# LIE STRUCTURE OF SMASH PRODUCTS

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Abstract. We investigate the conditions under which the smash product of an (ordinary or restricted) enveloping algebra and a group algebra is Lie solvable or Lie nilpotent.

## 1. Introduction

Let $A$ be an associative algebra over a field and regard $A$ as a Lie algebra via the Lie product defined by $[x, y] = xy - yx$, for every $x, y \in A$. Then $A$ is said to be Lie solvable (respectively, Lie nilpotent) if it is solvable (nilpotent) as a Lie algebra. The Lie structure of associative algebras has been extensively studied over the years and considerable attention has been especially devoted to group algebras (see e.g. [3, 4, 5, 13, 18, 19, 20]) and restricted enveloping algebras (see e.g. [15, 16, 17, 22, 23, 24]). In particular, Passi, Passman and Sehgal established in [13] when the group algebra $\mathbb{F}G$ of a group $G$ over a field $\mathbb{F}$ is Lie solvable and Lie nilpotent. Later, Riley and Shalev in [15] settled the same problems for the restricted enveloping algebra $u(L)$ of a restricted Lie algebra $L$ over a field of characteristic $p > 0$, under the assumption that $p > 2$ for Lie solvability.

Now suppose that a group $G$ acts by automorphisms on a restricted Lie algebra $L$ over a field $\mathbb{F}$ of positive characteristic. Then this action is naturally extended to the action of $\mathbb{F}G$ on $u(L)$ and one can form the smash product $u(L)\#\mathbb{F}G$. Necessary and sufficient conditions under which these smash products satisfy a nontrivial polynomial identity were provided by Bahturin and Petrogradsky in [1].

In the main results of this paper we determine the conditions under which $u(L)\#\mathbb{F}G$ is Lie solvable in odd characteristic (Theorem 3.1) or Lie nilpotent (Theorem 4.2). We also deal with smash products $U(L)\#\mathbb{F}G$, where $U(L)$ is the ordinary enveloping algebra of a Lie algebra over any field. In particular, we establish when $U(L)\#\mathbb{F}G$ is Lie solvable (in characteristic different than 2) or Lie nilpotent.

It is worth mentioning that smash products, sometimes referred to as semidirect products, arise very frequently in the theory of Hopf algebras. A classical example is a celebrated structure theorem of Cartier-Kostant-Milnor-Moore, asserting that

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every cocommutative Hopf algebra over an algebraically closed field of characteristic zero can be presented as a smash product of a group algebra and an enveloping algebra (see e.g. [12, §5.6]). Cocommutative Hopf algebras satisfying a polynomial identity have been investigated by Kochetov in [11]. As an application of our results, we show that a cocommutative Hopf algebra over a field of characteristic zero is Lie solvable if and only if it is commutative.

## 2. Preliminaries

We briefly recall the notion of smash product. We refer the reader to [12, §4.1] or [7, §6.1] for details. Let $H$ be a Hopf algebra and suppose that $A$ is a left $H$-module algebra via $\varphi : H \to \mathrm{End}_{\mathbb{F}}(A)$. For every $h \in H$ and $x \in A$ we set $h * x = \varphi(h)(x)$ and use the so-called Sweedler's notation $\Delta(h) = \sum h_1 \otimes h_2$ for the comultiplication of $H$. We recall that the smash product $A\#H$ is the vector space $A \otimes_{\mathbb{F}} H$ endowed with the following multiplication (we will write $a\#h$ for the element $a \otimes h$):

$$(a\#h)(b\#k) = \sum a(h_1 * b)\#h_2 k.$$

Now, let $G$ be a group and suppose that $G$ acts by automorphisms on an associative algebra $A$ over a field $\mathbb{F}$ via a group homomorphism $\varphi : G \to \mathrm{Aut}(A)$. By linearity, the Hopf algebra $\mathbb{F}G$ acts on $A$ making $A$ an $\mathbb{F}G$-module algebra and, conversely, every $\mathbb{F}G$-module algebra arises in such a way. Note that, as $\Delta(g) = g \otimes g$, in this case the multiplication in $A\#\mathbb{F}G$ is just given by $(a\#g)(b\#h) = a(g * b)\#gh$, for all $a, b \in A$ and $g, h \in G$.

In particular, if the group $G$ acts on a Lie algebra $L$ by Lie algebra automorphisms, then this action is naturally extended to the action on the universal enveloping algebra $U(L)$ and we can form the smash product $U(L)\#\mathbb{F}G$. Similarly, we will consider the smash product $u(L)\#\mathbb{F}G$ when $G$ acts by (restricted) Lie algebra automorphisms on a restricted Lie algebra $L$ over a field of characteristic $p > 0$.

