

Utility-sharing games: How to improve the efficiency with limited subsidies

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ABSTRACT

In this work, we consider the problem of improving the efficiency of utility-sharing games, by resorting to a limited amount of subsidies. Utility-sharing games model scenarios in which strategic and self-interested players interact with each other by selecting resources. Each resource produces a utility that depends on the number of players selecting it, as a non-negative, non-decreasing and concave function, and each of these players receives an equal share of this utility. As the players' selfish behavior may lead to pure Nash equilibria whose total utility is sub-optimal, previous work has resorted to subsidies, incentivizing the use of some resources, to contrast this phenomenon. We focus on the case in which the budget used to provide subsidies is bounded. We consider a class of mechanisms, called α -subsidy mechanisms, that allocate the budget in such a way that each player's payoff is re-scaled up to a factor $\alpha \geq 1$. We design a specific sub-class of α -subsidy mechanisms, that can be implemented efficiently and distributedly by each resource, and evaluate their efficiency by providing upper bounds on their price of anarchy. These bounds are parametrized by both α and the underlying utility functions and are shown to be best-possible for α -subsidy mechanisms. Finally, we apply our results to the particular case of monomial utility functions of degree $p \in (0, 1)$, and derive bounds on the price of anarchy that are parametrized by p and α .

1. Introduction

In several real-life contexts arising from economics, operations research and computer science, we face the necessity of allocating a set of utility-producing resources to agents, in such a way that the total utility is maximized. For example, we could consider a scenario, connected with management engineering, in which each resource models a project or a task to be completed, and each agent is an employee of a company, or a server in a content delivery network, that can be assigned to one of the tasks. It is reasonable to assume that the more the number of employees assigned to a task the more the quality of the completed task (or the lower the completion time, or the higher the probability that the task will be correctly completed). Indeed, the employees assigned to the same task can work as a team, and it is expected that the resulting quality improves as the working team includes new members.

When completed, each task generates a profit (i.e., a utility) that is proportional to the resulting quality, and this profit (or a percentage of it) is equally shared among the employees who contributed to the task. As the number of employees and tasks could be very high, the presence of a centralized coordinator imposing all the assignments might be impracticable. Therefore, a decentralized implementation of the system, where each worker autonomously decides which task she wants to contribute to, is a more reasonable

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choice. To describe the effects of decentralization, we consider a game representation of the system in which each worker acts as a player who aims at maximizing the fraction of the profit that she receives (i.e., her payoff). This creates an interplay of strategic behavior, in which players compete with each other by selecting the tasks (i.e., the resources) that maximize their payoff. This may lead to suboptimal outcomes, in which the total utility is lower than the one achievable by a central authority imposing an optimal assignment of players to resources.

Algorithmic game theory [1] offers several tools to describe how the strategic choices of the players may affect the total utility achieved by all resources. First, the notion of *pure Nash equilibrium* [2], that is an outcome in which no player can increase her payoff by unilaterally deviating to another strategic choice, is used to model stable solutions arising from selfish behavior. Then, the *Price of Anarchy* [3], which compares the total utility of any pure Nash equilibrium against the optimal total utility achievable in a centralized and coordinated environment, is adopted to quantify the lack of cooperation and coordination.

Our Contribution. Given the difficulties in coordinating the players' strategic behavior, a reasonable approach to convey them toward better pure Nash equilibria is that of providing subsidies encouraging the use of certain resources. Several works [4–11] showed the effectiveness of this idea, by designing ad-hoc subsidy allocation mechanisms that are able to improve the price of anarchy. The amount of subsidies that these mechanisms require, however, can be very high, thus limiting their applicability to most real-life contexts, where budgets are usually severely constrained.

In this work, we show how to improve the efficiency of decentralized allocation systems, when the total amount of subsidies available to each resource is somewhat constrained by the total utility that can be generated by the resource itself. We model allocation systems as a class of games, called *utility-sharing games*, which constitutes a subclass of the general framework of *monotone valid utility games* defined in [12], and is similar and/or equivalent to other game classes studied in [8,13–18]. In utility-sharing games, we have a finite set of players, a finite set of resources available to the players, and each resource is associated with a certain non-negative, non-decreasing and concave *utility function*, whose value is equally shared among the players selecting it. Each player aims at maximizing her *payoff*, given by the fraction of utility she receives.

The main novelty of this work is the design and the analysis of α -subsidy mechanisms (α -SMs), a new class of subsidy allocation mechanisms that, parametrized by a value $\alpha \geq 1$, allocate to each resource an amount of subsidies that is at most $\alpha - 1$ times the utility produced by the resource, so that the players' payoffs can be re-scaled up by a multiplicative factor α . The multiplicative limitation $\alpha - 1$ on subsidies makes sense in various scenarios derived from welfare economics, where subsidies are provided by a central authority (such as the state or a private company) to improve the performance of a certain multi-agent system (for example, consisting of a group of workers tasked with completing certain assignments). These subsidies can reasonably vary dynamically and proportionally to the efficiency achieved, i.e., up to $\alpha - 1$ times the value reached by the social utility function. Furthermore, we observe that the multiplicative limitation is not restrictive. In fact, given any subsidy scheme for an n -player game, we can compute a constant $\alpha \geq 1$ such that all subsidies are at most $\alpha - 1$ times the maximum achievable global efficiency (i.e., the optimal value of the social utility function). We can then use the proposed α -subsidy mechanism to distribute the subsidies (or part of them) by assigning to each resource a subsidy that is at most an $(\alpha - 1)$ -fraction of its (local) efficiency, thereby inducing agents to reach more efficient stable configurations than in the absence of subsidies.

The results of this work include the design of specific α -subsidy mechanisms and the analysis of their efficiency across several classes of utility-sharing games, quantified through tight bounds on the price of anarchy. In particular, we resort to a particular sub-class of α -SMs, called *optimal-congestion-based α -SMs*, that can be computed and executed in polynomial time (Proposition 2), and we provide upper bounds on the resulting price of anarchy that depend on α , the number of players n , and the class of utility functions of the game (Theorem 1); we also provide simpler bounds that depend on n and α only (Corollary 1). Conversely, we show that optimal-congestion-based mechanisms achieve the best-possible performance within the general class of α -SMs, that is, no α -SM can further lower our bounds on the price of anarchy (Theorem 2, Corollaries 2 and 3). We also show that the concavity assumption for the utilities, in addition to reflecting the diminishing marginal utility principle (which is common in various multi-agent systems), is somewhat necessary to ensure a bounded price of anarchy (Theorem 3). Finally, we apply our general results to the specific case of utility functions representable as monomials of fixed degree $p \in (0, 1)$ (Theorem 4).

We point out that, for any utility-sharing game and sufficiently large $\alpha \geq 1$, all pure Nash equilibria induced by optimal-congestion-based α -SMs maximize the total utility (Remark 1). Thus, our approach guarantees the same performance of the subsidy allocation mechanisms studied in [8], that, differently from ours, may fail under some budget limitations. Furthermore, for $\alpha = 1$, we re-obtain the tight bounds on the price of anarchy for utility-sharing games without subsidies (Remark 2), already shown in [8,17].

