Published for SISSA by Springer

Received: *July 29, 2024*
Accepted: *September 9, 2024*
Published: *October 4, 2024*

$\mathcal{N} = 4$ SYM line defect Schur index and semiclassical string

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Abstract: The giant graviton expansion of the line defect Schur index in four dimensional $\mathcal{N}4$ U(N) SYM was recently proposed in arXiv:2403.11543 to be captured in the dual string theory by counting fluctuations states of two half-infinite fundamental strings in $AdS_5 \times S^5$ ending on the line defect and D3 brane giant. However, agreement with the gauge theory data for the defect line index at finite N required the inclusion of ad hoc extra contributions with unclear origin. We discuss the large N leading order contribution of the giant graviton expansion of the defect line index by a direct analysis of semiclassical string partition function in a twisted background. We discuss supersymmetric boundary conditions in the presence of the D3 brane and evaluate the quadratic fluctuations effective action by introducing a suitable projection of fluctuation modes. We show that the extra contributions to the single giant graviton correction found in arXiv:2403.11543 correspond to a supersymmetric Casimir energy contribution.

Keywords: AdS-CFT Correspondence, D-Branes

ArXiv ePrint: 2407.06900

https://doi.org/10.1007/JHEP10(2024)046

JHEP10(2024)046

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1 Introduction

The superconformal index was introduced in [1–3] to encode all group theoretical information about protected short multiplets of four dimensional superconformal field theories on $S^1 \times S^3$. Its Schur specialization [4, 5] was defined for theories with (at least) $\mathcal{N} = 2$ supersymmetry where it computes the vacuum character of the chiral algebra in a protected sector [6]. The Schur index could be generalized in the presence of half-BPS defects, in particular line defects [7, 8].

The defect line Schur index is quite interesting from the perspective of AdS/CFT. Indeed, in many cases it is exactly computable and thus provides highly non-trivial tests of the correspondence making predictions for string theory in dual geometries associated with defects.

In this paper we consider the Schur index $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}$ of 4d $\mathcal{N} = 4$ U(N) SYM with two Wilson lines in the fundamental and antifundamental of U(N) located at the two poles of S^3. After conformal mapping the two Wilson lines build up a full defect line in $\mathbb{R}^4$. The line index depends on two fugacities, the universal one q (coupled to the Hamiltonian and the two spins of the isometry SO(4) of S^3) and a flavour fugacity η coupled to a R-symmetry generator.

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The line index is known exactly at large N. In this limit, it factorizes according to

$$\mathrm{I}_F^{\mathrm{U}(\infty)} = I_{\mathrm{F1}}\, \mathrm{I}_{\mathrm{sugra}}, \tag{1.1}$$

where $\mathrm{I}_{\mathrm{sugra}}$ is the large N limit of the undecorated Schur index. The labels of the two factors in (1.1) stem from their AdS/CFT origin. In particular, $\mathrm{I}_{\mathrm{sugra}}$ is the index of KK IIB supergravity modes while I_{F1} is the Schur index of fluctuations of a fundamental string wrapping AdS_2 in AdS_5 and having the Wilson lines as boundary [8].

At finite N, superconformal indices admit corrections with q-dependent weights $\sim q^N, q^{2N}, \ldots$ multiplied by non-trivial functions of the remaining fugacities.[1] In the gauge theory side of AdS/CFT these corrections come from trace relations in the gauge group, see e.g. [14].[2] On the gravity side they are due to BPS multiply wrapped branes with charge of order N [9, 18, 19] and usually called giant gravitons being a generalization of the configurations studied in the past in [20, 21]. In the regime $N \gg 1$ giant graviton corrections are exponentially suppressed.

In the specific case of the Schur index of 4d $\mathcal{N} = 4$ U(N) SYM, giant graviton corrections are due to D3 branes wrapped on contractible cycles in S^5 [22] with perfect agreement with gauge theory calculations of the index [23]. For the defect line index, it was recently proposed in [12] that giant graviton corrections come from fluctuations of two half fundamental strings wrapping AdS_2 in AdS_5 and ending on the Wilson lines, as well as on the D3 giant. This description was shown to be fully consistent with gauge theory computations at leading order in the giant graviton expansion [24].

However, the gravity calculation in [12] raised a puzzling issue. It was based on counting BPS states of fundamental string fluctuations in the presence of the D3 brane and required an extra factor equal to $1/(\eta q)$ at leading order and whose origin was unexplained. Schematically, the leading order correction had the following form

$$\delta\mathrm{I}_F^{\mathrm{U}(N)}(\eta; q) = (\text{D3 fluctuations}) \times \frac{1}{\eta q} \times (\text{F1 fluctuations})\, q^N + \cdots\,, \tag{1.2}$$

where the two factors in round bracket represent the naive expected result. Similar "extra" factors were shown to be needed in higher order terms in the giant graviton expansion, as well as in more general Schur correlators [25].

The aim of this paper is to derive (1.2) by exploiting the known representation of the superconformal index as a supersymmetric partition function, see e.g. [26, 27]. On gauge theory side, this amounts to study the Euclidean partition function of the theory on $S^1 \times S^3$ with suitable background fields associated with the chemical potentials in the index and preserving supersymmetry [28–30].[3] On gravity side, the same construction requires

[1] For the purposes of this section our discussion is necessarily schematic. More precisely, if only q is switched on the remaining factors are functions of q with a possible dependence on N that is at most polynomial. If more fugacities are present, there may be several expansions that are non-trivially equivalent [9–13].

[2] The giant graviton expansion of superconformal indices can be studied on the gauge theory by examining finite N dependence without reference to dual string interpretation. It was shown to come from counting invariants in generic unitary matrix models [15]. It can also be interpreted as an instanton expansion in gauge theory [16]. The expansion obtained by this approach with the wrapped D-brane expansion, term by term, but the full sum is equivalent, see [17].

[3] The very definition of the index involves Cartan charges that implement a Scherk-Schwarz dimensional reduction when 4d $\mathcal{N} = 4$ SYM is obtained from 10d $\mathcal{N} = 1$ SYM.

a corresponding twisted deformation of the $AdS_5 \times S^5$ background. In this way, the large N giant graviton correction to the superconformal index may be computed by a direct semiclassical calculation starting from the Green-Schwarz superstring action in static gauge moving in a suitable twisted background that accounts for the charges in the index.

This approach was recently successfully applied to derive the leading large N correction to several indices. In particular: (a) the superconformal index of the 6d (2,0) theory on $S^1 \times S^5$ from quantum M2 brane wrapped on $S^1 \times S^2$ in the dual M-theory twisted background $AdS_7 \times S^4$ [31], (b) the superconformal index of the $\mathcal{N} = 8$ supersymmetric (level-one) $\mathrm{U}(N) \times \mathrm{U}(N)$ ABJM theory from the quantum M5 brane wrapped on $S^1 \times S^5$ in twisted $AdS_4 \times S^7$ background [32], and (c) the Schur index of 4d $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM from quantum D3 brane wrapped on $S^1 \times S^3$ in twisted $AdS_5 \times S^5$ background [33].

The case of the defect line index is, however, substantially more difficult than the above examples. In fact, the large N limit of the undecorated Schur index computes the spectrum of supergravity states, while insertion of defects is non-trivial on string side, even at large N, as illustrated by the factor I_{F1} in (1.1). Thus, although giant graviton corrections to the undecorated index are captured by an extended object in supergravity background, the same correction in the defect index requires consideration of a more complex system — here the fundamental string(s) and the D3 giant. This requires to clarify novel issues related to the boundary conditions of the half-infinite fundamental strings on the D3 giant, a key point in the state counting analysis in [12]. We will see that, in our approach, this amounts to evaluating the effective action (or functional determinants) of quadratic fluctuations restricted by certain parity conditions or mode selection rules.[4]

The outcome of our analysis is the explicit semiclassical partition function from quadratic bosonic and fermionic fluctuation modes (X, θ) of the two fundamental half-strings in the presence of the spectator D3 giant, i.e.

$$Z = \int DX D\theta \; e^{-S[X,\theta]} = e^{-\Gamma^{\mathrm{D3}}_{1-\mathrm{loop}}}, \tag{1.3}$$

where the free energy $\Gamma^{\mathrm{D3}}_{1-\mathrm{loop}}$ is a function of the flavour fugacity η and inverse temperature β, i.e. the length of Euclidean thermal cycle with $q = e^{-\beta}$. The expression of $\Gamma_{1-\mathrm{loop}}$ contains two terms

$$\Gamma_{1-\mathrm{loop}}(\eta; \beta) = \beta\, E_c - \sum_{n=1}^{\infty} \frac{1}{n} f_{\mathrm{F}}(\eta^n, q^n), \qquad E_c = -1 + \frac{1}{\beta} \log \eta, \tag{1.4}$$

where E_c is the supersymmetric Casimir energy contribution dominating at small temperature while the second term is the usual many particle contribution built with a single particle index $f_{\mathrm{F}}(\eta; q)$ that agrees with state counting in [12]. We thus observe that Z reproduces (1.2) and explains the so far unclear origin of the prefactor $1/(\eta q)$ in terms of a Casimir contribution. Notice that in the case of the line index at large N, so without D3 giant, the Casimir energy vanishes, consistently with (1.1) and absence of extra corrections. The non vanishing value of E_c in the finite N corrections to the line index appears thus to be related to the partial breaking of supersymmetry in the presence of the D3 brane.

[4]The techniques we develop are expected to be applicable to more general situations such as those recently considered in [25, 34].

The fact that E_c plays an important role in this matching is peculiar and deserves some comments. At the level of gauge theory, the relation between the twisted supersymmetric partition function on $S^1 \times S^3$ and the index was shown in many cases to be[5]

$$Z(\beta,\mu) = e^{-\beta E_c(\mu)}\ \mathrm{I}(\beta\mu), \tag{1.5}$$

where μ denotes chemical potentials. The logarithm of the index is exponentially suppressed at large β and (1.5) corresponds to the structure in (1.4) and the index is obtained by removing the prefactor $e^{-\beta E_c}$ from Z. On the other hand, the correction (1.2) is not an index but instead its leading giant graviton correction. Besides, as we already remarked, it corresponds to a non trivial subleading semiclassical saddle in the presence of the D3 brane, here playing an external role. Comparison with (1.2) shows that the full expression of Z should be kept, including the E_c contribution. This is similar to what happens in the Wilson loop calculation in [31].[6] A full clarification of this issue presumably needs a non-leading order analysis and is beyond the scope of this paper.

Our analysis suggests that the study of finite N giant graviton-like corrections to defect indices by semiclassical string theory is an interesting approach, likely to be important in order to complement state counting methods and capture subtle effects like the middle factor in (1.2).

Plan of the paper In section 2 we summarize the available information about the large N limit and leading giant graviton correction to the Schur index and to the defect line index. In section 3 we discuss general aspects of the computation of superconformal indices and briefly discuss the evaluation of the large N line index by state counting. In section 4 we present in full details its semiclassical evaluation in string theory. Section 5 considers the leading giant graviton correction to the line index and the novel features associated with the boundary conditions of fundamental strings ending on the D3 giant. A few appendices contain technical material. In particular, in appendix C we analyze in some details the ultra-short multiplet of OSp(4*|4) and the structure of the small superconformal representations in the presence of the D3 brane.

2 Schur (line) index and leading giant graviton correction

The definition of the Schur index in 4d $\mathcal{N}=4$ U(N) SYM is

$$\mathrm{I}^{\mathrm{U}(N)}(\eta;q) = \mathrm{Tr}_{\mathrm{BPS}}[(-1)^{\mathrm{F}}\, q^{H+J+\bar{J}}\eta^R], \tag{2.1}$$

[5]Relation (1.5) was first observed in [35] where E_c was recognized as the supersymmetric version of the Casimir vacuum energy. The possibility of a non-trivial factor connecting the partition function and the index due to possible local counterterms was discussed in general in [30]. It was computed explicitly in [36] for backgrounds with $S^1 \times S^3$ topology admitting two supercharges of opposite R-charge, i.e. Hopf surfaces with two complex structure moduli, and proved to be scheme independent, as well as independent on continuous parameters in the action in [37]. Indeed it was related to the anomaly polynomial in even dimension on $S^1 \times S^{D-1}$ in [38]. For the generalization of E_c in the presence of 2d conformal defects see [39, 40].

