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Degenerate operators on the half-line

G. Metafune, L. Negro and C. Spina

Abstract. We study elliptic and parabolic problems governed by the singular elliptic operators

$$y^\alpha \left(D_{yy} + \frac{c}{y} D_y\right) - V(y), \quad \alpha \in \mathbb{R}$$

in $\mathbb{R}_+$, where V is a potential having nonnegative real part.

1. Introduction

In this paper, we study solvability and regularity of elliptic and parabolic problems associated with the degenerate operators

$$L = y^\alpha \left(D_{yy} + \frac{c}{y} D_y\right) - V \quad \text{and} \quad D_t - L$$

in the half-line $\mathbb{R}_+$.

Here, c, α are real numbers and $V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)$ is a potential having nonnegative real part. The operator $B = D_{yy} + \frac{c}{y} D_y$ is a Bessel operator and satisfies the scaling property

$$I_s^{-1} B I_s = s^2 B, \quad I_s u(y) = u(sy).$$

We study L in the weighted spaces $L^p_m := L^p\left(\mathbb{R}^+, y^m dy\right)$, $m \in \mathbb{R}$, and we characterize all m such that L generates a C_0-semigroup. When $V \geq 0$, we also prove that the generated semigroup is analytic and we show that it has maximal regularity, which means that both $D_t v$ and Lv have the same regularity as $(D_t - L)v$. In the case $V(y) = y^\alpha$, we finally characterize the domain of L.

We observe that the results already available for B, see [13, Section 3] and also [8–11,15] for the N-d version of B, imply the corresponding ones for $y^\alpha B$ in L^p_m by a change of variables, as described in Sect. 3. The change of variables varies the underlying measure and explains why we need the full scale of L^p_m spaces.

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More effort is needed to add the potential term. We consider first $B - V$ in $L^2(\mathbb{R}_+; y^c dy)$. We use form methods to construct an analytic semigroup, and we prove kernel bounds for complex times via Davies–Gaffney estimates and provide a core. Then, with the methods of Sect. 3, we deduce similar results for $y^\alpha B - V$ in $L^2(\mathbb{R}_+; y^{c-\alpha} dy)$. Next we prove that the semigroup can be extended to L^p_m under sharp conditions on p and m. Finally, we prove that for every $\epsilon > 0$ the family of operators

$$\left\{e^{z(y^\alpha B - V)} : \ z \in \Sigma_{\frac{\pi}{2}-\epsilon}, \ 0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)\right\}$$

is $\mathcal{R}$-bounded in L^p_m, which implies the maximal regularity of the semigroup when $V \ge 0$.

As a motivation for our investigation, we point out that, in the special case $V(y) = y^\alpha$, all the results above play a crucial role in [14] in the investigation of the degenerate operators

$$\mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} \left(D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2}\right).$$

Let us suppose, for simplicity, $b = 0$, $\alpha_1 = \alpha_2 := \alpha$. Assuming that $y^\alpha(\Delta_x u + B_y u) = f$ and taking the Fourier transform $\mathcal{F}u$ or $\hat{u}$ with respect to x, we obtain $y^\alpha |\xi|^2 \hat{u}(\xi, y) = -y^\alpha |\xi|^2 (y^\alpha |\xi|^2 - y^\alpha B_y)^{-1} \hat{f}(\xi, y)$. Therefore,

$$y^\alpha \Delta_x \mathcal{L}^{-1} = \mathcal{F}^{-1} \left(y^\alpha |\xi|^2 (y^\alpha |\xi|^2 - y^\alpha B_y)^{-1})\right) \mathcal{F}$$

and the boundedness of $y^\alpha \Delta_x \mathcal{L}^{-1}$ is equivalent to that of the multiplier

$$\xi \in \mathbb{R}^N \to y^\alpha |\xi|^2 (y^\alpha |\xi|^2 - y^\alpha By)^{-1}.$$

For this reason, we prove in Sect. 8 that certain multipliers associated with $y^\alpha B - V$ satisfy a vector-valued Mikhlin theorem. These results rely on square function estimates which we deduce from kernel bounds and the following equality, which allows to treat λ or $|\xi|^2$ as spectral parameters simultaneously

$$\left(\lambda - y^\alpha B + |\xi|^2 y^\alpha\right)^{-1} = \left(|\xi|^2 - B + \frac{\lambda}{y^\alpha}\right)^{-1} \frac{1}{y^\alpha}.$$

We restrict ourselves to $\alpha < 2$ and consider $y^\alpha B$ with Neumann boundary condition at 0, namely $\lim_{y \to 0} y^c D_y u(y) = 0$. This is equivalent to require $y^{\alpha-1} D_y u \in L^p_m$, see [12, Proposition 5.11]. The restriction $\alpha < 2$ is not really essential since one can deduce from it the case $\alpha > 2$, which requires a boundary condition at ∞, using the change of variables described in Sect. 3.

Besides this, our strategy can be easily adapted to different boundary conditions and to more general operators $y^\alpha \left(D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2}\right) - V$. We do this (in much more generality) in [14, Sections 7, 8].

The paper is organized as follows. In Sect. 2, we briefly recall the harmonic analysis background needed in the paper, such as square function estimates, $\mathcal{R}$-boundedness and a vector-valued multiplier theorem.

In Sect. 3, we exploit an elementary change of variables, in a functional analytic setting, to reduce our operators to the simpler case where $\alpha = 0$.

Section 4 is devoted to the study of the Bessel operator $y^\alpha B$. In Sects. 5, 6 and 7, we perturb the Bessel operator by adding the potential V and we prove real and complex kernel estimates, generation results and maximal regularity for $y^\alpha B - V$. Finally in Sect. 8, we treat the case $V(y) = y^\alpha$ and characterize the domain of $y^\alpha B - y^\alpha$.

Notation. For $m \in \mathbb{R}$, we consider the measure $y^m dy$ in $\mathbb{R}_+$ and we write L^p_m for $L^p(\mathbb{R}_+, y^m dy)$. Similarly, $W^{k,p}_m = \{u \in L^p_m : \partial^\alpha u \in L^p_m \quad |\alpha| \le k\}$. When we write $V \in L^q_{loc}(\mathbb{R}^+, y^m\, dy)$, we mean that $V \in L^q([0, b], y^m\, dy)$ for every $b < \infty$.

We use $\mathbb{C}^+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$, and for $|\theta| \le \pi$, we denote by Σ_θ the open sector $\{\lambda \in \mathbb{C} : \lambda \neq 0,\ |Arg(\lambda)| < \theta\}$.

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2. Harmonic analysis and maximal regularity

The study of maximal regularity of parabolic problems of the form $u_t = Au + f, u(0) = 0$, where A is the generator of an analytic semigroup on a Banach space X, consists in proving estimates like

$$\|u_t\|_p + \|Au\|_p \le \|f\|_p$$

where the L^p norm is that of $L^p([0, T[; X)$. This can be interpreted as closedness of $D_t - A$ on the intersection of the respective domains or, equivalently, boundedness of the operator $A(D_t - A)^{-1}$ in $L^p([0, T[; X)$.

Nowadays this strategy is well established and relies on Mikhlin vector-valued multiplier theorems. Let us state the relevant definitions and main results we need, referring the reader to [5,6,17] or [7].

Let $\mathcal{S}$ be a subset of $B(X)$, the space of all bounded linear operators on a Banach space X. $\mathcal{S}$ is $\mathcal{R}$-bounded if there is a constant C such that

$$\left\|\sum_i \varepsilon_i S_i x_i\right\|_{L^p(\Omega;X)} \le C \left\|\sum_i \varepsilon_i x_i\right\|_{L^p(\Omega;X)}$$

for every finite sum as above, where $(x_i) \subset X$, $(S_i) \subset \mathcal{S}$ and $\varepsilon_i : \Omega \to \{-1, 1\}$ are independent and symmetric random variables on a probability space Ω. The smallest constant C for which the above definition holds is the $\mathcal{R}$-bound of $\mathcal{S}$, denoted by $\mathcal{R}(\mathcal{S})$. It is well known that this definition does not depend on $1 \le p < \infty$ (however, the constant $\mathcal{R}(\mathcal{S})$ does) and that $\mathcal{R}$-boundedness is equivalent to boundedness when

X is an Hilbert space. When X is an L^p space (with respect to any σ-finite measure), testing $\mathcal{R}$-boundedness is equivalent to proving square functions estimates, see [7, Remark 2.9].

Proposition 2.1. *Let $\mathcal{S} \subset B(L^p(\Sigma))$, $1 < p < \infty$. Then, $\mathcal{S}$ is $\mathcal{R}$-bounded if and only if there is a constant $C > 0$ such that for every finite family $(f_i) \in L^p(\Sigma)$, $(S_i) \in \mathcal{S}$*

$$\left\| \left(\sum_i |S_i f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Sigma)} \le C \left\| \left(\sum_i |f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Sigma)}.$$

The best constant C for which the above square functions estimates hold satisfies $\kappa^{-1} C \le \mathcal{R}(\mathcal{S}) \le \kappa C$ for a suitable $\kappa > 0$ (depending only on p). The proposition above $\mathcal{R}$-boundedness follows from domination.

Corollary 2.2. *Let $\mathcal{S}, \mathcal{T} \subset B(L^p(\Sigma))$, $1 < p < \infty$ and assume that $\mathcal{T}$ is $\mathcal{R}$ bounded and that for every $S \in \mathcal{S}$ there exists $T \in \mathcal{T}$ such that $|Sf| \le |Tf|$ pointwise, for every $f \in L^p(\Sigma)$. Then, $\mathcal{S}$ is $\mathcal{R}$-bounded.*

Let $(A, D(A))$ be a densely defined, sectorial operator in a Banach space X; this means that $\rho(-A) \supset \Sigma_{\pi-\phi}$ for some $\phi < \pi$ and that $\lambda(\lambda + A)^{-1}$ is bounded in $\Sigma_{\pi-\phi}$. The infimum of all such ϕ is called the spectral angle of A and denoted by ϕ_A. Note that $-A$ generates a strongly continuous analytic semigroup if and only if $\phi_A < \pi/2$. The definition of $\mathcal{R}$-sectorial operator is similar, substituting boundedness of $\lambda(\lambda + A)^{-1}$ with $\mathcal{R}$-boundedness in $\Sigma_{\pi-\phi}$. As above one denotes by ϕ_A^R the infimum of all ϕ for which this happens; since $\mathcal{R}$-boundedness implies boundedness, we have $\phi_A \le \phi_A^R$.

The $\mathcal{R}$-boundedness of the resolvent characterizes the regularity of the associated inhomogeneous parabolic problem, as we explain now.