For a group $G$ we will denote by $G'$ the derived subgroup of $G$. If $L$ is a restricted Lie algebra over a field of characteristic $p > 0$, we recall that a subset $S$ of $L$ is said to be $p$-nilpotent if $S^{[p]^m} = \{x^{[p]^m} \mid x \in S\} = 0$, for some $m \geq 1$. We will also denote by $L'$ the derived subalgebra $[L, L]$ of $L$. The Lie structure of group algebras and restricted enveloping algebras has been investigated, respectively, by Passi, Passman and Sehgal in [13] and by Riley and Shalev in [15]. We quote their results for future reference:

**Theorem 2.1** ([13]). *Let $\mathbb{F}G$ be the group algebra of a group $G$ over a field $\mathbb{F}$ of characteristic $p \geq 0$. Then $\mathbb{F}G$ is Lie nilpotent if and only if one of the following conditions holds:*

(1) *$p = 0$ and $G$ is abelian;*
(2) *$p > 0$, $G$ is nilpotent and $G'$ is a finite $p$-group.*

**Theorem 2.2** ([13]). *Let $\mathbb{F}G$ be the group algebra of a group $G$ over a field $\mathbb{F}$ of characteristic $p \geq 0$. Then $\mathbb{F}G$ is Lie solvable if and only if one of the following conditions holds:*

1. *$p = 0$ and $G$ is abelian;*
2. *$p > 2$ and $G'$ is a finite $p$-group;*
3. *$p = 2$ and $G$ has a subgroup $N$ of index at most 2 such that $N'$ is a finite 2-group.*

**Theorem 2.3** ([15]). *Let $L$ be a restricted Lie algebra over a field of characteristic $p > 0$. Then $u(L)$ is Lie nilpotent if and only if $L$ is nilpotent and $L'$ is finite-dimensional and $p$-nilpotent.*

**Theorem 2.4** ([15]). *Let $L$ be a restricted Lie algebra over a field of characteristic $p > 2$. Then $u(L)$ is Lie solvable if and only if $L'$ is finite-dimensional and $p$-nilpotent.*

We mention that Theorem 2.4 does not hold in characteristic 2 and the characterization of Lie solvable restricted enveloping algebras has been completed only recently in [24].

We will make frequent use of the following result which was proved by Zalesskii and Smirnov in [28] and, independently, by Sharma and Srivastava in [21].

**Theorem 2.5.** *Let $R$ be a Lie solvable ring. Then the two-sided ideal of $R$ generated by $[[R,R],[R,R]],R]$ is associative nilpotent.*

## 3. Lie solvability

Let $G$ be a group and $L$ be a restricted Lie algebra over a field $\mathbb{F}$ of positive characteristic $p$. In this section, we establish when the smash product of $\mathbb{F}G$ and $u(L)$ is Lie solvable. We will denote by $\omega(G)$ and $\omega(L)$ the augmentation ideals of $\mathbb{F}G$ and $u(L)$, respectively. Let $S \subseteq L$. We will use the symbols $\langle S\rangle_{\mathbb{F}}$ and $\langle S\rangle_p$, respectively, for the $\mathbb{F}$-vector space and the restricted subalgebra generated by $S$. Moreover, we write $L'_p$ instead of $\langle L'\rangle_p$, and we denote by $Z(L)$ the center of $L$. If $G$ acts by automorphisms on $L$, we say that a subalgebra $H$ of $L$ is $G$-stable if $g * x \in H$ for every $g \in G$ and $x \in H$, and that $G$ acts trivially on $H$ if $g * x = x$ for every $g \in G$, $x \in H$.

**Theorem 3.1.** *Let $G$ be a group acting by automorphisms on a restricted Lie algebra $L$ over a field $\mathbb{F}$ of characteristic $p > 2$. Then $u(L)\#\mathbb{F}G$ is Lie solvable if and only if the following conditions hold:*

1. *$G'$ is a finite $p$-group;*
2. *$L$ contains a finite-dimensional $p$-nilpotent $G$-stable restricted ideal $P$ such that $L/P$ is abelian and $G$ acts trivially on $L/P$.*

*Proof.* Let $R = u(L)\#\mathbb{F}G$. First, we prove the sufficiency. Since $P$ is a $G$-stable restricted ideal of $L$, we have

$$u(L)\#\mathbb{F}G/(Pu(L)\#\mathbb{F}G) \cong u(L/P)\#\mathbb{F}G.$$

Since $P$ is finite-dimensional and $p$-nilpotent, $\omega(P)$ is an associative nilpotent ideal of $u(P)$, by [15]. Consequently, $Pu(L) = \omega(P)u(L) = u(L)\omega(P)$ is a nilpotent ideal of $u(L)$. Hence, $Pu(L)\#\mathbb{F}G$ is a nilpotent ideal of $R$. Thus, as $G$ acts trivially on $L/P$, it is enough to show that

$$u(L/P)\#\mathbb{F}G \cong u(L/P) \otimes_{\mathbb{F}} \mathbb{F}G$$

is Lie solvable. But this is indeed the case as $u(L/P)$ is commutative and $\mathbb{F}G$ is Lie solvable by Theorem 2.2.