[6]In [31], the large N expectation value of a suitable supersymmetric Wilson loop in the SYM theory on S^5 given by $\langle W\rangle = (2\sinh\frac{\beta}{2})^{-1}e^{N\beta}+\cdots$ was captured by a quantum M2 brane wrapping AdS_3. The factor $e^{N\beta}$ comes from its classical action, while the β-dependent prefactor is from fluctuations. In that case, one had, cf. (1.4), $\log(2\sinh\frac{\beta}{2}) = \frac{\beta}{2} - \sum_{n\geqslant 1}\frac{1}{n}e^{-\beta n}$ where the term $\frac{\beta}{2}$ is the supersymmetric Casimir contribution and it has to be kept.

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where H is the Hamiltonian, $J, \bar{J}$ are two spins,[7] and R is a generator of the $\mathrm{SU}(4)_R$ R-symmetry of the $\mathrm{PSU}(2,2|4)$ superconformal group. The variables q, η are the so-called universal and flavor fugacities. The Schur index admits the holonomy integral representation [41][8]

$$\mathrm{I}^{\mathrm{U}(N)}(\eta;q) = \oint_{|\boldsymbol{z}|=1} D^N \boldsymbol{z}\ \mathrm{PE}[f(\eta;q)\chi_{\Box}(\boldsymbol{z})\chi_{\Box}(\boldsymbol{z}^{-1})], \qquad f(\eta;q) = \frac{(\eta+\eta^{-1})\,q-2q^2}{1-q^2}, \quad (2.2)$$

where the function $f(\eta;q)$ is the single particle Schur index. Exact results for the Schur index were obtained in the case of $\mathrm{U}(N)$ gauge group in [42–44] for $\eta = 0$ and in [45] for $\eta \neq 0$, and generalized to B_n,C_n, D_n, G_2 groups in [46]. The leading large N correction to the Schur index reads[9]

$$\frac{\mathrm{I}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}^{\mathrm{U}(\infty)}(\eta;q)} = 1+\left[\eta^N G^+_{\mathrm{D3}}(\eta;q)+\eta^{-N}G^-_{\mathrm{D3}}(\eta;q)\right] q^N+\mathcal{O}(q^{2N}), \quad \mathrm{I}^{\mathrm{U}(\infty)}(\eta;q) = \frac{(q^2)_\infty}{(\eta q)_\infty(\eta^{-1}q)_\infty}, \tag{2.3}$$

where the function $G^\pm_{\mathrm{D3}}(\eta;q)$ is the single giant graviton contribution from wrapped D3 brane [22, 23] and admits the closed expression

$$G^+_{\mathrm{D3}}(\eta;q) = G^-_{\mathrm{D3}}(\eta^{-1};q) = -\eta^2 q\,\frac{\left(\frac{q}{\eta}\right)^3_\infty}{\vartheta(\eta^2,\frac{q}{\eta})}. \tag{2.4}$$

As we discussed in the Introduction, an interesting generalization of the Schur index (2.1) consists in decorating it by inserting BPS defect lines [7, 8] and exact results were obtained in the presence of an arbitrary number of 't Hooft or Wilson lines in [47–53].[10] The associated so-called Schur (line defect) correlators may be regarded as a supersymmetric partition function on $S^1 \times S^3$ with defect lines wrapping S^1 and located on a great circle of S^3 to preserve the same supersymmetry used in definition of the undecorated index [48]. Schur correlators are topological and do not depend on the precise position of the insertions (at fixed ordering of insertions). We will focus on the line defect 2-point function $\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}}(\eta;q) \equiv \mathrm{I}^{\mathrm{U}(N)}_{\Box,\bar{\Box}}(\eta;q)$ with two Wilson lines one in the fundamental and the other in the anti-fundamental. In the large N limit it takes the factorized form [8]

$$\mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{F}}(\eta;q) = \mathrm{I}_{\mathrm{F1}}(\eta;q)\ \mathrm{I}^{\mathrm{U}(\infty)}(\eta;q), \qquad \mathrm{I}_{\mathrm{F1}}(\eta;q) = \frac{1}{1-f(\eta;q)}\,. \tag{2.5}$$

The algebraic identity

$$\mathrm{I}_{\mathrm{F1}}(\eta;q) = \mathrm{PE}[f_{\mathrm{F1}}(\eta;q)], \qquad f_{\mathrm{F1}}(\eta;q) = -q^2+(\eta+\eta^{-1})\,q, \tag{2.6}$$

[7]The theory in $\mathbb{R}^4$ is radially quantized in $\mathbb{R}\times S^3$ and $J, \bar{J}$ are Cartan of $\mathrm{SO}(4) \simeq \mathrm{SU}(2)\times\mathrm{SU}(2)$ where $\mathrm{SO}(4)$ is the isometry group of the spatial part S^3.

[8]The measure is $D^N\boldsymbol{z} = \frac{1}{N!}\prod_{n=1}^N \frac{dz_n}{2\pi i\,z_n}\prod_{n\neq m}(1-\frac{z_n}{z_m})$. The character $\chi_\Box(\boldsymbol{z})$ of the $\mathrm{U}(N)$ fundamental representation is $\chi_\Box(\boldsymbol{z}) = \sum_{n=1}^N z_n$. The operation PE is plethystic exponentiation with respect to fugacities and holonomies.

[9]Notation for q-functions is $(a;q)_\infty = \prod_{k=0}^\infty(1-a\,q^k)$, $(a^\pm;q)_\infty = (a;q)_\infty(a^{-1};q)_\infty$, $(q)_\infty \equiv (q;q)_\infty = \prod_{k=1}^\infty(1-q^k)$, and $\vartheta(x,q) = -x^{-\frac{1}{2}}(q)_\infty(x;q)_\infty(qx^{-1};q)_\infty$.

[10]The expression corresponding to (2.2) with Wilson line defects in representations $R_1, R_2,\dots$ is $\mathrm{I}^{\mathrm{U}(N)}_{R_1,R_2,\dots}(\eta;q) = \oint_{|\boldsymbol{z}|=1} D^N\boldsymbol{z}\ \prod_{n\geqslant 1}\chi_{R_n}(\boldsymbol{z})\ \mathrm{PE}[f(\eta;q)\chi_\Box(z)\chi_\Box(z^{-1})]$.

has a clean AdS/CFT interpretation. The factor $\mathrm{I}_{\mathrm{F1}}(\eta;q)$ is expected to represent fluctuations of a fundamental string along $AdS_2 \subset AdS_5$ [54, 55] meeting the boundary of AdS_2 at the two poles of S^3 in $\partial AdS_5 = \mathbb{R}\times S^3$, where we place the two line operators are placed. This was checked in [8] where the expression $f_{\mathrm{F1}}(\eta;q)$ was shown to perfectly match the single particle index of fluctuations of the fundamental string.

The leading large N correction to the line index was first addressed in [12] on string side assuming it is given by fluctuations of two half-infinite strings ending on the D3 giant. At the single giant graviton order, it was later confirmed on gauge theory side at all orders in q in [24]. It takes a form similar to (2.3), and it is convenient to present it as a correction to the ratio

$$R_N(\eta;q) = \frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)} - \mathrm{I}_{\mathrm{F1}}\,\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}}. \tag{2.7}$$

The numerator of this ratio subtracts the contribution to $\mathrm{I}_{\mathrm{F1}}\mathrm{I}^{(N)}$ corresponding to the case when the two half-infinite strings do not end on the giant graviton and the difference is divided by the supergravity contribution, i.e. the undecorated index $\mathrm{I}^{\mathrm{U}(\infty)}$ at large N. At leading large N, the exact expression of the first correction reads

$$R_N(\eta;q) = \left(\mathscr{G}_{\mathrm{F}}^{+}(\eta;q)\,\eta^N + \mathscr{G}_{\mathrm{F}}^{-}(\eta;q)\,\eta^{-N}\right) q^N + \mathcal{O}(q^{2N}), \tag{2.8}$$

where the non-trivial correction factors $\mathscr{G}^{\pm}(\eta;q)$ are given by

$$\mathscr{G}_{\mathrm{F}}^{\pm}(\eta;q) = G_{\mathrm{D3}}^{\pm}(\eta;q)\times\frac{1}{\eta q}\,\mathrm{PE}[f_{\mathrm{F}}(\eta;q)]\,, \qquad f_{\mathrm{F}}(\eta;q) = 2\eta^{-1}q - 2q^2\,. \tag{2.9}$$

As observed in [12] by counting BPS contributions to the index, the function $f_{\mathrm{F}}(\eta;q)$ in (2.9) agrees with the single particle index from fluctuations of the two semi-infinite strings ending on the giant graviton. Factorization in (2.9) implies an important decoupling of the contributions from fluctuations of the D3 giant, at least at leading order in large N.

The extra factor $1/(\eta q)$ in (2.9) is a puzzling feature and so far has no explanation. As we mentioned in the Introduction, it seems important to clarify its origin because similar (more complicated) extra factors are present in higher order giant graviton contributions to the line index and its generalizations [12, 25].

3 Review of Gang-Koh-Lee calculation

We begin with a quick summary of the computation in [8] and a review of general aspects concerning the evaluation of superconformal indices on the two sides of AdS/CFT. The line index $\mathrm{I}_{\mathsf{F}}(q;\eta)$ in $\mathcal{N}=4$ SYM is defined starting with a preserved supercharge Q such that

$$\Delta \equiv \{Q, Q^\dagger\} = H - J - \bar{J} - r_1, \qquad [Q, H+J+\bar{J}] = 0, \tag{3.1}$$

where H is the energy associated with time direction in $\mathbb{R}\times S^3$ or conformal dimension of operators in $\mathbb{R}^4$. The two angular momenta $J, \bar{J}$ are Cartan of $\mathrm{SU}(2)_L$, $\mathrm{SU}(2)_R$. Finally $r_{1,2,3}$ are Cartan of R-symmetry SU(4). The line index (2.1) is

$$\mathrm{I}_{\mathsf{F}}(q;\eta) = \mathrm{Tr}_{\mathcal{H}_L}(-1)^F q^{H+J+\bar{J}}\eta^{r_3}, \tag{3.2}$$

where $\mathcal{H}_L$ is the Hilbert space on S^3 in the presence of line operator L and the trace is restricted to BPS states with $\Delta = 0$. In the gauge theory side, the actual calculation of the index can be done in two complementary ways. A direct method consists just in summing over BPS states with $\Delta = 0$. Alternatively, the index can be regarded as a supersymmetric partition function on $S^1_\beta \times S^3$ after Wick rotation $\tau = -it$ and compactification $\tau = \tau + \beta$. Fields are periodic along S^1_β and should be expanded in fluctuations around a background suitable for the line operators L and chemical potentials. In our context, this corresponds to the following twist in the action

$$\partial_\tau \to \partial_\tau - (j_L + j_R) + r_3 \frac{\log \eta}{\beta}. \tag{3.3}$$

These two approaches have a counterpart on the gravity side. The calculation in [8] is done by the direct method. The gravity dual in the presence of two oppositely charged Wilson lines in the fundamental and anti-fundamental is a fundamental string wrapping AdS_2 in AdS_5 with metric

$$ds^2_{AdS_5} = d\rho^2 + \cosh^2 \rho \, d\tau^2 + \sinh^2 \rho \, dS_3. \tag{3.4}$$

The string worldsheet (τ, ρ) is at fixed poles of S^3 $(1, 0, 0, 0)$ and $(-1, 0, 0, 0)$ (and at a point in S^5). It preserves the same symmetries of a single Wilson line in the CFT theory. The symmetry group of $\mathcal{N} = 4$ SYM is PSU$(2, 2|4)$ with bosonic SU$(2, 2) \times$ SU$(4)_R$. The first factor is the conformal group in 4d SU$(2, 2) \simeq$ SO$(4, 2)$. The second is SU$(4)_R \simeq$ SO$(6)_R$ and gives R-symmetry equal to isometries of S^5. The symmetry group of the line Wilson loop is discussed in [56]. The loop breaks part of the SO$(4, 2)$ generators $\{P_\mu, J_{\mu\nu}, D, K_\mu\}$. In fact, invariance of the line (along 0 direction) is preserved by $\{P_0, J_{ij}, D, K_0\}$. The generators J_{ij} span SO$(3) \simeq$ SU(2),[11] (rotations around the line). The other ones give rotation around the line, dilatation, and a special transformation. Generators close as

$$[P_0, K_0] = -2D, \qquad [P_0, D] = -P_0, \qquad [K_0, D] = K_0, \tag{3.5}$$

and span SU$(1, 1)$ = SL$(2, \mathbb{R})$. Thus conformal symmetry is reduced to

$$\mathrm{SO}(4, 2) \to \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(3) \simeq \mathrm{SO}(4^*) \tag{3.6}$$

where we recall that SO$(2n^*)$ is a non-compact real form of SO$(2n, \mathbb{C})$. The R-symmetry group is broken to SO$(6)_R \to$ SO$(5)_R \simeq$ USp$(4)_R$ by the choice of the versor in the coupling of the loop with $\mathcal{N} = 4$ SYM scalars. Finally the Wilson line is 1/2-BPS and preserves 16 of the supersymmetries of PSU$(2, 2|4)$. The AdS supergroup with 16 supercharges and bosonic part SO$(4^*) \times$ USp$(4)_R$ is OSp$(4^*|4)$ [57, 58]. Fluctuations of the fundamental string were computed in [59] and indeed can be arranged into 8 bosonic and 8 fermionic states in the OSp$(4^*|4)$ ultra-short multiplet [60]

$$(1, \mathbf{1}, \mathbf{5}) + (\tfrac{3}{2}, \mathbf{2}, \mathbf{4}) + (2, \mathbf{3}, \mathbf{1}), \tag{3.7}$$

where the first label h is eigenvalue of dilatation in SL$(2, \mathbb{R})$, the second label is dimension of SU(2) with angular momentum $\mathcal{J} = J + \bar{J}$, and the third label is the dimension of USp$(4)_R$.[12]

[11] Here and in the following we denote by $\simeq$ the standard 2–1 homomorphisms.