An analytic semigroup $(e^{-tA})_{t \ge 0}$ on a Banach space X with generator $-A$ has *maximal regularity of type L^q* ($1 < q < \infty$) if for each $f \in L^q([0, T]; X)$ the function $t \mapsto u(t) = \int_0^t e^{-(t-s)A}) f(s)\, ds$ belongs to $W^{1,q}([0, T]; X) \cap L^q([0, T]; D(A))$. This means that the mild solution of the evolution equation

$$u'(t) + Au(t) = f(t), \quad t > 0, \qquad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. A characterization of maximal regularity is available in UMD Banach spaces, through the $\mathcal{R}$-boundedness of the resolvent in a suitable sector $\omega + \Sigma_\phi$, with $\omega \in \mathbb{R}$ and $\phi > \pi/2$ or, equivalently, of the scaled semigroup $e^{-(A+\omega')t}$ in a sector around the positive axis. In the case of L^p spaces, it can be restated in the following form, see [7, Theorem 1.11]

Theorem 2.3. *Let $(e^{-tA})_{t \ge 0}$ be a bounded analytic semigroup in $L^p(\Sigma)$, $1 < p < \infty$, with generator $-A$. Then, $T(\cdot)$ has maximal regularity of type L^q if and only if the set $\{\lambda(\lambda + A)^{-1}, \lambda \in \Sigma_{\pi/2+\phi}\}$ is $\mathcal{R}$- bounded for some $\phi > 0$. In an equivalent way, if and only if there are constants $0 < \phi < \pi/2$, $C > 0$ such that for every finite sequence $(\lambda_i) \subset \Sigma_{\pi/2+\phi}$, $(f_i) \subset L^p$*

$$\left\|\left(\sum_i |\lambda_i(\lambda_i+A)^{-1}f_i|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Sigma)} \le C\left\|\left(\sum_i |f_i|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Sigma)}$$

or, equivalently, there are constants $0<\phi'<\pi/2$, $C'>0$ *such that for every finite sequence* $(z_i)\subset\Sigma_{\phi'}$, $(f_i)\subset L^p$

$$\left\|\left(\sum_i |e^{-z_iA}f_i|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Sigma)} \le C'\left\|\left(\sum_i |f_i|^2\right)^{\frac{1}{2}}\right\|_{L^p(\Sigma)}.$$

Finally, we state a version of the operator-valued Mikhlin multiplier theorem in the N-dimensional case, see [5, Theorem 3.25] or [7, Theorem 4.6].

Theorem 2.4. *Let* $1<p<\infty$, $M\in C^N(\mathbb{R}^N\setminus\{0\};B(L^p(\Sigma))$ *be such that the set*

$$\left\{|\xi|^{|\alpha|}D_\xi^\alpha M(\xi):\xi\in\mathbb{R}^N\setminus\{0\},\ |\alpha|\le N\right\}$$

is $\mathcal{R}$*-bounded. Then, the operator* $T_M=\mathcal{F}^{-1}M\mathcal{F}$ *is bounded in* $L^p(\mathbb{R}^N,L^p(\Sigma))$*, where* $\mathcal{F}$ *denotes the Fourier transform.*

We end this section with the following lemma on radially symmetric multipliers.

Lemma 2.5. *Let* $1<p<\infty$, $m\in C^N(\mathbb{R}_+;B(L^p(\Sigma))$ *be such that the set*

$$\left\{s^km^{(k)}(s):\ s\in\mathbb{R}_+,\ k\le N\right\}$$

is $\mathcal{R}$*-bounded. For* $a\in\mathbb{R}$*, let* $M(\xi)=m\left(|\xi|^a\right)$*. Then,* $M\in C^N(\mathbb{R}^N\setminus\{0\};B(L^p(\Sigma))$ *and*

$$\left\{|\xi|^{|\alpha|}D_\xi^\alpha M(\xi):\xi\in\mathbb{R}^N\setminus\{0\},\ |\alpha|\le N\right\}$$

is $\mathcal{R}$*-bounded and*

$$\mathcal{R}\left\{|\xi|^{|\alpha|}D_\xi^\alpha M(\xi):\xi\in\mathbb{R}^N\setminus\{0\},\ |\alpha|\le N\right\}\le C(N)\mathcal{R}\left\{s^km^{(k)}(s):\ s\in\mathbb{R}_+,\ k\le N\right\}.$$

Proof. Let us first observe that for any multi-index α with $0<|\alpha|\le N$ one has

$$D_\xi^\alpha M(\xi)=\sum_{i=1}^{|\alpha|}h_{i,\alpha}(\xi)m^{(i)}\left(|\xi|^a\right) \tag{1}$$

where $h_{i,\alpha}\in C^\infty(\mathbb{R}^N\setminus\{0\})$ are homogeneous functions of degree $ia-|\alpha|$. Obviously, (1) is valid for $|\alpha|=1$ since $\nabla M(\xi)=a\,m'(|\xi|^a)|\xi|^{a-2}\xi$ and follows by induction, and the derivatives of $h_{i,\alpha}$ are homogeneous of degree $ia-|\alpha|-1$.

The proof of the lemma now follows by Corollary (2.2) since from (1) one has for $f\in L^p(\Sigma)$

$$|\xi|^{|\alpha|}|D_\xi^\alpha M(\xi)f|\le|\xi|^{|\alpha|}\sum_{i=1}^{|\alpha|}|h_{i,\alpha}(\xi)||m^{(i)}\left(|\xi|^a\right)f|\le C\sum_{i=1}^{|\alpha|}|\xi|^{ia}|m^{(i)}\left(|\xi|^a\right)f|.$$

□

3. Degenerate operators and similarity transformations

We investigate when the operators

$$B = D_{yy} + \frac{c}{y} D_y, \qquad y^\alpha B = y^\alpha \left(D_{yy} + \frac{c}{y} D_y \right)$$

can be transformed one into the other by means of change of variables. Here, α, c are unrestricted real coefficients.

For $\beta \in \mathbb{R}$, $\beta \neq -1$ let

$$T_\beta u(y) := |\beta + 1|^{\frac{1}{p}} u(y^{\beta+1}), \quad y \in \mathbb{R}_+. \tag{2}$$

Observe that

$$T_\beta^{-1} = T_{-\frac{\beta}{\beta+1}}.$$

Proposition 3.1. *Let $1 \le p \le \infty$, $k, \beta \in \mathbb{R}$, $\beta \neq -1$. The following properties hold.*

(i) For every $m \in \mathbb{R}$, T_β maps isometrically $L^p_{\tilde{m}}$ onto L^p_m where

$$\tilde{m} = \frac{m - \beta}{\beta + 1}.$$

(ii) For every $u \in W^{2,1}_{loc}(\mathbb{R}_+)$, one has

1. *$y^\alpha T_\beta u = T_\beta (y^{\frac{\alpha}{\beta+1}} u)$, for any $\alpha \in \mathbb{R}$;*
2. *$D_y T_\beta u = T_\beta \left((\beta + 1) y^{\frac{\beta}{\beta+1}} D_y u \right)$,*
   $$D_{yy}(T_\beta u) = T_\beta \left((\beta + 1)^2 y^{\frac{2\beta}{\beta+1}} D_{yy} u + (\beta + 1)\beta y^{\frac{\beta-1}{\beta+1}} D_y u \right).$$

Proof. The proof of (i) follows after observing the Jacobian of $y \mapsto y^{\beta+1}$ is $|1+\beta| y^\beta$. Then, we compute

$$D_y T_\beta u(y) = |\beta + 1|^{\frac{1}{p}} \left((\beta + 1) y^\beta D_y u(y^{\beta+1}) \right) = T_\beta \left((\beta + 1) y^{\frac{\beta}{\beta+1}} D_y u \right)$$

and similarly

$$D_{yy} T_\beta u(y) = T_\beta \left((\beta + 1)^2 y^{\frac{2\beta}{\beta+1}} D_{yy} u + (\beta + 1)\beta y^{\frac{\beta-1}{\beta+1}} D_y u \right).$$

□

Proposition 3.2. *Let T_β be the isometry above defined. The following properties hold. For every $u \in W^{2,1}_{loc}(\mathbb{R}_+)$, one has*

$$T_\beta^{-1} \left(y^\alpha B \right) T_\beta u = \left((\beta + 1)^2 y^{\frac{\alpha+2\beta}{\beta+1}} \tilde{B} \right) u$$

where $\tilde{B}$ is the operator defined as in (1) *with parameter c replaced, respectively, by*

$$\tilde{c} = \frac{c + \beta\,(c + 1 + \beta)}{(\beta + 1)^2}.$$

Proof. Using Proposition 3.1, we can compute

$$\begin{aligned} &B T_\beta u(y) \\ &= T_\beta \left[(\beta+1)^2 y^{\frac{2\beta}{\beta+1}} D_{yy} u + (\beta+1)\beta y^{\frac{\beta-1}{\beta+1}} D_y u + c(\beta+1) y^{\frac{\beta-1}{\beta+1}} D_y u - b y^{-\frac{2}{\beta+1}} u \right] \\ &= T_\beta \left[y^{\frac{2\beta}{\beta+1}} \left((\beta+1)^2 D_{yy} u + \frac{(\beta+1)\,(\beta+c)}{y} D_y u - b \frac{u}{y^2} \right) \right] \\ &= T_\beta \left(y^{\frac{2\beta}{\beta+1}} \tilde{B} u \right) \end{aligned}$$

which implies

$$T_\beta^{-1} \left(y^\alpha B \right) T_\beta\, u = y^{\frac{\alpha+2\beta}{\beta+1}} \tilde{B} u.$$

□

4. The Bessel operator $y^\alpha B^n$

In this section, we consider for $\alpha < 2$, $c \in \mathbb{R}$ the operator

$$y^\alpha B = y^\alpha \left(D_{yy} + \frac{c}{y} D_y \right)$$

in the space L^p_m under Neumann boundary conditions.

According to Proposition 3.2, for $0 < (m+1)/p < c+1-\alpha$, we use the isometry

$$T_{-\frac{\alpha}{2}} : L^p_{\tilde{m}} \to L^p_m \quad T_{-\frac{\alpha}{2}} u(y) = \left|1 - \frac{\alpha}{2}\right|^{\frac{1}{p}} u(y^{1-\frac{\alpha}{2}}),$$

$\tilde{m} = \frac{m+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$, under which $y^\alpha B$ becomes isometrically equivalent to $T^{-1}_{-\frac{\alpha}{2}} \left(y^\alpha B \right) T_{-\frac{\alpha}{2}} = \left(1 - \frac{\alpha}{2}\right)^2 \tilde{B}$ where $\tilde{B} = D_{yy} + \frac{\tilde{c}}{y} D_y$, $\tilde{c} = \frac{c-\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$ and $0 < (\tilde{m}+1)/p < \tilde{c}+1$.

All the results for $y^\alpha B$ in L^p_m are then immediate consequence of those of $\tilde{B}$ in $L^p_{\tilde{m}}$ already proved in [13, Section 3] (see also [9–11,15] for analogous results in the multi-dimensional case).

If $1 < p < \infty$, we define

$$W^{2,p}_{\mathcal{N}}(\alpha, m) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}_+) : \ u,\ y^\alpha D_{yy} u,\ y^{\frac{\alpha}{2}} D_y u,\ y^{\alpha-1} D_y u \in L^p_m \right\}$$

and refer to [12] where these spaces are studied in detail in $\mathbb{R}^{N+1}_+$. The Neumann boundary condition, denoted by the pedix $\mathcal{N}$, is enclosed in the requirement $y^{\alpha-1} D_y u \in$

L^p_m. This last is redundant when $(m+1)/p > 1-\alpha$ and equivalent to $D_y u(y) \to 0$ as $y \to 0$, when $(m+1)/p < 1-\alpha$, see [12, Proposition 4.3].

Consequently, we write $y^\alpha B^n$ or, more pedantically $y^\alpha B^n_{m,p}$ if necessary, for the operator $y^\alpha B$ endowed with the domain $W^{2,p}_{\mathcal{N}}(\alpha, m)$. This time the suffix n reminds the Neumann boundary condition at $y=0$.