Now suppose that $R$ is Lie solvable. Since $\mathbb{F}G$ and $u(L)$ embed in $R$, we deduce by Theorems 2.2 and 2.4 that $G'$ is a finite $p$-group and $L'_p$ is finite-dimensional and $p$-nilpotent. Note that $L'_p$ is $G$-stable and

$$u(L)\#\mathbb{F}G/(L'_p u(L)\#\mathbb{F}G) \cong u(L/L'_p)\#\mathbb{F}G.$$

Moreover, $L'_p u(L)\#\mathbb{F}G$ is associative nilpotent. Therefore, we can replace $L$ with $L/L'_p$ and assume that $L$ is abelian. Indeed, suppose that there exists a finite-dimensional $p$-nilpotent $G$-stable restricted ideal $\bar{P} = P/L'_p$ of $\bar{L} = L/L'_p$ such that $G$ acts trivially on $\bar{L}/\bar{P}$. Then $L/P \cong \bar{L}/\bar{P}$ is abelian and clearly $P$ satisfies condition (2) of the statement. Note that, by [14], the ideal $[R,R]R$ is nil of bounded index, say $t$. Let $r$ be a positive integer such that $p^r \geq t$. Then the elements $[1\#g, x\#g^{-1}] = ((g-1)*x)\#1$ must be nilpotent of index at most $p^r$. Let $P$ be the space spanned by all elements $(g-1)*x$, where $x \in L$ and $g \in G$. Since $L$ is abelian, we have

$$(g*x - x)^p = (g*x)^p - x^{[p]} = g*x^{[p]} - x^{[p]} = (g-1)*x^{[p]} \in P.$$

Thus, $P$ is a $p$-nilpotent restricted ideal of $L$. Furthermore, $P$ is $G$-stable. Indeed, we have

$$h*((g-1)*x) = hg*x - h*x = (hgh^{-1}-1)*(h*x) \in P,$$

for every $g, h \in G$ and $x \in L$. In order to prove the necessity part, it is then enough to show that $P$ is finite-dimensional.

Let $J$ be the associative ideal of $R$ generated by $[[[R,R],[R,R]],R]$. Note that, by Theorem 2.5, $J$ is associative nilpotent. Now consider Lie commutators of the following type:

$$\begin{aligned}
\xi_1 :=& [x\#g^{-1}, 1\#g] = (1-g)*x\#1;\\
\xi_2 :=& [y\#1, 1\#h] = (1-h)*y\#h;\\
\xi :=& [\xi_1, \xi_2] = ((1-h)*y)((1-h)(1-g)*x)\#h;\\
[\xi, z\#1] =& ((1-h)*y)((1-h)(1-g)*x)((h-1)*z)\#h\\
=& -(1-h)*(y((1-g)*x)z)\#h.
\end{aligned}$$

Let $Q$ be the subspace spanned by all $(1-h)(1-g)*x$, where $x \in L$, $g \in G$ and $h \in G'$. Note that $Q$ is $G$-stable:

$$\begin{aligned} a*((1-h)(1-g)*x) &= (a(1-h)(1-g))*x \\ &= (1-aha^{-1})(1-aga^{-1})*(a*x) \in Q, \end{aligned}$$

for every $a, g \in G$, $h \in G'$ and $x \in L$. Furthermore, $J$ contains all the elements of the form

$$[\xi, z\#1](1\#h^{-1}) = (h-1)*(y((1-g)*x)z)\#1.$$

We deduce, by the PBW Theorem for restricted Lie algebras (see e.g. [26, Chapter 2, §5, Theorem 5.1]), that $(1-h)\omega(G)*L$ must be finite-dimensional, for every $h \in G'$. Since $G'$ is finite, we then conclude that $Q$ is indeed a $G$-stable $p$-nilpotent finite-dimensional restricted ideal of $L$. Since the ideal $Qu(L)\#\mathbb{F}G$ is associative nilpotent, we replace without loss of generality $L$ with $L/Q$ and assume that $G'$ acts trivially on $P$. In particular,

$$(g^2-g)*x = g(g-1)*x = g*x-x,$$

for every $g \in G'$ and $x \in L$. It follows by induction on $n$ that

$$g^n*x = ng*x-(n-1)x,$$

for every $g \in G'$, $x \in L$. In particular,

$$g^p*x = x,$$

for every $g \in G'$ and $x \in L$.

Next we consider Lie commutators of the following type:

$$\begin{aligned} [1\#g, x\#1] &= (g-1)*x\#g; \\ [z\#g, y\#1] &= z((g-1)*y)\#g; \\ \eta &= [(g-1)*x\#g, z((g-1)*y)\#g] \\ &= (g-1)*(xyz)\#g^2; \\ [\eta, u\#g^{-2}] &= (g-1)*(xyz((g+1)*u))\#1 \\ &= ((g-1)*x)((g-1)*y)((g-1)*z)((g^2-1)*u)\#1, \end{aligned}$$

where $x, y, z \in L$ and $g \in G'$. Let

$$I = \langle (g^2-1)*u \mid u \in L, g \in G' \rangle_{\mathbb{F}}.$$

Then $I$ is a $p$-nilpotent restricted ideal of $L$ contained in $P$. Moreover, for every $u \in L$, $g \in G'$ and $h \in G$ we have

$$h((g^2-1)*u) = (hg^2h^{-1})h*u - h*u = ((hgh^{-1})^2-1)*(h*u) \in I,$$

so that $I$ is $G$-stable. Note that $J$ contains all the elements

$$((g-1)*x)((g-1)*y)((g-1)*z)((g^2-1)*u)\#1,$$

where $x, y, z, u \in L$ and $g \in G'$. Since $J$ is nilpotent and $p > 2$, we deduce by the PBW Theorem that $I$ must be finite-dimensional. It follows that $Iu(L)\#\mathbb{F}G$

is associative nilpotent. Hence, without loss of generality, we replace $L$ with $L/I$ and assume that $g^2$ acts trivially on $L$, for every $g \in G'$. Put $m = \frac{p-1}{2}$. For every $g \in G'$ and $x \in L$, we have