Applications. The considered subsidy mechanism could have multiple applications in several domains of interest. An application is connected with the decentralized task assignment described in the introduction, where each worker (modeled as a player) chooses a task (represented by a resource) and a percentage of the profit generated by each completed task is equally shared among workers. In such a case, our mechanism could be applied to encourage the choice of certain tasks that are more convenient in terms of total profit, even with budget constraints on the amount of subsidies that can be disbursed.

Another interesting application arises from green economy, and it is related to the management of energy communities. Energy communities (or energy projects, or renewable energy cooperatives) [19] have recently emerged in the international energy market, and constitute a decentralized approach to sustainable energy generation, distribution and consumption. Local residents or organizations can decide to adhere to an energy community, so that they collectively manage and benefit from renewable energy resources within their community. The *Directive on common rules for the internal electricity market* (EU/2019/944) aims to support the adhesion to energy communities, with the aim of reducing CO2 emissions and energy waste. Our subsidy mechanism could be adopted to incentivize in a decentralized fashion the participation to energy communities and/or their local resources, in such a way that the

total energy produced (i.e., the total utility) is maximized; furthermore, our analysis of the price of anarchy could be used to give robust theoretical guarantees on the efficiency of the implemented mechanism.

Further Related Work. The first general game-theoretic model for decentralized resource allocation systems with payoff-maximizing players is that of monotone valid utility games [12], where the payoff functions satisfy some mild assumptions, such as monotonicity and submodularity w.r.t. the selected resources. In this seminal paper, a tight bound of 2 on the price of anarchy of monotone valid utility games is provided. Subsequently, several (sub)classes of monotone valid utility games have been introduced and studied.

A work that is closely related to ours is [17], which studies the price of anarchy of (an equivalent model of) utility-sharing games, and provides tight bounds that are parametrized by the number of players and the considered utility functions; tight bounds on the price of anarchy for more general settings in which the set of available resources is player-specific are also provided. Papers [8,14] model strategic project selection as specific instantiations of monotone valid utility games, and provide more specific bounds on the price of anarchy and other efficiency metrics (such as the price of stability [20]). [16,18] study the efficiency of specific monotone valid utility games where, differently from our model of utility-sharing games, the sharing rules do not necessarily split each resource utility in an equal way among the players selecting it.

The problem of determining mechanisms that improve the price of anarchy in utility-sharing games (as well as for their variants and/or generalizations) has been widely considered in the literature. In [8], it is shown how to assign a credit (i.e., a subsidy) to each project, so as to guarantee that any pure Nash equilibrium is an optimal strategy profile. A considerable amount of work [5–7,9–11] shows how to modify the payoffs of the players participating in utility-sharing games (e.g., via subsidies), with the purpose of improving the efficiency of pure Nash equilibria; in particular, tight bounds on the resulting price of anarchy, that depend on the considered class of utility functions, are provided.

Utility-sharing games are strictly related to the cost-minimization game-theoretic model of *congestion games* [21,22]. Congestion games are resource selection games with a finite set of cost-minimizing players and a finite set of resources, where each player selects a subset of resources (among a finite collection that is player-specific), and the cost of each selected resource is a function of the number of players selecting it. In particular, utility-sharing games can be seen as the payoff-maximization version of congestion games with symmetric players [23,24] (that is, all players can share all resources) and singleton strategies (that is, each player can select exactly one resource). The problem of measuring the price of anarchy of congestion games has been a hot-topic in algorithmic game theory in the last two decades [25–28], and several works have provided upper and lower bounds depending on the considered cost-functions [23,29–35] or the structure of the players' strategies [23,24,30,32,36–40].

Several works have also focused on the design and analysis of mechanisms to improve the price of anarchy. A significant class of mechanisms closely associated with profit-sharing games is that of cost-sharing protocols [20,41–45], which are employed in cost-minimization games to define the rules for allocating the cost of each resource among the players utilizing it. Other important and well-studied classes of mechanisms include the following: *taxation mechanisms* [7,46–52], where each player, instead of receiving a subsidy, is charged a tax; *Stackelberg strategies* [23,53–55], in which a fraction of the players can be controlled by a central authority; *coordination-mechanisms* [45,56,57], where the order in which players are processed is decided by a local policy implemented on each resource. In relation to the applications highlighted in this work, some of the aforementioned mechanisms have also been studied within the framework of green economics [58].

2. Model and definitions

Given an integer $k \geq 1$, let $[k] := \{1, 2, \dots, k\}$ denote the set of the first k positive integers. Given an integer $h \geq 0$, $\mathbb{N}_{\geq h}$ denotes the set of natural numbers higher or equal to h , and $\mathbb{N} := \mathbb{N}_{\geq 1}$ denotes the set of natural numbers.

Utility-Sharing Games. A *utility-sharing game* is formally defined as a tuple $SG = (N, R, (u_r)_{r \in R})$ where $N = [n]$ is a set of n players, $R = \{r_1, \dots, r_m\}$ is a set of m resources and $u_r : \mathbb{N}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a non-negative (*resource*) *utility function* associated with each resource $r \in R$. We further assume that (i) $u_r(0) = 0$, (ii) $u_r(x) > 0$ for some $x \in \mathbb{N}$, and u_r is (iii) non-decreasing and (iv) concave in $\mathbb{N}_{\geq 0}$; in particular, (i) holds to impose that a resource without players does not produce any utility, (ii) has been considered to avoid the presence of resources that do not produce any utility, (iii) holds to impose that the more players there are, the higher the utility, and (iv) holds to impose that the contribution of a player joining a resource does not increase as the number of players choosing that resource increases (*Diminishing Marginal Utility principle*), as is common in many multi-agent systems that generate utility.

A *strategy profile* (or *assignment*) $\sigma = (\sigma_i)_{i \in N}$ is a configuration in which each player i has selected resource $r = \sigma_i \in R$, and σ_i denotes the *strategy* of player i in σ . The *congestion* $n_r(\sigma) := |\{i \in N : \sigma_i = r\}|$ of resource r in strategy profile σ is the total amount of players selecting r in σ . Given a strategy profile σ , the *payoff* of player i is defined as $p_i(\sigma) := u_{\sigma_i}(n_{\sigma_i}(\sigma))/n_{\sigma_i}(\sigma)$. Informally, we assume that the utility achieved on each resource is equally shared among the players selecting it, and this fraction of utility determines the payoff of each player. The *total utility function* $SUM(\sigma) := \sum_{r \in R} u_r(n_r(\sigma))$ is equal to the sum of all resource utilities. We have that $SUM(\sigma) = \sum_{r \in R} u_r(n_r(\sigma)) = \sum_{r \in R} \sum_{i \in N : \sigma_i = r} \frac{u_{\sigma_i}(n_{\sigma_i}(\sigma))}{n_{\sigma_i}(\sigma)} = \sum_{r \in R} \sum_{i \in N : \sigma_i = r} p_i(\sigma) = \sum_{i \in N} p_i(\sigma)$, i.e., the total utility can be seen as the sum of all payoffs and thus represents a social welfare function proportional to the overall satisfaction of all players.