Remark 4.1. The restriction $\alpha < 2$ is not really essential since one can deduce from it the case $\alpha > 2$, which requires boundary condition at ∞, using the change of variables described in Sect. 3 or directly from the equality $T^{-1}_{-\frac{\alpha}{2}}\left(y^\alpha B\right)T_{-\frac{\alpha}{2}} = \left(1-\frac{\alpha}{2}\right)^2 \tilde{B}$ which is valid for any $\alpha \neq 2$. However, here and in what follows, we keep to it in order to simplify the exposition.

Theorem 4.2. *If* $0 < \frac{m+1}{p} < c+1-\alpha$, *then* $y^\alpha B^n$ *endowed with domain* $W^{2,p}_{\mathcal{N}}(\alpha, m)$ *generates a bounded positive analytic semigroup of angle* $\pi/2$ *on* $L^p\left(\mathbb{R}_+, y^m dy\right)$.

Proof. We use the identity $T^{-1}_{-\frac{\alpha}{2}}\left(y^\alpha B^n\right)T_{-\frac{\alpha}{2}} = \left(1-\frac{\alpha}{2}\right)^2 \tilde{B}^n$ and apply [13, Proposition 3.3] in $L^p_{\tilde{m}}$. Note that $D(y^\alpha B^n_{m,p}) = T_{-\frac{\alpha}{2}} D(\tilde{B}^n_{\tilde{m},p})$ which means

$$u \in D(y^\alpha B^n_{m,p}) \qquad \Longleftrightarrow \qquad v(y) := u(y^{\frac{2}{2-\alpha}}) \in D(\tilde{B}^n_{\tilde{m},p}).$$

□

Under the hypothesis of Theorem 4.2, the domain of $y^\alpha B^n$ consists of all functions in the maximal domain satisfying a Neumann condition at 0, see [12, Proposition 4.6, 4.7], that is

$$D(y^\alpha B^n_{m,p}) = \left\{u \in W^{2,p}_{loc}(\mathbb{R}_+) : u,\ y^\alpha Bu \in L^p_m \text{ and } \lim_{y\to 0} y^c D_y u = 0\right\}.$$

(The condition $\lim_{y\to 0} y^c D_y u = 0$ can be deleted in the range $0 < \frac{m+1}{p} \leq c-1$.) When $c \geq 1$, the domain can also be described involving a Dirichlet, rather than Neumann, boundary condition

$$D(y^\alpha B^n_{m,p}) = \left\{u \in W^{2,p}_{loc}(\mathbb{R}_+) : u,\ y^\alpha Bu \in L^p_m \text{ and } \lim_{y\to 0} y^{c-1} u = 0\right\}, \qquad \text{if } c > 1;$$

$$D(y^\alpha B^n_{m,p}) = \left\{u \in W^{2,p}_{loc}(\mathbb{R}_+) : u,\ y^\alpha Bu \in L^p_m \text{ and } \lim_{y\to 0} u \in \mathbb{C}\right\}, \qquad \text{if } c = 1.$$

We close this section by describing a core which does not depend on α, m, p and on the coefficients of the operator.

Proposition 4.3. *If* $0 < \frac{m+1}{p} < c+1-\alpha$, *then a core for* $y^\alpha B^n$ *is*

$$\mathcal{D} = \left\{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\right\}.$$

Proof. The proof immediately follows by observing that, by [13, Proposition 5.4], $\mathcal{D}$ is a core when $\alpha = 0$, that is for $\tilde{B}^n_{\tilde{m},p}$, and the isometry $T_{-\frac{\alpha}{2}}$ leaves invariant $\mathcal{D}$ since $\alpha < 2$. □

Remark 4.4. We point out that, by the proof of [13, Proposition 5.4] or by [12, Remark 4.14], it follows that if $u \in D(y^\alpha B^n_{m,p})$ has support in $[0,b]$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{D}$ such that supp $u_n \subseteq [0,b]$ and $u_n \to u$ in $D(y^\alpha B^n_{m,p})$.

5. The operator $B^n - V$

We start our investigation by adding a potential $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^c\,dy\right)$ to B^n. Here, we prove kernel bounds and construct a core.

5.1. Kernel bounds

For $c+1>0$ and $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^c\,dy\right)$, we prove upper bounds for the heat kernel of $B^n - V$, following the method used in [3, Sections 3, 4].

Setting $H^1_c = \{u \in L^2_c, u' \in L^2_c\}$, we recall that from [13, Section 2] the operator $B^n_{c,2}$ is associated with the nonnegative, symmetric and closed form in L^2_c

$$\mathfrak{a}(u,v) := \int_0^\infty D_y u D_y \overline{v}\, y^c dy, \qquad D(\mathfrak{a}) = H^1_c.$$

We consider the perturbed form $\mathfrak{a}_V$ in L^2_c defined by

$$\begin{aligned} \mathfrak{a}_V(u,v) &= \mathfrak{a}(u,v) + \langle Vu, v\rangle_{L^2_c} = \int_{\mathbb{R}_+} \left(D_y u D_y \overline{v} + V u \overline{v}\right) y^c\,dy \\ D(\mathfrak{a}_V) &= D(\mathfrak{a}) \cap L^2\left(\mathbb{R}^+, V y^c\,dy\right) \end{aligned} \tag{3}$$

and define $B^n - V$ in L^2_c as the operator associated with the form $\mathfrak{a}_V$

$$\begin{aligned} D(B^n - V) = \{u \in D(\mathfrak{a}_V) : \exists f \in L^2_c \text{ such that } \mathfrak{a}_V(u,v) \\ = \int_0^\infty f\overline{v} y^c\,dy \text{ for every } v \in D(\mathfrak{a}_V)\}, \\ B^n u - Vu = -f. \end{aligned}$$

The positivity of V implies that the norm induced by the form $\mathfrak{a}_V$ is stronger than the one induced by $\mathfrak{a}$: As an immediate consequence, one deduces that $\mathfrak{a}_V$ is closed. By standard theory on sesquilinear forms, we have the following result.

Proposition 5.1. *If $c+1>0$, $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^c\,dy\right)$, then $\mathfrak{a}_V$ is a nonnegative, symmetric and closed form in L^2_c. Its associated operator $-B^n + V$ is nonnegative and self-adjoint, and $B^n - V$ generates a contractive analytic semigroup $\left\{e^{z(B^n-V)} : z \in \mathbb{C}_+\right\}$ in L^2_c. Moreover:*

(i) The semigroup $\left(e^{t(B^n-V)}\right)_{t\ge 0}$ is sub-Markovian (i.e., it is positive and L^∞-contractive), and it is dominated by e^{tB^n}, that is

$$|e^{t(B^n-V)} f| \le e^{tB^n}|f|, \qquad t>0, \quad f \in L^2_c.$$

(ii) $\left(e^{t(B^n-V)}\right)_{t\ge 0}$ *is a semigroup of integral operators, and its heat kernel* p_V*, taken with respect to the measure* $\rho^c d\rho$*, satisfies*

$$0 \le p_V(t,y,\rho) \le C t^{-\frac{1}{2}} \rho^{-c} \left(\frac{\rho}{t^{\frac{1}{2}}} \wedge 1\right)^c \exp\left(-\frac{|y-\rho|^2}{\kappa t}\right).$$

Proof. The first claim follows from the property of $\mathfrak{a}_V$. $e^{t(B^n-V)}$ is sub-Markovian from [16, Corollary 2.17]. The domination property follows from [16, Corollary 2.21]. (ii) is a consequence of [2, Proposition 1.9] since $e^{t(B^n-V)}$ is dominated by the positive integral operator e^{tB^n} whose kernel satisfies the stated estimate, see [13, Proposition 2.8], where, however, the kernel is written with respect to the Lebesgue measure. □

To extend the above heat kernel estimates to the half-plane $\mathbb{C}_+$, we need the following lemma.

Lemma 5.2. *Let* $c+1>0$ *and for* $y_0, r > 0$

$$Q_c(y_0,r) := \int_{[y_0,y_0+r]} y^c \mathrm{d}y.$$

Then one has

$$Q_c(y_0,r) \simeq r^{c+1}\left(\frac{y_0}{r}\right)^c \left(\frac{y_0}{r}\wedge 1\right)^{-c}, \qquad r, y_0 > 0.$$

In particular, the function Q_c *satisfies, for some constants* $C \ge 1$*, the doubling condition*

$$\frac{Q_c(y_0,s)}{Q_c(y_0,r)} \le C\left(\frac{s}{r}\right)^{1\vee(c+1)}, \qquad \forall y_0>0, \quad 0<r<s.$$

Proof. A scaling argument immediately yields $Q_c(y_0,r) = r^{c+1} Q_c\left(\frac{y_0}{r},1\right)$, and we may therefore assume $r=1$. The local integrability of y^c implies that $Q_c(y_0,1)$ is continuous as a function of y_0 and moreover $Q_c(y_0,1) \to \int_{(0,1)} y^c \mathrm{d}y > 0$ as $y_0 \to 0$. Therefore, if $y_0 \le 1$, then

$$Q_c(y_0,1) \simeq 1.$$

On the other hand, if $y_0 > 1$, then $y \simeq y_0$ for any $y \in (y_0, y_0+1)$ which implies

$$Q_c(y_0,1) = \int_{(y_0,y_0+1)} y^c \mathrm{d}y \simeq y_0^c.$$

The last two inequalities yield $Q_c(y_0,1) \simeq (y_0)^c\,(y_0\wedge 1)^{-c}$. The doubling condition follows from the previous estimates and the fact that for $0<r<s$ one has

$$\frac{Q_c(y_0,s)}{Q_c(y_0,r)} \le C \begin{cases} \left(\frac{s}{r}\right)^{c+1}, & \text{if } \frac{y_0}{s} \le \frac{y_0}{r} \le 1; \\ \frac{s}{r}\left(\frac{s}{y_0}\right)^c, & \text{if } \frac{y_0}{s} \le 1 < \frac{y_0}{r}; \\ \frac{s}{r}, & \text{if } 1 \le \frac{y_0}{s} \le \frac{y_0}{r}. \end{cases}$$

(Note that in the range $\frac{y_0}{s} \le 1 < \frac{y_0}{r}$ one has $(\frac{s}{y_0})^c \le 1$ if $c<0$ and $(\frac{s}{y_0})^c \le \left(\frac{s}{r}\right)^c$ if $c \ge 0$.) □

Proposition 5.3. *Let* $c+1>0$, $0\le V\in L^1_{loc}\left(\mathbb{R}^+, y^c\,\mathrm{d}y\right)$. *The semigroup* $\left\{e^{z(B^n-V)}:\ z\in\mathbb{C}_+\right\}$ *consists of integral operators*

$$e^{z(B^n-V)}f(y)=\int_0^\infty p_V(z,y,\rho)f(\rho)\,\rho^c d\rho,\quad f\in L^2_c,\quad y>0.$$

Furthermore for every $\epsilon>0$, *there exist* $k_\epsilon, C_\epsilon>0$ *such that, for every* $z\in\Sigma_{\frac{\pi}{2}-\epsilon}$ *and* $y,\rho>0$,

$$|p_V(z,x,y)|\le C_\epsilon|z|^{-\frac{1}{2}}\rho^{-c}\left(\frac{\rho}{|z|^{\frac{1}{2}}}\wedge 1\right)^c\exp\left(-\frac{|y-\rho|^2}{\kappa_\epsilon|z|}\right).$$

Proof. Using the previous lemma, we rewrite Proposition 5.1 (ii) as

$$0\le p_V(t,y,\rho)\le C\frac{1}{Q_c(\rho,\sqrt{t})}\exp\left(-\frac{|y-\rho|^2}{\kappa t}\right).$$

