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ORIGINAL ARTICLE

On composition operators between weighted (LF)- and (PLB)-spaces of continuous functions

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Abstract

Let X be a locally compact Hausdorff topological space, let $\mathcal{V} = (v_{n,k})_{n,k\in\mathbb{N}}$ be a system of positive continuous functions on X and let φ be a continuous self-map on X. The composition operators $C_\varphi : f \mapsto f \circ \varphi$ on the weighted function (LF)-spaces $\mathcal{V}C(X)$ ($\mathcal{V}_0C(X)$, resp.) and on the weighted function (PLB)-spaces $\mathcal{A}C(X)$ ($\mathcal{A}_0C(X)$, resp.) are studied. We characterize when the operator C_φ acts continuously on such spaces in terms of the system $\mathcal{V}$ and the map φ, as well as we determine conditions on $\mathcal{V}$ and φ which correspond to various basic properties of the composition operator C_φ, like boundedness, compactness, and weak compactness. Our approach requires a study of the continuity, boundedness, (weak) compactness of the linear operators between (LF)-spaces and (PLB)-spaces.

KEYWORDS

(LF)-spaces, (PLB)-spaces, (weakly) compact operator, bounded operator, composition operator, weighted function spaces

1 | INTRODUCTION

The operators on topological vector spaces of continuous functions have been extensively studied for the last several decades. On any space of functions with some structure, there are two natural types of operators, that are the multiplication operator and the composition operator.

Composition and multiplication operators have been studied on weighted spaces of (vector-valued) continuous functions on a locally compact Hausdorff topological space X in various directions. We refer the reader to the survey paper [20]. See also the book [19]. The main question in this study is to characterize when such operators are well-defined and continuous.

Recently, some authors have considered the problem to characterize the well-posedness and the continuity of composition operators in the setting of (PLB)-spaces or (LF)-spaces of smooth functions on $\mathbb{R}^N$, like the space $\mathcal{O}_M(\mathbb{R}^N)$ of the slowly increasing smooth functions and the space $\mathcal{O}_C(\mathbb{R}^N)$ of the very slowly increasing smooth functions, see [2, 10]. While, in [8] it has been characterized when continuous multiplication operators on a weighted inductive limit of Banach spaces of continuous functions are power bounded, mean ergodic or uniformly mean ergodic. Moreover, in [17] the second author has analyzed the action of the multiplication (diagonal) operators between weighted sequence (LF)-spaces, studying general properties like boundedness, compactness, and dynamics.

Motivated by these recent results, we study in this paper the composition operators on weighted function (LF)-spaces and on weighted function (PLB)-spaces. Our first aim is to characterize when the composition operator is well-defined and continuous. We also determine conditions which correspond to various basic properties of the composition operator, like boundedness, compactness and weak compactness in the sense of [18, p. 98]. To this end, we first study the continuity, boundedness, and (weak) compactness of the linear operators between (LF)-spaces and between (PLB)-spaces. This leads us to consider the problem of factorization of continuous linear operators acting on (LF)-spaces and on (PLB)-spaces. In the case of inductive limits, the factorization through the steps of a continuous linear operator is well-known and due to Grothendieck. For projective limits, unlike the case of operators between Fréchet spaces, it is not known if continuous linear operators between (PLB)-spaces factorize through the steps. So, we show that also for the (PLB)-spaces, the factorization is still valid.

The paper is organized as follows. In Section 2, we collect the necessary definitions and give the characterization of the continuity, boundedness, (weak) compactness for linear operator between (LF)-spaces and between (PLB)-spaces. In Section 3, we first introduce the weighted function (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ and the weighted function (PLB)-spaces $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$. Thereafter, we recall some known results about their topological properties, like completeness and regularity. Then, we characterize the continuity, boundedness, (weak) compactness for composition operators C_φ acting on pairs of weighted (LF)-spaces $(\mathcal{V}C(X), \mathcal{W}C(X))$ ($(\mathcal{V}_0C(X), \mathcal{W}_0C(X))$, resp.), and on pairs of weighted (PLB)-spaces $(\mathcal{A}_\mathcal{V}C(X), \mathcal{A}_\mathcal{W}C(X))$ ($(\mathcal{A}_{0,\mathcal{V}}C(X), \mathcal{A}_{0,\mathcal{W}}C(X))$, resp.) in terms of the weights $\mathcal{V}, \mathcal{W}$ and the function φ. Finally, in Section 4, we present some applications.

2 | DEFINITIONS AND GENERAL RESULTS ON (LF)- AND (PLB)-SPACES

Let E and F be two locally convex Hausdorff spaces (briefly, lcHs for locally convex Haudorff space). We denote by $\mathcal{L}(E,F)$ the space of all continuous linear operators from E into F. In particular, $\mathcal{L}_s(E,F)$ ($\mathcal{L}_b(E,F)$, resp.) denotes $\mathcal{L}(E,F)$ endowed with the strong operator topology τ_s ($\mathcal{L}(E,F)$ endowed with the topology τ_b of the uniform convergence on bounded subsets of E, resp.). In case $F = E$, we simply write $\mathcal{L}(E)$, $\mathcal{L}_s(E)$ and $\mathcal{L}_b(E)$.

Let T be a linear operator from E into F. The operator T is called *bounded* if T maps some 0-neighborhood of E into a bounded subset of F, while it is called *compact* (*weakly compact*, resp.) if T maps some 0-neighborhood of E into a relatively compact (relatively weakly compact, resp.) subset of F. We observe that if T is a bounded or (weakly) compact operator from E into F, then it is necessarily continuous, that is, $T \in \mathcal{L}(E,F)$.

In the following, we collect some results on operators acting between (LF)-spaces or (PLB)-spaces. We first consider the case of (LF)-spaces. To do this, we recall some necessary definitions and properties.

A lcHs E is called an (LF)-space if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of Fréchet spaces with $E_n \hookrightarrow E_{n+1}$ continuously such that $E = \cup_{n\in\mathbb{N}} E_n$ and the topology of E coincides with the finest locally convex topology for which each inclusion $E_n \hookrightarrow E$ is continuous. In such a case, we simply write $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$. The sequence $\{E_n\}_{n\in\mathbb{N}}$ is called a *defining inductive spectrum* for E. In this paper, we point out that (LF)-spaces are Hausdorff by definition. The space $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ is called an (LB)-space if E_n is a Banach space for all $n \in \mathbb{N}$. An (LF)-space $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ is called *regular* if every bounded set B in E is contained and bounded in E_n for some $n \in \mathbb{N}$. Every complete (LF)-space is always regular. Next, we introduce other useful regularity conditions.

Let $E = \operatorname{ind}_n E_n$ be an (LF)-space and τ denote the locally convex topology of E. The (LF)-space E is said to satisfy the *condition (M)* ((M_0), resp.) *of Retakh* if there exists an increasing sequence $\{U_n\}_{n\in\mathbb{N}}$ of subsets of E such that U_n is an absolutely convex 0-neighborhood of E_n for all $n \in \mathbb{N}$ for which

$$\forall n \in \mathbb{N}\ \exists m \geq n\ \forall \mu \geq m :\ \tau_\mu \text{ and } \tau_m \text{ induce the same topology on } U_n,$$

$$(\forall n \in \mathbb{N}\ \exists m \geq n\ \forall \mu \geq m :\ \sigma(E_\mu, E'_\mu) \text{ and } \sigma(E_m, E'_m) \text{ induce the same topology on } U_n,\ \text{resp.})$$

where τ_n denotes the locally convex topology of E_n for all $n \in \mathbb{N}$. An (LF)-space satisfying condition (M) ((M_0), resp.) is called *acyclic* (*weakly acyclic*, resp.). Every acyclic (LF)-space is weakly acyclic and also complete (see [22, Corollary 6.5]).

An (LF)-space $E = \operatorname{ind}_n E_n$ is called *compactly retractive* (*weakly compactly retractive*, resp.) if every compact (weakly compact, resp.) set K in E is contained and compact (weakly compact, resp.) in E_n for some $n \in \mathbb{N}$. An (LF)-space $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ is called *boundedly retractive* if every bounded set B in E is contained in some step E_n and the topologies of E and E_n coincide on B, while E is called *sequentially retractive* (*weakly sequentially retractive*, resp.) if every convergent sequence (weakly convergent sequence, resp.) in E is contained in some step E_n and converges (weakly converges, resp.) there. We observe that, in view of Grothendieck's factorization theorem [12, p. 147], these conditions do not depend on the defining inductive spectrum of E.

Some of these notions are related to each other, as it is shown in the following theorem due to Wengenroth [22].

Theorem 1 [22, Theorem 6.4]. *For an (LF)-space $E = \operatorname{ind}_n E_n$, the following conditions are equivalent:*

(1) E satisfies condition (M);
(2) E is boundedly retractive;
(3) E is compactly retractive;
(4) E is sequentially retractive.

Every weakly compactly retractive (LF)-space is clearly weakly sequentially retractive and hence, regular by [13, Theorem 1]. The converse is also valid when the (LF)-space satisfies the condition (M_0) according to the following result.

Theorem 2 [13, Theorem 2]. *Let $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ be an (LF)-space satisfying the condition (M_0). Then, the following conditions are equivalent:*

(1) E is regular;
(2) E is weakly compactly retractive;
(3) E is weakly sequentially retractive.