Pure Nash Equilibria and Price of Anarchy. All players aim at maximizing their payoffs, regardless of the others. As a reasonable outcome of selfish behavior, we consider the notion of *pure Nash equilibrium* [2]. Given a strategy profile σ , a player $i \in N$ and a resource r , let (σ_{-i}, r) denote the strategy profile σ' in which player i chooses resource r (that is, $\sigma'_i = r$), and all other players choose the same resource as in σ (that is, $\sigma'_h = \sigma_h$ for any $h \in N \setminus \{i\}$). A strategy profile σ is a *pure Nash equilibrium* if, for any player i and resource $r \in R$, we have $p_i(\sigma) \geq p_i(\sigma_{-i}, r)$, that is, no player improves her payoff by deviating to another resource. We note that

utility-sharing games always have at least one pure Nash equilibrium, since they are isomorphic to congestion games with possibly decreasing costs, which, according to [22], always have a pure Nash equilibrium.

Given a utility-sharing game SG, let $SP(SG)$ denote the set of strategy profiles of SG, and $PNE(SG)$ denote the set of pure Nash equilibria of SG; furthermore, let $OPT(SG) = \max_{\sigma \in SP(SG)} \text{SUM}(\sigma)$ denote the maximum total utility achievable in SG.

A universal metric to measure the impact of selfishness on the total utility is the *price of anarchy* [3]. Given a utility-sharing game SG, the *price of anarchy* (PoA) of SG is defined as $\text{PoA}(SG) = \max_{\sigma \in PNE(SG)} \frac{OPT(SG)}{\text{SUM}(\sigma)}$, that is, the worst-case ratio between the optimal total utility and that achieved by any pure Nash equilibrium of SG. We observe that the lower the price of anarchy the lower the impact of selfish behavior in terms of total utility.

Subsidy Mechanisms. We assume that each resource can implement a local policy that, by means of a subsidy, may increase the payoffs of players selecting it, up to a maximum factor $\alpha \geq 1$. We refer to this set of local policies as α -subsidy mechanisms (α -SMs). In particular, an α -subsidy mechanism Π_α takes as input a utility-sharing game SG and returns a new utility-sharing game $SG_{\Pi_\alpha} = (N, R, (u'_r)_{r \in R})$ that will be played in place of SG, where each utility function u'_r is called *perceived utility* of resource r , and verifies the following properties: (i) $u'_r(x) = u_r(x) + \theta_r(x)$ for any $x \in \mathbb{N}_{\geq 0}$, for an opportune non-negative *subsidy function* θ_r ; (ii) The subsidy function θ_r is locally and dynamically computed by resource r , based solely on the knowledge of the initial game SG and the congestion $n_r(\sigma)$ of r in the strategy profile σ reached by the game dynamics; (iii) $\theta_r(x) \leq (\alpha - 1) \cdot u_r(x)$, that is, $u'_r(x) \leq \alpha \cdot u_r(x)$, for any $x \in \mathbb{N}_{\geq 0}$. We observe that the perceived utilities u'_r lead to a new players' payoff $p'_i(\sigma) = u'_{\sigma_i}(n_{\sigma_i}(\sigma)) / n_{\sigma_i}(\sigma) = (u_{\sigma_i}(n_{\sigma_i}(\sigma)) + \theta_{\sigma_i}(n_{\sigma_i}(\sigma))) / n_{\sigma_i}(\sigma)$, for any strategy profile σ of SG_{Π_α} and player $i \in N$. Differently from the original utility function u_r , we do not require the perceived utility function u'_r to satisfy the properties of monotonicity or concavity as a hypothesis. Furthermore, we observe that the presence of subsidy mechanisms still guarantees the existence of a pure Nash equilibrium in the modified utility-sharing game SG_{Π_α} (as noted above for standard utility-sharing games without any subsidy mechanism).

In the following, we provide some justifications for the above properties characterizing α -SMs. Property (i) states that the perceived utility is equal to the overall value given by the utility and the additional subsidy (determined by θ_r) assigned by the mechanism to each resource; then, both the utility and the subsidy are shared by the players selecting the resource, thus leading to new players' payoffs. Property (ii) is motivated by the fact that the mechanism can be reasonably executed in a distributed way, where each resource uses its local information on the congestion, without knowing the congestion of the other resources. Finally, property (iii) is motivated by scenarios in which the subsidies, because of some budget constraints, are limited by a factor α of the utility achievable by each resource.

Given an α -SM Π_α for SG, the *price of anarchy of SG under Π_α* is defined as $\text{PoA}(SG, \Pi_\alpha) = \max_{\sigma \in PNE(SG_{\Pi_\alpha})} \frac{OPT(SG)}{\text{SUM}(\sigma)}$, where $PNE(SG_{\Pi_\alpha})$ is the set of pure Nash equilibria of the game SG_{Π_α} induced by Π_α , $OPT(SG)$ is the optimal total utility of the initial game SG and $\text{SUM}(\sigma)$ is the total utility computed according to the utility-functions of the initial game SG. Informally, the price of anarchy of a game SG under an α -SM Π_α measures how inefficient a pure Nash equilibrium of the game modified by Π_α is, compared with the optimal total utility, but considering as total utility functions those of the initial game SG. The modelling choice for which subsidies are not taken into account in the total utility appearing in the price of anarchy is motivated by the realistic scenario in which players receive money as subsidies, and then the same amount of money is lost by the central authority (e.g., a governmental entity) who disburses them. Thus, the overall contribution to the social welfare of subsidies is null, that is, the social welfare continues to be equal to the total utility without subsidies.

3. Computation and efficiency of α -SMs

In this section, we first define a specific α -SM that is polynomial-time computable and executable, and then we measure its efficiency by providing bounds on the resulting price of anarchy. We first provide some preliminary notation. We say that a function $f : \mathbb{N}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is *payoff-regular* if the function u defined as $u(x) := x \cdot f(x)$ is a well-defined utility function satisfying all assumptions of the model (that is, f determines the payoff associated with utility function u); for our purposes, we can assume without loss of generality that $f(0) = 0$. Given a class of payoff-regular functions $\mathcal{D}$, let $\mathcal{L}(\mathcal{D}) := \{u : u(x) = w \cdot x \cdot f(x), f \in \mathcal{D}, w > 0\}$ denote the family of utility functions whose related payoff functions, up to a scaling factor, belong to $\mathcal{D}$. We observe that all functions in $\mathcal{L}(\mathcal{D})$ are well-defined utilities satisfying the assumptions of the model. The following proposition shows some useful properties of payoff-regular functions.

Proposition 1. *Each payoff-regular function is non-increasing and positive in $\mathbb{N}$.*

Proof. Let $f : \mathbb{N}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a payoff-regular function and $u(x) = x f(x)$ be the corresponding utility function. We first show that f is non-increasing in $\mathbb{N}$. Let $\alpha_h := u(h) - u(h-1)$ for any $h \in \mathbb{N}$. Given $x \in \mathbb{N}$, we have:

$$f(x) = \frac{u(x)}{x} = \frac{\sum_{h=1}^x \alpha_h}{x} \geq \frac{\sum_{h=1}^{x+1} \alpha_h}{x+1} = \frac{u(x+1)}{x+1} = f(x+1),$$

where the intermediate inequality holds since the quantities α_h are non-increasing in $h \in \mathbb{N}$ by the concavity of the utility function u , and then the arithmetic mean $\frac{\sum_{h=1}^x \alpha_h}{x}$ of the first x terms of type α_h is non-increasing in x , too. We conclude that $f(x) \geq f(x+1)$ for any $x \in \mathbb{N}$, and this shows the non-increasing monotonicity.