Furthermore by [4, Theorem 3.3], $e^{t(B^n-V)}$ satisfies the Davies–Gaffney estimates

$$|\langle e^{t(B^n-V)}f_1, f_2\rangle|\le\exp\left(-\frac{r^2}{4t}\right)\|f_1\|_{L^2_c}\|f_2\|_{L^2_c}$$

for all $t>0$, U_1,U_2 open subsets of $(0,+\infty)$, $r:=d(U_1,U_2)=\min\{|x-y|:x\in U_1, y\in U_2\}$ and f_i in $L^2(U_i, y^c\mathrm{d}y)$. By [4, Corollary 4.4] and Lemma 5.2, we then obtain for $z\in\Sigma_{\frac{\pi}{2}-\epsilon}$ and $y,\rho>0$

$$\begin{aligned}|p_V(z,y,\rho)|&\le C_\epsilon\frac{1}{\left(Q_c(y,\sqrt{|z|})\right)^{\frac{1}{2}}\left(Q_c(\rho,\sqrt{|z|})\right)^{\frac{1}{2}}}\exp\left(-\frac{|y-\rho|^2}{\kappa_\epsilon|z|}\right)\\&\le C'_\epsilon|z|^{\frac{c+1}{2}}\left(\frac{y}{\sqrt{|z|}}\right)^{-\frac{c}{2}}\left(1\wedge\frac{y}{\sqrt{|z|}}\right)^{\frac{c}{2}}\\&\quad\left(\frac{\rho}{\sqrt{|z|}}\right)^{-\frac{c}{2}}\left(1\wedge\frac{\rho}{\sqrt{|z|}}\right)^{\frac{c}{2}}\exp\left(-\frac{|y-\rho|^2}{\kappa_\epsilon|z|}\right).\end{aligned}$$

This is an equivalent form (after modifying the constant in the exponential) of the estimate in the statement, by [13, Lemma 10.2] with $\gamma_1=\gamma_2=-\frac{c}{2}$. □

Remark 5.4. We remark that in [4], the authors work in an abstract metric measure space (M,d,μ) and assume that the heat kernel p associated with a semigroup e^{-zL}, where L is a nonnegative self-adjoint operator on $L^2(M,d\mu)$, is continuous with respect to the space variables. In such a case, in fact,

$$\underset{x\in U_1,y\in U_2}{sup}|p(z,x,y)|=\sup\{\int_M e^{-zL}f_1\overline{f_2}\,d\mu,\quad\|f_1\|_{L^1(U_1,d\mu)}=\|f_2\|_{L^1(U_2,d\mu)}=1\}.$$

In our setting, the continuity assumption on p can be avoided since the proofs of [4, Theorem 4.1, Corollary 4.4] hold only assuming that for a.e. $x, y \in M$

$$p(z,x,y) = \lim_{s\to 0}\int_M e^{-zL} f_1\overline{f_2}\, d\mu = \lim_{s\to 0}\frac{1}{\mu(B(x,s)\mu(B(y,s)}$$
$$\int_{B(x,s)\times B(y,s)} p(z,\bar{x},\bar{y}), d\mu(\bar{x})d\mu(\bar{y}),$$

where $f_1 = \frac{\chi_{B(x,s)}}{\mu(B(x,s))}$, $f_2 = \frac{\chi_{B(y,s)}}{\mu(B(y,s))}$. This holds, outside a set of zero measure, when the measure μ is doubling, by the Lebesgue differentiation theorem.

5.2. A core for $B^n - V$

We prove that under mild hypotheses the set

$$\mathcal{D} = \left\{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\right\}$$

is a core for $B^n - V$ in L^2_c. Note that this is true when $V = 0$, by Proposition 4.3. We need some elementary lemmas. Unless explicitly stated, we only assume that $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^c\,\mathrm{d}y\right)$.

Lemma 5.5. *Assume that* $0 \le V \in L^2_{loc}(\mathbb{R}_+, y^c\,\mathrm{d}y)$. *Then,* $D(\mathfrak{a}_V) = H^1_c \cap L^2\left(\mathbb{R}^+, Vy^c\,\mathrm{d}y\right)$ *is dense in* H^1_c.

Proof. By Proposition 4.3, $\mathcal{D}$ is dense in $D(B^n)$ with respect to the graph norm. Moreover, since $V \in L^2_c$ locally, $\mathcal{D} \subset D(\mathfrak{a}_V)$. The claim follows from the density of $D(B^n)$ in H^1_c. □

Lemma 5.6. *Let* $u \in H^1_c$ *such that* $Vu \in L^2_c$. *Then,* $u \in D(B^n)$ *if and only if* $u \in D(B^n - V)$. *Moreover,*

$$(B^n - V)u = Bu - Vu.$$

Proof. Let $u \in D(B^n)$. Then, $u \in D(\mathfrak{a})$ and there exists $f \in L^2_c$ such that

$$\mathfrak{a}(u,v) = \int_0^\infty D_y u D_y \overline{v} y^c\,\mathrm{d}y = \int_0^\infty f\overline{v}y^c\,\mathrm{d}y$$

for every $v \in H^1_c$. Setting $g = f + Vu \in L^2_c$, we have

$$\mathfrak{a}_V(u,v) = \int_0^\infty (D_y u D_y \overline{v} + Vu\overline{v})y^c\,\mathrm{d}y = \int_0^\infty (f + Vu)\overline{v}y^c\,\mathrm{d}y$$

for every $v \in H^1_c$ and, in particular, for every $v \in D(\mathfrak{a}_V) \subseteq H^1_c$. Therefore $u \in D(B^n - V)$. Conversely, if $u \in D(B^n - V)$, then $u \in D(\mathfrak{a}_V)$ and there exists $g \in L^2_c$ such that

$$\mathfrak{a}_V(u,v) = \int_0^\infty (D_y u D_y \overline{v} + Vu\overline{v})y^c\,\mathrm{d}y = \int_0^\infty g\overline{v}y^c\,\mathrm{d}y$$

for every $v \in D(\mathfrak{a}_V)$. Setting $f = g - Vu \in L^2_c$, we have that

$$\mathfrak{a}(u, v) = \int_0^\infty f \overline{v} y^c \, dy$$

for every $v \in D(\mathfrak{a}_V)$, hence for every $v \in H^1_c$, by Lemma 5.5. □

Lemma 5.7. *Let $u \in D(B^n - V)$ and η be a smooth function such that $\eta = 1$ for $0 \le y \le 1$ and $\eta = 0$ for $y \ge 2$. Then, $\eta u \in D(B^n - V)$ and*

$$(B^n - V)(\eta u) = \eta (B^n - V)u + 2 D_y \eta D_y u + u D_{yy} \eta + cu \frac{D_y \eta}{y}.$$

Proof. Let $u \in D(B^n - V)$, then $\eta u \in D(\mathfrak{a}_V)$ and, setting $f = (B^n - V)u$,

$$\begin{aligned}
\mathfrak{a}_V(\eta u, v) &= \int_0^\infty (D_y(\eta u) D_y \overline{v} + V \eta u \overline{v}) y^c \, dy \\
&= \int_0^\infty (D_y u D_y(\eta \overline{v}) + V u \eta \overline{v} + u D_y \eta D_y \overline{v} - D_y u D_y \eta \overline{v}) y^c \, dy \\
&= -\int_0^\infty \eta f \overline{v} y^c \, dy - \int_0^\infty D_y u D_y \eta \overline{v} y^c \, dy + \int_0^\infty u D_y \eta D_y \overline{v} y^c \, dy \\
&= -\int_0^\infty \eta f \overline{v} y^c \, dy - \int_0^\infty D_y u D_y \eta \overline{v} y^c \, dy - \int_0^\infty \overline{v} D_y(u D_y \eta y^c) \, dy \\
&= -\int_0^\infty \eta f \overline{v} y^c \, dy - 2 \int_0^\infty D_y u D_y \eta \overline{v} y^c \, dy \\
&\quad - \int_0^\infty \overline{v} u D_{yy} \eta y^c \, dy - \int_0^\infty \frac{cu}{y} \overline{v} D_y \eta y^c \, dy
\end{aligned}$$

for every $v \in D(\mathfrak{a}_V)$. □

Lemma 5.8. *Let $u \in D(B^n - V)$. Then, there exists $(u_k) \subseteq D(B^n - V)$ with compact support such that $(u_k) \to u$ in $D(B^n - V)$.*

Proof. Let η be a smooth function such that $\eta = 1$ for $0 \le y \le 1$ and $\eta = 0$ for $y \ge 2$. Setting $\eta_k(y) = \eta\left(\frac{y}{k}\right)$, by Lemma 5.7, $u_k = \eta_k u \in D(B^n - V)$ and

$$(B^n - V)(\eta_k u) = \eta_k (B^n - V)u + 2 D_y \eta_k D_y u + u D_{yy} \eta_k + \frac{cu}{y} D_y \eta_k.$$

Then, $u_k \to u$, $\eta_k (B^n - V)u \to (B^n - V)u$ in L^2_c by dominated convergence and, since $D_y \eta_k = 0$ in $[0, 1]$,

$$\left| D_y \eta_k D_y u + u D_{yy} \eta_k + \frac{cu}{y} D_y \eta_k \right| \le C \left(\frac{|u|}{k} + \frac{|u|}{k^2} + \frac{|D_y u|}{k} \right) \chi_{[k, \infty[} \to 0.$$

□

Lemma 5.7 shows that functions with compact support are a core for $B^n - V$. To show that $\mathcal{D}$ is a core, we need more information on the behavior near $y = 0$ of functions in the domain of $B^n - V$.

We start by recalling some well-known facts about the modified Bessel functions I_ν and K_ν which constitute a basis of solutions of the modified Bessel equation

$$z^2\frac{d^2 v}{dz^2} + z\frac{dv}{dz} - (z^2+\nu^2)v = 0, \quad Re\, z > 0.$$

We recall that for $Re\, z > 0$ one has

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+1+m)}\left(\frac{z}{2}\right)^{2m}, \quad K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu},$$

where limiting values are taken for the definition of K_ν when ν is an integer. The basic properties of these functions we need are collected in the following lemma, see, e.g., [1, Sections 9.6 and 9.7].

Lemma 5.9. *For $\nu > -1$, I_ν is increasing and K_ν is decreasing (when restricted to the positive real half-line). Moreover, they satisfy the following properties if $z \in \Sigma_{\pi/2-\varepsilon}$.*

(i) $I_\nu(z) \neq 0$ *for every* $\mathrm{Re}\, z > 0$.

(ii) $I_\nu(z) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu$, *as* $|z| \to 0$, $\quad I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}}(1+O(|z|^{-1}))$, *as* $|z| \to \infty$.

(iii) If $\nu \neq 0$, $K_\nu(z) \approx \frac{\nu}{|\nu|}\frac{1}{2}\Gamma(|\nu|)\left(\frac{z}{2}\right)^{-|\nu|}$, $\quad K_0(z) \approx -\log z$, $\quad$ *as* $|z| \to 0$

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}e^{-z}, \quad \textit{as } |z| \to \infty.$$

(iv) $I'_\nu(z) = I_{\nu+1}(z) + \frac{\nu}{z}I_\nu(z)$, $K'_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z}K_\nu(z)$, *for every* $\mathrm{Re}\, z > 0$.

Note that

$$|I_\nu(z)| \simeq C_{\nu,\epsilon}(1 \wedge |z|)^{\nu+\frac{1}{2}}\frac{e^{Re z}}{\sqrt{|z|}}, \qquad z \in \Sigma_{\frac{\pi}{2}-\epsilon} \tag{4}$$

for suitable constants $C_{\nu,\epsilon} > 0$ which may be different in lower an in the upper estimate.

