INSPIRE].

[11] M. Beccaria, S. Giombi and A. Tseytlin, *Non-supersymmetric Wilson loop in $\mathcal{N} = 4$ SYM and defect 1d CFT*, JHEP **03** (2018) 131 [arXiv:1712.06874] [INSPIRE].

[12] M. Beccaria and A.A. Tseytlin, *On non-supersymmetric generalizations of the Wilson-Maldacena loops in $N = 4$ SYM*, Nucl. Phys. B **934** (2018) 466 [arXiv:1804.02179] [INSPIRE].

[13] M. Cooke, A. Dekel and N. Drukker, *The Wilson loop CFT: Insertion dimensions and structure constants from wavy lines*, J. Phys. A **50** (2017) 335401 [arXiv:1703.03812] [INSPIRE].

[14] S. Giombi, R. Roiban and A.A. Tseytlin, *Half-BPS Wilson loop and AdS_2/CFT_1*, Nucl. Phys. B **922** (2017) 499 [arXiv:1706.00756] [INSPIRE].

[15] P. Liendo, C. Meneghelli and V. Mitev, *Bootstrapping the half-BPS line defect*, JHEP **10** (2018) 077 [arXiv:1806.01862] [INSPIRE].

[16] M. Beccaria, S. Giombi and A.A. Tseytlin, *Correlators on non-supersymmetric Wilson line in $\mathcal{N} = 4$ SYM and AdS_2/CFT_1*, JHEP **05** (2019) 122 [arXiv:1903.04365] [INSPIRE].

[17] D.H. Correa, V.I. Giraldo-Rivera and G.A. Silva, *Supersymmetric mixed boundary conditions in AdS_2 and $DCFT_1$ marginal deformations*, JHEP **03** (2020) 010 [arXiv:1910.04225] [INSPIRE].

[18] N.B. Agmon and Y. Wang, *Classifying Superconformal Defects in Diverse Dimensions Part I: Superconformal Lines*, arXiv:2009.06650 [INSPIRE].

[19] G. Cuomo, Z. Komargodski and A. Raviv-Moshe, *Renormalization Group Flows on Line Defects*, arXiv:2108.01117 [INSPIRE].

[20] R.A. Brandt, F. Neri and M.-a. Sato, *Renormalization of Loop Functions for All Loops*, Phys. Rev. D **24** (1981) 879 [INSPIRE].

[21] C. Hoyos, *A defect action for Wilson loops*, JHEP **07** (2018) 045 [arXiv:1803.09809] [INSPIRE].

[22] I.R. Klebanov, S.S. Pufu and B.R. Safdi, *F-Theorem without Supersymmetry*, JHEP **10** (2011) 038 [arXiv:1105.4598] [INSPIRE].

[23] S. Giombi and I.R. Klebanov, *Interpolating between a and F*, JHEP **03** (2015) 117 [arXiv:1409.1937] [INSPIRE].

JHEP01(2022)056

[24] L. Fei, S. Giombi, I.R. Klebanov and G. Tarnopolsky, *Generalized F-Theorem and the ϵ Expansion*, *JHEP* **12** (2015) 155 [arXiv:1507.01960] [INSPIRE].

[25] N. Kobayashi, T. Nishioka, Y. Sato and K. Watanabe, *Towards a C-theorem in defect CFT*, *JHEP* **01** (2019) 039 [arXiv:1810.06995] [INSPIRE].

[26] I. Affleck and A.W.W. Ludwig, *Universal noninteger 'ground state degeneracy' in critical quantum systems*, *Phys. Rev. Lett.* **67** (1991) 161 [INSPIRE].

[27] D. Friedan and A. Konechny, *On the boundary entropy of one-dimensional quantum systems at low temperature*, *Phys. Rev. Lett.* **93** (2004) 030402 [hep-th/0312197] [INSPIRE].

[28] R. Brüser, S. Caron-Huot and J.M. Henn, *Subleading Regge limit from a soft anomalous dimension*, *JHEP* **04** (2018) 047 [arXiv:1802.02524] [INSPIRE].

[29] D. Grabner, N. Gromov and J. Julius, *Excited States of One-Dimensional Defect CFTs from the Quantum Spectral Curve*, *JHEP* **07** (2020) 042 [arXiv:2001.11039] [INSPIRE].

[30] P. Ferrero and C. Meneghelli, *Bootstrapping the half-BPS line defect CFT in $N = 4$ supersymmetric Yang-Mills theory at strong coupling*, *Phys. Rev. D* **104** (2021) L081703 [arXiv:2103.10440] [INSPIRE].

[31] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **582** (2000) 155 [hep-th/0003055] [INSPIRE].

[32] N. Drukker and D.J. Gross, *An Exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[33] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[34] D. Correa, P. Pisani, A. Rios Fukelman and K. Zarembo, *Dyson equations for correlators of Wilson loops*, *JHEP* **12** (2018) 100 [arXiv:1811.03552] [INSPIRE].

JHEP01(2022)056