$$g * x = (g^{p-2m})) * x = g^p * ((g^m)^2 * x) = g^p * x = x.$$

This means that $G'$ acts trivially on $L$. Consequently, there is an induced action of $G/G'$ on $L$ given by $(G'g) * x = g * x$, for every $g \in G$ and $x \in L$. Moreover, we clearly have $P = \omega(G) * L = \omega(G/G') * L$ and then, in order to prove our claim, we can replace $G$ by $G/G'$. For every $g, h, a \in G$ and $x, y, z \in L$ we have:

$$\begin{aligned}
[1\#g, x\#g^{-1}a] &= (g-1) * x\#a; \\
[1\#h, y\#h^{-1}] &= (h-1) * y\#1; \\
\zeta := {} & [(g-1) * x\#a, (h-1) * y\#1] \\
&= ((g-1) * x)((a-1)(h-1) * y)\#a; \\
[\zeta, z\#1] &= ((g-1) * x)((a-1)(h-1) * y)((a-1) * z)\#a \in J.
\end{aligned}$$

Since $J$ is nilpotent and $p > 2$, we deduce that $N = \omega^2(G) * L$ is a finite-dimensional $p$-nilpotent $G$-stable restricted ideal of $L$. We can then replace $L$ by $L/N$ and assume that $G$ acts trivially on $P$. Finally, we have

$$\begin{aligned}
[1\#a, x\#a^{-1}g] &= (a-1) * x\#g; \\
[z\#h, y\#1] &= z((h-1) * y)\#h; \\
\theta := {} & [(a-1) * x\#g, z((h-1) * y)\#h] \\
&= ((a-1) * x)((h-1) * y)((g-1) * z)\#gh; \\
[\theta, u\#g^{-1}h^{-1}] &= ((a-1)*x)((h-1)*y)((g-1) * z)((gh-1) * u)\#1 \in J, \quad (3.1)
\end{aligned}$$

for every $a, g, h \in G$ and $x, y, z, u \in L$. We observe from $\omega(G)^2 * L = 0$ that

$$(gh-1) * x = (g-1) * x + (h-1) * x, \tag{3.2}$$

for every $g, h \in G$ and $x \in L$. Suppose, if possible, that $P$ is infinite-dimensional. If $(g-1) * L$ is infinite-dimensional for some $g \in G$, then there exists a sequence $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ of elements of $L$ such that the set

$$\{(g-1) * x_i, (g-1) * y_j, (g-1) * z_k \,|\, i, j, k = 1, 2, \ldots\}$$

is linearly independent. Similarly, as in Equation (3.1), we can see that

$$A_i = ((g-1) * x_i)((g-1) * y_i)((g-1) * z_i)((g^2-1) * z_i)\#1 \in J,$$

for every $i$. However, by Equation (3.2) and the PBW Theorem, it follows that $A_1 A_2 \cdots A_n \neq 0$ for every $n$, contradicting the fact the $J$ is nilpotent. Thus, $(g-1) * L$ is finite-dimensional, for every $g \in G$. Moreover, using Equations (3.1) and (3.2), we can see that $\omega(G) * x$ is finite-dimensional, for every $x \in L$. Now, we claim that there exist $u \in L$ and $g, h \in G$ such that $(g-1) * u$ and $(h-1) * u$ are linearly independent. Let $g \in G$ be such that $(g-1) * L \neq 0$ and consider

$\chi(g) = \{y \in L | (g-1) * y = 0\}$. Let $H$ be a complement of the subspace $\chi(g)$ in $L$. Then

$$\dim H = \dim L/\chi(g) = \dim (g-1) * L < \infty.$$

Note that $\omega(G)*H$ is finite-dimensional. Then, since $(g-1)*L$ is finite-dimensional and $P = \omega(G) * L$ is infinite-dimensional, there exist $h \in G$ and $y \in \chi(g)$ such that $(h-1) * y \notin (g-1) * L$. Let $x \in L \setminus \chi(g)$ and put $u = x + y$. It is clear that $(g-1) * u$ and $(h-1) * u$ are linearly independent, as claimed. Now, set $g_1 = g, h_1 = h, u_1 = u, V_0 = 0, Z_1 = \{(g_1 - 1) * u_1, (h_1 - 1) * u_1\}$. Suppose that we have already defined $g_1, h_1, \ldots, g_n, h_n \in G$ and $u_1, u_2, \ldots, u_n \in L$ such that the set

$$Z_i = \{(g_i - 1) * u_i, (h_i - 1) * u_i\}$$

is linearly independent modulo $V_{i-1}$, for every $i = 1, 2, \ldots, n$, where

$$V_i = \sum_{k=1}^{i} (g_k - 1) * L + \sum_{k=1}^{i} (h_k - 1) * L.$$

Note that $V_n$ and all the spaces $(g-1) * L$ are finite-dimensional whereas $P$ is infinite-dimensional. Arguing in a similar way as above, we can find $g_{n+1}, h_{n+1} \in G$ and $u_{n+1} \in L$ such that the set

$$Z_{n+1} = \{(g_{n+1} - 1) * u_{n+1}, (h_{n+1} - 1) * u_{n+1}\}$$

is linearly independent modulo $V_n$. Let $m$ denote the nilpotency index of $J$. Since $P$ is infinite-dimensional, we can find $a_1, \ldots, a_m \in G$ and $x_1, \ldots, x_m \in L$ such that the elements $(a_1 - 1) * x_1, \ldots, (a_m - 1) * x_m$ are linearly independent modulo $V_m$. By (3.1), for every $j = 1, 2, \ldots, m$, we have

$$B_i = ((a-1) * x_i)((g_i - 1) * u_i)((h_i - 1) * u_i)((g_i h_i - 1) * u_i) \in J.$$