The following estimates of the resolvent operator of $B^n - V$ are a consequence of the domination property stated in Proposition 5.1.

Proposition 5.10. *Let $c + 1 > 0$ and $\lambda > 0$. Then, for every $f \in L^2_c$,*

$$(\lambda - B^n + V)^{-1} f = \int_0^\infty G(\lambda, y, \rho) f(\rho)\rho^c d\rho$$

with

$$0 \leq G(\lambda, y, \rho) \leq G^n(\lambda, y, \rho)$$

where

$$G^n(\lambda, y, \rho) := \begin{cases} y^{\frac{1-c}{2}} \rho^{\frac{1-c}{2}} I_{\frac{c-1}{2}}(\sqrt{\lambda}\, y) K_{\frac{|1-c|}{2}}(\sqrt{\lambda}\, \rho) & y \le \rho \\ [1.5ex] y^{\frac{1-c}{2}} \rho^{\frac{1-c}{2}} I_{\frac{c-1}{2}}(\sqrt{\lambda}\, \rho) K_{\frac{|1-c|}{2}}(\sqrt{\lambda}\, y) & y \ge \rho, \end{cases} \tag{5}$$

is the integral kernel (taken with respect to the measure $\rho^c d\rho$) of the operator $(\lambda - B^n)^{-1}$.

Proof. Writing $(\lambda - B^n + V)^{-1} = \int_0^\infty e^{-\lambda t} e^{t(B^n - V)} \mathrm{d}t$ and using property (i) of Proposition 5.1, we get that

$$|(\lambda - B^n + V)^{-1} f| \le (\lambda - B^n)^{-1} |f|, \qquad \lambda > 0, \quad f \in L^2_c.$$

This yields the domination $G(\lambda, y, \rho) \le G^n(\lambda, y, \rho)$. (The existence of the kernel follows by [2, Proposition 1.9] as in Proposition 5.1.) Formula (5) is proved in [13, Proposition 2.4]. □

We now prove local pointwise estimates for functions in the domain of $B^n - V$.

Proposition 5.11. *Let $c + 1 > 0$. Then, there exists $C > 0$, independent of V, such that for every $u \in D(B^n - V)$ and $0 < y < 1$*

(i) if $-1 < c < 3$

$$|u(y)| \le C \left(\|u\|_{L^2_c} + \|(B - V)u\|_{L^2_c} \right),$$

(ii) if $c = 3$

$$|u(y)| \le C \left(\|u\|_{L^2_c} + \|(B - V)u\|_{L^2_c} \right) |\log y|^{\frac{1}{2}},$$

(iii) if $c > 3$

$$|u(y)| \le C \left(\|u\|_{L^2_c} + \|(B - V)u\|_{L^2_c} \right) y^{\frac{3-c}{2}}.$$

Proof. Let $u \in D(B^n - V)$ and $f = u - (B^n - V)u \in L^2_c$ so that $u = (I - B^n + V)^{-1} f$. Let us distinguish between the following cases and always take $0 < y < 1$.

(i) If $-1 < c < 1$, Lemma 5.9 implies that for $y \le 1$

$$G(1, y, \rho) \simeq \begin{cases} 1, & \rho < 1, \\ \rho^{-\frac{c}{2}} e^{-\rho}, & 1 < \rho. \end{cases}$$

Then, one has

$$\begin{aligned} |u(y)| &\le \int_0^\infty G(1, y, \rho) |f(\rho)| \rho^c d\rho \le C \left(\int_0^1 |f(\rho)| \rho^c d\rho + \int_1^\infty \rho^{-\frac{c}{2}} e^{-\rho} |f(\rho)|\, \rho^c d\rho \right) \\ &\le C \left(\|f\|_{L^2_c((0,1))} + \|\rho^{-\frac{c}{2}} e^{-\rho}\|_{L^2_c((1,\infty))} \|f\|_{L^2_c((1,\infty))} \right) \le C \|f\|_{L^2_c}. \end{aligned}$$

(ii) If $c=1$, Lemma 5.9 gives for $y \le 1$

$$G(1,y,\rho) \simeq \begin{cases} |\log y| \le |\log \rho|, & \rho < y < 1, \\ |\log \rho|, & y < \rho < 1, \\ \rho^{-\frac{1}{2}} e^{-\rho}, & 1 < \rho. \end{cases}$$

Then, analogously

$$\begin{aligned} |u(y)| &\le C\left(\int_0^1 |\log \rho||f(\rho)|\rho d\rho + \int_1^\infty \rho^{\frac{1}{2}} e^{-\rho}|f(\rho)|d\rho\right) \\ &\le C\left(\|\log \rho\|_{L^2_c((0,1))}\|f\|_{L^2_c((0,1))} + \|\rho^{-\frac{1}{2}} e^{-\rho}\|_{L^2_c((1,\infty))}\|f\|_{L^2_c((1,\infty))}\right) \le C\|f\|_{L^2_c}. \end{aligned}$$

(iii) Let now $1<c$. Then, Lemma 5.9 implies that for $y \le 1$

$$G(1,y,\rho) \simeq \begin{cases} y^{1-c} \le \rho^{1-c}, & \rho < y < 1, \\ \rho^{1-c}, & y < \rho < 1, \\ \rho^{-\frac{c}{2}} e^{-\rho}, & 1 < \rho. \end{cases}$$

If $c<3$, one has

$$\begin{aligned} |u(y)| &\le C\left(\int_0^1 \rho^{1-c}|f(\rho)|\rho^c d\rho + \int_1^\infty \rho^{-\frac{c}{2}} e^{-\rho}|f(\rho)|\rho^c d\rho\right) \\ &\le C\left(\|\rho^{1-c}\|_{L^2_c((0,1))}\|f\|_{L^2_c((0,1))} + \|\rho^{-\frac{c}{2}} e^{-\rho}\|_{L^2_c((1,\infty))}\|f\|_{L^2_c((1,\infty))}\right) \le C\|f\|_{L^2_c}. \end{aligned}$$

If $c=3$, then we get

$$\begin{aligned} |u(y)| &\le C\left(y^{-2}\int_0^y |f(\rho)|\rho^3 d\rho + \int_y^1 \rho^{-2}|f(\rho)|\rho^3 d\rho + \int_1^\infty \rho^{-\frac{3}{2}} e^{-\rho}|f(\rho)|\rho^3 d\rho\right) \\ &\le C\|f\|_{L^2_c}\left(y^{-2}\left(\int_0^y \rho^3 d\rho\right)^{\frac{1}{2}} + \left(\int_y^1 \rho^{-4}\rho^3 d\rho\right)^{\frac{1}{2}} + \|\rho^{-\frac{3}{2}} e^{-\rho}\|_{L^2_c((1,\infty))}\right) \\ &\le C\|f\|_{L^2_c}\left(1+|\log y|^{\frac{1}{2}}\right) \end{aligned}$$

and finally if $c>3$

$$\begin{aligned} |u(y)| &\le C\left(y^{1-c}\int_0^y |f(\rho)|\rho^c d\rho + \int_y^1 \rho^{1-c}|f(\rho)|\rho^c d\rho + \int_1^\infty \rho^{-\frac{c}{2}} e^{-\rho}|f(\rho)|\rho^c d\rho\right) \\ &\le C\|f\|_{L^2_c}\left(y^{1-c}\left(\int_0^y \rho^c d\rho\right)^{\frac{1}{2}} + \left(\int_y^1 \rho^{2-2c}\rho^c d\rho\right)^{\frac{1}{2}} + \|\rho^{-\frac{c}{2}} e^{-\rho}\|_{L^2_c((1,\infty))}\right) \\ &\le C\|f\|_{L^2_c} y^{\frac{3-c}{2}}. \end{aligned}$$

□

We can now show that, under stronger assumptions, the potential term V can be seen as a perturbation of B^n near 0, that is $Vu \in L^2_c$ for every $u \in D(B_n)$ having compact support. In particular, we prove that $\mathcal{D}$ is a core for $B^n - V$.

Proposition 5.12. *Let $c+1>0$ and assume that*

(i) $c<3$ *and* $V \in L^2_{loc}\left(\mathbb{R}^+, y^c dy\right)$ *or*

(ii) $c=3$ *and* $V|\log y|^{\frac{1}{2}} \in L^2_{loc}\left(\mathbb{R}^+, y^c dy\right)$ *or*

(iii) $c>3$ *and* $V y^{\frac{3-c}{2}} \in L^2_{loc}\left(\mathbb{R}^+, y^c dy\right)$.

If $\mathcal{C}_r := \{u \in L^2_c : \text{supp}\, u \subseteq [0,r]\}$, *then* $D(B^n - V)\cap \mathcal{C}_r = D(B^n)\cap \mathcal{C}_r$ *with equivalence of norms*

$$\|u\|_{D(B^n-V)} \simeq \|u\|_{D(B^n)}, \quad \forall u \in D(B^n)\cap \mathcal{C}_r.$$

Finally,

$$\mathcal{D} = \left\{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\right\}$$

is a core for $B^n - V$.

Proof. Let $u \in \mathcal{C}_r$. Then, the hypotheses on V and Proposition 5.11 imply that $Vu \in L^2_c$ and $\|Vu\|_{L^2_c} \le C\|u-(B-V)u\|_{L^2_c}$. Then, by Lemma 5.6 $u \in D(B^n - V)$ if and only if $u \in D(B^n)$. This shows the equality $D(B^n - V)\cap \mathcal{C}_r = D(B^n)\cap \mathcal{C}_r$. Using Proposition 5.11 again, we also have $\|Vu\|_{L^2_c} \le C_1\|u - Bu\|_{L^2_c}$ for any $u \in D(B^n)\cap \mathcal{C}_r$, which proves the equivalence of the graph norms. Finally, let $u \in D(B^n - V)$. We have to prove that u can be approximated in the graph norm with functions belonging to $\mathcal{D}$. Using Lemma 5.8, we may suppose, without any loss of generality, that $\text{supp}\, u \subseteq (0,r)$. Then, by Proposition 4.3, there exist $(u_n) \subset \mathcal{D}$ such that $u_n \to u$ in the graph norm $\|\cdot\|_{D(B^n)}$. We may also assume, after multiplying by a suitable cutoff function, that $\text{supp}\, u_n \subseteq (0,2r)$ for every n. Then, the previous point implies that $Vu_n \to Vu$ in L^2_c, too. □

6. The operator $y^\alpha B^n - V$ in $L^2_{c-\alpha}$

We consider now for $c \in \mathbb{R}$, $\alpha < 2$, and $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\, dy\right)$ the operator

$$y^\alpha B^n - V = y^\alpha\left(D_{yy} + \frac{c}{y}D_y\right) - V$$

in the space $L^2_{c-\alpha}$. As in Sect. 4, we use the isometry $T_{-\frac{\alpha}{2}}u(y) = \left|1-\frac{\alpha}{2}\right|^{\frac{1}{p}} u(y^{1-\frac{\alpha}{2}})$,

$$T_{-\frac{\alpha}{2}} : L^2_{\tilde{c}} \to L^2_{c-\alpha}, \qquad \tilde{c} = \frac{c-\frac{\alpha}{2}}{1-\frac{\alpha}{2}},$$

under which $y^\alpha B - V$ becomes similar to

$$T^{-1}_{-\frac{\alpha}{2}}\left(y^\alpha B - V\right)T_{-\frac{\alpha}{2}} = \left(1-\frac{\alpha}{2}\right)^2\left(\tilde{B} - \tilde{V}\right)$$

where $\tilde{B} = D_{yy} + \frac{\tilde{c}}{y}D_y$ and $\tilde{V}(y) = (1-\frac{\alpha}{2})^{-2}V\left(y^{\frac{2}{2-\alpha}}\right) \in L^1_{loc}\left(\mathbb{R}^+, y^{\tilde{c}}\, dy\right)$.