We now show that f is positive on $\mathbb{N}$. We first claim that $f(1) > 0$. Suppose by contradiction that this is not the case, and let $x > 1$ be the smallest natural number such that $f(x) > 0$, or equivalently $u(x) > 0$. Then we have $\alpha_x = u(x) - u(x-1) = u(x) > 0 = u(x-1) - u(x-2) = \alpha_{x-1}$, where $u(x) > 0$ and $u(x-1) = u(x-2) = 0$ hold by the minimality of x . But $\alpha_x > \alpha_{x-1}$ contradicts the

concavity of the utility function. Therefore, $f(1) > 0$. Since $f(1) > 0$, and $u(x) = x \cdot f(x)$ is non-decreasing in x by definition of utility function, we have $0 < f(1) = 1 \cdot f(1) \leq x \cdot f(x)$ for any $x \in \mathbb{N}$, and this necessarily implies the positivity of $f(x)$ for any $x \in \mathbb{N}$. $\square$

3.1. Computation

Fix $\alpha \geq 1$. Given a game $SG = (N, R, (u_r)_{r \in R})$ and an optimal strategy profile σ^* of the original utility-sharing game SG, let $\Pi_\alpha(\sigma^*)$ be an optimal-congestion-based α -SM for SG that returns a new utility-sharing game $SG_{\Pi_\alpha(\sigma^*)} = (N, R, (u'_r)_{r \in R})$ with perceived utility functions u'_r defined as

$$u'_r(x) = \begin{cases} u_r(x), & \text{if } x > n_r(\sigma^*), \\ \alpha \cdot u_r(x), & \text{if } x \leq n_r(\sigma^*), \end{cases} \quad (1)$$

for any $x \in \mathbb{N}_{\geq 0}$. Such an α -SM is called *optimal-congestion-based*. We observe that optimal-congestion-based α -SMs encourage the use of resources whose congestion in the optimal strategy profile is at least that in the played strategy profile, and this is done by increasing the utility of such resources by a factor of α .

In the following proposition, obtained by applying in a straightforward way a result of [59], we show that optimal-congestion-based α -SMs, under mild assumptions, can be computed and executed in polynomial time.

Proposition 2. *Given $\alpha \geq 1$ and a utility-sharing game SG with utility functions in $\mathcal{L}(D)$, with D containing payoff-regular functions only, we can compute and execute in polynomial time an optimal-congestion-based α -SM for SG.*

Proof. We observe that, once an optimal strategy profile σ^* of SG is computed, then the α -SM $\Pi_\alpha(\sigma^*)$ can be executed in polynomial time. Thus, to show the claim, it is sufficient to show that an optimal strategy profile σ^* for SG can be computed in polynomial time. To compute an optimal strategy profile σ^* of SG, we apply a greedy algorithm, denoted by GREEDY, which processes the players in the order induced by their indices. Each player $i \in N$, when processed, is assigned to the resource r that maximizes the marginal increase in total utility resulting from assigning i to r at that step (i.e., the quantity $u_r(k_{i-1,r} + 1) - u_r(k_{i-1,r})$, where $k_{i-1,r}$ denotes the congestion of resource r after the previous $i - 1$ assignments), with ties broken arbitrarily.

GREEDY obviously runs in polynomial time. By exploiting the concavity of each function u_r and a result of [59] (who applied an equivalent greedy algorithm to a similar concave maximization problem) we have that the strategy profile returned by GREEDY is optimal. $\square$

3.2. Efficiency

In the following, we provide an upper bound on the price of anarchy under optimal-congestion-based α -SMs, that is parametrized by the considered class of utility functions; furthermore, we show that, under mild assumptions on the considered utility functions, no α -SM can improve the efficiency achieved by optimal-congestion-based α -SMs, that is, the latter are essentially the best possible α -subsidy mechanisms.

PoA Upper Bounds. We first give some preliminary notation. Given a payoff-regular function f , let

$$\beta_{n,\alpha}(f) := \sup_{x,y \in \mathbb{N}_{\geq 0} : n \geq x \geq y \geq 0, x > 0} \frac{y \left(f(y) - \frac{1}{\alpha} f(x) \right)}{x f(x)};$$

we observe that $\beta_{n,\alpha}(f) \geq 1 - 1/\alpha$ (indeed, it is sufficient to set $x = y$ in the argument of the supremum to get a value equal to $1 - 1/\alpha$); furthermore, the denominator appearing in the right-hand part of the definition is always non-zero, as each utility function $f(x)$ is positive in $x > 0$ (by Proposition 1). Given a class of payoff-regular functions $\mathcal{D}$, let $\beta_{n,\alpha}(\mathcal{D}) := \sup_{f \in \mathcal{D}} \beta_{n,\alpha}(f)$ and $\beta_\alpha(\mathcal{D}) := \sup_{n \in \mathbb{N}} \beta_{n,\alpha}(\mathcal{D})$.

Theorem 1. *Let $\mathcal{D}$ be a class of payoff-regular functions. Given a game SG with at most $n \geq 2$ players and utility functions in $\mathcal{L}(\mathcal{D})$, and given an optimal-congestion-based α -SM $\Pi_\alpha(\sigma^*)$ for SG, we have that*

$$\text{PoA}(SG, \Pi_\alpha(\sigma^*)) \leq \frac{1}{\alpha} + \beta_{n,\alpha}(\mathcal{D}) \leq \frac{1}{\alpha} + \beta_\alpha(\mathcal{D}).$$

Proof. Let $SG = (N, R, (u_r)_{r \in R})$ be a utility-sharing game with utility functions in $\mathcal{L}(\mathcal{D})$, and let σ^* be a social optimum of SG. Let σ be a pure Nash equilibrium of the game $SG_{\Pi_\alpha(\sigma^*)}$ obtained after applying the optimal-congestion-based α -SM $\Pi_\alpha(\sigma^*)$. We have that each utility function u_{r_j} of SG can be written as $u_{r_j}(x) := w_j \cdot x \cdot f_j(x)$, with $w_j > 0$ and $f_j \in \mathcal{D}$. Let o_j and k_j be the congestions of each resource $r_j \in R$ in σ^* and σ , respectively.

For each $r_j \in R$ such that $k_j \geq o_j$, we have that

$$o_j f_j(o_j) - \frac{1}{\alpha} o_j f_j(k_j) \leq k_j f_j(k_j) \beta_{n,\alpha}(\mathcal{D}), \quad (2)$$

by definition of $\beta_{n,\alpha}(\mathcal{D})$. On the other hand, for any $r_j \in R$ such that $o_j > k_j$, we have that

$$o_j f_j(o_j) - (o_j - k_j) f_j(k_j + 1) \leq o_j f_j(o_j) - (o_j - k_j) f_j(o_j) = k_j f_j(o_j) \leq k_j f_j(k_j), \quad (3)$$

where $k_j f_j(k_j) \geq k_j f_j(k_j + 1) \geq k_j f_j(o_j)$ trivially holds if $k_j = 0$, and holds even if $k_j > 0$, since f is payoff-regular (thus, non-increasing in $\mathbb{N}$ by Proposition 1) and $k_j + 1 \leq o_j$ (by the integrality of k_j and o_j).