According to Theorems 1 and 2, every sequentially retractive (LF)-space is also weakly sequentially retractive.

The characterization of the continuity of operators between (LF)-spaces is well-known and due to Grothendieck. The characterization of boundedness as well as the compactness of operators acting between (LF)-spaces has been given in [17] as follows (see also [7, Proposition 5], where the (LB)-case is considered). We include also the weakly compact case. The proof is analogous to that of the compactness and is left to the reader.

Proposition 1 [17, Proposition 2.3]. *Let $E = \operatorname{ind}_n E_n$ and $F = \operatorname{ind}_n F_n$ be two (LF)-spaces. Let $T : E \to F$ be a linear operator. Then, the following assertions hold true:*

(1) Assume that F is regular. Then, the operator T is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T : E_m \to F_n$ is bounded;
(2) Assume that F satisfies the condition (M). Then, the operator T is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T : E_m \to F_n$ is compact;
(3) Assume that F is weakly compactly retractive. Then, the operator T is weakly compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T : E_m \to F_n$ is weakly compact.

A lcHs E is called a (PLB)-space if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of (LB)-spaces with $E_{n+1} \hookrightarrow E_n$ continuously such that $E = \cap_{n\in\mathbb{N}} E_n$ and the topology of E is the coarsest locally convex topology for which each inclusion $E \hookrightarrow E_n$ is continuous. In such a case, we simply write $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$. Clearly, a (PLB)-space $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ is complete whenever E_n is a complete (LB)-space for an infinite number of indices n.

Unlike the case of operators between Fréchet spaces, in general continuous linear operators between (PLB)-spaces do not factorize through the steps. Under suitable condition on the (PLB)-spaces, the factorization is still valid. Indeed, along the lines of [11, Lemma 4], we get the following result:

Proposition 2. *Let $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ be a (PLB)-space such that the continuous inclusion $E \hookrightarrow E_n$ has dense range for all $n \in \mathbb{N}$. Let $F = \operatorname{proj}_{k\in\mathbb{N}} F_k$ be a (PLB)-space such that F_k is a complete (LB)-space for all $k \in \mathbb{N}$. If $T : E \to F$ is a continuous linear operator, then for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the operator T admits a unique continuous extension T_k^n from E_n into F_k.*

Proof. Let τ denote the topology of E and for all $n \in \mathbb{N}$, let τ_n denote the topology of E_n. We claim that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the operator $T : (E, \tau_n) \to F_k$ is continuous. To prove the claim, we proceed by arguing by contradiction.

Suppose that there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the operator $T : (E, \tau_n) \to F_k$ is not continuous. Accordingly, for all $n \in \mathbb{N}$ there exists a 0-neighborhood U_n in F_k such that the operator $T : (E, \tau_n) \to (F_k, p_{U_n})$ is not continuous, where the range is the space equipped with the gauge functional p_{U_n} as a seminorm. Since F_k is an (LB)-space, there exists a 0-neighborhood U in F_k such that U is absorbed by U_n for all $n \in \mathbb{N}$ (see [14, Proposition 2.7.9]). Now, the continuity of the operator $T : (E, \tau) \to F_k$ implies that there exists a 0-neighborhood V in (E, τ) such that $T(V) \subseteq U$. Since E is the projective limit of the (LB)-spaces E_n, there exist some $n_0 \in \mathbb{N}$ and a continuous seminorm p on E_{n_0} such that $V_{n_0} := \{x \in E : p(x) \leq 1\} \subseteq V$. This yields that the operator $T : (E, \tau_{n_0}) \to (F_k, p_U)$ is continuous. On the other hand, the fact that U is absorbed by U_{n_0} implies that the inclusion $(F_k, p_U) \hookrightarrow (F_k, p_{U_{n_0}})$ is continuous. Accordingly, $T : (E, \tau_{n_0}) \to (F_k, p_{U_{n_0}})$ is necessarily continuous. But this is a contradiction.

According to what was proved above, for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the operator $T : (E, \tau_n) \to F_k$ is continuous. Since E is a dense subspace of E_n and F_k is complete, the operator $T : (E, \tau_n) \to F_k$ admits a unique continuous extension T_k^n from E_n into F_k. □

Remark 3.

(1) Due to the definition of the projective limit topology, the condition of Proposition 2 is clearly also sufficient.
(2) The proof of Proposition 2 ensures that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such the operator $T : (E, \tau_n) \to F_k$ is continuous also in the case, where $E = \text{proj}_{n\in\mathbb{N}} E_n$ is a (PLB)-space with no dense inclusion in E_n for any $n \in \mathbb{N}$.

Also for operators between (PLB)-spaces, we can give the characterization of the boundedness and of the (weak) compactness as follows.

Proposition 3. *Let $E = \text{proj}_{n\in\mathbb{N}} E_n$ be a (PLB)-space such that the inclusion $E \hookrightarrow E_n$ has dense range for all $n \in \mathbb{N}$. Let $F = \text{proj}_{k\in\mathbb{N}} F_k$ be a (PLB)-space such that F_k is a complete (LB)-space for all $k \in \mathbb{N}$. A linear operator $T : E \to F$ is bounded ((weakly) compact, resp.) if, and only if, there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator T admits a unique linear extension $T_k^n : E_n \to F_k$ which is bounded ((weakly) compact, resp.).*

Proof. The condition is clearly sufficient. So, we suppose that $T : E \to F$ is bounded ((weakly) compact, resp.), thereby implying that $T \in \mathcal{L}(E, F)$ necessarily. Accordingly, there exists a 0-neighborhood V of E such that $T(V)$ is a bounded (relatively (weakly) compact, resp.) set in F. Since E is the projective limit of the (LB)-spaces E_n, there exist $n_0 \in \mathbb{N}$ and a 0-neighborhood U of E_{n_0} such that $E \cap U \subseteq V$. Consequently, $T(E \cap U)$ is a bounded (relatively (weakly) compact, resp.) set in F and hence, $T(E \cap U)$ is a bounded (relatively (weakly) compact, resp.) set in F_k for all $k \in \mathbb{N}$. This implies that the operator $T : (E, \tau_{n_0}) \to F_k$ is bounded ((weakly) compact, resp.) for all $k \in \mathbb{N}$ and hence, continuous. Since E is a dense subspace of E_{n_0} and each F_k is a complete (LB)-space, for all $k \in \mathbb{N}$ the operator $T : (E, \tau_{n_0}) \to F_k$ admits a unique extension $T_k^{n_0} : E_{n_0} \to F_k$ which is clearly bounded ((weakly) compact, resp.). □

Remark 4.

(1) We observe that Proposition 3 covers also the case that each E_n is a Banach space (hence, E is a Fréchet space) or each F_k is a Banach space (hence, F is a Fréchet space).
(2) The proof of Proposition 3 ensures that there exists $n_0 \in \mathbb{N}$ such that the operator $T : (E, \tau_{n_0}) \to F_k$ is bounded ((weakly) compact, resp.) for all $k \in \mathbb{N}$ also in the case that $E = \text{proj}_{n\in\mathbb{N}} E_n$ is a (PLB)-space with no dense inclusion in E_n for any $n \in \mathbb{N}$.

3 | COMPOSITION OPERATORS BETWEEN WEIGHTED (LF)- AND (PLB)-SPACES OF CONTINUOUS FUNCTIONS

3.1 | Weighted (LF)- and (PLB)-spaces of continuous functions

Throughout this paper, X will be denote a locally compact (Hausdorff) topological space.

For all $n \in \mathbb{N}$, let $V_n = \left(v_{n,k}\right)_{k\in\mathbb{N}}$ be a sequence of (strictly) positive continuous functions, called *weights*, on X. We denote by $\mathcal{V}$ the sequence $(V_n)_{n\in\mathbb{N}}$ and we assume that these two conditions are satisfied:

(1) $v_{n,k}(x) \le v_{n,k+1}(x)$ for all $n, k \in \mathbb{N}$ and $x \in X$;
(2) $v_{n,k}(x) \ge v_{n+1,k}(x)$ for all $n, k \in \mathbb{N}$ and $x \in X$.

The sequence $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is said to be a *system of weights*. In the following, for all $k \in \mathbb{N}$ we set $V^k = (v_{n,k})_{n\in\mathbb{N}}$.

Denote by $C(X)$ the space of all $\mathbb{K}$-valued continuous functions on X, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. We recall that a function $g \in C(X)$ is said to vanish at infinity if for every $\varepsilon > 0$ there exists a compact subset K of X such that $|g(x)| \le \varepsilon$ for all $x \in X \setminus K$. Given a single weight v on X, we can define:

$$C_v(X) := \{f \in C(X) : \|f\|_v := \|vf\|_\infty < \infty\},$$

$$C_{v_0}(X) := \{f \in C(X) : v|f| \text{ vanishes at infinity}\}.$$

The space $(C_v(X), \|\cdot\|_v)$ is a Banach space and $(C_{v_0}(X), \|\cdot\|_v)$ is a closed subspace of $C_v(X)$. Furthermore, if we endow the space $C(X)$ with the compact open topology, then the inclusion $C_v(X) \hookrightarrow C(X)$ is continuous for every weight v on X. Indeed, for a fixed weight v on X and a compact subset K of X, we have that $\max_{x\in K} |f(x)| = \max_{x\in K} \frac{1}{v(x)} v(x)|f(x)| \le c_K \|f\|_v$ for all $f \in C_v(X)$, where $c_K := \max_{x\in K} \frac{1}{v(x)} < \infty$ being $\frac{1}{v} \in C(X)$.