Defining

$$D(y^\alpha B^n - V) := T_{-\frac{\alpha}{2}}\left(D(\tilde{B^n} - \tilde{V})\right),$$

one obtains that when $c > -1+\alpha$, $y^\alpha B^n - V$ generates a contractive analytic semigroup $\left\{e^{z(y^\alpha B^n - V)} : z \in \mathbb{C}_+\right\}$ in $L^2_{c-\alpha}$ which satisfies

$$e^{z(y^\alpha B - V)} = T_{-\frac{\alpha}{2}}\left(e^{z(1-\frac{\alpha}{2})^2\left(\tilde{B} - \tilde{V}\right)}\right) T^{-1}_{-\frac{\alpha}{2}}. \tag{6}$$

We state the properties obtained so far, together with a density result which is a restating of Proposition 5.12 under the isometry $T_{-\frac{\alpha}{2}}$.

Proposition 6.1. *Let $c+1-\alpha > 0$ and $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)$. Then, the operator $y^\alpha B^n - V$ generates a contractive analytic semigroup in $L^2_{c-\alpha}$. If, in addition,*

(i) $c < 3-\alpha$ and $V \in L^2_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)$ or

(ii) $c = 3-\alpha$ and $V|\log y|^{\frac{1}{2}} \in L^2_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)$ or

(iii) $c > 3-\alpha$ and $Vy^{\frac{3-c-\alpha}{2}} \in L^2_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)$,

then

$$\mathcal{D} = \left\{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\right\}$$

is a core for $y^\alpha B^n - V$ in $L^2_{c-\alpha}$.

Remark 6.2. If $V(y) = y^\alpha$, then V always satisfies (ii) and (iii) when $c \ge 3-\alpha$. Instead, if $c < 3-\alpha$, we need $c+1-|\alpha| > 0$.

Let $\mathfrak{a}_{\tilde{V}}$ be the form in $L^2_{\tilde{c}}$, defined in (3), associated with $\tilde{B}^n - \tilde{V}$. In $L^2_{c-\alpha}$, we introduce the form $\mathfrak{a}_{\alpha,V}$ which is the image of $\mathfrak{a}_{\tilde{V}}$ under the isometry $T_{0,-\frac{\alpha}{2}}$, that is

$$\mathfrak{a}_{\alpha,V}(u,v) := \mathfrak{a}_{\tilde{V}}\left(T^{-1}_{-\frac{\alpha}{2}}u, T^{-1}_{-\frac{\alpha}{2}}v\right) = \int_{\mathbb{R}_+}\left(y^\alpha D_y u D_y\overline{v} + Vu\overline{v}\right) y^{c-\alpha}\,dy,$$
$$D(\mathfrak{a}_{\alpha,V}) := T_{-\frac{\alpha}{2}} D(\mathfrak{a}_{\tilde{V}}) = \left\{u \in L^2_{c-\alpha} : u' \in L^2_c\right\} \cap L^2\left(\mathbb{R}^+, Vy^{c-\alpha}\,dy\right). \tag{7}$$

To keep consistency of notation, we often write $\mathfrak{a}_{0,V} = \mathfrak{a}_V$. By construction, $y^\alpha B^n - V$ is the operator associated with the form $\mathfrak{a}_{\alpha,V}$ in $L^2_{c-\alpha}$

$$D(y^\alpha B^n - V) = \{u \in D(\mathfrak{a}_{\alpha,V}) : \exists f \in L^2_{c-\alpha} \quad \text{such that}$$
$$\mathfrak{a}_{\alpha,V}(u,v) = \int_0^\infty f\overline{v}y^{c-\alpha}\,dy \text{ for every } v \in D(\mathfrak{a}_{\alpha,V})\},$$
$$y^\alpha B^n u - Vu = -f.$$

The next lemma, which follows from the considerations above, will be used later to relate the resolvents of $y^\alpha B^n - y^\alpha$ and $B^n - y^{-\alpha}$.

Lemma 6.3. *Let $\mathfrak{a}_{\alpha,y^\alpha}$ and $\mathfrak{a}_{y^{-\alpha}}$ be the sesquilinear forms associated, respectively, with the operator $y^\alpha B^n - y^\alpha$ in $L^2_{c-\alpha}$ and $B^n - y^{-\alpha}$ in L^2_c. Then,*

$$\mathfrak{a}_{\alpha,y^\alpha}(u,v) = \int_{\mathbb{R}_+} \left(D_y u D_y \overline{v} + u\overline{v}\right) y^c \, dy,$$
$$\mathfrak{a}_{y^{-\alpha}}(u,v) = \int_{\mathbb{R}_+} \left(D_y u D_y \overline{v} + y^{-\alpha} u\overline{v}\right) y^c \, dy.$$

on the common form domain

$$D(\mathfrak{a}_{\alpha,y^\alpha}) = D(\mathfrak{a}_{y^{-\alpha}}) = \left\{u \in L^2_{c-\alpha} \cap L^2_c : u' \in L^2_c\right\}$$

Note that the above operators act in different Hilbert spaces; in particular, their domains are different. However, the form domains coincide.

7. The operator $y^\alpha B^n - V$ in L^p_m

Here, we investigate properties of $y^\alpha B - V, \alpha < 2$, in L^p_m when $0 < \frac{m+1}{p} < c+1-\alpha$.

We introduce the family of integral operators $(S^\beta_\alpha(t))_{t>0}$ on L^p_m

$$S^\beta_\alpha(t) f(y) := t^{-\frac{1}{2}} \int_{\mathbb{R}_+} \left(\frac{\rho}{t^{\frac{1}{2-\alpha}}} \wedge 1\right)^{-\beta+\frac{\alpha}{2}} \exp\left(-\frac{|y^{1-\frac{\alpha}{2}} - \rho^{1-\frac{\alpha}{2}}|^2}{\kappa t}\right) f(\rho)\rho^{-\frac{\alpha}{2}} \, d\rho$$

and note that

$$S^\beta_\alpha(t) = T_{-\frac{\alpha}{2}} \circ S^{\tilde{\beta}}_0(t) \circ T^{-1}_{-\frac{\alpha}{2}}, \qquad \tilde{\beta} = \frac{\beta - \frac{\alpha}{2}}{1 - \frac{\alpha}{2}}.$$

As usual $T_{-\frac{\alpha}{2}} u(y) = \left|1 - \frac{\alpha}{2}\right|^{\frac{1}{p}} u(y^{1-\frac{\alpha}{2}})$ is an isometry from $L^p_{\tilde{m}}$ onto L^p_m, $\tilde{m} = \frac{m+\frac{\alpha}{2}}{1-\frac{\alpha}{2}}$. Here, κ is a positive constant, but we omit the dependence on it. The following result has been proved for $\alpha = 0$ in [13].

Lemma 7.1. *Let $m \in \mathbb{R}$, and let $p \in (1,\infty)$ such that $0 < \frac{m+1}{p} < 1 - \alpha - \beta$. The families $\left(S^\beta_\alpha(t)\right)_{t\geq 0}$ and $\{\Gamma(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S^\beta_\alpha(t)\, dt, \lambda > 0\}$ are $\mathcal{R}$-bounded in L^p_m.*

Proof. Since the $\mathcal{R}$-boundedness is preserved under isometries, from $S^\beta_\alpha(t) = T_{-\frac{\alpha}{2}} \circ S^{\tilde{\beta}}_0(t) \circ T^{-1}_{-\frac{\alpha}{2}}$ we may assume that $\alpha = 0$. (Note that $0 < \frac{m+1}{p} < -\beta+1-\alpha$ is equivalent to $0 < \frac{\tilde{m}+1}{p} < -\tilde{\beta} + 1$.) The first result is then a consequence of [13, Theorem 7.7]. The family

$$\Gamma(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S^\beta_\alpha(t)\, dt, \quad \lambda > 0$$

is $\mathcal{R}$-bounded by [7, Corollary 2.14]. □

We can now prove our main results for the operator $y^\alpha B - V$.

Theorem 7.2. *Let* $0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\,dy\right)$. *For any* $p \in (1,\infty)$ *such that* $0 < \frac{m+1}{p} < c+1-\alpha$, *the semigroup* $e^{z(y^\alpha B^n - V)}$ *initially defined on* $L^2_{c-\alpha}$ *extends to a bounded analytic semigroup on* L^p_m *of angle* $\pi/2$ *which consists of integral operators. Moreover, the generated semigroup has maximal regularity and the following properties hold.*

(i) For every $\epsilon > 0$, *there exist* $C = C(\epsilon,\alpha) > 0$ *(independent of* V*) such that*

$$\left|e^{z(y^\alpha B^n - V)} f\right| \le C S_\alpha^{-c}(|z|)|f|, \quad f \in L^p_m, \quad |\arg z| < \frac{\pi}{2} - \epsilon.$$

(ii) For every $\epsilon > 0$, *the families of operators*

$$\left\{e^{z(y^\alpha B^n - V)} : z \in \Sigma_{\frac{\pi}{2}-\epsilon},\ 0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)\right\},$$
$$\left\{\lambda\left(\lambda - y^\alpha B^n + V\right)^{-1} : \lambda \in \Sigma_{\pi-\epsilon} : 0 \le V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right)\right\}$$

are $\mathcal{R}$*-bounded in* L^p_m.

Proof. By Proposition 5.3 and (6), (i) holds for any $f \in L^2_{c-\alpha}$. The boundedness of $e^{z(y^\alpha B^n - V)}$ in L^p_m follows from the previous lemma, and (i) extends to L^p_m. The semigroup law is inherited from $L^2_{c-\alpha}$ via a density argument, and we have only to prove the strong continuity at 0. Using the isometry $T_{-\frac{\alpha}{2}}$, we may suppose that $\alpha = 0$. Let $f, g \in C_c^\infty(0,\infty)$. Then as $z \to 0$, $z \in \Sigma_{\frac{\pi}{2}-\epsilon}$,

$$\int_0^\infty (e^{z(B^n - V)} f)\, g\, y^m \,dy = \int_0^\infty (e^{z(B^n - V)} f)\, g\, y^{m-c} y^c \,dy$$
$$\to \int_0^\infty f g y^{m-c} y^c \,dy = \int_0^\infty f g y^m \,dy,$$

by the strong continuity of $e^{z(B^n - V)}$ in L^2_c. By density and uniform boundedness of the family $(e^{z(B^n - V)})_{z \in \Sigma_{\frac{\pi}{2}-\epsilon}}$, this holds for every $f \in L^p_m$, $g \in L^{p'}_m$. The semigroup is then weakly continuous, hence strongly continuous.