At this stage, from (3.2) and the PBW Theorem, we conclude that

$$B_1 B_2 \cdots B_m \neq 0,$$

a contradiction. Thus $P$ is finite-dimensional, completing the proof. □

**Remark 3.2.** In the statement of Theorem 3.1, the assumption on the characteristic cannot be removed. In fact, by Theorem 2.2 and the main theorem in [24], in characteristic 2 the Lie solvability of $u(L)\#\mathbb{F}G$ forces neither $G'$ to be a finite 2-group nor $L'$ be finite-dimensional and 2-nilpotent (implying that condition (2) of Theorem 3.1 cannot hold in this case). Moreover, unlike the odd characteristic case, in characteristic 2 the tensor product of two Lie solvable algebras is not necessarily Lie solvable. For example, consider the three-dimensional Heisenberg Lie algebra $L$ with a non-trivial $p$-map on the center and the symmetric group $S_3$ on three elements acting trivially on $L$. Thus, in characteristic 2, $u(L) \otimes_{\mathbb{F}} \mathbb{F}G$ may not be Lie solvable when $u(L)$ and $\mathbb{F}G$ are Lie solvable.

We recall that an associative algebra $A$ is said to be *strongly Lie solvable* if $\delta^{(n)}(R) = 0$ for some $n$, where $\delta^{(0)}(A) = A$ and $\delta^{(n+1)}(A)$ is the associative ideal $[\delta^{(n)}(A), \delta^{(n)}(A)]A$ of $A$. In view of a result of Jennings (see [10], Theorem 5.6), this is equivalent to require that $A$ contains a nilpotent ideal $N$ such that $A/N$ is commutative. Certainly, strong Lie solvability implies Lie solvability, but the converse is not true in general. For instance, in characteristic 2, Lie solvable group algebras or restricted enveloping algebras need not be strongly Lie solvable (cf. [22] and Section V.6 of [18]). However, using the aforementioned result of Jennings, it is easy to see that in any positive characteristic, if $A = u(L)\#\mathbb{F}G$ satisfies the conditions of the statement of Theorem 3.1, then it is strongly Lie solvable. Indeed, take

$$N = Pu(L)\#\mathbb{F}G + u(L)\#\omega(G')\mathbb{F}G$$

and observe that $N$ is a nilpotent ideal of $A$ such that $A/N$ is commutative. Thus, we obtain the following generalization of a result of Sehgal (see [18, §V.6]) and of Corollary 1 of [22]:

**Corollary 3.3.** *Let $G$ be a group acting by automorphisms on a restricted Lie algebra $L$ over a field $\mathbb{F}$ of characteristic $p > 2$. Then $u(L)\#\mathbb{F}G$ is Lie solvable if and only if it is strongly Lie solvable.*

## 4. Lie nilpotency

We now consider the Lie nilpotency. We keep the same notation as in Section 3. Longer Lie commutators are interpreted using the left-normed convention.

If $u(L)\#\mathbb{F}G$ is Lie nilpotent and the ground field has characteristic $p > 2$, then we already know from Theorem 3.1 that $\omega(G) * L$ is finite-dimensional. However, in order to extend such a conclusion also in characteristic 2, we need the following:

**Lemma 4.1.** *Let $G$ be a group acting by automorphisms on a restricted Lie algebra $L$ over a field $\mathbb{F}$ of characteristic $p > 0$. If $u(L)\#\mathbb{F}G$ is Lie nilpotent then $L$ contains a finite-dimensional $p$-nilpotent $G$-stable restricted ideal $P$ such that $L/P$ is abelian and $G$ acts trivially on $L/P$.*

*Proof.* By Theorem 2.1, we have that $G'$ is a finite $p$-group and, by Theorem 2.3, $L'_p$ is finite-dimensional and $p$-nilpotent. Moreover, $L'_p$ is $G$-stable and so $L'_p u(L)\#\mathbb{F}G$ is an associative nilpotent ideal of $u(L)\#\mathbb{F}G$. Therefore, without loss of generality we can replace $L$ by $L/L'_p$ and assume that $L$ is abelian. We can show, similarly as in the proof of Theorem 3.1, that $P = \omega(G) * L$ is a $G$-stable $p$-nilpotent restricted ideal of $L$. We can further assume that $G'$ acts trivially on $P$. It follows that

$$[1\#g, x_1\#g, \ldots, x_n\#g] = ((g-1) * x_1)((g-1) * x_2) \cdots ((g-1) * x_n),$$

for every $g \in G'$ and $x_1, x_2, \ldots, x_n \in L$. Since $u(L)\#\mathbb{F}G$ is Lie nilpotent, the PBW Theorem forces that $\dim (g-1) * L < \infty$, for every $g \in G'$. As $G'$ is finite, it then follows that $Q = \omega(G') * L$ is a finite-dimensional $G$-stable restricted ideal of $L$.