Given a system $\mathcal{V}$ of weights on X and $n \in \mathbb{N}$, due to condition (1) both the sequences $\{C_{v_{n,k}}(X)\}_{k\in\mathbb{N}}$ and $\{C_{(v_{n,k})_0}(X)\}_{k\in\mathbb{N}}$ of Banach spaces form a projective spectrum. Hence, for all $n \in \mathbb{N}$ the following *weighted spaces of continuous functions*, defined by

$$CV_n(X) := \left\{f \in C(X) : \|f\|_{v_{n,k}} := \|v_{n,k} f\|_\infty < \infty \ \forall k \in \mathbb{N}\right\},$$

$$C(V_n)_0(X) := \left\{f \in C(X) : v_{n,k}|f| \text{ vanishes at infinity } \forall k \in \mathbb{N}\right\},$$

are Fréchet spaces with respect to the lc-topology generated by the sequence $(\|\cdot\|_{v_{n,k}})_{k\in\mathbb{N}}$ of norms. We observe that $C(V_n)_0(X)$ is a dense subset of $C_{(v_{n,k})_0}(X)$ for all $n, k \in \mathbb{N}$.

According to condition (2), $CV_n(X)$ ($C(V_n)_0(X)$, resp.), is continuously included in $CV_{n+1}(X)$ ($C(V_{n+1})_0(X)$, resp.) for all $n \in \mathbb{N}$. So, we can define the following *weighted (LF)-spaces of continuous functions*

$$\mathcal{V}C(X) := \operatorname*{ind}_n CV_n(X) \quad \text{and} \quad \mathcal{V}_0C(X) := \operatorname*{ind}_n C(V_n)_0(X).$$

Obviously, $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ are continuously included in $C(X)$ for every system of weights $\mathcal{V}$ on X.

In [4], Bierstedt and Bonet characterized the regularity of the (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ in terms of the system $\mathcal{V}$ of weights. In order to state such results, we recall some necessary definitions.

Let $\mathcal{V} = (v_{n,k})_{n,k\in\mathbb{N}}$ be a system of weights on X. The system $\mathcal{V}$ of weights is said to satisfy the *condition (WQ)* (or is of *type (WQ)*) if

$$\forall n \in \mathbb{N}\ \exists \mu,\ m \in \mathbb{N}\ \forall k,\ N \in \mathbb{N}\ \exists K \in \mathbb{N},\ S > 0 : \forall x \in X\ v_{m,k}(x) \le S(v_{n,\mu}(x) + v_{N,K}(x)).$$

The system $\mathcal{V}$ of weights is said to satisfy the *condition (Q)* (or is of *type (Q)*) if

$$\forall n \in \mathbb{N}\ \exists \mu,\ m \in \mathbb{N}\ \forall k,\ N \in \mathbb{N},\ R > 0\ \exists K \in \mathbb{N},\ S > 0 : \forall x \in X\ v_{m,k}(x) \le \frac{1}{R} v_{n,\mu}(x) + S v_{N,K}(x).$$

The characterization of the regularity of the (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ is contained in the following result.

Theorem 5. *Let X be a locally compact Hausdorff topological space and $\mathcal{V}$ be a system of weights on X. Then, the following assertions hold true:*

(1) *([4, Proposition 4 and Theorem 7]) $\mathcal{V}C(X)$ is regular if, and only if, $\mathcal{V}$ satisfies condition (WQ) if, and only if, $\mathcal{V}C(X)$ is complete;*
(2) *([4, Theorem 3]) $\mathcal{V}_0C(X)$ is regular if, and only if, $\mathcal{V}$ satisfies condition (Q) if, and only if, $\mathcal{V}_0C(X)$ is complete.*

We recall also that Bierstedt and Bonet in [4, Theorem 3] characterized when the (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ satisfy the condition (M).

Theorem 6. *Let X be a locally compact Hausdorff topological space and $\mathcal{V}$ be a system of weights on X. Then, the following assertions are equivalent:*

(1) $\mathcal{V}C(X)$ satisfies condition (M);
(2) $\mathcal{V}_0C(X)$ satisfies condition (M);
(3) $\mathcal{V}$ satisfies condition (Q).

We refer the reader to [3, 4] for more details.

Given a system $\mathcal{V}$ of weights on X and $k \in \mathbb{N}$, due to condition (2) both the sequences $\{C_{v_{n,k}}(X)\}_{n\in\mathbb{N}}$ and $\{C_{(v_{n,k})_0}(X)\}_{n\in\mathbb{N}}$ of Banach spaces form an inductive spectrum. Hence, for all $k \in \mathbb{N}$ we can define the following weighted (LB)-spaces of continuous functions:

$$\mathcal{A}_kC(X) := \operatorname*{ind}_n C_{v_{n,k}}(X) \quad \text{and} \quad (\mathcal{A}_k)_0C(X) := \operatorname*{ind}_n C_{(v_{n,k})_0}(X).$$

By [5], the space $\mathcal{A}_kC(X)$ is always a complete, hence, a regular (LB)-space for all $k \in \mathbb{N}$. The (LB)-space $(\mathcal{A}_k)_0C(X)$ need not be regular. The regularity is ensured by a stronger condition on the system $\mathcal{V}$ of weights. In order to see this, we recall the following.

Given a sequence of decreasing weights $V = \{v_n\}_{n\in\mathbb{N}}$ on X, we say that V is *regularly decreasing* if, given $n \in \mathbb{N}$, there exists $m \geq n$ so that, for every $\varepsilon > 0$ and every $k \geq m$ there exists $\delta = \delta(k, \varepsilon) > 0$ such that $v_k(x) \geq \delta v_n(x)$ whenever $v_m(x) \geq \varepsilon v_n(x)$. In other words, V is regularly decreasing if, and only if, given $n \in \mathbb{N}$, there exists $m \geq n$ such that, on each subset of X on which the quotient $\frac{v_m}{v_n}$ is bounded away from zero, also all quotients $\frac{v_k}{v_n}$, $k \geq m$, are bounded away from zero.

By [5, Corollary 2.7], for any $k \in \mathbb{N}$, the (LB)-space $\mathcal{A}_kC(X)$ satisfies condition (M) if, and only if, it is (strongly) boundedly retractive if, and only if, the sequence $V^k = \left(v_{n,k}\right)_{n\in\mathbb{N}}$ is regularly decreasing. While, by [5, Theorem 2.6], for any $k \in \mathbb{N}$, the (LB)-space $(\mathcal{A}_k)_0C(X)$ is regular if, and only if, it is complete if, and only if, it satisfies condition (M) if, and only if, it is (strongly) boundedly retractive, and this is in turn equivalent to the fact that the sequence $V^k = \left(v_{n,k}\right)_{n\in\mathbb{N}}$ is regularly decreasing.

Due to condition (1), both the sequences $\{\mathcal{A}_kC(X)\}_{k\in\mathbb{N}}$ and $\{(\mathcal{A}_k)_0C(X)\}_{k\in\mathbb{N}}$ of (LB)-spaces form a projective spectrum. Hence, we can define the following *weighted (PLB)-spaces of continuous functions:*

$$\mathcal{A}C(X) := \operatorname*{proj}_k \mathcal{A}_kC(X) \quad \text{and} \quad \mathcal{A}_0C(X) := \operatorname*{proj}_k (\mathcal{A}_k)_0C(X).$$

Obviously, $\mathcal{A}C(X)$ and $\mathcal{A}_0C(X)$ are continuously included in $C(X)$ for every system $\mathcal{V}$ of weights. We also observe that $\mathcal{A}_0C(X)$ is a dense subset of $(\mathcal{A}_k)_0C(X)$ for all $k \in \mathbb{N}$. We refer the reader to [1] for more details.

3.2 | Composition operators

Let E and F be two lcHs of $\mathbb{K}$-valued functions defined on X. Let φ be a function from X into X. If $f \circ \varphi \in F$ for all $f \in E$, then we can consider the composition operator $C_\varphi : E \to F$, $f \mapsto f \circ \varphi$. The operator C_φ is clearly linear. In case that $C_\varphi \in \mathcal{L}(E, F)$, the function φ is said to be a *symbol* for the pair (E, F). If $E = F$, we say simply that φ is a symbol for E.

In the following, we denote by $C(X, X)$ the space of all continuous functions $\varphi : X \to X$, by $C_b(X)$ the space of all $\mathbb{K}$-valued bounded continuous functions on X and $C_0(X)$ the space of all $\mathbb{K}$-valued bounded continuous functions on X vanishing at infinity. The spaces $C_b(X)$ and $C_0(X)$ are Banach space with respect to the supremum norm $\|\cdot\|_\infty$ on X. We point out that every $\varphi \in C(X, X)$ is a symbol for the space $C(X)$, endowed with the compact open topology.

In this section, we study the composition operator C_φ acting between the (LF)- and (PLB)-spaces introduced in Section 3.1. The first aim is to establish what continuous functions $\varphi : X \to X$ are symbols for the pairs $(\mathcal{V}C(X), \mathcal{W}C(X))$ and $(\mathcal{A}_\mathcal{V}C(X), \mathcal{A}_\mathcal{W}C(X))$. In order to do this, we study the composition operators C_φ between weighted Banach spaces of continuous functions. So, we recall that a continuous map $\varphi : X \to X$ is called *proper* if the preimage of every compact set K in X is also a compact set in X.