The $\mathcal{R}$-boundedness of $e^{z(y^\alpha B^n - V)}$ follows then by domination from Lemma 7.1, see Corollary 2.2. To prove the $\mathcal{R}$-boundedness of the resolvent family, for $\lambda \in \Sigma_{\pi-\epsilon} \setminus \{0\}$ let $\theta = \frac{|\arg\lambda|}{\arg\lambda}\left(\frac{\pi}{2} - \frac{\epsilon}{2}\right)$ so that $\mu := e^{-i\theta}\lambda \in \Sigma_{\frac{\pi}{2}-\frac{\epsilon}{2}}$. Then,

$$\begin{aligned}
\left|\lambda\left(\lambda - y^\alpha B^n + V\right)^{-1} f\right| &= \left|\mu\left(\mu - e^{-i\theta}(y^\alpha B^n - V)\right)^{-1} f\right| \\
&= \left|\int_0^\infty \mu e^{-\mu t} e^{-i\theta t(y^\alpha B^n - V)} f \,dt\right| \\
&\le C \int_0^\infty |\mu| e^{-Re\,\mu t} S_\alpha^{-c}(t)|f| \,dt \\
&\le C \int_0^\infty |\lambda| e^{-|\lambda| \sin\frac{\epsilon}{2} t} S_\alpha^{-c}(t)|f| \,dt.
\end{aligned}$$

The $\mathcal{R}$-boundedness of the second family in (ii) now follows from [7, Corollary 2.14] and the maximal regularity of the semigroup from Theorem 2.3. □

In our investigation of degenerate Nd problems, see [14], we need also a weaker version of the result above for potentials having nonnegative real part. We formulate it in the next proposition.

Proposition 7.3. *Let $V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\, \mathrm{d}y\right)$ be a potential having nonnegative real part. Then, for any $1 < p < \infty$ such that $0 < \frac{m+1}{p} < c+1-\alpha$, $y^\alpha B^n - V$ generates a C_0-semigroup on L^p_m. The generated semigroup consists of integral operators, and the following estimates hold*

$$\left|e^{t(y^\alpha B^n - V)} f\right| \le e^{t y^\alpha B^n} |f|, \quad f \in L^p_m, \quad t \ge 0$$

In particular, the families of operators

$$\left\{e^{t(y^\alpha B^n - V)} : t \ge 0,\ V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right),\ Re\ V \ge 0\right\},$$
$$\left\{\lambda\left(\lambda - y^\alpha B^n + V\right)^{-1} : \lambda > 0,\ V \in L^1_{loc}\left(\mathbb{R}^+, y^{c-\alpha}\right),\ Re\ V \ge 0\right\}$$

are $\mathcal{R}$-bounded in L^p_m.

Proof. Using the isometry $T_{0,-\frac{\alpha}{2}}$, we may assume that $\alpha = 0$. Let us treat first the symmetric case in L^2_c. The generation results can be proved as in Proposition 5.1 (where we assumed $V \ge 0$). If $\mathfrak{a}$ is the form associated with B^n, then $B^n - V$ is associated with $\mathfrak{a}_V := \mathfrak{a}(u, v) + \langle V u, v\rangle_{L^2_c}$ and, by the standard theory on sesquilinear forms, $B^n - V$ generates a C_0-semigroup on L^2_c.

The domination properties follow from [16, Theorem 2.21]. Let $u, v \in D(\mathfrak{a}_V) = D(\mathfrak{a}) \cap L^2\left(\mathbb{R}^+, |V| y^c\, \mathrm{d}y\right)$ such that $u\bar{v} \ge 0$. Since e^{tB^n} is positive, one has $\mathrm{Re}\ \mathfrak{a}(u, v) \ge \mathfrak{a}(|u|, |v|)$. Moreover,

$$\mathrm{Re}\ \mathfrak{a}_V(u, v) = \mathrm{Re}\ \mathfrak{a}(u, v) + \int_0^\infty \mathrm{Re}\ V\, u\bar{v}\, y^c \mathrm{d}y \ge \mathrm{Re}\ \mathfrak{a}(|u|, |v|)$$

which by [16, Theorem 2.21] again implies the stated domination of the generated semigroups. (One easily verifies that $D(\mathfrak{a}_V)$ is an ideal of $D(\mathfrak{a})$ since this last is an ideal in itself, by the positivity of e^{tB^n}, see [16, Proposition 2.20].) The extrapolation on L^p_m follows as in Theorem 7.2. The domination of the resolvent is a straightforward consequence of that of the semigroup. The $\mathcal{R}$-boundedness of the semigroup follows by domination from the $\mathcal{R}$-boundedness of $(e^{tB^n})_{t\ge 0}$ proved in Theorem 7.2. The $\mathcal{R}$-boundedness of the resolvent follows as in Theorem 7.2. □

8. The operator $y^\alpha B^n - y^\alpha$

We end the paper by thoroughly investigating the special case $V(y) = y^\alpha$, keeping $\alpha < 2$. We prove, in particular, that the domain of $y^\alpha B - V$ is $D(y^\alpha B) \cap D(V)$, under slightly more restrictive hypotheses than those of Theorem 7.2.

As explained in Introduction, this case plays a crucial role in [14] in the investigation of the degenerate operators

$$\mathcal{L} = y^{\alpha_1}\Delta_x + y^{\alpha_2}\left(D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}\right), \qquad \alpha_1, \alpha_2 \in \mathbb{R}$$

in the spaces $L^p\left(\mathbb{R}^{N+1}_+, y^m dxdy\right)$. In particular, we prove in Propositions 8.3 and 8.4 that the multipliers

$$\xi \in \mathbb{R}^N \to N_\lambda(\xi) = \lambda(\lambda - y^\alpha By + y^\alpha|\xi|^2)^{-1},$$
$$\xi \in \mathbb{R}^N \to M_\lambda(\xi) = y^\alpha|\xi|^2(\lambda - y^\alpha By + y^\alpha|\xi|^2)^{-1}$$

satisfy the hypothesis of Theorem 2.4.

We start with the following lemma.

Lemma 8.1. *Assume that $c+1>0$ and $c+1-\alpha>0$; that is, B^n generates a C_0-semigroup in L^2_c and $y^\alpha B^n$ generates a C_0-semigroup in $L^2_{c-\alpha}$. If $\lambda \in \mathbb{C}^+$ and $\mu > 0$, then*

$$\left(\lambda - y^\alpha B^n + \mu y^\alpha\right)^{-1} f = \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{f}{y^\alpha}\right), \qquad \forall f \in C_c^\infty((0,\infty)).$$

Proof. Under the assumptions, $y^\alpha B^n - \mu y^\alpha$ and $B^n - \lambda y^{-\alpha}$ generate a semigroup on $L^2_{c-\alpha}$ and L^2_c, respectively, see Theorem 7.2. Since $Re\,\lambda > 0$, $\mu > 0$, both resolvents are well defined but map to different spaces.

Let $\mathfrak{a}_{\alpha,\mu y^\alpha}$, $\mathfrak{a}_{\lambda y^{-\alpha}}$ be the forms associated with $y^\alpha B^n - \mu y^\alpha$ in $L^2_{c-\alpha}$ and $B^n - \lambda y^{-\alpha}$ in L^2_c

$$\mathfrak{a}_{\alpha,\mu y^\alpha}(u,v) = \int_{\mathbb{R}_+}\left(D_y u D_y \overline{v} + \mu u\overline{v}\right) y^c\, \mathrm{d}y, \qquad \mathfrak{a}_{\lambda y^{-\alpha}}(u,v) = \int_{\mathbb{R}_+}\left(D_y u D_y \overline{v} + \lambda y^{-\alpha} u\overline{v}\right) y^c\, \mathrm{d}y.$$

By Lemma 6.3, they are defined on the common domain

$$\mathcal{F} := \left\{u \in L^2_{c-\alpha} \cap L^2_c : u' \in L^2_c\right\}$$

Given $f \in C_c^\infty((0,\infty))$, let $u := \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{f}{y^\alpha}\right)$. In order to prove that the equality $u = (\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} f$ holds, we have to show that $u \in \mathcal{F}$ and that for every $v \in \mathcal{F}$, u satisfies the weak equality

$$\int_0^\infty f\overline{v}y^{c-\alpha}\,\mathrm{d}y = \int_0^\infty \lambda u\overline{v}y^{c-\alpha}\,\mathrm{d}y + \mathfrak{a}_{\alpha,\mu y^\alpha}(u,v) = \int_0^\infty (\lambda y^{-\alpha}u\overline{v} + D_y u D_y\overline{v} + \mu u\overline{v})y^c\,\mathrm{d}y. \tag{8}$$

By construction, u is in the domain of $B^n - \lambda y^{-\alpha}$ which is contained in $\mathcal{F}$ and satisfies

$$\int_0^\infty \frac{f}{y^\alpha}\overline{v}y^c\,dy = \int_0^\infty \mu u\overline{v}y^c\,dy + \mathfrak{a}_{\alpha,\lambda y^{-\alpha}}(u,v)$$
$$= \int_0^\infty (\mu u\overline{v} + D_y u D_y\overline{v} + \lambda y^{-\alpha}u\overline{v})y^c\,dy,$$

which is the same as (8). □

In the next results, we relate the resolvent of $y^\alpha B^n - y^\alpha$ with that of $B^n - \frac{1}{y^\alpha}$. We shall assume both the conditions $0 < \frac{m+1}{p} < c+1-\alpha$ and $-\alpha < \frac{m+1}{p} < c+1-\alpha$ (that is $\alpha^- < \frac{m+1}{p} < c+1-\alpha$). The first guarantees that $y^\alpha B^n$ is a generator in L^p_m and the second that B^n is a generator in $L^p_{m+\alpha p}$.

Corollary 8.2. *Assume that* $\alpha^- < \frac{m+1}{p} < c+1-\alpha$. *If* $\lambda \in \mathbb{C}^+$ *and* $\mu > 0$, *then*

(i) for every $f \in L^p_m$

$$\left(\lambda - y^\alpha B^n + \mu y^\alpha\right)^{-1} f = \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{f}{y^\alpha}\right) \in L^p_{m+\alpha p}\cap L^p_m;$$

(ii) the operator $y^\alpha\,(\lambda - y^\alpha B^n + \mu y^\alpha)^{-1}$ *is bounded in* L^p_m;

(iii) the operator $\frac{1}{y^\alpha}\left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}$ *is bounded in* $L^p_{m+\alpha p}$.

Proof. Equality (i) is proved in Lemma 8.1 for any $f \in C_c^\infty((0,\infty))$. Since $(\lambda - y^\alpha B^n + \mu y^\alpha)^{-1}$ is bounded form L^p_m into itself and $\left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}\left(\frac{\cdot}{y^\alpha}\right)$ is bounded from L^p_m to $L^p_{m+\alpha p}$, by density, (i) holds for every $f \in L^p_m$. Parts (ii), (iii) are consequence of (i).

□

In the next propositions, we prove the boundedness of the multipliers N_λ and M_λ. We start with M_λ, used in [14] to characterize the domain of $\mathcal{L} = y^\alpha(\Delta_x + B_y)$.

Proposition 8.3. *Assume that* $\alpha^- < \frac{m+1}{p} < c+1-\alpha$ *and let for* $\lambda \in \mathbb{C}^+$, $\xi \neq 0$

$$M_\lambda(\xi) = |\xi|^2 y^\alpha\left(\lambda - y^\alpha B^n + |\xi|^2 y^\alpha\right)^{-1} \in \mathcal{B}(L^p_m).$$

Then, the family $\left\{|\xi|^{|\beta|}D^\beta_\xi(M_\lambda)(\xi) : \xi \in \mathbb{R}^N\setminus\{0\},\ |\beta| \le N, \lambda \in \mathbb{C}^+\right\}$ *is* $\mathcal{R}$*-bounded in* L^p_m.

Proof. Let $m_\lambda(\mu) = \mu y^\alpha\,(\lambda - y^\alpha B^n + \mu y^\alpha)^{-1}$, $\mu > 0$.