Now, since each utility function u_{r_j} of SG can be written as $u_{r_j}(x) = w_j \cdot x \cdot f_j(x)$, the value of the perceived utility function u'_{r_j} achieved when playing σ in $\text{SG}_{\Pi_\alpha(\sigma^*)}$ is equal to $u'_{r_j}(k_j) = w'_j(k_j) \cdot k_j \cdot f_j(k_j)$, where $w'_j(x)$ is a weight defined as follows for any $x \in \mathbb{N}_{\geq 0}$:

$$w'_j(x) = \begin{cases} w_j, & \text{if } x > o_j, \\ \alpha \cdot w_j, & \text{if } x \leq o_j. \end{cases} \quad (4)$$

Analogously, the payoff function of each player in the new game $\text{SG}_{\Pi_\alpha(\sigma^*)}$ when playing the equilibrium σ is equal to $w'_j(k_j) \cdot f_j(k_j)$. Thus, since σ is an equilibrium in game $\text{SG}_{\Pi_\alpha(\sigma^*)}$ and all payoff-regular functions are non-increasing in $\mathbb{N}$ (by [Proposition 1](#)), for each pair of resources (r_j, r_h) with $k_j > 0$, we have that

$$w'_j(k_j)f_j(k_j) - w'_h(k_h+1)f_h(k_h+1) \geq 0. \quad (5)$$

Furthermore, as $\sum_{r_j \in R} k_j = |N| = \sum_{r_j \in R} o_j$, we have that

$$\sum_{j: k_j > o_j} (k_j - o_j) = \sum_{h: k_h < o_h} (o_h - k_h). \quad (6)$$

By (6) we can create a directed bipartite graph that connects each resource r_j with $k_j > o_j$ to a resource r_h with $k_h < o_h$ via a directed edge (r_j, r_h) , such that the out-degree of r_j is $k_j - o_j > 0$ and the in-degree of r_h is $o_h - k_h > 0$. Thus, by summing inequality (5) over all edges (r_j, r_h) of this graph, we obtain the following inequality:

$$\sum_{j: k_j > o_j} (k_j - o_j) w'_j(k_j) f_j(k_j) - \sum_{h: k_h < o_h} (o_h - k_h) w'_h(k_h + 1) f_h(k_h + 1) \geq 0. \quad (7)$$

By combining (2), (3) and (7), and by exploiting the definition of w'_j given in (4), we get the following inequalities:

$$\sum_j o_j w_j f_j(o_j) \leq \sum_j o_j w_j f_j(o_j) + \frac{1}{\alpha} \left(\sum_{j: k_j > o_j} (k_j - o_j) w'_j(k_j) f_j(k_j) - \sum_{j: k_j < o_j} (o_j - k_j) w'_j(k_j + 1) f_j(k_j + 1) \right) \quad (8)$$

$$= \sum_j o_j w_j f_j(o_j) + \frac{1}{\alpha} \left(\sum_{j: k_j \geq o_j} (k_j - o_j) w_j f_j(k_j) - \sum_{j: k_j < o_j} (o_j - k_j) \alpha w_j f_j(k_j + 1) \right) \quad (9)$$

$$= \sum_{j: k_j \geq o_j} w_j \left(o_j f_j(o_j) - \frac{1}{\alpha} o_j f_j(k_j) + \frac{1}{\alpha} k_j f_j(k_j) \right) + \sum_{j: k_j < o_j} w_j \left(o_j f_j(o_j) - (o_j - k_j) f_j(k_j + 1) \right) \quad (10)$$

$$\leq \sum_{j: k_j \geq o_j} w_j \left(\beta_{n,\alpha}(D) k_j f_j(k_j) + \frac{1}{\alpha} k_j f_j(k_j) \right) + \sum_{j: k_j < o_j} w_j k_j f_j(k_j) \quad (11)$$

$$\leq \sum_{j: k_j \geq o_j} w_j \left(\frac{1}{\alpha} + \beta_{n,\alpha}(D) \right) k_j f_j(k_j) + \sum_{j: k_j < o_j} w_j \left(\frac{1}{\alpha} + \beta_{n,\alpha}(D) \right) k_j f_j(k_j) \quad (12)$$

where (8) follows from (7), (9) holds since $w'_j(k_j) = w_j$ for $k_j > o_j$ and $w'_j(k_j + 1) = \alpha w_j$ for $k_j < o_j$, (10) follows from (2) and (3), and (11) holds since $1/\alpha + \beta_{n,\alpha}(D) \geq 1$ (indeed, we already observed at the beginning of this section that $1 - 1/\alpha \leq \beta_{n,\alpha}(D)$).

Therefore, from (12) we have that $\sum_{r_j \in R} o_j w_j f_j(o_j) \leq \left(\frac{1}{\alpha} + \beta_{n,\alpha}(D) \right) \sum_{r_j \in R} k_j w_j f_j(k_j)$, and we can conclude that

$$\text{PoA}(\text{SG}, \Pi_\alpha(\sigma^*)) = \frac{\sum_{r_j \in R} u_{r_j}(o_j)}{\sum_{r_j \in R} u_{r_j}(k_j)} = \frac{\sum_{r_j \in R} o_j w_j f_j(o_j)}{\sum_{r_j \in R} k_j w_j f_j(k_j)} \leq \frac{1}{\alpha} + \beta_{n,\alpha}(D).$$

□

Remark 1 (Optimal PoA via α -SMs). Given a class of payoff-regular functions D and a game $\text{SG} = (N, R, (u_r)_{r \in R})$ with utility functions in $\mathcal{L}(D)$, if we apply to SG an optimal-congestion-based α -SM with $\alpha > \frac{\max_{r \in R, l \in [n]} (u_r(l)/l)}{\min_{r \in R, l \in [n]} (u_r(l)/l)}$, we have that the resulting pure Nash equilibrium σ necessarily coincides with the optimal strategy profile σ^* which the considered α -SM is based on (the proof of this property is deferred to the appendix). Thus, the price of anarchy becomes 1, and we re-obtain the optimal performance of the subsidy mechanisms considered in [8].

The following corollary of [Theorem 1](#) provides a tight bound on the price of anarchy that depends on the number of players only, under mild assumptions on the considered utility functions.