We can then replace $L$ with $L/Q$ in order to assume that $G'$ acts trivially on $L$. In turn, this allows to replace $G$ by $G/G'$ and assume that $G$ is abelian. We have

$$[y\#1, 1\#g_1] = (1 - g_1) * y\#g_1;$$
$$[y\#1, 1\#g_1, \ldots, 1\#g_n] = (1 - g_n)\cdots(1 - g_1) * y\#g_1 \cdots g_n,$$

for every $y \in L$ and $g_1, \ldots, g_n \in G$. Since $u(L)\#\mathbb{F}G$ is Lie nilpotent, we deduce by the PBW Theorem that there exists a minimal $m$ such that $\omega(G)^m * L = 0$. We now prove by induction that $\omega(G) * L$ is finite-dimensional. We may assume $m \geq 2$. Note that $\omega(G)^i * L$ is a $G$-stable $p$-nilpotent restricted ideal of $L$, for every $1 \leq i \leq m$. We have:

$$[1\#g_1, y\#g_1^{-1}g_2] = (g_1 - 1) * y\#g_2,$$

for every $g_1, g_2 \in G$ and $y \in \omega(G)^{m-2} * L$. Since $\omega(G)^m * L = 0$, we have

$$[1\#g_1, y_1\#g_1^{-1}g_2, \ldots, y_n\#g_n^{-1}g_{n+1}]=((g_1 - 1) * y_1)\cdots((g_n - 1) * y_n)\#g_{n+1},$$

for every $g_1, g_2, \ldots, g_n, g_{n+1} \in G$ and $y_1, y_2, \ldots, y_n \in \omega(G)^{m-2} * L$. Since $u(L)\#\mathbb{F}G$ is Lie nilpotent, we deduce by the PBW Theorem that $\omega(G)^{m-1} * L$ must be finite-dimensional. We can now replace $L$ with $L/(\omega(G)^{m-1} * L)$ and assume $\omega(G)^{m-1} * L = 0$. By induction on $m$, we deduce that $\omega(G) * L$ is finite-dimensional, as required. □

We are now in position to prove the main result of this section. Let a group $G$ act by automorphisms on an $\mathbb{F}$-vector space $V$. One says that $G$ acts nilpotently on $V$ if there exists a chain $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ of $G$-stable subspaces of $V$ such that the induced action of $G$ on each factor $V_i/V_{i-1}$ is trivial. Note that this is tantamount to saying that $\omega(G)^m * V = 0$ for some $m$, where $V$ is regarded as an $\mathbb{F}G$-module in the natural way.

**Theorem 4.2.** *Let $G$ be a group acting by automorphisms on a restricted Lie algebra $L$ over a field $\mathbb{F}$ of characteristic $p > 0$. Then $u(L)\#\mathbb{F}G$ is Lie nilpotent if and only if the following conditions are satisfied:*

1. *$G$ is nilpotent and $G'$ is a finite $p$-group;*
2. *$L$ is nilpotent;*
3. *$G$ acts nilpotently on $L$;*
4. *$L$ has a finite-dimensional $p$-nilpotent $G$-stable restricted ideal $P$ such that $L/P$ is abelian and $G$ acts trivially on $L/P$.*

*Proof.* First we prove the necessity. Clearly, $L$ is nilpotent and, by Theorem 2.1, $G$ is nilpotent and $G'$ is a finite $p$-group. By Lemma 4.1, $L$ has a finite-dimensional $p$-nilpotent $G$-stable restricted ideal $P$ such that $L/P$ is abelian and $G$ acts trivially on $L/P$. Thus, it is enough to prove that that $G$ acts nilpotently on $P$. If $P = 0$, we are done. So, we suppose $P \neq 0$. Let us first prove the claim when $G$ is abelian.

We have

$$[y\#1, 1\#g_1] = (1 - g_1) * y\#g_1;$$
$$[y\#1, 1\#g_1, \ldots, 1\#g_n] = (1 - g_n)\cdots(1 - g_1) * y\#g_1\cdots g_n,$$

for every $y \in L$ and $g_1, \ldots, g_n \in G$. We deduce that $\omega(G)^m * L = 0$, for some integer $m$.

Suppose now that $G$ is nilpotent of class $c \geq 2$ and set $N = \gamma_c(G)$, the last nontrivial term of the descending central series of $G$. The argument above shows that $N$ acts nilpotently on $L$. In particular, there exists a smallest integer $r$ such that $\omega(N)^{r+1} * (Z(L) \cap P) = 0$. Since $L$ is nilpotent, we have $Z(L) \cap P \neq 0$. Hence, there exists a nonzero element $y \in \omega(N)^r * (Z(L) \cap P)$ such that $y^{[p]} = 0$. Then we have $h * y = y$, for all $h \in N$. Let $H$ be the $G$-stable restricted ideal of $L$ generated by $y$, that is

$$H = \langle g * y \mid g \in G \rangle_{\mathbb{F}}.$$

Note that

$$h * (g * y) = (hg) * y = (gh) * y = g * (h * y) = g * y,$$

for every $h \in N$ and $g \in G$. This means that $N$ acts trivially on $H$ and so we get an induced action of $G/N$ on $H$ given by $(Ng) * z = g * z$, for every $z \in H$ and $g \in G$. Consider the smash product $u(H)\#\mathbb{F}(G/N)$. By induction on the nilpotence class $c$ of $G$, we deduce that there exists an integer $k$ such that $\omega(G/N)^k * H = 0$. Hence,