Using Lemma 2.5, it suffices to show that the family $\left\{\mu^k D^k_\mu(m_\lambda)(\mu) : \mu > 0,\ k \le N, \lambda \in \mathbb{C}^+\right\}$ is $\mathcal{R}$-bounded in L^p_m.

The map $Tf = f/y^\alpha$ is an isometry of L^p_m onto $L^p_{m+\alpha p}$ and by Corollary 8.2,

$$m_\lambda(\mu) = T^{-1}\mu\left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1} T.$$

The family

$$\left\{\mu^k D^k_\mu(\Gamma_\lambda)(\mu) : \mu > 0,\ k \le N, \lambda \in \mathbb{C}^+\right\}, \qquad \Gamma_\lambda(\mu) = \mu\left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}$$

is $\mathcal{R}$-bounded in $L^p_{m+\alpha p}$. Indeed,

$$\Gamma_\lambda(\mu) = \int_0^\infty \mu e^{-\mu t} e^{t(B^n - \frac{\lambda}{y^\alpha})}\,\mathrm{d}t$$

and $\left\{e^{t(B^n-\frac{\lambda}{y^\alpha})} : t \ge 0,\ \lambda \in \mathbb{C}^+\right\}$ is $\mathcal{R}$-bounded in $L^p_{m+\alpha p}$, by Theorem 7.3. The $\mathcal{R}$-boundedness of the derivatives follows either by the resolvent equation or by differentiating the last equation under the integral and using [7, Corollary 2.14]. In fact, if $h(\mu, t) = \mu e^{-\mu t}$, then

$$\mu^k \int_0^\infty |D^k_\mu h(\mu, t)|\mathrm{d}t \le C_k, \quad \mu > 0.$$

□

Next we deal with N_λ which is crucial in [14] for the proof that $\mathcal{L} = y^\alpha(\Delta_x + B_y)$ generates an analytic semigroup.

Proposition 8.4. *Assume that* $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$ *and let for* $\lambda \in \mathbb{C}^+$, $\xi \ne 0$

$$N_\lambda(\xi) = (\lambda - y^\alpha B^n + |\xi|^2 y^\alpha)^{-1} \in \mathcal{B}\left(L^p_m\right).$$

Then, the family

$$\left\{|\xi|^{|\beta|} D^\beta_\xi(\lambda N_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\},\ |\beta| \le N, \lambda \in \mathbb{C}^+\right\}$$

is $\mathcal{R}$*-bounded in* L^p_m.

Proof. For $\mu > 0$, let $n_\lambda(\mu) = (\lambda - y^\alpha B^n + \mu y^\alpha)^{-1}$. Using Lemma 2.5, we have to show that the family

$$\left\{\mu^k D^k_\mu(n_\lambda)(\mu) : \mu > 0,\ k \le N,\ \lambda \in \mathbb{C}^+\right\} \tag{9}$$

is $\mathcal{R}$-bounded in L^p_m.

Theorem 7.2 with $V(y) = \mu y^\alpha$ and Proposition 8.3 imply that the families

$$\left\{\lambda n_\lambda(\mu) : \mu > 0,\ \lambda \in \mathbb{C}^+\right\}, \qquad \left\{\mu y^\alpha n_\lambda(\mu) : \mu > 0,\ \lambda \in \mathbb{C}^+\right\} \tag{10}$$

are $\mathcal{R}$-bounded in L^p_m.

We have that $n_\lambda(\cdot) \in C^1\left(\mathbb{R}_+, \mathcal{B}\left(L^p_m\right)\right)$ and

$$D_\mu(n_\lambda(\mu)) = -n_\lambda(\mu) y^\alpha n_\lambda(\mu). \tag{11}$$

Indeed setting $A = \lambda - y^\alpha B^n_y$, $V = y^\alpha$, we have

$$\begin{aligned}
\frac{n_\lambda(\mu+h) - n_\lambda(\mu)}{h} &= \frac{(A+(\mu+h)V)^{-1} - (A+\mu V)^{-1}}{h} \\
&= (A+\mu V)^{-1} \frac{(A+\mu V)(A+(\mu+h)V)^{-1} - I}{h} \\
&= -(A+\mu V)^{-1}\, V\, (A+(\mu+h)V)^{-1}
\end{aligned}$$

which tends to $-n_\lambda(\mu)\, y^\alpha\, n_\lambda(\mu)$ as $h \to 0$ in the norm of $\mathcal{B}\left(L^p_m\right)$ since, by Corollary 8.2,

$$\mu \mapsto V(A+\mu)V)^{-1} = \mu y^\alpha \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1} \frac{1}{y^\alpha}$$

is continuous from $(0, \infty)$ to $\mathcal{B}\left(L^p_m\right)$. This shows (11) and then $n_\lambda(\cdot) \in C^\infty\left(\mathbb{R}_+, \mathcal{B}\left(L^p_m\right)\right)$ and

$$D^k_\mu(n_\lambda(\mu)) = a_k n_\lambda(\mu) \left(y^\alpha n_\lambda(\mu)\right)^k, \qquad a_1 = -1, \quad a_{k+1} = -(k+1)a_k. \tag{12}$$

Formula (12) follows by induction after observing that since $y^\alpha n_\lambda(\mu)$ and its derivative $D_\mu\,(y^\alpha n_\lambda) = -\,(y^\alpha\, n_\lambda(\mu))^2$ commute, then

$$D_\mu\left(y^\alpha n_\lambda(\mu)\right)^k = k D_\mu\left(y^\alpha n_\lambda(\mu)\right)\left(y^\alpha n_\lambda(\mu)\right)^{k-1} = -k\left(y^\alpha n_\lambda(\mu)\right)^{k+1}.$$

The $\mathcal{R}$-boundedness of the family (9) then follows from the $\mathcal{R}$-boundedness of the families (10) since

$$\mu^k D^k_\mu(\lambda n_\lambda(\mu)) = a_k \lambda n_\lambda(\mu)\left(\mu y^\alpha n_\lambda(\mu)\right)^{k+1}.$$

□

In order to characterize the domain of $y^\alpha B^n - y^\alpha$, we denote by

$$D(y^\alpha) = \left\{u \in L^p_m :\ y^\alpha u \in L^p_m\right\}$$

the domain of the potential $V = y^\alpha$ in L^p_m. Recalling that Theorem 4.2 assures that $D(y^\alpha B^n) = W^{2,p}_{\mathcal{N}}(\alpha, m)$, we consider, for $0 < \frac{m+1}{p} < c + 1 - \alpha$, the Banach space

$$W^{2,p}_{\mathcal{N}}(\alpha, m) \cap D(y^\alpha) = \left\{u \in W^{2,p}_{loc}(\mathbb{R}_+) :\ u,\ y^\alpha u,\ y^\alpha D_{yy}u,\ y^{\frac{\alpha}{2}} D_y u,\ y^{\alpha-1} D_y u \in L^p_m\right\}$$

endowed with norm $\|y^\alpha Bu\|_{L^p_m} + \|y^\alpha u\|_{L^p_m} + \|u\|_{L^p_m}$.

Theorem 8.5. *Let $\alpha < 2$, $\mu > 0$, $c \in \mathbb{R}$. Then, for any $1 < p < \infty$ such that $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$ the operator $L = y^\alpha B^n - \mu y^\alpha$ with domain $W^{2,p}_{\mathcal{N}}(\alpha, m) \cap D(y^\alpha)$ generates a bounded analytic semigroup in L^p_m which has maximal regularity. Moreover,*

$$\mathcal{D} = \left\{u \in C_c^\infty([0,\infty)) : u \text{ constant in a neighborhood of } 0\right\}$$

is a core for $y^\alpha B^n - \mu y^\alpha$.

Proof. The generation properties as well as the maximal regularity follow from Theorem 7.2. Without any loss of generality, we may assume that $\mu = 1$. We prove preliminarily that $\mathcal{D}$ is dense in $W^{2,p}_{\mathcal{N}}(\alpha, m) \cap D(y^\alpha) = D(y^\alpha B^n) \cap D(y^\alpha)$. Let $u \in W^{2,p}_{\mathcal{N}}(\alpha, m) \cap D(y^\alpha)$; up to using a standard cutoff argument we may suppose that supp $u \subseteq [0, b]$ for some $b > 0$. Using Remark 4.4, let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that supp $u_n \subseteq [0, b]$ and $u_n \to u$ in $W^{2,p}_{\mathcal{N}}(\alpha, m)$. Then by [12, Proposition 3.2 (ii)]

$$\|y^\alpha(u_n - u)\|_{L^p_m} \le C\|y^{\alpha+1}(D_y u_n - D_y u)\|_{L^p_m} \le Cb^2\|y^{\alpha-1}D_y(u_n - u)\|_{L^p_m}$$

which tends to 0 as $n \to \infty$. This proves the density of $\mathcal{D}$.

Let us now characterize the domain. By definition, $D(y^\alpha B^n - y^\alpha) = (1 - y^\alpha B^n + y^\alpha)^{-1}\left(L^p_m\right)$. Let $u = (1 - y^\alpha B^n + y^\alpha)^{-1} f$ with $f \in L^p_m$. Using Corollary 8.2 (ii), we obtain

$$\|y^\alpha u\|_{L^p_m} + \|y^\alpha Bu\|_{L^p_m} \le C\left(\|(y^\alpha B - y^\alpha)u\|_{L^p_m} + \|u\|_{L^p_m}\right) \tag{13}$$

which proves the inclusion $D(y^\alpha B^n - y^\alpha) \subseteq D(y^\alpha B^n) \cap D(y^\alpha)$. To prove the reverse property, we observe that since the graph norm of $y^\alpha B^n - y^\alpha$ is clearly weaker than the norm of $D(y^\alpha B^n) \cap D(y^\alpha)$, inequality (13) again shows that they are equivalent on $D(y^\alpha B^n - y^\alpha)$, in particular on $\mathcal{D}$ which is dense in $D(y^\alpha B^n) \cap D(y^\alpha)$, by the previous step. Therefore, $D(y^\alpha B^n - y^\alpha) = D(y^\alpha B^n) \cap D(y^\alpha)$ and in particular $\mathcal{D}$ is a core. □

We remark that Theorem 7.2 assures that $y^\alpha B^n - y^\alpha$ generates a semigroup on L^p_m under the milder assumption $0 < \frac{m+1}{p} < c + 1 - \alpha$ and $c + 1 > 0$. However, the hypothesis $(m + 1)/p + \alpha > 0$ must be added when $\alpha < 0$ in order that $\mathcal{D} \subset D(y^\alpha)$.

The same method yields the domain of $B^n - \frac{1}{y^\alpha}$, using Corollary 8.2 (iii) with m replaced by $m - \alpha p$.

Corollary 8.6. *If $\alpha^+ < \frac{m+1}{p} < c + 1$, then the domain of $B^n - \frac{1}{y^\alpha}$ is $W^{2,p}_{\mathcal{N}}(0, m) \cap D(\frac{1}{y^\alpha})$.*

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G. Metafune, L. Negro and C. Spina
Dipartimento di Matematica
e Fisica "Ennio De Giorgi"
Università del Salento
C. P. 193,73100 Lecce
Italy
E-mail: giorgio.metafune@unisalento.it

C. Spina
E-mail: chiara.spina@unisalento.it

L. Negro
Dipartimento di Matematica
Università degli Studi di Salerno
Via Giovanni Paolo II, 132
84084 Fisciano SA
Italy
E-mail: luigi.negro@unisalento.it

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