[12] Values of h can be checked to agree with the explicit mass spectrum in [59]. For instance, for scalars one finds three massive fluctuations in AdS_5 with $m^2 = 2$ and five massless fluctuations in S^5 with $m^2 = 0$. For scalars, we use $h = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$ where here for AdS_{1+1} we have $d = 1$. So $m^2 = 0$ gives $h = 1$ and $m^2 = 2$ gives $h = 2$ as in (3.7).

Following the direct counting approach, the trace (3.2) is evaluated in [8] by summing over fluctuations modes after restriction to BPS states. For the states in the **5** of SO(5) one has

$$\mathbf{5}:(r_1,r_3)=(1,1),(1,-1),(-1,-1),(-1,1),(0,0), \qquad h=1,\mathcal{J}=0, \tag{3.8}$$

so we have two bosonic BPS states with $(r_1,r_3)=(1,1),(1,-1)$. For the states in the **4** of SO(5) one has

$$\mathbf{4}:(r_1,r_3)=(1,0),(-1,0),(0,1),(0,-1), \qquad h=\tfrac{3}{2},\mathcal{J}=\tfrac{1}{2}, \tag{3.9}$$

so we have one fermionic BPS state with $(r_1,r_3)=(1,0)$. From the singlet of SO(5) we don't have BPS states. In total, the index gets the following three contributions

$$\begin{aligned}
(1,0,5)\ \text{with}\ (r_1,r_3)&=(1,1), &&\to(-1)^F q^{h+\mathcal{J}}\eta^{r_3}=q\eta,\\
(1,0,5)\ \text{with}\ (r_1,r_3)&=(1,-1), &&\to(-1)^F q^{h+\mathcal{J}}\eta^{r_3}=q\eta^{-1},\\
(\tfrac{3}{2},\tfrac{1}{2},4)\ \text{with}\ (r_1,r_3)&=(1,0), &&\to(-1)^F q^{h+\mathcal{J}}\eta^{r_3}=-q^2,
\end{aligned} \tag{3.10}$$

summig up to the single particle index $f_{\rm F1}(\eta;q)$ in (2.6).

In the following sections, we will begin by obtaining this result by the indirect approach (still on gravity side), i.e. by evaluating the string semiclassical partition function in a suitable background. Later, we will address finite N corrections in the same framework, but in the presence of the D3 giant.

4 Large N line index from AdS_2 string in twisted background

Let us introduce the following twists in the $AdS_5\times S^5$ background associated with the Cartan generators in the index. A first twist takes into account $\mathcal{J}=J+\bar{J}$ and is a rotation of an angle in $S^3\subset AdS_5$. A second one is for r_3 in the index and amounts to two opposite rotations of two angles in S^5 in toroidal parametrization, cf. section 5 of [33]. The twisted background is thus

$$\begin{aligned}
ds^2_{\widetilde{AdS}_5}&=d\rho^2+\cosh^2\rho\, d\tau^2+\sinh^2\rho\, d\widetilde{S}_3, &(4.1)\\
d\widetilde{S}_3&=d\psi_1^2+\sin^2\psi_1 d\psi_2^2+\sin^2\psi_1\sin^2\psi_2(d\psi_3+i\alpha_1 d\tau)^2, &(4.2)\\
ds^2_{\widetilde{S}^5}&=dn_1^2+n_1^2d\varphi_1^2+dn_2^2+n_2^2(d\varphi_2-i\alpha_2 d\tau)^2+dn_3^2+n_3^2(d\varphi_3+i\alpha_2 d\tau)^2, &(4.3)\\
&n_1^2+n_2^2+n_3^2=1.
\end{aligned}$$

The specific value of the twist angles α_1,α_2 will be discussed later. We take as classical solution the N pole or S pole in S^3 so $\psi_1=0,\pi$, with both giving the same. The choice of a specific point in S^5 is irrelevant by rotational symmetry.

The fundamental string is wrapped on $AdS_{2,\beta}$ with coordinates $(\xi^1,\xi^2)=(\tau\equiv\tau+\beta,\rho)$. The bosonic action is

$$S_{\rm NG}=\mathsf{T}\int d^2\xi\sqrt{G}, \qquad G_{ab}=\partial_a X^M\partial_b X^N G_{MN}(X)=G_{ab}(\xi)\,d\xi^a d\xi^b, \tag{4.4}$$

where $\mathsf{T}=\frac{\sqrt{\lambda}}{2\pi}\gg 1$ controls the semiclassical expansion. Given a classical solution $X^M=X^M(\xi)$ $(M=1,\dots,10)$ we adopt a static gauge where two of the X^M coordinates are equal to the world-volume coordinates ξ^a $(a = 1, 2)$ and a κ-symmetry gauge for fermions. The remaining 8 bosonic and 8 fermionic fluctuations produce a β-dependent 1-loop prefactor in the partition function Z

$$\mathrm{Z} = \int DXD\theta\, e^{-S[X,\theta]} = \mathcal{Z}_1\, e^{-\mathsf{T}\,\bar{S}_{\rm cl}}[1+\mathcal{O}(\mathsf{T}^{-1})], \qquad S_{\rm cl} = \mathsf{T}\bar{S}_{\rm cl}, \tag{4.5}$$

$$\mathcal{Z}_{1-\rm loop} = e^{-\Gamma_{1-\rm loop}}, \qquad \Gamma_{1-\rm loop} = \frac{1}{2}\sum_k (-1)^{\mathsf{F}_k} \log\det\Delta_k, \tag{4.6}$$

where Δ_k is the differential operator governing quadratic fluctuations of the k-th field. In our case, the classical action vanishes because the Wilson line is BPS [59].[13] The aim is thus to compute the non-trivial piece $\Gamma_{1-\rm loop} = \Gamma_{1-\rm loop}(\eta;\beta)$.

4.1 Scalar fluctuations

4.1.1 AdS_5 sector

To study bosonic fluctuations in AdS_5 we introduce new variables to parametrize $S^3 \subset AdS_5$

$$X_1 = \cos\psi_1, \qquad X_2 = \sin\psi_1\,\cos\psi_2, \tag{4.7}$$

$$X_3 = \sin\psi_1\,\sin\psi_2\,\cos\psi_3, \qquad X_4 = \sin\psi_1\,\sin\psi_2\,\sin\psi_3, \tag{4.8}$$

with $X_1^2 + \cdots + X_4^2 = 1$. We have at quadratic order $(I = 2, 3, 4)$

$$ds^2_{\widetilde{AdS_5}} = d\rho^2 + \cosh^2\rho\, d\tau^2 + \sinh^2\rho\,[dX_I^2 - \alpha_1^2(X_3^2+X_4^2)d\tau^2 + 2i\alpha_1(X_3dX_4 - X_4dX_3)d\tau]. \tag{4.9}$$

Denoting by $a = 1, 2$ the indices of the world-sheet coordinates $\xi^a = (\tau, \rho)$, we obtain

$$\begin{aligned} ds^2_{\widetilde{AdS_5}} &= d\rho^2 + [\cosh^2\rho - \alpha_1^2\sinh^2\rho(X_3^2+X_4^2) + 2i\alpha_1\sinh^2\rho(X_3\dot{X}_4 - X_4\dot{X}_3)]\,d\tau^2 \\ &\quad + \sinh^2\rho\,\partial_a X_I\partial_b X_I\, d\xi^a d\xi^b + \text{off diagonal terms}\,. \end{aligned} \tag{4.10}$$

Expanding the metric by $\delta\sqrt{G^{(0)}+G^{(1)}} = \sqrt{G^{(0)}} + \frac{1}{2}\sqrt{G^{(0)}}\,(G^{(0)})^{ab}G^{(1)}_{ab} + \cdots$, we get

$$\delta S_{\rm NG} = \frac{1}{2}\mathsf{T}\int d^2\xi\sqrt{g}[g^{ab}\sinh^2\rho\,\partial_a X_I\partial_b X_I - \alpha_1^2\tanh^2\rho\,(X_3^2+X_4^2) + 2i\alpha_1\tanh^2\rho\,(X_3\dot{X}_4 - X_4\dot{X}_3)], \tag{4.11}$$

where $g_{ab} \equiv G^{(0)}_{ab}$ is the AdS_2 metric

$$ds^2_{AdS_2} = g_{ab}d\xi^a d\xi^b = d\rho^2 + \cosh^2\rho\, d\tau^2, \qquad g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \cosh^2\rho \end{pmatrix}, \qquad \sqrt{g} = \cosh\rho. \tag{4.12}$$

Notice that for small ρ one has $ds^2 \simeq d\rho^2 + d\tau^2$ or $S^1 \times S^1_\beta$ and the thermal cycle is not contractible. With the field redefinition

$$X_I = \frac{1}{\sinh\rho}Y_I, \tag{4.13}$$

[13] With the usual regularization of the boundary, this follows from $\int_0^{-\log\varepsilon}\cosh\rho\, d\rho = \frac{1}{2\varepsilon} + 0 + \mathcal{O}(\varepsilon)$.

we get

$$g^{ab}\sinh^2\rho\,\partial_a X_I\partial_b X_I=\sinh^2\rho\partial_\rho\left(\frac{Y_I}{\sinh\rho}\right)\partial_\rho\left(\frac{Y_I}{\sinh\rho}\right)+\frac{1}{\cosh^2\rho}\dot{Y}_I\dot{Y}_I, \tag{4.14}$$

and integration by parts leads to

$$\delta S_{\rm NG}=\frac{1}{2}\beta\mathsf{T}\int d^2\xi\sqrt{g}\left[g^{ab}\partial_a Y_I\partial_b Y_I+2\,Y_I^2-\frac{\alpha_1^2}{\cosh^2\rho}(Y_3^2+Y_4^2)+2i\frac{\alpha_1}{\cosh^2\rho}(Y_3\dot{Y}_4-Y_4\dot{Y}_3)\right]. \tag{4.15}$$

The twist α_1 gives ρ dependent mass terms (and mixing contributions) for two of the three scalars. This is same as coupling the complex scalar Y_3+iY_4 to a constant gauge field in τ direction. Indeed, let us define

$$Z=\frac{1}{\sqrt{2}}(Y_3+iY_4). \tag{4.16}$$

We can write

$$\dot{Y}_3\dot{Y}_3+\dot{Y}_4\dot{Y}_4+Y_3^2+Y_4^2-\alpha_1^2(Y_3^2+Y_4^2)+2i\alpha_1(Y_3\dot{Y}_4-Y_4\dot{Y}_3)=2(\dot{\bar{Z}}+\alpha_1\bar{Z})(\dot{Z}-\alpha_1 Z)+4\bar{Z}Z, \tag{4.17}$$

and thus

$$\delta S_{\rm NG}=\mathsf{T}\int d^2\xi\sqrt{g}\left[\frac{1}{2}\left(g^{ab}\partial_a Y_2\partial_b Y_2+2Y_2^2\right)+g^{ab}D_a\bar{Z}D_bZ+2\bar{Z}Z\right], \tag{4.18}$$

where $D_aZ=\partial_aZ-A_aZ$, $D_a\bar{Z}=\partial_a\bar{Z}+A_a\bar{Z}$ with the constant gauge field along τ

$$A_a=(\alpha_1,0). \tag{4.19}$$

The squared mass in AdS_2 is same $m^2=2$ for all three $(1+2)$ fluctuations. The α_1 twist enters only through the constant gauge field.