$$0 = (Ng_1 - 1)\cdots(Ng_k - 1) * z = (g_1 - 1)\cdots(g_k - 1) * z,$$

for every $g_1, \ldots, g_k \in G$ and $z \in H$. We conclude that

$$\omega(G)^k * H = 0. \tag{4.1}$$

We now proceed by induction on $\dim P$ to show that $G$ acts nilpotently on $P$. Since $H$ is $G$-stable, we note that $G$ acts on $L/H$. Then consider the smash product $u(L/H)\#\mathbb{F}G$. By the induction hypothesis, there exists an integer $m$ such that

$$\omega(G)^m * (P/H) \subseteq H. \tag{4.2}$$

Thus, Equations (4.1) and (4.2) together imply that $\omega(G)^{m+k} * P = 0$, as required.

Now we prove the converse. Let $I = Pu(L)\#\mathbb{F}G$. Since $Pu(L)$ is associative nilpotent, we note that $I$ is an associative nilpotent ideal of $R = u(L)\#\mathbb{F}G$. We argue by induction on $\dim P$ to show that $R$ is Lie nilpotent. If $\dim P = 0$ the claim is trivial. Let $k$ be the smallest integer such that $\omega(G)^{k+1} * (Z(L) \cap P) = 0$. Let $z$ be a non-zero element in $\omega(G)^k * (Z(L) \cap P)$ such that $z^{[p]} = 0$. Note that $g * z = z$, for all $g \in G$. By the induction hypothesis, we have that $u(L/\langle z \rangle_p)\#\mathbb{F}G$ is Lie nilpotent. Note that

$$u(L/\langle z \rangle_p)\#\mathbb{F}G \cong \frac{u(L)\#\mathbb{F}G}{\langle z \rangle_p u(L)\#\mathbb{F}G}.$$

Thus, there exists an integer $n$ such that $\gamma_n(R)\subseteq\langle z\rangle_p u(L)\#\mathbb{F}G$, where $\gamma_n(R)$ is the $n$-th term of the descending central series of $R$ regarded as a Lie algebra. Let $u \in u(L)$ and $g \in G$. Since $G$ acts trivially on $L/P$, we have $g * u = u$ modulo $Pu(L)$. Moreover, $L/P$ is abelian. So,

$$\begin{aligned}[zu\#g, v\#h] &= zu(g*v)\#gh - v(h*zu)\#hg \\ &= zuv\#gh - vzu\#hg \pmod{I^2} \\ &= zuv\#[g,h] \pmod{I^2}.\end{aligned}$$

By induction we get:

$$[zu_1\#g_1, u_2\#g_2, \ldots u_t\#g_t] = zu_1\cdots u_t\#[g_1,\ldots,g_t] \pmod{I^2},$$

for every $u_1,\ldots,u_t \in u(L)$ and $g_1,\ldots,g_t \in G$. Since $\mathbb{F}G$ is Lie nilpotent, we deduce that $\gamma_\ell(R)\subseteq I^2$, for some $\ell$. This means that $R/I^2$ is Lie nilpotent and it follows from Proposition 4.3 in [27] that $R$ is Lie nilpotent. □

We recall that an associative algebra $A$ is *strongly Lie nilpotent* if $A^{(m)} = 0$ for some $m$, where $A^{(1)} = A$ and $A^{(m+1)}$ is the associative ideal $[A^{(m)}, A]A$ of $A$. Of course, if $A$ is strongly Lie nilpotent then it is Lie nilpotent, but the converse is not true in general (see [8]). However, the combination of our previous result and Theorem 6.5 in [10] shows that $u(L)\#\mathbb{F}G$ is indeed strongly nilpotent when $L$ and $G$ satisfy conditions (1)-(4) of Theorem 4.2. Therefore we have the following corollary, which generalizes a result of Passi, Passman and Sehgal for group algebras (see [13] or [18, §V.6] ) and a result of Riley and Shalev for restricted enveloping algebras (see [15, Corollary 6.3]):

**Corollary 4.3.** *Let $G$ be a group acting by automorphisms on a restricted Lie algebra $L$ over a field $\mathbb{F}$ of characteristic $p > 0$. Then $u(L)\#\mathbb{F}G$ is Lie nilpotent if and only if it is strongly Lie nilpotent.*

## 5. Ordinary enveloping algebras and concluding remarks

In view of Theorems 3.1 and 4.2, it is a rather easy matter to deal with the Lie properties of smash products of ordinary enveloping algebras and group algebras. Indeed, we have

**Theorem 5.1.** *Let $G$ be a group acting by automorphisms on a Lie algebra $L$ over a field $\mathbb{F}$ of characteristic $p \geq 0$.*

1) *If $p = 0$, then $U(L)\#\mathbb{F}G$ is Lie solvable if and only $G$ and $L$ are abelian and $G$ acts trivially on $L$.*
2) *If $p > 2$, then $U(L)\#\mathbb{F}G$ is Lie solvable if and only $L$ is abelian, $G'$ is a finite $p$-group and $G$ acts trivially on $L$.*
3) *If $p > 0$, then $U(L)\#\mathbb{F}G$ is Lie nilpotent if and only if $L$ is abelian, $G$ is nilpotent, $G'$ is a finite $p$-group and $G$ acts trivially on $L$.*