Proposition 4. *Let v, w be two weights on X and $\varphi \in C(X,X)$. Then the following assertions hold true:*

(1) The composition operator $C_\varphi : C_v(X) \to C_w(X)$, $f \mapsto f \circ \varphi$, is well-defined (and so continuous) if, and only if, $\frac{w}{v\circ\varphi} \in C_b(X)$;
(2) If φ is a proper map, then the composition operator $C_\varphi : C_{v_0}(X) \to C_{w_0}(X)$, $f \mapsto f \circ \varphi$, is well-defined (and so continuous) if, and only if, $\frac{w}{v\circ\varphi} \in C_b(X)$.

Proof.

(1) If $\frac{w}{v\circ\varphi} \in C_b(X)$, the operator $C_\varphi : C_v(X) \to C_w(X)$ is clearly well-defined and continuous.
Conversely, suppose that $C_\varphi : C_v(X) \to C_w(X)$ is well-defined. Since the function $\frac{1}{v}$ belongs to $C_v(X)$, it follows that $C_\varphi\left(\frac{1}{v}\right) = \frac{1}{v\circ\varphi} \in C_w(X)$ and hence, $\frac{w}{v\circ\varphi} \in C_b(X)$.

(2) If $\frac{w}{v\circ\varphi} \in C_b(X)$, to show that C_φ is well-defined it suffices to establish that $(vf)\circ\varphi$ vanishes at infinity for any $f \in C_{v_0}(X)$. Indeed, if $(vf)\circ\varphi$ vanishes at infinity for $f \in C_{v_0}(X)$ and $C := \|\frac{w}{v\circ\varphi}\|_\infty$, then for a fixed $\varepsilon > 0$ there exists a compact set K in X such that $|(vf)(\varphi(x))| < \frac{\varepsilon}{C}$ for every $x \in X \setminus K$. Hence, for every $x \in X \setminus K$ we get

$$|w(x)(f\circ\varphi)(x)| = \frac{w(x)}{(v\circ\varphi)(x)}|(vf)(\varphi(x))| < C\frac{\varepsilon}{C} = \varepsilon.$$

This means that $w(f\circ\varphi)$ vanishes at infinity, that is, $C_\varphi(f) \in C_{w_0}(X)$.
Now, to prove the claim observe that for fixed $f \in C_{v_0}(X)$ and $\varepsilon > 0$, there exists a compact set K in X such that $|v(x)f(x)| < \varepsilon$ for every $x \in X \setminus K$. Since φ is proper, the preimage of K, that is, $H := \varphi^{-1}(K)$, is a compact set in X such that $|(vf)(\varphi(x))| < \varepsilon$ for every $x \in X \setminus H$, being $\varphi(x) \notin K$. This shows that the condition is satisfied.
Conversely, suppose that $C_\varphi : C_{v_0}(X) \to C_{w_0}(X)$ is well-defined and hence, continuous by the Closed Graph Theorem [16, Section 35, p. 57]. Accordingly, there exists $C > 0$ such that $\|C_\varphi(f)\|_w \le C\|f\|_v$ for all $f \in C_{v_0}(X)$. Now, for a fixed $x \in X$, let $f_x \in C(X)$ such that $0 \le f_x(y) \le \frac{1}{v(y)}$ for all $y \in X$, supp f_x is a compact set in X and $f_x(\varphi(x)) = \frac{1}{(v\circ\varphi)(x)}$. Then

$$\frac{w(x)}{(v\circ\varphi)(x)} = w(x)f_x(\varphi(x)) \le \|C_\varphi(f_x)\|_w \le C\|f_x\|_v \le C.$$

Since $x \in X$ is arbitrary, it follows that the function $\frac{w}{v\circ\varphi} \in C_b(X)$. □

Remark 7. Let v, w be two weights on X and $\varphi \in C(X,X)$. Let $T : C(X) \to C(X)$ be the operator defined by $T(f) := \frac{w}{v\circ\varphi}(f\circ\varphi)$ for $f \in C(X)$. Then, $T \in \mathcal{L}(C_b(X))$ if, and only if, $\frac{w}{v\circ\varphi} \in C_b(X)$. Indeed, it suffices to observe that $T = M_w \circ C_\varphi \circ M_{\frac{1}{v}}$ and then apply Proposition 4(1), being $M_{\frac{1}{v}} : C_v(X) \to C_b(X)$, $f \mapsto \frac{1}{v}f$, ($M_w : C_w(X) \to C_b(X)$, $f \mapsto wf$, resp.) an isometric surjective operator.

Suppose that φ is a proper map. So, by arguing as above and due to Proposition 4(2), we obtain that $T \in \mathcal{L}(C_0(X))$ if, and only if, $\frac{w}{v\circ\varphi} \in C_b(X)$, being again $M_{\frac{1}{v}} : C_{v_0}(X) \to C_0(X)$, $f \mapsto \frac{1}{v}f$, ($M_w : C_{w_0}(X) \to C_0(X)$, $f \mapsto wf$, resp.) an isometric surjective operator.

Now, we can characterize the symbols φ for the pairs of (LF)-spaces $(\mathcal{V}C(X), \mathcal{W}C(X))$ and $(\mathcal{V}_0C(X), \mathcal{W}_0C(X))$.

Theorem 8. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X,X)$. Then, the following properties are equivalent:*

(1) $C_\varphi(\mathcal{V}C(X)) \subseteq \mathcal{W}C(X)$;
(1') $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is continuous;

(2) For all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ for which

$$\sup_{x\in X} \frac{w_{n,k}(x)}{v_{m,l}(\varphi(x))} < \infty. \tag{1}$$

Furthermore, if φ is a proper map, the previous assertions are equivalent to:

(3) $C_\varphi(\mathcal{V}_0C(X)) \subseteq \mathcal{W}_0C(X)$;
(3') $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{W}_0C(X)$ is continuous.

Proof. Clearly, (1') implies (1) ((3') implies (3), resp.) and (1) implies (1') ((3) implies (3'), resp.) by the Closed Graph theorem [16, Section 35, p. 57]. Indeed, consider a net $(f_i)_i \subset \mathcal{V}C(X)$ convergent to f in $\mathcal{V}C(X)$ such that $(C_\varphi(f_i))_i$ converges to g in $\mathcal{W}C(X)$. Since the inclusion $\mathcal{V}C(X) \hookrightarrow C(X)$ is continuous, the net $(f_i)_i$ converges to f in $C(X)$, thereby implying that $(C_\varphi(f_i))_i$ converges to $f\circ\varphi$ in $C(X)$ because $C_\varphi \in \mathcal{L}(C(X))$. The same argument yields that $(C_\varphi(f_i))_i$ converges to g in $C(X)$. So, it follows that $g = f\circ\varphi$ and hence, $g = C_\varphi(f)$. This proves that the graph of C_φ is closed. The proof of (3) implies (3') follows by arguing in a similar way.

We prove that (1') is equivalent to (2).

(1')⇔(2). The composition operator $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is continuous if, and only if, for all $m \in \mathbb{N}$ the composition operator $C_\varphi : CV_m(X) \to \mathcal{W}C(X)$ is continuous. From Grothendieck's factorization theorem [12, p.147], C_φ is then continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $C_\varphi : CV_m(X) \to CW_n(X)$ is continuous. But the composition operator C_φ between the Fréchet spaces $CV_m(X)$ and $CW_n(X)$ is continuous if, and only if, for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ and $C > 0$ such that

$$\|C_\varphi(f)\|_{w_{n,k}} \le C\|f\|_{v_{m,l}}, \quad \forall f \in CV_m(X). \tag{2}$$

If $CV_m(X)$ is dense in $C_{v_{m,l}}(X)$ for all $l \in \mathbb{N}$, then it follows that the operator C_φ admits a unique continuous linear extension $(C_\varphi)_k^l$ from $C_{v_{m,l}}(X)$ into $C_{w_{n,k}}(X)$. Since $C_\varphi \in \mathcal{L}(C(X))$ and the spaces $C_{v_{m,l}}(X)$ and $C_{w_{n,k}}(X)$ are continuously included in $C(X)$, necessarily $(C_\varphi)_k^l = C_\varphi$, that is, $C_\varphi : C_{v_{m,l}}(X) \to C_{w_{n,k}}(X)$ is continuous. Due to Proposition 4(1), this is equivalent to require that the function $\frac{w_{n,k}}{v_{m,l}\circ\varphi}$ belongs to $C_b(X)$. This means that the condition (1) is satisfied.

If $CV_m(X)$ is not dense in $C_{v_{m,l}}(X)$ for all $l \in \mathbb{N}$, to get the result we argue as follows. For a fixed $x \in X$, let $f_x \in C(X)$ such that $0 \le f_x(y) \le \frac{1}{v_{m,l}(y)}$ for all $y \in X$, supp f_x is a compact set in X and $f_x(\varphi(x)) = \frac{1}{(v_{m,l}\circ\varphi)(x)}$. Then, $f_x \in CV_m(X)$ because f_x has compact support. Moreover, $\|f_x\|_{v_{m,l}} = 1$. So, by Equation (2) it follows that

$$\frac{w_{n,k}(x)}{(v_{m,l}\circ\varphi)(x)} = w_{n,k}(x)f_x(\varphi(x)) \le \|C_\varphi(f_x)\|_{w_{n,k}} \le C\|f_x\|_{v_{m,l}} \le C.$$

Since $x \in X$ is arbitrary, it follows that $\frac{w_{n,k}}{v_{m,l}\circ\varphi} \in C_b(X)$, that is, the condition (1) is satisfied.