4.1.2 S^5 sector

In unflavored case $\eta=1$ we switch-off the twist in S^5 and get five massless scalars in (thermal) AdS_2 as follows from [59]. In flavored case, we replace in (4.3) the explicit parametrization

$$\begin{aligned}
&(n_1\sin\varphi_1,n_1\cos\varphi_1)=(U_1,U_2),\\
&(n_2\sin\varphi_2,n_2\cos\varphi_2)=(U_3,U_4),\\
&(n_3\sin\varphi_3,n_3\cos\varphi_3)=(U_5,U_6),\qquad U_1^2+\cdots+U_6^2=1,\\
&\varphi_1=\arctan\frac{U_1}{U_2},\qquad \varphi_2=\arctan\frac{U_3}{U_4},\qquad \varphi_3=\arctan\frac{U_5}{U_6},\\
&n_1=\sqrt{U_1^2+U_2^2},\qquad n_2=\sqrt{U_3^2+U_4^2},\qquad n_3=\sqrt{U_5^2+U_6^2}.
\end{aligned} \tag{4.20}$$

Solving U_1 in terms of the other fluctuation fields $U_2,\ldots,U_6$, we obtain

$$ds^2_{\widetilde{S}^5}=\sum_{I=2}^{6}dU_I^2+2i\alpha_2(U_3dU_4-U_4dU_3-U_5dU_6+U_6dU_5)\,d\tau-\alpha_2^2(U_3^2+U_4^2+U_5^2+U_6^2)\,d\tau^2. \tag{4.21}$$

Introducing two complex scalars

$$W_+ = \frac{U_3 + iU_4}{\sqrt{2}}, \qquad W_- = \frac{U_5 + iU_6}{\sqrt{2}}, \tag{4.22}$$

gives (following similar steps as in AdS sector)

$$\delta S_{\rm NG} = \mathsf{T} \int d^2\xi \sqrt{g} \left[\frac{1}{2} \left(g^{ab} \partial_a U_2 \partial_b U_2 \right) + \sum_{s=\pm} g^{ab} D_a \overline{W}_s D_b W_s \right], \tag{4.23}$$

where the covariant derivative couples $W_\pm$ to $\pm A'_a$ which is the following constant gauge field along Euclidean time direction τ

$$A'_a = (\alpha_2, 0). \tag{4.24}$$

In summary, in the sector of scalar fluctuations we have a massless scalar plus two complex scalars coupling to A' with opposite charge.

4.2 Fermionic fluctuations

We have 8 fermionic fields with conformal dimension $h = \frac{3}{2}$ in the ultra-short OSp(4*|4) representation. The detailed coupling to a constant gauge field can be derived from the fermionic part of the Green-Schwarz action [61], see also [62].

Fermionic action The fermionic action is ($\bar\theta = \theta^{\rm T} C$)[14]

$$L_F = i \left(\sqrt{g}\, g^{ab} \delta^{IJ} - \varepsilon^{ab} s^{IJ} \right) \bar\theta^I \rho_a D_b^{JK} \theta^K, \qquad \rho_a = \Gamma_{\underline{m}} e_a^{\underline{m}}, \qquad e_a^{\underline{m}} = E_\mu^{\underline{m}} \partial_a X^\mu, \tag{4.25}$$

where $I, J = 1, 2$, $s^{IJ} = \mathrm{diag}(1, -1)$, and ρ_a are projections of the 10-d Dirac matrices. The fermionic fields are two 10d MW spinors θ^I with same chirality. X^μ are the string coordinates, i.e. given functions of τ and σ for a particular classical solution.

The covariant derivative takes the following form

$$\begin{aligned} D_a^{IJ} &= \delta^{IJ} \mathsf{D}_a + \mathcal{F}_a \varepsilon^{IJ}, \\ \mathsf{D}_a &= \partial_a + \tfrac{1}{4} \omega_a^{\underline{mn}} \Gamma_{\underline{mn}}, \qquad \omega_a^{\underline{mn}} = \partial_a X^\mu \omega_\mu^{\underline{mn}}, \end{aligned} \tag{4.26}$$

where the flux term is

$$\mathcal{F}_\mu = -\frac{1}{8 \cdot 5!} F_{\mu_1 \dots \mu_5} \Gamma^{\mu_1 \dots \mu_5} \Gamma_\mu. \tag{4.27}$$

We fix κ-symmetry by

$$\theta^1 = \theta^2 = \theta. \tag{4.28}$$

Then

$$\begin{aligned} L_F &= L_F^{\rm kin} + L_F^{\rm flux}, \\ L_F^{\rm kin} &= 2i \sqrt{g} g^{ab} \bar\theta \rho_a (\partial_b + \tfrac{1}{4} \omega_b^{\underline{mn}} \Gamma_{\underline{mn}}) \theta, \qquad L_F^{\rm flux} = -2i \varepsilon^{ab} \bar\theta \rho_a \mathcal{F}_b \theta. \end{aligned} \tag{4.29}$$

[14] There were several factors of 2 missing or notation not fully explained in some earlier papers but end results were correct, see footnote 34 in revised version of [63].

Specialization to our background In real time $\tau = it$

$$\begin{aligned} ds^2_{\widetilde{AdS}_5} &= d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\widetilde{S}_3, \\ d\widetilde{S}_3 &= d\psi_1^2 + \sin^2\psi_1 d\psi_2^2 + \sin^2\psi_1 \sin^2\psi_2 (d\psi_3 - \alpha_1 dt)^2, \\ ds^2_{\widetilde{S}^5} &= dn_1^2 + n_1^2 d\varphi_1^2 + dn_2^2 + n_2^2(d\varphi_2 + \alpha_2 dt)^2 + dn_3^2 + n_3^2(d\varphi_3 - \alpha_2 dt)^2. \end{aligned} \tag{4.30}$$

In classical solution parametrized by (t, ρ) all other coordinates are zero with the exception of $n_1 = 1$. Let us introduce angles for the $\boldsymbol{n}$ 3-vector

$$n_1 = \cos\chi_1, \qquad n_2 = \sin\chi_1 \cos\chi_2, \qquad n_3 = \sin\chi_1 \sin\chi_2. \tag{4.31}$$

The metric is

$$\begin{aligned} ds^2_{\widetilde{AdS}_5} &= -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\widetilde{S}_3, \\ d\widetilde{S}_3 &= d\psi_1^2 + \sin^2\psi_1 d\psi_2^2 + \sin^2\psi_1 \sin^2\psi_2 (d\psi_3 - \alpha_1 dt)^2, \\ ds^2_{\widetilde{S}^5} &= d\chi_1^2 + \sin^2\chi_1 d\chi_2^2 + \cos^2\chi_1 d\varphi_1^2 + \sin^2\chi_1 \cos^2\chi_2 (d\varphi_2 + \alpha_2 dt)^2 + \sin^2\chi_1 \sin^2\chi_2 (d\varphi_3 - \alpha_2 dt)^2. \end{aligned} \tag{4.32}$$

Let us label 10d coordinates as

t	ρ	ψ_1	ψ_2	ψ_3	χ_1	χ_2	φ_1	φ_2	φ_3
0	1	2	3	4	5	6	7	8	9

Vielbein $E^{\underline{m}} = E^{\underline{m}}_\mu dX^\mu$ (with flat 10d metric in $(-, +, +, \ldots, +)$ signature) are

$$\begin{aligned} & E^{\underline{0}} = \cosh\rho\, dt \qquad E^{\underline{1}} = d\rho, \qquad E^{\underline{2}} = \sinh\rho\, d\psi_1, \qquad E^{\underline{3}} = \sinh\rho \sin\psi_1 d\psi_2, \\ & E^{\underline{4}} = \sinh\rho \sin\psi_1 \sin\psi_2 (d\psi_3 - \alpha_1 dt), \qquad E^{\underline{5}} = d\chi_1, \qquad E^{\underline{6}} = \sin\chi_1 d\chi_2, \qquad E^{\underline{7}} = \cos\chi_1 d\varphi_1, \\ & E^{\underline{8}} = \sin\chi_1 \cos\chi_2 (d\varphi_2 + \alpha_2 dt), \qquad E^{\underline{9}} = \sin\chi_1 \sin\chi_2 (d\varphi_3 - \alpha_2 dt). \end{aligned} \tag{4.33}$$

On classical solution

$$\begin{aligned} & E^{\underline{0}}_0 = \cosh\sigma, \quad E^{\underline{1}}_1 = 1, \quad E^{\underline{2}}_2 = \sinh\sigma, \quad E^{\underline{3}}_3 = 0, \quad E^{\underline{4}}_4 = 0 \\ & E^{\underline{5}}_5 = 1, \quad E^{\underline{6}}_6 = 0, \quad E^{\underline{7}}_7 = 1, \quad E^{\underline{8}}_8 = 0, \quad E^{\underline{9}}_9 = 0, \end{aligned} \tag{4.34}$$

and the non zero induced 2-bein $e^{\underline{m}}_a$ are

$$e^{\underline{0}}_0 = \cosh\sigma, \quad e^{\underline{1}}_1 = 1 \qquad \to \qquad \rho_0 = \cosh\sigma\, \Gamma_{\underline{0}}, \quad \rho_1 = \Gamma_{\underline{1}}. \tag{4.35}$$

Spin connection and twists The spin connection $\omega^{\underline{mn}}_\mu$ is obtained from the Cartan equation

$$dE^{\underline{m}} + \omega^{\underline{mn}} \wedge E^{\underline{n}} = 0, \qquad E^{\underline{n}} = E^{\underline{n}}_\mu dX^\mu. \tag{4.36}$$

The twist in ψ_3 has a peculiar effect [64]. Let us follow the α_1 part in

$$\begin{aligned} dE^{\underline{4}} &= -\alpha_1 \sinh\rho \sin\psi_1 \cos\psi_2\, d\psi_2 \wedge dt + \cdots = -\alpha_1 \cos\psi_2\, E^{\underline{3}} \wedge dt + \cdots \\ &= -\omega^{\underline{43}}_0 \wedge E^{\underline{3}} + \cdots, \qquad \omega^{\underline{43}}_0 = -\alpha_1 \cos\psi_2 + \cdots, \end{aligned} \tag{4.37}$$

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where dots are terms vanishing on the classical solution. A similar effect is associated with α_2 twist. From

$$
\begin{aligned}
dE^{\underline{8}} &= \alpha_2 \cos\chi_1 \cos\chi_2 d\chi_1 \wedge dt + \cdots = \alpha_2 \cos\chi_1 \cos\chi_2 \, E^{\underline{5}} \wedge dt + \cdots , \\
dE^{\underline{9}} &= -\alpha_2 \sin\chi_1 \cos\chi_2 d\chi_2 \wedge dt + \cdots = -\alpha_2 \cos\chi_2 E^{\underline{6}} \wedge dt + \cdots ,
\end{aligned}
\tag{4.38}
$$

we get

$$
\omega_0^{\underline{85}} = \alpha_2 \cos\chi_1 \cos\chi_2 + \cdots , \qquad \omega_0^{\underline{96}} = -\alpha_2 \cos\chi_2 + \cdots , \tag{4.39}
$$

both non vanishing on classical solution. The other relevant spin connection components can be computed directly for the induced metric

$$
de^{\underline{m}} + \omega^{\underline{mn}} \wedge e^{\underline{n}} = 0, \qquad e^{\underline{n}} = e_a^{\underline{n}} d\xi^a. \tag{4.40}
$$

Since $e^{\underline{0}} = \cosh\sigma d\tau$ and $e^{\underline{1}} = d\sigma$, the unique case is

$$
d(\cosh\sigma d\tau) + \omega^{\underline{01}} \wedge d\sigma = 0 \qquad \rightarrow \qquad \omega^{\underline{01}} = \sinh\sigma \, d\tau. \tag{4.41}
$$