Corollary 1. Let SG be a utility-sharing game with at most $n \geq 2$ players and utility functions in a class $\mathcal{L}(D)$, with D containing payoff-regular functions only. For any optimal-congestion-based α -SM $\Pi_\alpha(\sigma^*)$ for SG, we have that

$$\text{PoA}(\text{SG}, \Pi_\alpha(\sigma^*)) \leq 1 + \frac{1}{\alpha} \left(1 - \frac{1}{n} \right) \leq 1 + \frac{1}{\alpha}.$$

Proof. To prove the claim it is sufficient to show that

$$\frac{y\left(f(y) - \frac{1}{\alpha}f(x)\right)}{xf(x)} \leq 1 - \frac{1}{\alpha n} \quad \forall x, y : n \geq x \geq y \geq 0, x > 0. \quad (13)$$

Indeed, by combining the previous inequality with [Theorem 1](#), we get

$$\text{PoA}(\text{SG}) \leq \frac{1}{\alpha} + \beta_{n,\alpha}(D) \leq \frac{1}{\alpha} + \left(1 - \frac{1}{\alpha n}\right) = 1 + \frac{1}{\alpha} \left(1 - \frac{1}{n}\right).$$

If $y = 0$, inequality (13) trivially holds. If $y \geq 1$, we have that

$$\frac{y(f(y) - \frac{1}{\alpha}f(x))}{xf(x)} = \frac{yf(y) - \frac{1}{\alpha}yf(x)}{xf(x)} \leq \frac{xf(x) - \frac{1}{\alpha}yf(x)}{xf(x)} = 1 - \frac{y}{\alpha x} \leq 1 - \frac{1}{\alpha n}, \quad (14)$$

where the first inequality follows from the fact that the functions of the type $t \cdot f(t)$ are necessarily non-decreasing (as they belong to a class $\mathcal{L}(D)$ that contains, by assumptions, proper utility functions only) and (14) holds since $y \geq 1$ and $x \leq n$. $\square$

PoA Lower Bounds. In the following theorem we show that no α -SM can outperform the worst-case efficiency achieved by the class of optimal-congestion-based α -SMs.

Theorem 2. *Let D be a class of payoff-regular functions. For any $\epsilon > 0$ and $n \in \mathbb{N}_{\geq 2}$, there exists a utility-sharing game SG with at most n players such that, for any α -SM Π_α for SG, the price of anarchy of SG under Π_α is higher than $1/\alpha + \beta_{n,\alpha}(D) - \epsilon$.*

Proof. Fix $\epsilon > 0$ and $n \in \mathbb{N}_{\geq 2}$. By definition of supremum, we have that there exist $x, y \in \mathbb{N}_{\geq 0}$ with $n \geq x \geq y$ and $x > 0$, and $f \in D$ such that

$$\frac{1}{\alpha} + \frac{y(f(y) - \frac{1}{\alpha}f(x))}{xf(x)} > 1 + \beta_{n,\alpha}(D) - \epsilon. \quad (15)$$

Consider an instance $\text{SG} = (N, R, (u_r)_{r \in R})$ with $x = |N| \leq n$ players and a set R of $x - y + 1$ resources. Set $R' := R \setminus \{r_1\} = \{r_2, \dots, r_{x-y+1}\}$. The utility function of resource r_1 is $u_{r_1}(t) = w_1 \cdot t \cdot f(t)$, with $w_1 = 1$, while the utility function of each resource $r_j \in R'$ is $u_{r_j}(t) = u_{r_2}(t) = w_2 \cdot t \cdot f(t)$, with $w_2 = \frac{f(x)}{\alpha f(1)}$.

Let Π_α be an arbitrary α -SM for SG, and let u'_{r_j} denote the utility-function of resource r_j in the new game SG_{Π_α} induced by Π_α . Let σ be the strategy profile in which all players choose resource r_1 . We observe that, for any player $i \in N$ and for any other resource $r_j \in R'$, it holds that

$$p_i(\sigma_{-i}, r_j) = u'_{r_j}(1) \leq \alpha u_{r_j}(1) = \alpha w_2 f(1) = f(x) = \frac{xf(x)}{x} = \frac{u_{r_1}(x)}{x} = p_i(\sigma).$$

Then we have that the strategy profile σ is a pure Nash equilibrium of the game SG_{Π_α} , independently of the adopted α -SM Π_α .

The total utility of strategy profile σ is

$$\text{SUM}(\sigma) = \sum_{r \in R} u_r(n_r(\sigma)) = u_{r_1}(x) = xf(x). \quad (16)$$

Let σ^* be the strategy profile in which y players choose resource r_1 , while each of the remaining $x - y$ players chooses a different resource in R' .

The total utility of σ^* is

$$\text{SUM}(\sigma^*) = \sum_{r \in R} u_r(n_r(\sigma^*)) = u_{r_1}(y) + (x - y)u_{r_2}(1) = yf(y) + (x - y)\frac{1}{\alpha}f(x), \quad (17)$$

where the second equality holds since $u_r = u_{r_2}$ for each $r \in R'$. Therefore, from (16) and (17), we have that

$$\begin{aligned} \text{PoA}(\text{SG}, \Pi_\alpha) &\geq \frac{\text{SUM}(\sigma^*)}{\text{SUM}(\sigma)} \\ &= \frac{yf(y) + (x - y)\frac{1}{\alpha}f(x)}{xf(x)} \\ &= \frac{1}{\alpha} + \frac{y\left(f(y) - \frac{1}{\alpha}f(x)\right)}{xf(x)} \\ &> \frac{1}{\alpha} + \beta_{n,\alpha}(D) - \epsilon, \end{aligned} \quad (18)$$

where the last inequality follows from (15). $\square$

We also have the following corollary of [Theorem 2](#), that provides a lower bound that does not depend on the maximum number of players.

Corollary 2. *Let D be a class of payoff-regular functions. For any $\epsilon > 0$, there exists a utility-sharing game SG such that, for any α -SM Π_α for SG, the price of anarchy of SG under Π_α is higher than $1/\alpha + \beta_\alpha(D) - \epsilon$.*

Proof. Let $\epsilon > 0$. By definition of supremum, there exists $n \in \mathbb{N}$ such that $\beta_{n,\alpha}(D) > \beta_\alpha(D) - \epsilon/2$. Thus, by applying [Theorem 2](#) with $\epsilon' = \epsilon/2$, we have that there exists a utility-sharing game SG with at most n players such that, for any α -SM Π_α for SG, the price of anarchy of SG under Π_α is at least

$$\text{PoA}(\text{SG}, \Pi_\alpha) > \frac{1}{\alpha} + \beta_{n,\alpha}(D) - \frac{\epsilon}{2} > \frac{1}{\alpha} + \beta_\alpha(D) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = \frac{1}{\alpha} + \beta_\alpha(D) - \epsilon,$$

where the first inequality follows from [Theorem 2](#) and the second one follows from $\beta_{n,\alpha}(D) > \beta_\alpha(D) - \epsilon/2$. $\square$

The following corollary of [Theorem 2](#) and [Corollary 2](#) shows that the upper bound provided in [Corollary 1](#) is tight.

Corollary 3. (i) For any $n \in \mathbb{N}_{\geq 2}$, there exists a utility-sharing game SG with at most n players such that, for any α -SM Π_α for SG, the price of anarchy of SG under Π_α is at least $1 + \frac{1}{\alpha} \left(1 - \frac{1}{n}\right)$. (ii) Furthermore, for any $\epsilon > 0$, there exists a game SG such that the price of anarchy of SG under Π_α is higher than $1 + \frac{1}{\alpha} - \epsilon$.