Conversely, if the condition (1) is satisfied, then Proposition 4(1) ensures that $C_\varphi \in \mathcal{L}(C_{v_{m,l}}(X), C_{w_{n,k}}(X))$ and hence, (2) is satisfied. This implies that $C_\varphi \in \mathcal{L}(CV_m(X), CW_n(X)) \subset \mathcal{L}(CV_m(X), \mathcal{W}C(X))$ and hence, $C_\varphi \in \mathcal{L}(\mathcal{V}C(X), \mathcal{W}C(X))$ as $m \in \mathbb{N}$ is arbitrary.

The proof of (3')⇔(2) is analogous and so it is omitted. We only observe that, in such a case, each Fréchet space $C(V_m)_0(X)$ is dense in $C_{(v_{m,l})_0}(X)$ for all $l \in \mathbb{N}$ and so we can apply directly Proposition 4(2). □

The characterization of the symbols for the pairs of (PLB)-spaces $(\mathcal{A}_\mathcal{V}C(X), \mathcal{A}_\mathcal{W}C(X))$ and $(\mathcal{A}_{0,\mathcal{V}}C(X), \mathcal{A}_{0,\mathcal{W}}C(X))$ is given in the following result.

Theorem 9. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X, X)$. Then, the following properties are equivalent:*

(1) $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is continuous;
(2) For all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ for which

$$\sup_{x\in X} \frac{w_{n,k}(x)}{v_{m,l}(\varphi(x))} < \infty. \tag{3}$$

If, in addition, $\mathcal{V}$ satisfies condition (Q), the assertion (1) is equivalent to

(1') $C_\varphi(\mathcal{A}_\mathcal{V}C(X)) \subseteq \mathcal{A}_\mathcal{W}C(X)$.

Furthermore, if φ is a proper map and the sequence $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$, the assertions (1) and (2) are equivalent to:

(3) $C_\varphi : \mathcal{A}_{0,\mathcal{V}}C(X) \to \mathcal{A}_{0,\mathcal{W}}C(X)$ is continuous.

If, in addition, $\mathcal{V}$ satisfies condition (WQ), the assertion (3) is equivalent to

(3') $C_\varphi(\mathcal{A}_{0,\mathcal{V}}C(X)) \subseteq \mathcal{A}_{0,\mathcal{W}}C(X)$.

Proof. Arguing as done in the proof of Theorem 8, (1) and (1') ((3) and (3'), resp.) are equivalent by the Closed Graph theorem [16, §35, pp.57-58]. Indeed, under the assumption that $\mathcal{V}$ satisfies condition (Q) ((WQ), resp.), the space $\mathcal{A}_\mathcal{V}C(X)$ is ultrabornological (see [1, Theorems 3.5] and [22, Theorem 3.3.4] ($\mathcal{A}_{0,\mathcal{V}}C(X)$ is ultrabornological by [1, Theorem 3.7], resp.) and $\mathcal{A}_\mathcal{W}C(X)$ ($\mathcal{A}_{0,\mathcal{W}}C(X)$, resp.) is a webbed space. So, we can apply the Closed Graph theorem.

We prove that (1) is equivalent to (2).

(1)⇔(2). By Remark 3(2), the composition operator $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is continuous if, and only if, for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that the operator $C_\varphi : (\mathcal{A}_\mathcal{V}C(X), \tau_l) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is continuous, where τ_l denotes the lc-topology of the (LB)-space $\mathcal{A}_{l,\mathcal{V}}C(X)$. In the case the inclusion $\mathcal{A}_\mathcal{V}C(X) \hookrightarrow \mathcal{A}_{l,\mathcal{V}}C(X)$ has dense range for all $l \in \mathbb{N}$, recalling that each $\mathcal{A}_{k,\mathcal{W}}C(X)$ is a complete (LB)-space, the composition operator $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is then continuous if, and only if, for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that the operator $C_\varphi : \mathcal{A}_{l,\mathcal{V}}C(X) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is continuous. But the composition operator $C_\varphi : \mathcal{A}_{l,\mathcal{V}}C(X) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is continuous if, and only if, for all $m \in \mathbb{N}$ the composition operator $C_\varphi : C_{v_{m,l}}(X) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is continuous, as $\mathcal{A}_{l,\mathcal{V}}C(X)$ is an (LB)-space. Since $\mathcal{A}_{k,\mathcal{W}}C(X)$ is also an (LB)-space, from Grothendieck's factorization theorem [12, p. 147], C_φ is continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $C_\varphi : C_{v_{m,l}}(X) \to C_{w_{n,k}}(X)$ is continuous. By Proposition 4(1), this is equivalent to require that the condition (3) is satisfied.

If $\mathcal{A}_\mathcal{V}C(X)$ is not a dense subspace of $\mathcal{A}_{l,\mathcal{V}}C(X)$ for all $l \in \mathbb{N}$, to get the result we argue as follows.

For a fixed $m \in \mathbb{N}$, let $B := \{f_x \in C(X) : \forall y \in X\ 0 \le f_x(y) \le \frac{1}{v_{m,l}(y)},\ f_x(\varphi(x)) = \frac{1}{v_{m,l}(\varphi(x))},\ \text{supp}\, f_x \text{ compact}\}$. Then, B is clearly a subset of $\mathcal{A}_\mathcal{V}C(X)$, as the support of each f_x is compact. Moreover, B is contained in $C_{v_{m,l}}(X)$ and bounded there. Indeed, for every $x \in X$ we have

$$\|f_x\|_{v_{m,l}} = \sup_{y\in X} v_{m,l}(y)|f_x(y)| = 1.$$

Accordingly, B is a bounded subset of $(\mathcal{A}_\mathcal{V}C(X), \tau_l)$. The continuity of C_φ from $(\mathcal{A}_\mathcal{V}C(X), \tau_l)$ into $\mathcal{A}_{k,\mathcal{W}}C(X)$ implies that $C_\varphi(B)$ is also a bounded subset of $\mathcal{A}_{k,\mathcal{W}}C(X)$ and hence, there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\|C_\varphi(f_x)\|_{w_{n,k}} \le C, \quad \forall x \in X.$$

Therefore, it follows that

$$\frac{w_{n,k}(x)}{(v_{m,l} \circ \varphi)(x)} = w_{n,k}(x) f_x(\varphi(x)) \le \|C_\varphi(f_x)\|_{w_{n,k}} \le C, \quad \forall x \in X,$$

that is, $\frac{w_{n,k}}{v_{m,l}\circ\varphi} \in C_b(X)$. Conversely, if the condition (3) is satisfied, then Proposition 4(1) implies that the operator $C_\varphi \in \mathcal{L}(C_{v_{m,l}}(X), C_{w_{n,k}}(X))$ and hence, the thesis follows.

The proof of (3)⇔(2) is analogous and so it is omitted. We only observe that under the assumption on each W^k, the (LB)-spaces $(\mathcal{A}_{k,\mathcal{W}})_0C(X)$ are complete and that $\mathcal{A}_{0,\mathcal{V}}C(X)$ is dense in each (LB)-space $(\mathcal{A}_{k,\mathcal{V}})_0C(X)$. So, we can apply directly Propositions 2 and 4(2). □

A similar characterization holds for the boundedness of composition operators between the pairs of (LF)-spaces $(\mathcal{V}C(X), \mathcal{W}C(X))$ and $(\mathcal{V}_0C(X), \mathcal{W}_0C(X))$ and also for the pairs of (PLB)-spaces $(\mathcal{A}_\mathcal{V}C(X), \mathcal{A}_\mathcal{W}C(X))$ and $(\mathcal{A}_{0,\mathcal{V}}C(X), \mathcal{A}_{0,\mathcal{W}}C(X))$.

For the (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$, the following holds.

Theorem 10. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X,X)$. Assume that $\mathcal{W}C(X)$ ($\mathcal{W}_0C(X)$, resp.) is regular. Then, $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$*

$$\sup_{x\in X} \frac{w_{n,k}(x)}{v_{m,l}(\varphi(x))} < \infty. \tag{4}$$

Furthermore, if φ is a proper map, then the previous assertions are equivalent to $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{W}_0C(X)$ is bounded.

Proof. By Proposition 1(1), $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $C_\varphi : CV_m(X) \to CW_n(X)$ is bounded. Using Proposition 3 (see also Remark 4), this holds if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator $C_\varphi : (CV_m(X), \tau_{m,l}) \to C_{w_{n,k}}(X)$ is continuous, where $\tau_{m,l}$ denotes the lc-topology of $C_{v_{m,l}}(X)$, and hence, there exists $C > 0$ such that

$$\|C_\varphi(f)\|_{w_{n,k}} \le C\|f\|_{v_{m,l}}, \quad \forall f \in CV_m(X). \tag{5}$$

In view of Equation (5), we can argue as in the proof of Theorem 8 to conclude that this is equivalent to require that Equation (4) is satisfied.