The kinetic part of the action is thus

$$
\begin{aligned}
L_F^{\text{kin}} &= 2i \sqrt{g} g^{ab} \bar\theta \rho_a (\partial_b + \tfrac{1}{4}\omega_b^{\underline{mn}} \Gamma_{\underline{mn}})\theta = 2i\,\bar\theta \Big[-\frac{1}{\cosh\sigma} \rho_0 (\partial_0 + \tfrac{1}{4}\omega_0^{\underline{mn}} \Gamma_{\underline{mn}}) + \cosh\sigma \rho_1 \partial_1 \Big] \theta \\
&= 2i\bar\theta[-\Gamma_{\underline{0}}(\partial_0 - \tfrac{1}{2}\alpha_1 \Gamma_{\underline{43}} + \tfrac{1}{2}\alpha_2 \Gamma_{\underline{85}} - \tfrac{1}{2}\alpha_2 \Gamma_{\underline{96}}) + \cosh\sigma \Gamma_{\underline{1}} \partial_1 + \tfrac{1}{2}\sinh\sigma \Gamma_{\underline{1}}]\,\theta.
\end{aligned}
\tag{4.42}
$$

Let us finish by evaluating the mass terms. In the conventions of [61], the 5-form is

$$
F_5 = 2\,(1+\star)\,\text{vol}_{AdS_5} = 2(E^{\underline{0}} \wedge \cdots \wedge E^{\underline{4}} + E^{\underline{5}} \wedge \cdots \wedge E^{\underline{9}}), \tag{4.43}
$$

where twists are inside three of the vielbeins. Slashing it with curved Γ matrices gives

$$
\begin{aligned}
F_{\mu_1 \dots \mu_5} \Gamma^{\mu_1 \dots \mu_5} &= 2 \cdot 5!\,(E_0^{\underline{0}} \cdots E_4^{\underline{4}} \Gamma^{01234} + (01234 \to 56789)) = 2\cdot 5!\,(\Gamma^{\underline{01234}} + \Gamma^{\underline{56789}}) \\
&= 2\cdot 5!\,\Gamma^{\underline{01234}}(1-\widehat\Gamma), \qquad \widehat\Gamma = \Gamma^{\underline{012\cdots 9}}.
\end{aligned}
\tag{4.44}
$$

Chirality constraint is $\widehat\Gamma\theta = \theta$, so

$$
\mathcal{F}_a \rho_b \theta = -\frac{1}{4}\Gamma^{\underline{01234}} \rho_b (1+\widehat\Gamma)\theta = -\frac{1}{2}\Gamma^{\underline{01234}}\rho_b\theta, \tag{4.45}
$$

and

$$
L_F^{\text{flux}} = i\varepsilon^{ab}\bar\theta\rho_a \Gamma^{\underline{01234}}\rho_b\theta. \tag{4.46}
$$

This is

$$
L_F^{\text{flux}} = i\varepsilon^{ab}\bar\theta\rho_a\Gamma^{\underline{01234}}\rho_b\theta = i\cosh\sigma\bar\theta(\Gamma_{\underline{0}}\Gamma^{\underline{01234}}\Gamma_{\underline{1}} - \Gamma_{\underline{1}}\Gamma^{\underline{01234}}\Gamma_{\underline{0}})\theta = -2i\cosh\sigma\bar\theta\,\Gamma^{\underline{234}}\theta, \tag{4.47}
$$

and

$$
L_F = 2i\bar\theta[\Gamma^{\underline{0}}(\partial_0 - \tfrac{1}{2}\alpha_1\Gamma_{\underline{43}} + \tfrac{1}{2}\alpha_2\Gamma_{\underline{85}} - \tfrac{1}{2}\alpha_2\Gamma_{\underline{96}}) + \cosh\sigma\Gamma^{\underline{1}}\partial_1 + \tfrac{1}{2}\sinh\sigma\Gamma^{\underline{1}} - \cosh\sigma\,\Gamma^{\underline{234}}]\,\theta. \tag{4.48}
$$

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Fluctuation	Y_1	$Z,\bar{Z}$	A_2	$W_+,\overline{W}_+$	$W_-,\overline{W}_-$	$\psi_{1,2}$	$\psi_{3,4}$	$\psi_{5,6,7,8}$
h	2	2	1	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
κ	0	α_1	0	α_2	$-\alpha_2$	$\frac{1}{2}+\alpha_2$	$\frac{1}{2}-\alpha_2$	0

Table 1. Conformal dimension and κ-shifts for the scalar fluctuations in AdS_5, S^5, and for the eight fermionic modes.

The last term is a "σ_3" constant mass term since $\cosh\sigma = \sqrt{g}$. In other words, this is same as in [59] up to the coupling to the constant gauge field (all $\pm$ signs are uncorrelated)

$$L_F = 2\sqrt{g}\bar{\theta}[i\,\rho^a\widehat{\nabla}_a + \tfrac{1}{2}(\pm\alpha_1 \pm \alpha_2 \pm \alpha_2)\not{A} \pm 1]\theta, \qquad A = (0,1). \tag{4.49}$$

Signs are eigenvalues of commuting $i\Gamma^{\underline{34}}, i\Gamma^{\underline{234}}, i\Gamma^{\underline{85}}, i\Gamma^{\underline{96}}$. Since $\alpha_1 = 1$ we have $8+8$ possibilities, with mass $M = \pm 1$ and charge

$$\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm(\tfrac{1}{2}+\alpha_2), \pm(\tfrac{1}{2}-\alpha_2), \tag{4.50}$$

corresponding to the 8 fermionic terms in (4.58). Notice also that the mass value is the right one

$$|M| = 1 = \tfrac{3}{2} - \tfrac{1}{2} = h - \tfrac{1}{2}. \tag{4.51}$$

4.3 Reproducing the line index

Fields coupled to a constant gauge field in τ direction receive a shift in the mode number of their Fourier expansion on the Euclidean time circle. Following notation of [31] we denote by κ this shift. The spectrum and shifts of all fluctuation fields are shown in table 1. Adapting to AdS_2 the discussion in [31], the partition function of a scalar with conformal dimension h in thermal AdS_2 gives the following one-loop determinant, cf. also appendix A,

$$\Gamma^{(h,\kappa)} = \frac{1}{2}\log\det\Delta^{(h,\kappa)} = \beta E_c^{(h,\kappa)} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{nh}(q^{n\kappa}+q^{-n\kappa})}{1-q^n}, \qquad q = e^{-\beta}, \tag{4.52}$$

where E_c is the supersymmetric Casimir energy contribution. Notice that this can be written

$$\exp(-\Gamma^{(h,\kappa)}) = e^{-\beta E_c^{(h,\kappa)}}\,\mathrm{PE}\left[-\frac{1}{2}\frac{q^h(q^\kappa+q^{-\kappa})}{1-q}\right]. \tag{4.53}$$

The expression of the supersymmetric Casimir energy E_c is, see [65] and appendix B,

$$\begin{aligned} E_c &= \frac{1}{2}\zeta_E(-1), \qquad \zeta_E(z) = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}Z(\beta), \\ Z &= \frac{1}{2}\widetilde{Z}(h+\kappa) + \frac{1}{2}\widetilde{Z}(h-\kappa), \qquad \widetilde{Z}(\Delta) = \frac{q^\Delta}{1-q}. \end{aligned} \tag{4.54}$$

We evaluate the integral in terms of Hurwitz zeta function[15]

$$\frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}\frac{e^{-\beta\Delta}}{1-e^{-\beta}} = \zeta(z,\Delta), \tag{4.55}$$

and then

$$\frac{1}{2}\zeta'(-1,\Delta) = \frac{1}{24}(-1+6\Delta-6\Delta^2). \tag{4.56}$$

Taking half the sum with $\Delta = h \pm \kappa$ gives

$$E_c^{(h,\kappa)} = \frac{1}{24}(-1+6h-6h^2-6\kappa^2). \tag{4.57}$$

From the data in table 1, we need to compute the combination[16]

$$\Gamma_{\text{one-loop}} = \Gamma^{(2,0)} + 2\Gamma^{(2,\alpha_1)} + \Gamma^{(1,0)} + 4\Gamma^{(1,\alpha_2)} - \left[2\Gamma^{(\frac{3}{2},\frac{1}{2}+\alpha_2)} + 2\Gamma^{(\frac{3}{2},\frac{1}{2}-\alpha_2)} + 4\Gamma^{(\frac{3}{2},\frac{1}{2})}\right], \tag{4.58}$$

with a similar sum over fields for total $E_{c,\text{one-loop}}$. Comparing with (3.3), we replace in (4.58)

$$\alpha_1 = 1, \qquad q^{\alpha_2} = \eta. \tag{4.59}$$

This gives, cf. (4.53),

$$\mathrm{I}_{\mathsf{F}}(\eta;q) = \exp(-\Gamma_{\text{one-loop}}) = \mathrm{PE}[-q^2 + (\eta+\eta^{-1})\,q] = \frac{1-q^2}{(1-\eta q)(1-\eta^{-1}q)}, \tag{4.60}$$

in agreement with the single particle index (2.6). The total one-loop Casimir contribution vanishes $E_{c,\text{one-loop}} = 0$. We have thus reproduced the large N line index by a semiclassical computation in twisted background.

5 Giant graviton correction to $\mathrm{I}_{\mathsf{F}}(\eta;q)$

According to the proposal in [12], the leading large N corrections to the line index are captured by fluctuations of two half-infinite fundamental (F) strings with worldsheet ending on one of the two Wilson lines and D3 giant graviton. The exact expression in (2.8) shows that D3 brane and F string fluctuations decouple at this order. The role of the D3 giant is to assign specific boundary conditions for F string fluctuations.

The D3 brane worldvolume is $S^1_\beta \times S^3$ where $S^3 \subset S^5$ and is at $\rho = 0$. The F string is pointlike in S^5 and three fluctuations are along S^3 of the D3 brane. These should obey Neumann boundary conditions at $\rho = 0$. All other fluctuations should have Dirichlet boundary conditions at $\rho = 0$. This way we have two worldsheets each corresponding to an edge on the D3 brane and the other on the Wilson line at AdS boundary $\rho \to \infty$, cf. figure 1.

[15] This is value of the integral from definition of Hurwitz zeta function or as Mellin transform. Alternatively, one may expand $1/(1-e^{-\beta})$ and gets the same from $\sum_{n=0}^{\infty}(n+\Delta)^{-z} = \zeta(z,\Delta)$.

[16] Fermions have here periodic boundary condition in Euclidean time. Their contribution to $\Gamma_{1-\text{loop}}$ is same as for scalars, up to a minus sign.

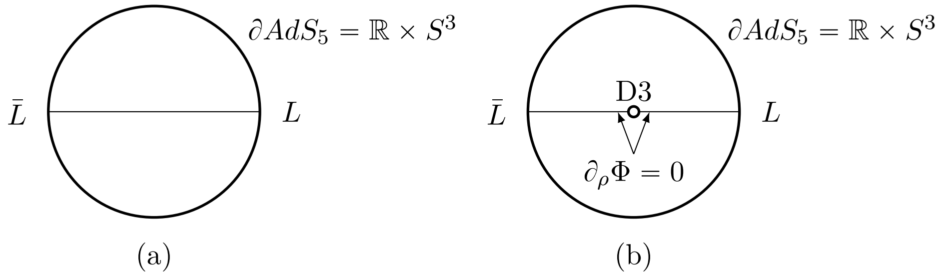

Figure 1. (a) without D3 brane: one AdS_2 worldsheet with edges ending on Wilson lines; (b) in the presence of the D3 brane: two AdS_2 worldsheets each having an edge on the D3 brane and the other on a Wilson line. In the figure, Φ is any of the three S^5 scalar fluctuations along S^3 of the D3 brane.

The square of the corresponding one-loop determinant should reproduce the correction in (2.8), i.e. the factor

$$\exp(-2\Gamma^{\rm D3}_{\rm one-loop}) = \frac{1}{\eta q}\,\mathrm{PE}[2\eta^{-1}q - 2q^2]. \tag{5.1}$$

This is same as

$$\Gamma^{\rm D3}_{\rm one-loop} = -\frac{1}{2}\log\left[\frac{1}{\eta q}\,\mathrm{PE}[2\eta^{-1}q-2q^2]\right] = -\frac{1}{2}(1+\alpha_2)\,\beta - \sum_{n=1}^{\infty}\frac{1}{n}(\eta^{-n}q^n - q^{2n}), \tag{5.2}$$

where we used (4.59). This suggests that the elusive (and unclear) factor $\frac{1}{\eta q}$ found in [12] is related to a non-vanishing Casimir energy contribution due to reduced supersymmetry in the presence of the D3 brane.