*Proof.* 1) Suppose that $R = U(L)\#\mathbb{F}G$ is Lie solvable. Since $R$ contains isomorphic copies of $U(L)$ and $\mathbb{F}G$, both $U(L)$ and $\mathbb{F}G$ must be Lie solvable. It then follows from Proposition 6.2 of [15] and Theorem 2.2 that $L$ and $G$ are both abelian. It remains to prove that the action of $G$ on $L$ is trivial. Note that by [2] the ideal $[R, R]R$ is nil. Then, the element

$$z = [1\#g, x\#g^{-1}] = (g * x - x)\#1 \in [R, R]R$$

must be nilpotent. Hence, there is a positive integer $n$ such that

$$(g * x - x)^n\#1 = z^n = 0.$$

Since $U(L)$ is a domain, we have $x - g * x = 0$, as desired. The converse obviously follows from the fact that the tensor product of two commutative algebras is a commutative algebra.

2) Suppose that $U(L)\#\mathbb{F}G$ is Lie solvable. Let $\hat{L}$ be the restricted Lie algebra consisting of all primitive elements of the $\mathbb{F}$-Hopf algebra $U(L)$. Then:

$$\hat{L} = \sum_{k\geq 0} L^{p^k} \subseteq U(L).$$

Here, $L^{p^k}$ is the $\mathbb{F}$-vector space spanned by all $x^{p^k}$, where $x \in L$. It is known that $U(L) = u(\hat{L})$ (see e.g. [25, §1.1, Corollary 1.1.4]). Furthermore, $G$ acts on $\hat{L}$ by restricted automorphisms and so Theorem 3.1 applies. In particular, $G'$ is a finite $p$-group. Moreover, since $U(L)$ is a domain, the only $p$-nilpotent restricted ideal of $\hat{L}$ is zero. We deduce, by Theorem 3.1, that $L$ is abelian and $G$ acts trivially on $L$. To prove the converse, we note that $U(L)$ is commutative and $\mathbb{F}G$ is Lie solvable (by Theorem 2.2). Then, $U(L)\#\mathbb{F}G \cong U(L) \otimes_{\mathbb{F}} \mathbb{F}G$ is Lie solvable.

3) Suppose that $U(L)\#\mathbb{F}G$ is Lie nilpotent. Arguing as in part 2) of the proof, we see that $U(L) = u(\hat{L})$. Therefore, by Theorem 4.2, $G$ is nilpotent and $G'$ is a finite $p$-group. Moreover, since $U(L)$ is a domain, $L$ is abelian and $G$ acts trivially on $L$. The converse follows from the fact that $U(L)$ is commutative, $\mathbb{F}G$ is Lie nilpotent by Theorem 2.1, and $U(L)\#\mathbb{F}G \cong U(L) \otimes_{\mathbb{F}} \mathbb{F}G$. □

**Remark 5.2.** The second assertion of Theorem 5.1 does not hold in characteristic 2. Indeed, $U(L)\#\mathbb{F}G$ can be Lie solvable in characteristic 2 even if $L$ is not abelian (see Corollary 6.2 in [24]) or $G'$ is not a finite 2-group (cf. Theorem 2.2).

As an application of Theorem 5.1 we get the following:

**Corollary 5.3.** *Let $H$ be a cocommutative Hopf algebra over a field $\mathbb{F}$ of characteristic zero. Then $H$ is Lie solvable if and only if $H$ is commutative.*

*Proof.* Since Lie solvability is a multilinear identity, we can assume that $\mathbb{F}$ is algebraically closed. Now, by the Cartier–Kostant–Milnor–Moore Theorem (see e.g. [12, §5.6] ), every cocommutative Hopf algebra over an algebraically closed field of characteristic zero is isomorphic as an algebra to the smash product $U(L)\#\mathbb{F}G$, where $L$ is the Lie algebra of primitive elements of $H$ and $G$ is the group of the group-like elements of $H$. Therefore, by Theorem 5.1, we infer that $H$ is Lie solvable if and only if it is isomorphic to the tensor product of two commutative algebras (and so itself is commutative), yielding the claim. $\square$

As the following example shows, in general the previous result is false without the assumption of cocommutativity:

**Example 5.4.** Consider the 4-dimensional Sweedler's Hopf algebra $H$ over a field $\mathbb{F}$ of characteristic zero. We recall that $H$ is generated as an $\mathbb{F}$-algebra by two elements $c$ and $x$ satisfying the relations

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx.$$

Then $\{1, c, x, cx\}$ is a basis of $H$, and $H$ has a structure of a non-cocommutative Hopf algebra with comultiplication induced by $\Delta(c) = c \otimes c$ and $\Delta(x) = c \otimes x + x \otimes 1$, counit given by $\epsilon(c) = 1$ and $\epsilon(x) = 0$, and antipode given by $S(c) = c^{-1}$ and $S(x) = -cx$. (For details we refer the reader to Section 4.6 of [7].) We have $[H, H] = \langle x, cx \rangle_{\mathbb{F}}$ and $[[H, H], [H, H]] = 0$, so that $H$ is Lie solvable but not commutative.

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