A similar argument shows that the same characterization holds for the boundedness of $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{W}_0C(X)$. We only observe that each Fréchet space $C(V_m)_0(X)$ is dense in $C_{(v_{m,l})_0}(X)$ for all $l \in \mathbb{N}$. So, we can apply directly Proposition 4(2). □

A similar characterization is valid for the (PLB)-case.

Theorem 11. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X,X)$. Then, $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$*

$$\sup_{x\in X} \frac{w_{n,k}(x)}{v_{m,l}(\varphi(x))} < \infty. \tag{6}$$

Furthermore, if φ is a proper map and the sequence $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$, then the previous assertions are equivalent to $C_\varphi : \mathcal{A}_{0,\mathcal{V}}C(X) \to \mathcal{A}_{0,\mathcal{W}}C(X)$ is bounded.

Proof. By Proposition 3 (see Remark 4(2)), the operator $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator $C_\varphi : (\mathcal{A}_\mathcal{V}C(X), \tau_l) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is bounded, where τ_l denotes the lc-topology of the (LB)-space $\mathcal{A}_{l,\mathcal{V}}C(X)$. In the case the inclusion $\mathcal{A}_\mathcal{V}C(X) \hookrightarrow \mathcal{A}_{l,\mathcal{V}}C(X)$ has dense range for all $l \in \mathbb{N}$, being each (LB)-space $\mathcal{A}_{k,\mathcal{W}}C(X)$ complete, the composition operator $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is then bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator $C_\varphi : \mathcal{A}_{l,\mathcal{V}}C(X) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is bounded. So, by Proposition 1(1), this holds if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $C_\varphi : C_{v_{m,l}}(X) \to C_{w_{n,k}}(X)$ is bounded, that is, continuous. Now, by Proposition 4(1), this is equivalent to require that Equation (6) is satisfied.

If the inclusion $\mathcal{A}_\mathcal{V}C(X) \hookrightarrow \mathcal{A}_{l,\mathcal{V}}C(X)$ has no dense range for all $l \in \mathbb{N}$, to get the result we argue as follows.

Fix $k \in \mathbb{N}$. The fact that $C_\varphi : (\mathcal{A}_\mathcal{V}C(X), \tau_l) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is bounded implies that there exists a 0-neighborhood U in $\mathcal{A}_{l,\mathcal{V}}C(X)$ such that $C_\varphi(U \cap \mathcal{A}_\mathcal{V}C(X))$ is a bounded set in $\mathcal{A}_{k,\mathcal{W}}C(X)$. Since $\mathcal{A}_{k,\mathcal{W}}C(X)$ is a regular (LB)-space, there exists $n \in \mathbb{N}$ such that $C_\varphi(U \cap \mathcal{A}_\mathcal{V}C(X))$ is contained in $C_{w_{n,k}}(X)$ and bounded there. Now, for a fixed $m \in \mathbb{N}$, let $B := \{f_x \in C(X) : \forall y \in X\ 0 \le f_x(y) \le \frac{1}{v_{m,l}(y)},\ f_x(\varphi(x)) = \frac{1}{v_{m,l}(\varphi(x))},\ \mathrm{supp}\, f_x \text{ compact}\}$. Then, B is clearly a subset of $\mathcal{A}_\mathcal{V}C(X)$ because each f_x has compact support. Moreover, B is contained in $C_{v_{m,l}}(X)$ and bounded there (see the proof of Theorem 9). Accordingly, B is a bounded subset of $(\mathcal{A}_\mathcal{V}C(X), \tau_l)$. Hence, there exists $\lambda > 0$ such that $B \subset \lambda(U \cap \mathcal{A}_\mathcal{V}C(X))$, thereby implying that $C_\varphi(B)$ is also a bounded subset of $C_{w_{n,k}}(X)$. So, there exists $n \in \mathbb{N}$ and $C > 0$ such that

$$\|C_\varphi(f_x)\|_{w_{n,k}} \le C, \quad \forall x \in X.$$

As in the proof of Theorem 9, it follows that $\frac{w_{n,k}}{v_{m,l}\circ\varphi} \in C_b(X)$, that is, the condition (6) is satisfied. Conversely, if the condition (6) is satisfied, then Proposition 4(1) implies that the operator $C_\varphi \in \mathcal{L}(C_{v_{m,l}}(X), C_{w_{n,k}}(X))$ and hence, the thesis follows.

A similar argument shows that the same characterization holds for the boundedness of $C_\varphi : \mathcal{A}_{0,\mathcal{V}}C(X) \to \mathcal{A}_{0,\mathcal{W}}C(X)$. We only observe that under the assumption on each W^k, the (LB)-spaces $(\mathcal{A}_{k,\mathcal{W}})_0C(X)$ are complete and that $\mathcal{A}_{0,\mathcal{V}}C(X)$ is dense in each $(\mathcal{A}_{k,\mathcal{V}})_0C(X)$. So, we can apply directly Proposition 4(2). □

To describe the (weak) compactness of the composition operators, we recall some results on (weak) compactness of weighted composition operators acting on the Banach spaces $C_b(X)$ and $C_0(X)$. To state such results, if $u \in C_b(X)$ and $\varepsilon > 0$, we set $N(u) := \{x \in X : u(x) \neq 0\}$ and $N(u,\varepsilon) = \{x \in X : |u(x)| \geq \varepsilon\}$. Clearly, $N(u) = \cup_{\varepsilon>0} N(u,\varepsilon)$.

We first recall a result due to Singh and Summers [21] which characterizes the (weak) compactness of weighted composition operators acting on the Banach space $C_b(X)$ with X a completely regular Hausdorff topological space (see [15] in case X is a compact Hausdorff space).

Lemma 1 [21, Corollary 2.2]. *Let X be a completely regular Hausdorff topological space. Let $\phi \in C(X,X)$ and $u \in C_b(X)$. Then, the following properties are equivalent:*

(1) uC_ϕ is compact on $C_b(X)$;
(2) uC_ϕ is weakly compact on $C_b(X)$;
(3) $\phi(N(u,\varepsilon))$ is finite for every $\varepsilon > 0$.

Regarding weighted composition operators acting on the Banach space $C_0(X)$, with X a locally compact Hausdorff topological space, we recall the following characterization of the (weak) compactness due to Chan [9] (see also [19, Corollary 2.5]).

Lemma 2 [9, Theorem 2.1]. *Let X be a locally compact Hausdorff topological space. Let $\phi \in C(X,X)$ be a proper map and $u \in C_b(X)$. Then, the following properties are equivalent:*

(1) uC_ϕ is compact on $C_0(X)$;
(2) uC_ϕ is weakly compact on $C_0(X)$;
(3) (i) $u \in C_0(X)$ and (ii) ϕ is locally constant on $N(u)$.

Remark 12. Condition (3)(ii) in Lemma 2 is equivalent to condition that $\phi(K)$ is finite for every compact subset K of $N(u)$. When X is compact, such condition is equivalent to condition (3) in Lemma 1.

Now, we can determine when a composition operator between the pairs of (LF)-spaces $(\mathcal{V}C(X), \mathcal{W}C(X))$ and $(\mathcal{V}_0C(X), \mathcal{W}_0C(X))$ is (weakly) compact.

Theorem 13. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X,X)$. Assume that $CV_m(X)$ is a dense subset of $C_{v_{m,l}}(X)$ for all $l,m \in \mathbb{N}$ and that $\mathcal{W}C(X)$ satisfies condition (M). Then, $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the set*

$$\{\varphi(x) : w_{n,k}(x) \geq \varepsilon v_{m,l}(\varphi(x))\} \tag{7}$$

is finite for every $\varepsilon > 0$.

Proof. By Proposition 1(2), the operator $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $C_\varphi : CV_m(X) \to CW_n(X)$ is compact. Using Proposition 3, this holds if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator C_φ has a unique compact linear extension $(C_\varphi)^l_k$ from $C_{v_{m,l}}(X)$ into $C_{w_{n,k}}(X)$. Since $C_\varphi \in \mathcal{L}(C(X))$ and the spaces $C_{v_{m,l}}(X)$ and $C_{w_{n,k}}(X)$ are continuously included in $C(X)$, necessarily $(C_\varphi)^l_k = C_\varphi$. So, $C_\varphi : C_{v_{m,l}}(X) \to C_{w_{n,k}}(X)$ is compact. But, by Remark 7, the composition operator between the weighted Banach spaces $C_{v_{m,l}}(X)$

and $C_{w_{n,k}}(X)$ is compact if, and only if, the operator $T : C_b(X) \to C_b(X)$ defined by $T(f) := w_{n,k}\left(\frac{1}{v_{m,l}} \circ \varphi\right)(f \circ \varphi)$ is compact. Observe that such an operator T is in the form uC_ϕ, with $u := \frac{w_{n,k}}{v_{m,l} \circ \varphi} \in C_b(X)$ and $\phi := \varphi$. Hence, due to Lemma 1, this is equivalent to require that $\phi(N(u, \varepsilon))$ is finite for every $\varepsilon > 0$, that is, that the set in Equation (7) is finite for every $\varepsilon > 0$. □

Arguing in a similar way as done in the proof of the theorem above and due to Lemma 2, one shows the following result.

Theorem 14. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X, X)$ be a proper map. Assume that $\mathcal{W}_0C(X)$ satisfies condition (M). Then, $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{W}_0C(X)$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the function $\frac{w_{n,k}}{v_{m,l} \circ \varphi} \in C_0(X)$ and φ is locally constant on X.*

In the case of the (PLB)-spaces, the following holds true.