What we now need is to replace the standard effective action Γ on thermal AdS_2 in (4.52) by an expression that takes into account only fluctuations modes with the assigned boundary condition on the D3 brane. This is done in next section by a normal mode decomposition.

Remark These are not boundary conditions at $\rho \to \infty$. All fields will be outside the Breitenlohner-Freedman window [66] and admit a regular quantization with $\Phi \sim \exp(-h_\Phi \rho)$ at infinity. Instead, boundary conditions on the D3 brane at $\rho = 0$ refer to parity of states under $\rho \to -\rho$. In this respect, at least for scalars, even parity states obey Neumann boundary conditions at the D3 brane, while odd parity states obey Dirichlet boundary conditions. The case of fermionic fluctuations will be discussed later in full details.

5.1 One-loop thermal determinant in AdS_2 and boundary conditions at $\rho = 0$

5.1.1 Scalar field

In the normal mode method [67, 68],[17] the determinant of kinetic operator for a free scalar field in AdS_2 with dual conformal dimension h (and mass $m^2 = h(h-1)$) is written exploiting Weierstrass representation

$$\det(-\nabla^2 + h(h-1)) = e^{-P(h)}\prod_I (h - h_I), \tag{5.3}$$

[17] The method was developed for non-trivial black hole background where normal modes are actually dissipative quasi-normal modes. This is not the case here where we have genuine normal modes corresponding to Euclidean zero modes with suitable boundary conditions.

where h_I are special values of the analytically continued conformal dimension h to the complex plane that correspond to Euclidean zero modes of the kinetic operator. The entire function $\exp(-P(h))$ can be fixed by matching the zeta-function regularization at large h to the local heat kernel curvature expansion expression in this limit.[18]

In thermal AdS_2 (and Euclidean time τ) a zero mode has time dependence $\sim e^{-\Omega\tau}$ and periodicity in thermal cycle requires

$$\Omega = i\,\omega_n, \qquad \omega_n = \frac{2\pi n}{\beta}. \tag{5.4}$$

This means a pole in $\Gamma(1+\frac{i\beta}{2\pi}\Omega)$ in the variable Ω. If frequencies are symmetric under $\Omega \to -\Omega$ one has

$$\begin{aligned} \det(-\nabla^2 + h(h-1))^{-1} &= e^{P(h)} \prod_I \frac{\beta}{4\pi^2}\Omega_I \Gamma\left(\frac{i\beta}{2\pi}\Omega_I\right)\Gamma\left(-\frac{i\beta}{2\pi}\Omega_I\right) \\ &= \mathcal{N}\, e^{P(h)} \prod_{n>0} \left(n + \frac{i\beta}{2\pi}\Omega_I\right)^{-1} \left(n - \frac{i\beta}{2\pi}\Omega_I\right)^{-1}, \end{aligned} \tag{5.5}$$

(where we included in $\mathcal{N}$ the contribution from constant solutions with zero Ω and additional UV divergent factors). This may be written concisely as

$$\det(-\nabla^2 + m^2) = \mathcal{N}\, e^{P(h)} \prod_{n>0,I} (i\omega_n(\beta) - \Omega_I(h))(-i\omega_n(\beta) - \Omega_I(h)). \tag{5.6}$$

Normal mode frequencies are found by solving the Klein-Gordon equation and imposing boundary conditions at infinity, corresponding to h. In the rather special case of AdS_2, normal mode frequencies are parametrized by a single non-negative integer [69–71]

$$\{\Omega_I\} = \{\Omega_p = \pm(h+p), \quad p = 0, 1, 2, \dots\}. \tag{5.7}$$

To see this in our locally AdS_2 specific metric, let us write the Euclidean wave equation as

$$\cosh\rho\, \partial_\rho(\cosh\rho\, \partial_\rho \varphi) + \partial_\tau^2 \varphi - m^2 \cosh^2\rho\, \varphi = 0, \qquad m^2 = h(h-1). \tag{5.8}$$

We consider

$$\phi(\rho,\tau) = \phi(\rho)\, e^{\Omega\,\tau}, \tag{5.9}$$

and thus $\partial_\tau^2 = \Omega^2$. The general solution for $\varphi(\rho)$ is

$$\varphi(\rho) = (1-\tanh^2\rho)^{1/4}\left[C_1\, P^{h-\frac{1}{2}}_{\Omega-\frac{1}{2}}(\tanh\rho) + C_2\, Q^{h-\frac{1}{2}}_{\Omega-\frac{1}{2}}(\tanh\rho)\right], \tag{5.10}$$

where P, Q are associated Legendre functions. For integer $h \geqslant 1$, the solution vanishing at $\rho \to \infty$ has $C_1 = 0$ and it goes correctly as $e^{-h\rho}$. At $r \to 0$ we have then

$$\varphi(\rho) = C'\left[\cos\frac{\pi(h+\Omega)}{2} + \frac{2\Gamma\left(\frac{2-h+\Omega}{2}\right)\Gamma\left(\frac{1+h+\Omega}{2}\right)}{\Gamma\left(\frac{1-h+\Omega}{2}\right)\Gamma\left(\frac{h+\Omega}{2}\right)} \sin\frac{\pi(h+\Omega)}{2}\,\rho + \mathcal{O}(\rho^2).\right]. \tag{5.11}$$

[18]It is a polynomial in h, obtained by computing a finite number of integrals over spacetime of local curvature invariants.

Notice that this is smooth at $\rho = 0$ for any Ω, contrary to what happens in higher dimension, as pointed out in [72]. We now require $\varphi(0) = 0$ or $\varphi'(0) = 0$. This condition ensures that the resulting spectrum is discrete in spatial direction and not continuous as one would expect in a non-compact hyperbolic space, see for instance [73]. More physically, it corresponds to Dirichlet boundary conditions at the two boundaries of AdS_2.[19] This gives (5.7) where φ is even/odd for even/odd values of p.

Using these frequencies in (5.6) gives quickly[20]

$$\prod_{n>0,p\geqslant 0}(i\omega_n(\beta)-\Omega_p(h))(-i\omega_n(\beta)-\Omega_p(h)) = \prod_{n>0,p\geqslant 0}[\tfrac{2\pi i n}{\beta}-(p+h)]^2[-\tfrac{2\pi i n}{\beta}-(p+h)]^2$$

$$\to \prod_{n>0,p\geqslant 0}\left(1+\frac{\beta^2(p+h)^2}{4\pi^2 n^2}\right)^2. \tag{5.12}$$

We now use

$$\prod_{n>0}\left(1+\frac{a^2}{n^2}\right) = \frac{e^{\pi a}}{2\pi a}(1-e^{-2\pi a}), \tag{5.13}$$

and get

$$\det(-\nabla^2+m^2) = \mathcal{N}\prod_{p\geqslant 0}(1-q^{p+h})^2, \qquad q = e^{-\beta}. \tag{5.14}$$

Hence, expanding in q, summing over p, and denoting by a prime the non-Casimir part

$$\frac{1}{2}\log\det(-\nabla^2+m^2)' = \sum_{p\geqslant 0}\log(1-q^{p+h}) = -\sum_{p\geqslant 0}\sum_{n>0}\frac{1}{n}q^{n(p+h)} = -\sum_{n>0}\frac{1}{n}\frac{q^{nh}}{1-q^n}, \tag{5.15}$$

which reproduces (4.52) in neutral case $\kappa = 0$.

Charged case For a charged scalar field coupled to a constant gauge field in τ direction the discussion is almost unchanges. If $\partial_\tau \to \partial_\tau - \kappa$ in the wave equation, this means that Ω enters the previous calculation with the replacement $\Omega \to \Omega - \kappa$. Normal mode frequencies are then

$$\Omega_p = \kappa \pm (h+p), \quad p = 0,1,2,\ldots. \tag{5.16}$$

Thus,

$$\begin{aligned}&\prod_{n>0,I}(i\omega_n(\beta)-\Omega_I(h))(-i\omega_n(\beta)-\Omega_I(h))\\ &=\prod_p[\tfrac{2\pi i n}{\beta}-(h+\kappa+p)][-\tfrac{2\pi i n}{\beta}-(h+\kappa+p)][\tfrac{2\pi i n}{\beta}+h-\kappa+p][-\tfrac{2\pi i n}{\beta}+h-\kappa+p],\end{aligned} \tag{5.17}$$

[19]Euclidean AdS_2 (with unit radius) is the one sheet $X_0 \geqslant 1$ of $X_0^2 - X_E^2 - X_1^2 = 1$ covered once by coordinates $(X_0, X_1, X_E) = (\sec\sigma\cosh\tau, \tan\sigma, \sec\sigma\sinh\tau)$ with $\sigma \in (-\pi/2, \pi/2)$ and $\tau \in \mathbb{R}$. The induced metric is $ds^2 = -dX_0^2 + dX_E^2 + dX_1^2 = \frac{1}{\cos^2\sigma}(d\tau^2 + d\sigma^2)$, with two boundaries at $\sigma = \pm\frac{\pi}{2}$. With the change of variable $\sinh\rho = \tan\sigma$ the metric becomes $ds^2 = \cosh^2\rho\, d\tau^2 + d\rho^2$ and we should take here $\rho \in (-\infty, \infty)$, see for instance [74]. This is in contrast with higher dimensional AdS where $X_1 \to X_i = R\sinh\rho\, \Omega_i$ with $\sum_i \Omega_i^2 = 1$ and sign of X_i can be any while $\rho \in \mathbb{R}^+$ is radius.

[20]We absorbing in $P(h)$ various divergent factors. These may be treated carefully, but the aim of this derivation is to show how the non-Casimir part of (4.52) emerges. The remaining Casimir contribution can be quickly fixed later as explained in appendix B.

and this gives the non-Casimir part of (4.52) for a full complex field (two real components)

$$\log\det(-\nabla^2+m^2)' = -2\times\frac{1}{2}\sum_{n>0}\frac{1}{n}\frac{q^{n(h+\kappa)}+q^{n(h-\kappa)}}{1-q^n}. \tag{5.18}$$

Neumann/Dirichlet boundary conditions Now, suppose we keep in (5.7) only modes with even/odd p, corresponding to Neumann/Dirichlet conditions at $\rho=0$. This is a restriction on p summation in e.g. (5.15) and for even/odd p we get

$$\begin{aligned}\text{even } p: \sum_{k\geqslant 0}\sum_{n>0}\frac{1}{n}q^{n(2k+h)} &= -\sum_{n>0}\frac{1}{n}\frac{q^{nh}}{1-q^{2n}},\\ \text{odd } p: \sum_{k\geqslant 0}\sum_{n>0}\frac{1}{n}q^{n(2k+1+h)} &= -\sum_{n>0}\frac{1}{n}\frac{q^{n(h+1)}}{1-q^{2n}}.\end{aligned} \tag{5.19}$$

In conclusion, we can split (4.52) into contributions from fields with Neumann/Dirichlet boundary conditions as

$$\begin{aligned}\Gamma_{\rm N}^{(h,\kappa)} &= \beta E_{c,{\rm N}}^{(h,\kappa)} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{nh}(q^{n\kappa}+q^{-n\kappa})}{1-q^{2n}},\\ \Gamma_{\rm D}^{(h,\kappa)} &= \beta E_{c,{\rm D}}^{(h,\kappa)} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n(h+1)}(q^{n\kappa}+q^{-n\kappa})}{1-q^{2n}}.\end{aligned} \tag{5.20}$$

The Casimir energies can be computed as before and (4.57) splits into

$$E_{c,{\rm N}}^{(h,\kappa)} = \frac{1}{24}(-2+6h-3h^2-3\kappa^2), \qquad E_{c,{\rm D}}^{(h,\kappa)} = \frac{1}{24}(1-3h^2-3\kappa^2). \tag{5.21}$$

In the following, for charged scalars it will be convenient to separate the two contributions with $\pm\kappa$ in (5.18) and write (hat denoting one sign of κ)

$$\widehat{\Gamma}_{\rm N}^{(h,\kappa)} = \beta\widehat{E}_{c,{\rm N}}^{(h,\kappa)} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n(h+\kappa)}}{1-q^{2n}}, \qquad \widehat{\Gamma}_{\rm D}^{(h,\kappa)} = \beta\widehat{E}_{c,{\rm D}}^{(h,\kappa)} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n(h+1+\kappa)}}{1-q^{2n}}, \tag{5.22}$$

and

$$\widehat{E}_{c,{\rm N}}^{(h,\kappa)} = \frac{1}{48}(-2+6h-3h^2+6\kappa(1-h)-3\kappa^2), \qquad \widehat{E}_{c,{\rm D}}^{(h,\kappa)} = \frac{1}{48}(1-3h^2-6\kappa\, h-3\kappa^2). \tag{5.23}$$