Proof. To show (i), it is sufficient to consider the same lower bound instance SG as in [Theorem 2](#), with $x = n$, $y = 1$ and f defined as follows: $f(0) = 0$ and $f(t) = 1/t$ for any $t \in \mathbb{N}$. By using the same proof and starting from [\(18\)](#), we get that, for any α -SM Π_α , the price of anarchy of SG under Π_α is

$$\begin{aligned} \text{PoA}(\text{SG}, \Pi_\alpha) &\geq \frac{yf(y) + (x-y)\frac{1}{\alpha}f(x)}{xf(x)} \\ &= 1 + (n-1)\frac{1}{n\alpha} \\ &= 1 + \frac{1}{\alpha} \left(1 - \frac{1}{n}\right), \end{aligned} \tag{19}$$

and this shows the claim of part (i). Regarding part (ii), let $\epsilon > 0$. By reconsidering the proof of part (i), and by fixing $n \in \mathbb{N}$ in such a way that $n > 1/(\alpha\epsilon)$, [\(19\)](#) becomes

$$\text{PoA}(\text{SG}, \Pi_\alpha) \geq 1 + \frac{1}{\alpha} \left(1 - \frac{1}{n}\right) > 1 + \frac{1}{\alpha} - \epsilon,$$

and this shows the claim of part (ii). $\square$

Remark 2 (PoA without subsidies). If $\alpha = 1$, an optimal-congestion-based α -SM does not change the utility functions of the original game. Thus, the tight bounds provided in [Theorems 1–2](#) and [Corollaries 1–3](#) with $\alpha = 1$ are also tight bounds on the price of anarchy of utility-sharing games without the use of any subsidy mechanism, that is, we can re-obtain the tight bounds on the price of anarchy provided in [\[8,17\]](#).

Finally, in the following theorem, we show that the concavity assumption for the utility functions is, to some extent, necessary to achieve a bounded price of anarchy.

Theorem 3. For any $M \geq 1$, there exists a utility-sharing game SG with two players and two resources, where the utility functions are convex, such that for any α -SM Π_α , the price of anarchy of SG under Π_α is at least M .

Proof. Fix $M \geq 1$, and consider an instance $\text{SG} = (N, R, (u_r)_{r \in R})$ with two players and a set $R = \{r_1, r_2\}$ of two resources. The utility functions are defined as follows¹: $u_{r_1}(1) = \alpha$, $u_{r_1}(2) = 2\alpha$, $u_{r_2}(1) = 1$ and $u_{r_2}(2) = \alpha M$. One can easily observe that the strategy profile σ in which both players select resource r_1 is a pure Nash equilibrium, independently of the considered α -SM for SG. Indeed, for any $i \in N$, we have

$$p'_i(\sigma) = u'_{r_1}(2)/2 \geq u_{r_1}(2)/2 = \alpha = \alpha \cdot u_{r_2}(1) \geq u'_{r_2}(1) = p'_i(\sigma_{-i}, r_2),$$

that is, player i does not have any incentive to deviate in favor of resource r_2 . Now, let σ^* be the strategy profile in which both players select resource r_2 . We have that

$$\text{PoA}(\text{SG}, \Pi_\alpha) \geq \frac{\text{SUM}(\sigma^*)}{\text{SUM}(\sigma)} = \frac{2\alpha M}{2\alpha} = M,$$

and this shows the claim. $\square$

4. The case of monomial utility functions

In the following result, we apply [Theorems 1](#) and [2](#) to characterize the price of anarchy under optimal-congestion-based α -SMs of games with monomial utility-functions of fixed degree $p \in (0, 1)$. We exclude the limit cases $p \in \{0, 1\}$, as they are implicitly treated in the previous results. Indeed, for $p = 1$, all payoff functions are constant, and the upper bound on the price of anarchy provided by [Theorem 1](#) equals 1, that is, the outcome is optimal. For $p = 0$, [Corollaries 1](#) and [3](#) apply and yield a tight bound of $1 + 1/\alpha$ on the price of anarchy.

¹ The definition for $x \geq 3$ is omitted, as the maximum congestion is 2.

Theorem 4. Given $p \in (0, 1)$, a utility-sharing game SG with utility functions of type $u(x) = w \cdot x^p$ for some $w > 0$ and an optimal-congestion-based α -SM $\Pi_\alpha(\sigma^*)$ for SG, we have $\text{PoA}(\text{SG}, \Pi_\alpha(\sigma^*)) \leq \frac{1-t}{\alpha} + t^p$, with $t = \min \left\{ (\alpha p)^{\frac{1}{1-p}}, 1 \right\}$. Furthermore, no α -SM can achieve, in general, a better price of anarchy.

Proof. Let f be the function defined as $f(x) = x^{p-1}$, and let $D(p) = \{f\}$. We observe that $\mathcal{L}(D(p))$ coincides with the class of monomials defined as $u(x) := w \cdot x^p$ for some weight $w > 0$. Thus, by Theorem 1 we have that

$$\text{PoA}(\text{SG}) \leq \frac{1}{\alpha} + \beta_{n,\alpha}(D(p)),$$

and by Theorem 2 we have that no α -SM can improve the above upper bound. Then, we proceed with the explicit calculation of $\beta_{n,\alpha}(D(p))$. We have that

$$\begin{aligned} \beta_{n,\alpha}(D(p)) &= \sup_{n \in \mathbb{N}} \left(\sup_{x, y \in \mathbb{N}_{\geq 0} : n \geq x \geq y \geq 0, x > 0} \left(\frac{y \left(f(y) - \frac{1}{\alpha} f(x) \right)}{x f(x)} \right) \right) \\ &= \sup_{x, y \in \mathbb{N}_{\geq 0} : x \geq y \geq 0, x > 0} \left(\frac{y \left(\frac{1}{y^{1-p}} - \frac{1}{\alpha} \cdot \frac{1}{x^{1-p}} \right)}{x \cdot \frac{1}{x^{1-p}}} \right) \\ &= \sup_{x, y \in \mathbb{N}_{\geq 0} : x \geq y \geq 0, x > 0} \left(\frac{y \left(\frac{y^p}{y} - \frac{x^p}{\alpha x} \right)}{x \cdot \frac{x^p}{x}} \right) \\ &= \sup_{x, y \in \mathbb{N}_{\geq 0} : x \geq y \geq 0, x > 0} \left(\frac{y^p - x^p \cdot \frac{y}{\alpha x}}{x^p} \right) \\ &= \sup_{x, y \in \mathbb{N}_{\geq 0} : x \geq y \geq 0, x > 0} \left(\left(\frac{y}{x} \right)^p - \frac{y}{\alpha x} \right) \\ &= \sup_{0 \leq t \leq 1} \left(t^p - \frac{t}{\alpha} \right), \end{aligned} \tag{20}$$

where we set $t = \frac{y}{x}$ in the last equality. Because of (20), it is sufficient maximizing $g(t) = t^p - \frac{t}{\alpha}$ over $t \in [0, 1]$ to obtain an upper bound on the price of anarchy of SG; we observe that $g(t)$ is strictly concave, thus it admits a unique maximum point in $[0, 1]$. To maximize $g(t)$ in the interval $[0, 1]$, we first study its derivative:

$$\frac{\partial}{\partial t} \left(t^p - \frac{t}{\alpha} \right) = 0 \Leftrightarrow p t^{p-1} = \frac{1}{\alpha} \Leftrightarrow t = \left(\frac{1}{\alpha p} \right)^{\frac{1}{p-1}} = (\alpha p)^{\frac{1}{1-p}}.$$

Now, if the critical point (i.e., at which the derivative is zero) lies within interval $(0, 1)$, this point necessarily coincides with the maximum point of $g(t)$ on $[0, 1]$, since g is strictly concave. Otherwise, the critical point of g is at least 1 and then $g(t)$ is necessarily increasing in $t \in [0, 1]$ (again because of strict concavity). Hence, in this case, $t = 1$ is the maximum point of $g(t)$ on $[0, 1]$. We conclude that the maximum point of $g(t)$ in interval $[0, 1]$ is $t^* = \min \left\{ (\alpha p)^{\frac{1}{1-p}}, 1 \right\}$.