Theorem 15. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X, X)$. Assume that $\mathcal{A}_\mathcal{V}C(X)$ is a dense subset of $\mathcal{A}_{l,\mathcal{V}}C(X)$ for all $l \in \mathbb{N}$ and that $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$. Then, $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the set*

$$\{\varphi(x) : w_{n,k}(x) \geq \varepsilon v_{m,l}(\varphi(x))\} \tag{8}$$

is finite for every $\varepsilon > 0$.

Proof. By Proposition 3, the operator $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator C_φ has a unique compact linear extension $(C_\varphi)^l_k$ from $\mathcal{A}_{l,\mathcal{V}}C(X)$ into $\mathcal{A}_{k,\mathcal{W}}C(X)$. Since $C_\varphi \in \mathcal{L}(C(X))$ and the spaces $\mathcal{A}_{l,\mathcal{V}}C(X)$ and $\mathcal{A}_{k,\mathcal{W}}C(X)$ are continuously included in $C(X)$, it follows that $(C_\varphi)^l_k = C_\varphi$. So, $C_\varphi : \mathcal{A}_{l,\mathcal{V}}C(X) \to \mathcal{A}_{k,\mathcal{W}}C(X)$ is compact. Using Proposition 1(2) (observe that each (LB)-space $\mathcal{A}_{k,\mathcal{W}}C(X)$ satisfies condition (M) as W^k is regularly decreasing by assumption), this holds if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $C_\varphi(C_{v_{m,l}}(X)) \subset C_{w_{n,k}}(X)$ and the restriction $C_\varphi : C_{v_{m,l}}(X) \to C_{w_{n,k}}(X)$ is compact. Taking Remark 7 into account, we can apply Lemma 1 to conclude that this fact is equivalent to require that the set in Equation (8) is finite for every $\varepsilon > 0$. □

The same arguments used in the proof of Theorem 15, combined with Lemma 2, lead us to characterize the compactness of the composition operators acting between the (PLB)-spaces $\mathcal{A}_{0,\mathcal{V}}C(X)$ and $\mathcal{A}_{0,\mathcal{W}}C(X)$.

Theorem 16. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X, X)$ be a proper map. Assume that $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$. Then, $C_\varphi : \mathcal{A}_{0,\mathcal{V}}C(X) \to \mathcal{A}_{0,\mathcal{W}}C(X)$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the function $\frac{w_{n,k}}{v_{m,l} \circ \varphi} \in C_0(X)$ and φ is locally finite on X.*

Using Propositions 1 and 3, Lemmas 1 and 2, and taking Theorem 2 and the comments thereafter into account, we get the characterization of the weak compactness, by arguing as done in the proof of Theorems 13 and 15.

Theorem 17. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X, X)$. Assume that $CV_m(X)$ is a dense subset of $C_{v_{m,l}}(X)$ for all $l, m \in \mathbb{N}$ and that $\mathcal{W}C(X)$ ($\mathcal{W}_0C(X)$, resp.) satisfies condition (M). Then, $C_\varphi : \mathcal{V}C(X) \to \mathcal{W}C(X)$ is compact if, and only if, it is weakly compact.*

Furthermore, if φ is a proper map, then also $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{W}_0C(X)$ is compact if, and only if, it is weakly compact.

Analogously for the (PLB)-spaces.

Theorem 18. *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on X and $\varphi \in C(X, X)$. Assume that $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$. If $\mathcal{A}_V C(X)$ is a dense subset of $\mathcal{A}_{l,V}C(X)$ for all $l \in \mathbb{N}$, then $C_\varphi : \mathcal{A}_\mathcal{V}C(X) \to \mathcal{A}_\mathcal{W}C(X)$ is compact if, and only if, it is weakly compact.*

Furthermore, if φ is a proper map, then also $C_\varphi : \mathcal{A}_{0,\mathcal{V}}C(X) \to \mathcal{A}_{0,\mathcal{W}}C(X)$ is compact if, and only if, it is weakly compact.

4 | EXAMPLES

4.1 | Weighted (LF)-spaces and (PLB)-spaces of continuous functions

In order to construct concrete examples, we consider the following setting as in [4, Section 5]. Let X be a locally compact Hausdorff topological space, let $v, w : X \to \mathbb{R}$ be continuous functions such that $0 < v(x) \leq 1$ and $1 \leq w(x)$ for all $x \in X$. Let $(r_n)_{n\in\mathbb{N}}$ and $(\rho_k)_{k\in\mathbb{N}}$ be strictly increasing sequences of positive numbers such that $r_n \to r$ and $\rho_k \to \rho$, with $r, \rho > 0$ or $r, \rho = +\infty$. For all $n, k \in \mathbb{N}$, we set

$$v_{n,k}(x) := v(x)^{r_n} w(x)^{\rho_k}, \quad \forall x \in X, \tag{9}$$

and $\mathcal{V} := (v_{n,k})_{n,k\in\mathbb{N}}$. We recall that if $\rho = \infty$, then $\mathcal{V}$ satisfies condition (Q) (see [4, Section 5, Example 3]) and hence, both the (LF)-spaces $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ satisfy condition (M). On the other hand, if $\frac{v}{w}$ vanishes at infinity in X, then $\mathcal{V}C(X) = \mathcal{V}_0C(X)$ holds algebraically and topologically (see [4, Section 5, Proposition 1 and Example 4]) and $\mathcal{V}$ satisfies condition (M) if, and only if, it satisfies condition (WQ) (see [4, Section 5, Lemma 2]). The condition $\frac{v}{w}$ vanishes at infinity in X also implies that $\mathcal{A}C(X) = \mathcal{A}_0C(X)$ algebraically and topologically, as it is easy to show. Furthermore, if v vanishes at infinity in X, then for all $k \in \mathbb{N}$ the sequence $V^k = (v_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing, as the function $\frac{v_{n+1,k}}{v_{n,k}} = (v)^{r_{n+1}-r_n}$ vanishes at infinity in X for every $n \in \mathbb{N}$ (see [5]).

Due to the results of Section 3, we can state the following facts.

Theorem 19. *Let $\mathcal{V} = (v_{n,k})_{n,k\in\mathbb{N}}$ with $v_{n,k}$ defined as in Equation (9) and let $\varphi : X \to X$ be a continuous functions. Then, the following assertions hold true:*

(1) $C_\varphi : \mathcal{V}C(X) \to \mathcal{V}C(X)$ is continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ for which

$$\sup_{x\in X} \frac{v(x)^{r_n} w(x)^{\rho_k}}{v(\varphi(x))^{r_m} w(\varphi(x))^{\rho_l}} < \infty.$$

Moreover, if φ is a proper map, then the previous condition is equivalent to $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{V}_0C(X)$ is continuous;

(2) Suppose that $\rho = \infty$ and $\frac{v}{w}$ vanishes at infinity in X. Then, $C_\varphi : \mathcal{V}C(X) \to \mathcal{V}C(X)$ is bounded if, and only if, $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{V}_0C(X)$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_{x\in X} \frac{v(x)^{r_n} w(x)^{\rho_k}}{v(\varphi(x))^{r_m} w(\varphi(x))^{\rho_l}} < \infty;$$

(3) Suppose that $\rho = \infty$ and $\frac{v}{w}$ vanishes at infinity in X. If φ is a proper map, then $C_\varphi : \mathcal{V}C(X) \to \mathcal{V}C(X)$ is compact if, and only if, $C_\varphi : \mathcal{V}_0C(X) \to \mathcal{V}_0C(X)$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the set $\{\varphi(x) : v_{n,k}(x) \geq \varepsilon v_{m,l}(\varphi(x))\}$ is finite for every $\varepsilon > 0$.

Theorem 20. *Let $\mathcal{V} = (v_{n,k})_{n,k\in\mathbb{N}}$ with $v_{n,k}$ defined as in Equation (9) and let $\varphi : X \to X$ be a continuous functions. If $\frac{v}{w}$ vanishes at infinity in X, then the following assertions hold true:*

(1) $C_\varphi : \mathcal{A}C(X) \to \mathcal{A}C(X)$ is continuous if, and only if, $C_\varphi : \mathcal{A}_0C(X) \to \mathcal{A}_0C(X)$ is continuous if, and only if, for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ for which

$$\sup_{x\in X} \frac{v(x)^{r_n} w(x)^{\rho_k}}{v(\varphi(x))^{r_m} w(\varphi(x))^{\rho_l}} < \infty;$$

(2) $C_\varphi : \mathcal{A}C(X) \to \mathcal{A}C(X)$ is bounded if, and only if, $C_\varphi : \mathcal{A}_0C(X) \to \mathcal{A}_0C(X)$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$

$$\sup_{x\in X} \frac{v(x)^{r_n} w(x)^{\rho_k}}{v(\varphi(x))^{r_m} w(\varphi(x))^{\rho_l}} < \infty;$$

(3) Suppose that φ is a proper map and that v vanishes at infinity in X. Then, $C_\varphi : \mathcal{A}C(X) \to \mathcal{A}C(X)$ is compact if, and only if, $C_\varphi : \mathcal{A}_0C(X) \to \mathcal{A}_0C(X)$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the set $\{\varphi(x) : v_{n,k}(x) \geq \varepsilon v_{m,l}(\varphi(x))\}$ is finite for every $\varepsilon > 0$.