5.1.2 Spin-$\frac{1}{2}$ field

Let us now consider a spin-$\frac{1}{2}$ fermion in our AdS_2 induced metric (4.12). Zero modes are solutions of $K\theta=0$ where (τ_i are Pauli matrices)

$$K = \tau_1\partial_\tau + \tau_3(\cosh\rho\,\partial_1 + \tfrac{1}{2}\sinh\rho) - iM\tau_2\cosh\rho. \tag{5.24}$$

We may redefine

$$\theta \equiv \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = (\cosh\rho)^{-1/2}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{5.25}$$

to get

$$K = \frac{1}{\cosh\rho}\tau_1\partial_\tau + \tau_3\partial_1 - iM\tau_2. \tag{5.26}$$

Let us find the normal mode frequencies by considering

$$\theta(\rho,\tau) = \theta(\rho)\, e^{\Omega\tau}, \tag{5.27}$$

and solving

$$\left(\frac{\Omega}{\cosh\rho}\tau_1 + \tau_3\partial_1 - iM\tau_2\right)\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0. \tag{5.28}$$

$$\frac{\Omega}{\cosh\rho}\chi_2 + \chi_1' - M\chi_2 = 0, \qquad \frac{\Omega}{\cosh\rho}\chi_1 - \chi_2' + M\chi_1 = 0. \tag{5.29}$$

Let us set $M = h - \frac{1}{2}$ and

$$\begin{pmatrix}\chi_1(\rho)\\ \chi_2(\rho)\end{pmatrix} = U\begin{pmatrix}A(\rho)\\ B(\rho)\end{pmatrix}, \qquad U = \begin{pmatrix}i & 1\\ -i & 1\end{pmatrix}. \tag{5.30}$$

The two equations (5.29) take the slightly more decoupled form

$$A' + \left(h - \frac{1}{2}\right)A - i\frac{\Omega}{\cosh\rho}B = 0, \qquad -B' + \left(h - \frac{1}{2}\right)B + i\frac{\Omega}{\cosh\rho}A = 0. \tag{5.31}$$

In particular, we have

$$B = -i\frac{\cosh\rho}{\Omega}\left(A' + (h - \tfrac{1}{2})A\right), \tag{5.32}$$

and the second order differential equation for $A(\rho)$

$$A'' + \tanh\rho A' + \left[\frac{\Omega^2}{\cosh^2\rho} - (h - \tfrac{1}{2})(h - \tfrac{1}{2} - \tanh\rho)\right]A = 0. \tag{5.33}$$

The solution corresponding to $\theta(\rho)$ vanishing at $\rho\to\infty$ as $e^{-\rho h}$ is — using (5.32) and up to an overall normalization constant —

$$\begin{aligned} A(\rho) &= \frac{(1-\tanh\rho)^{h/2}(1+\tanh\rho)^{\frac{1-h}{2}}}{(1-\tanh^2\rho)^{1/4}}\, {}_2F_1(-\Omega, \Omega, h, \tfrac{1-\tanh\rho}{2}),\\ B(\rho) &= -\frac{i\Omega}{2h}\frac{(1-\tanh\rho)^{-1+h/2}(1+\tanh\rho)^{-\frac{1}{2}-\frac{h}{2}}}{(\cosh\rho)^{5/2}}\, {}_2F_1(1-\Omega, 1+\Omega, 1+h, \tfrac{1-\tanh\rho}{2}). \end{aligned} \tag{5.34}$$

The explicit form of $\theta(\rho)$ is then

$$\theta(\rho) = \frac{i}{\sqrt{\cosh\rho}}\, U\begin{pmatrix}A(\rho)\\ B(\rho)\end{pmatrix}, \tag{5.35}$$

and one can check that it is real. Besides, at small ρ we find for $h = \frac{3}{2}$ the expansions

$$\begin{aligned} \theta_1(\rho) &= -\frac{1}{1+2\Omega}\left(\cos\frac{\pi\Omega}{2} + \sin\frac{\pi\Omega}{2}\right) + \frac{\Omega-1}{2\Omega-1}\left(\cos\frac{\pi\Omega}{2} - \sin\frac{\pi\Omega}{2}\right)\rho + \mathcal{O}(\rho^2),\\ \theta_2(\rho) &= \frac{1}{1-2\Omega}\left(\cos\frac{\pi\Omega}{2} - \sin\frac{\pi\Omega}{2}\right) - \frac{\Omega+1}{2\Omega+1}\left(\cos\frac{\pi\Omega}{2} + \sin\frac{\pi\Omega}{2}\right)\rho + \mathcal{O}(\rho^2). \end{aligned} \tag{5.36}$$

Fluctuation	h	SU(2)	R	$\mathrm{SO}(3)_R$	mode	
φ_-	$n-1$	$\mathbf{1}$	-1	$\mathbf{1}$	$n \geqslant 2$	$n-2$
ψ_-	$n-\frac{1}{2}$	$\mathbf{2}$	$-\frac{1}{2}$	$\mathbf{2}$	$n \geqslant 2$	$n-2$
φ_0	n	$\mathbf{1}$	0	$\mathbf{3}$	$n \geqslant 2$	$n-1$
ϕ	n	$\mathbf{3}$	0	$\mathbf{1}$	$n \geqslant 1$	$n-2$
ψ_+	$n+\frac{1}{2}$	$\mathbf{2}$	$+\frac{1}{2}$	$\mathbf{2}$	$n \geqslant 1$	$n-1$
φ_+	$n+1$	$\mathbf{1}$	$+1$	$\mathbf{1}$	$n \geqslant 0$	n

Table 2. Fluctuation modes in small superconformal representation in the presence of the D3 brane.

We get even/odd components according to

$$\begin{aligned} &\theta_1(0) \neq 0 \text{ and } \theta_2(0) = 0 \quad : \ \Omega \bmod 4 = \tfrac{1}{2}, -\tfrac{3}{2}, \quad \Omega \neq \tfrac{1}{2}, \\ &\theta_1(0) = 0 \text{ and } \theta_2(0) \neq 0 \quad : \ \Omega \bmod 4 = -\tfrac{1}{2}, \tfrac{3}{2}, \quad \Omega \neq -\tfrac{1}{2}. \end{aligned} \tag{5.37}$$

These values correspond to the same formula we found in bosonic case, i.e. $\Omega = \pm(h+p)$, $p = 0, 1, 2, \ldots$ where here $h = \frac{3}{2}$. The first line in (5.37) corresponds to odd modes $p \in 2\mathbb{N}+1$, while the second line is for even modes $p \in 2\mathbb{N}$. If we refer to the parity of θ_1, we can say that even modes are of Dirichlet type, while odd modes are of Neumann type. The second component θ_2 has opposite behaviour.

5.2 Boundary conditions on D3 giant and supersymmetry

Let us now use supersymmetry to consistently pair the parities of bosonic and fermionic modes. Using notation of [12] for worldsheet fluctuations, the states in short multiplet of OSp(4*|4) in (3.7) are denoted

$$\varphi(1, \mathbf{1}, \mathbf{5}) + \psi(\tfrac{3}{2}, \mathbf{2}, \mathbf{4}) + \phi(2, \mathbf{3}, \mathbf{1}). \tag{5.38}$$

In our notation, bosonic fluctuation are

$$S^5 : \varphi \equiv (A_2, W_+, \overline{W}_+, W_-, \overline{W}_-), \qquad AdS_5 : \phi \equiv (Y_1, Z, \bar{Z}). \tag{5.39}$$

In the presence of the D3 brane, the R-symmetry reduces to $\mathrm{SO}(5)_R \to \mathrm{SO}(2)_R \times \mathrm{SO}(3)_R$ and the five S^5 fluctuations split into $3+1+1$ scalars $\varphi = (\varphi_0, \varphi_+, \varphi_-)$ transforming respectively in the $\mathbf{3}, \mathbf{1}, \mathbf{1}$ of $\mathrm{SO}(3)_R$. Under the supersymmetry preserved by the D3 brane the superconformal representation (5.38) splits into small representations summarized in table 2[21] The structure of the small superconformal representation is discussed in full details in appendix C and reads

$$\varphi_- \xrightarrow{\ Q\ } \psi_- \begin{array}{c} \overset{Q}{\nearrow} \\ \underset{Q}{\searrow} \end{array} \begin{array}{c} B^+\varphi_0 \\ \\ \phi \end{array} \begin{array}{c} \overset{Q}{\searrow} \\ \underset{Q}{\nearrow} \end{array} B^+\psi_+ \xrightarrow{\ Q\ } (B^+)^2\varphi_+ \tag{5.40}$$

where Q are the supercharges preserved by the D3 brane and B^+ is the raising operator

[21]This same as table 1 in [12] adapted to our notation.

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of $SL(2,\mathbb{R})$ generating conformal descendants. Each application of B^+ shifts forward the mode number. The mode of each state is $h - h_0$ where h_0 are the dilatation eigenvalues in (5.38), so $h_0 = 1$ for $\varphi_0, \varphi_\pm$, $h_0 = \frac{3}{2}$ for $\psi_\pm$, and $h_0 = 2$ for ϕ. Bosonic boundary conditions on the D3 brane are

$$\varphi_0 \ : \ \text{Neumann}, \qquad \varphi_\pm, \phi \ : \ \text{Dirichlet}, \tag{5.41}$$

corresponding to three S^5 fluctuations longitudinal to the D3 brane, while all other bosonic fluctuations (two in S^5 and three in AdS_5) are transverse. As already noted in [12] this requires n to be odd in the full multiplet in table 2.

Fermionic fluctuations ψ_- have then odd mode number and are of Neumann type according to the discussion after (5.37). Instead, fluctuations ψ_+ have even mode number and are thus of Dirichlet type. Let us now consider the fermion determinant computed restricting θ_1 to have Neumann b.c. in the presence of the constant gauge field along τ direction. The product in (5.6) reads (recall exclusion of $\Omega = \pm 1/2$, denoted with a prime)

$$\begin{aligned}
&\prod_{n\in\mathbb{Z},p\in\mathbb{Z}} [\tfrac{2\pi i n}{\beta} - (\tfrac{1}{2} + 4p + \kappa)]' [\tfrac{2\pi i n}{\beta} - (-\tfrac{3}{2} + 4p + \kappa)] \\
&\prod_{n\in\mathbb{Z},p\geqslant 0} [\tfrac{2\pi i n}{\beta} - (\tfrac{3}{2} + 4p + 3 + \kappa)][\tfrac{2\pi i n}{\beta} + (\tfrac{3}{2} + 4p + 2 - \kappa)][\tfrac{2\pi i n}{\beta} - (\tfrac{3}{2} + 4p + 1 + \kappa)][\tfrac{2\pi i n}{\beta} + (\tfrac{3}{2} + 4p - \kappa)] \\
&= \prod_{n\in\mathbb{Z},p\geqslant 0} [\tfrac{2\pi i n}{\beta} - (\tfrac{3}{2} + 2p + 1 + \kappa)][\tfrac{2\pi i n}{\beta} + (\tfrac{3}{2} + 2p - \kappa)].
\end{aligned} \tag{5.42}$$

Thus, we have same expression as the $+\kappa$ contribution for a half-boson (with $h = \frac{3}{2}$) with Dirichlet boundary conditions plus the $-\kappa$ contribution for a half-boson with Neumann boundary condition. The corresponding total contributions is

$$\widehat{\Gamma}_{\rm D}^{(h,\kappa)} + \widehat{\Gamma}_{\rm N}^{(h,-\kappa)}. \tag{5.43}$$

Similarly, we can consider the fermion determinant restricting θ_1 to have Dirichlet boundary condition. In this case we need the product

$$\begin{aligned}
&\prod_{n\in\mathbb{Z},p\in\mathbb{Z}} [\tfrac{2\pi i n}{\beta} - (-\tfrac{1}{2} + 4p + \kappa)]' [\tfrac{2\pi i n}{\beta} - (\tfrac{3}{2} + 4p + \kappa)] \\
&= \prod_{n\in\mathbb{Z},p\geqslant 0} [\tfrac{2\pi i n}{\beta} - (\tfrac{3}{2} + 2p + \kappa)][\tfrac{2\pi i n}{\beta} + (\tfrac{3}{2} + 2p + 1 - \kappa)],
\end{aligned} \tag{5.44}$$

that gives

$$\widehat{\Gamma}_{\rm N}^{(h,\kappa)} + \widehat{\Gamma}_{\rm D}^{(h,-\kappa)}. \tag{5.45}$$

To summarize, the structure of the small representation in table 2 implies that fermions will contribute by the sum of the two combinations in (5.43) and (5.45).