We conclude that $\beta_{n,\alpha}(D(p)) = t^p - \frac{t}{\alpha}$, with $t = \min \left\{ (\alpha p)^{\frac{1}{1-p}}, 1 \right\}$. Thus, by applying Theorems 1 and 2 with the above value of $\beta_{n,\alpha}(D(p))$ the claim follows. $\square$

Remark 3 (PoA without subsidies (cont.)). For $\alpha = 1$, the bounds provided in Theorem 4 are also tight for the price of anarchy of utility-sharing games with monomial profit functions without the use of any α -SM, that is, we can re-obtain the tight bounds on the price of anarchy provided in [17].

In the following example we show an application of Theorem 4 with $p = \frac{1}{2}$.

Example 1. By applying Theorem 4, we have that the price of anarchy under optimal-congestion-based α -SMs of games SG with monomial utility functions of type $u(x) = w \cdot x^{1/2}$ is equal to $\frac{\alpha^2+4}{4\alpha}$ for $\alpha < 2$, and equal to 1 for $\alpha \geq 2$. In Fig. 1, we see how the price of anarchy varies over $\alpha \geq 1$, and we compare it with the case of general functions.

5. Future works

In this work, we have shown how to reduce the price of anarchy in a large class of resource-selection games with utility-maximizing players, by resorting to a limited amount of subsidies that can be distributed among resources and is subsequently shared among the players who use them.

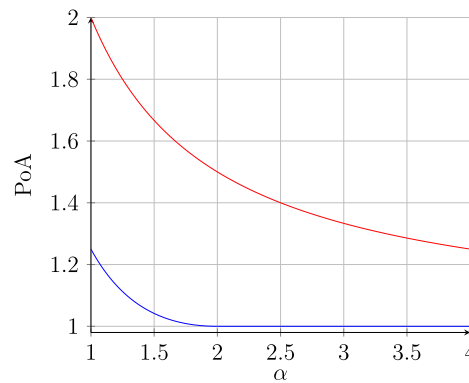


Fig. 1. The red (upper) line represents, for any $\alpha \geq 1$, the price of anarchy under optimal-congestion-based α -SMs when the underlying utility functions are not specified (Corollaries 1 and 3), while the blue (lower) line represents the price of anarchy of games whose utility functions are monomials of degree $1/2$ (a case of Theorem 4). From this comparison, we observe that the bounds on the price of anarchy that depend on the specific class of utility functions (Theorems 1 and 2) may be definitely more precise than the bounds which depend on α only (Corollaries 1 and 3).

Our work leaves several research directions on the problem of improving the efficiency in utility sharing games via limited subsidies.

First of all, our subsidy mechanisms dynamically depend on the actual congestion of each resource. Thus, it would be interesting to show how the efficiency can be improved if subsidies do not depend on the current game configuration. It would also be interesting to explore cases where computing the social optimum, on which the subsidy mechanism is based, is not feasible (e.g., if the number of players is unknown), and the design and analysis of (static or dynamic) subsidy mechanisms that do not rely on such information deserves further investigation. Furthermore, still with the aim of improving the efficiency of the considered games, it would be also interesting to consider other mechanisms besides subsidy disbursements (e.g., Stackelberg strategies [54]).

Another research direction could be that of finding subsidy (or taxing) mechanisms with limited budget for more general variants of utility-sharing games, where players can also be cost-minimizers (as in congestion games [22]) and/or have different weights and/or can select different subsets of resources.

Finally, by leveraging the theoretical findings obtained in this work, we have shown that our mechanisms improve the performance of several multi-agent systems related to resource selection. For instance, returning to the example of decentralized energy community management (described in the introduction), where we define the social function as the total energy produced by the various communities, and the fraction $\alpha - 1$ determines the maximum amount of incentives that can be provided based on the energy produced, our α -subsidy mechanisms ensure an efficiency that, reformulating the statement of Corollary 3, is at least $\alpha/(1 + \alpha)$ times the optimal efficiency. However, these performance guarantees are theoretical. Therefore, it would be valuable to apply our mechanisms directly to real-world case studies and experimentally assess their actual efficiency.

CRediT authorship contribution statement

Vittorio Bilò: Writing – review & editing, Investigation, Writing – original draft, Formal analysis, Methodology, Conceptualization; **Lucaleonardo Bove:** Writing – review & editing, Investigation, Writing – original draft, Formal analysis, Methodology, Conceptualization; **Cosimo Vinci:** Writing – review & editing, Investigation, Writing – original draft, Formal analysis, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Remark 1

Assume by contradiction that there exists a pure Nash equilibrium σ that is different from σ^* , when adopting an optimal-congestion-based α -SM based on σ^* and with

$$\alpha > \frac{\max_{r \in R, h \in [n]} u_r(h)/h}{\min_{r \in R, h \in [n]} u_r(h)/h} = \frac{\max_{r \in R} u_r(1)}{\min_{r \in R} u_r(n)/n}.$$

Thus, there are necessarily two resources r and s such that $n_r(\sigma) > n_r(\sigma^*)$ and $n_s(\sigma) < n_s(\sigma^*)$. By the pure Nash equilibrium conditions, we have that the payoff of any player selecting resource r is at least that she could obtain when deviating to resource s , that is:

$$\frac{u_r(n_r(\sigma))}{n_r(\sigma)} = \frac{u'_r(n_r(\sigma))}{n_r(\sigma)} \tag{A.1}$$

$$\geq \frac{u'_s(n_s(\sigma) + 1)}{(n_s(\sigma) + 1)} \tag{A.2}$$

$$= \alpha \cdot \frac{u_s(n_s(\sigma) + 1)}{(n_s(\sigma) + 1)} \tag{A.3}$$

$$> \left(\frac{u_r(n_r(\sigma))/n_r(\sigma)}{u_s(n_s(\sigma) + 1)/(n_s(\sigma) + 1)} \right) \cdot \frac{u_s(n_s(\sigma) + 1)}{(n_s(\sigma) + 1)} \tag{A.4}$$

$$= \frac{u_r(n_r(\sigma))}{n_r(\sigma)},$$

where (A.1) holds by $n_r(\sigma) > n_r(\sigma^*)$, (A.2) holds by the pure Nash equilibrium conditions, (A.3) holds by $n_s(\sigma) < n_s(\sigma^*)$, and (A.4) holds because of the considered lower bound on α . Thus, by $u_r(n_r(\sigma))/n_r(\sigma) > u_r(n_r(\sigma))/n_r(\sigma)$ we get a contradiction.

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