4.2 | Sequence (LF)-spaces

Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a system of weights on $\mathbb{N}$, where $V_n = (v_{n,k})_{k\in\mathbb{N}}$. Then, $C_{v_{n,k}}(\mathbb{N}) = \ell^\infty(v_{n,k})$ and $C_{(v_{n,k})_0}(\mathbb{N}) = c_0(v_{n,k})$ for any $n, k \in \mathbb{N}$ and hence, the Fréchet spaces $CV_n(\mathbb{N})$ and $C(V_n)_0(\mathbb{N})$ coincide with the *echelon spaces* $\lambda_\infty(V_n)$ and $\lambda_0(V_n)$, respectively. Setting $l_\infty(\mathcal{V}) := \mathrm{ind}_n\, \lambda_\infty(V_n)$ and $l_0(\mathcal{V}) := \mathrm{ind}_n\, \lambda_0(V_n)$, we have that $\mathcal{V}C(\mathbb{N}) = l_\infty(\mathcal{V})$ and $\mathcal{V}_0C(\mathbb{N}) = l_0(\mathcal{V})$.

Set $\omega = \mathbb{K}^{\mathbb{N}}$ and given a function $\varphi : \mathbb{N} \to \mathbb{N}$, that is, an $\mathbb{N}$-valued sequence $\varphi = (\varphi_i)_{i\in\mathbb{N}} \subset \mathbb{N}$, we can define the composition operator $C_\varphi : \omega \to \omega$ by $(x_i)_{i\in\mathbb{N}} \mapsto (x_{\varphi_i})_{i\in\mathbb{N}}$.

Due to the results of Section 3, we can characterize the continuity, the boundedness and the (weak) compactness of the composition operator acting between the pairs of sequence (LF)-spaces $(l_p(\mathcal{V}), l_p(\mathcal{W}))$, for $p = 0, \infty$. To do this, we observe the following facts.

Remark 21.

(1) Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a function. Then, φ is proper if, and only if, $\lim_{i\to\infty} \varphi_i = \infty$, as it is easy to show.
(2) Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a proper map and v, w be two weights on $\mathbb{N}$. Then, the set $\{\varphi_i : w(i) \geq \varepsilon v(\varphi_i)\}$ is finite for every $\varepsilon > 0$ if, and only if,

$$\lim_{i\to\infty} \frac{w(i)}{v(\varphi_i)} = 0.$$

Indeed, the sequence $\left(\frac{w(i)}{v(\varphi_i)}\right)_{i\in\mathbb{N}}$ does not converge to 0 if, and only if, there exists $\varepsilon > 0$ such that for all $i \in \mathbb{N}$ there exists $j > i$ such that $\frac{w(j)}{v(\varphi_j)} \geq \varepsilon$. Since φ is a proper map, this necessarily implies that the set $\left\{\varphi_j : w(j) \geq \varepsilon v(\varphi_j)\right\}$ contains infinite elements.
(3) Let w be a weight on $\mathbb{N}$. Then, every function $\varphi : \mathbb{N} \to \mathbb{N}$ is locally constant on $N(w) = \mathbb{N}$, being $\mathbb{N}$ endowed with the discrete topology.

Theorem 22. *Let $\mathcal{V}, \mathcal{W}$ be two system of weights on $\mathbb{N}$ and $\varphi : \mathbb{N} \to \mathbb{N}$ be a function. Then, the following assertions hold true:*

(1) $C_\varphi : l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ for which

$$\sup_{i\in\mathbb{N}} \frac{w_{n,k}(i)}{v_{m,l}(\varphi_i)} < \infty.$$

Moreover, if φ is a proper map, then the previous condition is equivalent to $C_\varphi : l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is continuous;
(2) Assume that $l_\infty(\mathcal{W})$ ($l_0(\mathcal{W})$, resp.) is regular. Then, $C_\varphi : l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_{i\in\mathbb{N}} \frac{w_{n,k}(i)}{v_{m,l}(\varphi_i)} < \infty.$$

Furthermore, if φ is a proper map, then the previous condition is equivalent to $C_\varphi : l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is bounded;

(3) *Assume that* $\lambda_\infty(V_m)$ *is a dense subset of* $\ell_\infty(v_{m,l})$ *for all* $l, m \in \mathbb{N}$ *and that* $l_\infty(\mathcal{W})$ ($l_0(\mathcal{W})$, *resp.*) *satisfies condition (M). Then,* $C_\varphi : l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ *is (weakly) compact if, and only if, there exists* $n \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *there exists* $l \in \mathbb{N}$ *such that for all* $k \in \mathbb{N}$ *the set*

$$\{\varphi_i : w_{n,k}(i) \geq \varepsilon v_{m,l}(\varphi_i)\}$$

is finite for every $\varepsilon > 0$.

Furthermore, if φ *is a proper map, then the previous condition is equivalent to* $C_\varphi : l_0(\mathcal{V}) \to l_0(\mathcal{W})$ *is (weakly) compact.*

We refer the reader to [17] for analogous results on diagonal (multiplication) operators acting on the sequence (LF)-space $l_p(V)$, with $1 \leq p \leq \infty$ or $p = 0$.

4.3 | Sequence (PLB)-spaces

Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a system of weights on $\mathbb{N}$, where $V_n = (v_{n,k})_{k\in\mathbb{N}}$. Then, the (LB)-spaces $\mathcal{A}_k C(\mathbb{N})$ and $(\mathcal{A}_k)_0 C(\mathbb{N})$ coincide with the *co-echelon spaces* $a_\infty(V^k) := \operatorname{ind}_n \ell^\infty(v_{n,k})$ and $a_0(V^k) := \operatorname{ind}_n c_0(v_{n,k})$, respectively. Setting $a_\infty(\mathcal{V}) := \operatorname{proj}_k a_\infty(V^k)$ and $a_0(\mathcal{V}) := \operatorname{proj}_k a_0(V^k)$, we have that $\mathcal{A}C(\mathbb{N}) = a_\infty(\mathcal{V})$ and $\mathcal{A}_0 C(\mathbb{N}) = a_0(\mathcal{V})$.

In [6, Corollary 2.8], it has been shown that the co-echelon spaces $a_\infty(V^k)$ is always a complete (LB)-space. On the other hand, by [6, Theorem 3.4], $a_\infty(V^k)$ satisfies condition (M) if, and only if, the sequence $V^k = (v_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing. While, in [6, Theorem 3.7] it is proved that the co-echelon space $a_0(V^k)$ is regular if, and only if, it is complete if, and only if, it satisfies condition (M) if, and only if, the sequence $V^k = (v_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing.

Taking Remark 21 and the results in Section 3 into account, we can characterize the continuity, the boundedness and the (weak) compactness of the composition operator acting between the pair of sequence (PLB)-spaces $(a_p(\mathcal{V}), a_p(\mathcal{W}))$, $p = 0, \infty$.

Theorem 23. *Let* $\mathcal{V}, \mathcal{W}$ *be two system of weights on* $\mathbb{N}$ *and* $\varphi : \mathbb{N} \to \mathbb{N}$ *be a function. Then, the following assertions hold true:*

(1) $C_\varphi : a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ *is continuous if, and only if, for all* $k \in \mathbb{N}$ *there exists* $l \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *for which*

$$\sup_{i\in\mathbb{N}} \frac{w_{n,k}(i)}{v_{m,l}(\varphi_i)} < \infty.$$

Moreover, if φ *is a proper map and the sequence* $W^k = (w_{n,k})_{n\in\mathbb{N}}$ *is regularly decreasing for all* $k \in \mathbb{N}$, *then the previous condition is equivalent to* $C_\varphi : a_0(\mathcal{V}) \to a_0(\mathcal{W})$ *is continuous;*

(2) $C_\varphi : a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ *is bounded if, and only if, there exists* $l \in \mathbb{N}$ *such that for all* $k \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$

$$\sup_{i\in\mathbb{N}} \frac{w_{n,k}(i)}{v_{m,l}(\varphi_i)} < \infty.$$

Moreover, if φ *is a proper map and the sequence* $W^k = (w_{n,k})_{n\in\mathbb{N}}$ *is regularly decreasing for all* $k \in \mathbb{N}$, *then the previous condition is equivalent to* $C_\varphi : a_0(\mathcal{V}) \to a_0(\mathcal{W})$ *is bounded;*

(3) *Assume that* $a_\infty(\mathcal{V})$ *is a dense subset of* $a_\infty(V^l)$ *for all* $l \in \mathbb{N}$ *and that the sequence* $W^k = (w_{n,k})_{n\in\mathbb{N}}$ *is regularly decreasing for all* $k \in \mathbb{N}$. *Then, the operator* $C_\varphi : a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ *is (weakly) compact if, and only if, there exists* $l \in \mathbb{N}$ *such that for all* $k \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *the set*

$$\{\varphi_i : w_{n,k}(i) \geq \varepsilon v_{m,l}(\varphi_i)\}$$

is finite for every $\varepsilon > 0$.

Moreover, if φ *is a proper map, then the previous condition is equivalent to* $C_\varphi : a_0(\mathcal{V}) \to a_0(\mathcal{W})$ *is (weakly) compact.*

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CONFLICT OF INTEREST STATEMENTS

The authors declare no conflicts of interest.

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