5.3 Total one-loop contribution in the presence of the D3 brane

The previous expression (4.58) valid in the absence of the D3 brane may be written using for fermionic contributions depending on α_2 the split expressions (5.22)

$$\begin{aligned}
\Gamma_{\rm one-loop} = &\underbrace{\Gamma^{(2,0)} + 2\Gamma^{(2,1)}}_{AdS_5} + \underbrace{\Gamma^{(1,0)} + 4\Gamma^{(1,\alpha_2)}}_{S^5} \\
&- 2\left[\widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2}+\alpha_2)} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2}-\alpha_2)} + \widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2}-\alpha_2)} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2}+\alpha_2)} + 2\widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2})} + 2\widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2})}\right],
\end{aligned} \tag{5.46}$$

where we recall that for the purpose of counting states $\widehat{\Gamma}$ gives 1/2 while Γ gives 1 so we have $8_B + 8_F$ contributions.

We now compute (5.46) by using Dirichlet boundary conditions for the 3 AdS scalar fluctuations and for 2 of the S^5 ones. The remaining 3 fluctuations in S^5 have Neumann boundary condition. For fermions, we don't have a natural $AdS_5 + S^5$ splitting, but our previous analysis of normal modes suggests the following modification of (5.46)

$$\begin{aligned}\Gamma^{\rm D3}_{\rm one-loop} = {}& \underbrace{\Gamma^{(2,0)}_{\rm D} + 2\Gamma^{(2,1)}_{\rm D}}_{AdS_5} + \underbrace{\Gamma^{(1,0)}_{\rm N} + 2\Gamma^{(1,\alpha_2)}_{\rm N} + 2\Gamma^{(1,\alpha_2)}_{\rm D}}_{S^5} \\ & - 2\Big[\widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2}+\alpha_2)}_{\rm N} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2}-\alpha_2)}_{\rm D} + \widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2}-\alpha_2)}_{\rm D} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2}+\alpha_2)}_{\rm N} \\ & + \widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2})}_{\rm D} + \widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2})}_{\rm N} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2})}_{\rm D} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2})}_{\rm N}\Big], \end{aligned} \tag{5.47}$$

where the fermionic contribution is a sum of terms of the form (5.43) and (5.45). One readily checks that (5.47) gives indeed (5.2) using (5.20) and (5.21). In fact,

$$\Gamma^{(2,0)}_{\rm D} = -\frac{q^3}{1-q^2}, \qquad \Gamma^{(2,1)}_{\rm D} = -\frac{q^2(1+q^2)}{2(1-q^2)}, \qquad \Gamma^{(1,0)}_{\rm N} = -\frac{q}{1-q^2}, \tag{5.48}$$

$$\Gamma^{(1,\alpha_2)}_{\rm N} = -\frac{q}{2(1-q^2)}(\eta+\eta^{-1}), \qquad \Gamma^{(1,\alpha_2)}_{\rm D} = -\frac{q^2}{2(1-q^2)}(\eta+\eta^{-1}), \tag{5.49}$$

$$\widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2}+\alpha_2)}_{\rm N} = -\frac{q^2}{2(1-q^2)}\,\eta, \qquad \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2}-\alpha_2)}_{\rm D} = -\frac{q^2}{2(1-q^2)}\,\eta^{-1}, \tag{5.50}$$

$$\widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2}-\alpha_2)}_{\rm D} = -\frac{q^3}{2(1-q^2)}\,\eta^{-1}, \qquad \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2}+\alpha_2)}_{\rm N} = -\frac{q}{2(1-q^2)}\,\eta, \tag{5.51}$$

$$\widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2})}_{\rm D} + \widehat{\Gamma}^{(\frac{3}{2},\frac{1}{2})}_{\rm N} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2})}_{\rm D} + \widehat{\Gamma}^{(\frac{3}{2},-\frac{1}{2})}_{\rm N} = -\frac{q(1+q)^2}{2(1-q^2)}, \tag{5.52}$$

and total is

$$\begin{aligned}\Gamma^{\rm D3}_{\rm 1-loop} = {}& -\frac{1}{1-q^2}[q^3 + q^2(1+q^2) + q + q(\eta+\eta^{-1}) + q^2(\eta+\eta^{-1}) \\ & - q^2\eta - q^2\eta^{-1} - q^3\eta^{-1} - q\eta - q(1+q)^2] = q^2 - q\,\eta^{-1}. \end{aligned} \tag{5.53}$$

The separate and total Casimir energies are

$$E^{(2,0)}_{c,\rm D} = -\tfrac{11}{24}, \qquad E^{(2,1)}_{c,\rm D} = -\tfrac{7}{12}, \qquad E^{(1,0)}_{c,\rm N} = \tfrac{1}{24}, \tag{5.54}$$

$$E^{(1,\alpha_2)}_{c,\rm N} = \tfrac{1}{24} - \tfrac{1}{8}\alpha_2^2, \qquad E^{(1,\alpha_2)}_{c,\rm D} = -\tfrac{1}{12} - \tfrac{1}{8}\alpha_2^2, \tag{5.55}$$

$$\widehat{E}^{(\frac{3}{2},\frac{1}{2}+\alpha_2)}_{c,\rm N} = -\tfrac{1}{24} - \tfrac{1}{8}\alpha_2 - \tfrac{1}{16}\alpha_2^2, \qquad \widehat{E}^{(\frac{3}{2},-\frac{1}{2}-\alpha_2)}_{c,\rm D} = -\tfrac{1}{24} + \tfrac{1}{8}\alpha_2 - \tfrac{1}{16}\alpha_2^2, \tag{5.56}$$

$$\widehat{E}^{(\frac{3}{2},\frac{1}{2}-\alpha_2)}_{c,\rm D} = -\tfrac{11}{48} + \tfrac{1}{4}\alpha_2 - \tfrac{1}{16}\alpha_2^2 \qquad \widehat{E}^{(\frac{3}{2},-\frac{1}{2}+\alpha_2)}_{c,\rm N} = \tfrac{1}{48} - \tfrac{1}{16}\alpha_2^2, \tag{5.57}$$

$$\widehat{E}^{(\frac{3}{2},\frac{1}{2})}_{c,\rm D} + \widehat{E}^{(\frac{3}{2},\frac{1}{2})}_{c,\rm N} + \widehat{E}^{(\frac{3}{2},-\frac{1}{2})}_{c,\rm D} + \widehat{E}^{(\frac{3}{2},-\frac{1}{2})}_{c,\rm N} = -\tfrac{7}{24}, \tag{5.58}$$

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and

$$\begin{aligned}E^{\rm D3}_{c,1-{\rm loop}} &= -\tfrac{11}{24} - 2\times\tfrac{7}{12} + \tfrac{1}{24} + 2\times(\tfrac{1}{24} - \tfrac{1}{8}\alpha_2^2) + 2\times(-\tfrac{1}{12} - \tfrac{1}{8}\alpha_2^2)\\ &\quad -2\Big[-\tfrac{1}{24} - \tfrac{1}{8}\alpha_2 - \tfrac{1}{16}\alpha_2^2 - \tfrac{1}{24} + \tfrac{1}{8}\alpha_2 - \tfrac{1}{16}\alpha_2^2 - \tfrac{11}{48} + \tfrac{1}{4}\alpha_2 - \tfrac{1}{16}\alpha_2^2 + \tfrac{1}{48} - \tfrac{1}{16}\alpha_2^2 - \tfrac{7}{24}\Big]\\ &= -\tfrac{1}{2} - \tfrac{1}{2}\alpha_2. \end{aligned} \tag{5.59}$$

Quite remarkably, in the presence of the D3 brane, the total value of the supersymmetric Casimir energy E_c is not zero but provides instead the linear in β term required in (5.2). It is thus responsible for the puzzling factor $1/(\eta q)$ in (1.2) and (2.9), first discovered in [12].

Acknowledgments

I thank Arkady Tseytlin for many valuable discussions and clarifications, and Alejandro Cabo-Bizet for useful comments. Financial support from the INFN grant GAST is acknowledged.

A Thermal one-loop scalar determinant in AdS_2

Let us focus on the non-Casimir part of the thermal scalar determinant in AdS_2. The effective action before thermal quotient is [75].

$$\log\det(-\nabla^2 + m^2) = -\int_{s_{\rm UV}}^{\infty}\frac{ds}{s}\int d^2\xi\,\sqrt{g}\,K(\xi,\xi;s), \qquad \mathrm{vol}(AdS_2) = -2\pi, \tag{A.1}$$

where K is the heat kernel. Let $D(\xi,\xi')$ be the geodesic distance. In Poincare' patch $\xi = (x,y)$ we have

$$ds^2 = \frac{dx^2+dy^2}{y^2}, \qquad \sqrt{g} = \frac{1}{y^2}, \qquad \cosh D(\xi,\xi') = 1 + \frac{(y-y')^2+(x-x')^2}{2yy'}. \tag{A.2}$$

The heat kernel for a massive scalar field is [76]

$$K(\xi,\xi';s) = K(D(\xi,\xi');s) = \frac{\sqrt{2}\,e^{-(m^2+\frac{1}{4})s}}{(4\pi s)^{3/2}}\int_{D(\xi,\xi')}^{\infty}\frac{du\,u\,e^{-\frac{u^2}{4s}}}{\sqrt{\cosh u - \cosh D(\xi,\xi')}}. \tag{A.3}$$

In thermal AdS_2 we follow [77] and consider the group action (we don't consider an angular twist)

$$\gamma\cdot\xi = (q^{-1}x, q^{-1}y). \tag{A.4}$$

According to the method of images, we need

$$\begin{aligned}\widetilde{K}(\xi,\xi;s) &= \sum_{n\in\mathbb{Z}} K(D(\xi,\gamma^n\cdot\xi);s),\\ \cosh D(\xi,\gamma^{-n}\cdot\xi) &= 1 + \frac{(x^2+y^2)(1-q^n)^2}{2y^2q^n}\end{aligned} \tag{A.5}$$

Let us introduce on the quotient $AdS_2/\mathbb{Z}$ polar coordinates on the fundamental domain (see for instance [77] for AdS_3 case)

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad r\in[1,e^{\beta}], \quad \phi\in[0,\pi], \qquad \sqrt{g} = \frac{1}{r\sin^2\phi} \tag{A.6}$$

$$\cosh D(r,\phi) = 1 + \frac{(1-q^n)^2}{2q^n}\frac{1}{\sin^2\phi}. \tag{A.7}$$

We need to evaluate

$$\int \frac{dr d\phi}{r\sin^2\phi}\int_{D(r,\phi)}^{\infty}\frac{du\, u\, e^{-\frac{u^2}{4s}}}{\sqrt{\cosh u-\cosh D(r,\phi)}} = 2\int_1^{q^{-1}}\frac{dr}{r}\int_0^{\pi/2}\frac{d\phi}{\sin^2\phi}\int_{D(r,\phi)}^{\infty}\frac{du\, u\, e^{-\frac{u^2}{4s}}}{\sqrt{\cosh u-\cosh D(r,\phi)}} \tag{A.8}$$

When $\phi \in (0,\pi/2)$ we have

$$\begin{aligned} &D \geqslant \operatorname{arccosh}\left[1+\frac{(1-q^n)^2}{2q^n}\right] = n\beta, \\ &\frac{d\phi}{\sin^2\phi} = \frac{1}{\sqrt{2}q^{n/2}(-1+q^{-n})}\frac{\sinh D}{\sqrt{\cosh D-\cosh(n\beta)}}dD. \end{aligned} \tag{A.9}$$

By exchanging order of integration we obtain

$$\begin{aligned} &\int_{n\beta}^{\infty} dD\frac{\sinh D}{\sqrt{\cosh D-\cosh(n\beta)}}\int_D^{\infty} du\frac{u\, e^{-\frac{u^2}{4s}}}{\sqrt{\cosh u-\cosh D}} \\ &= \int_{n\beta}^{\infty} du\, u\, e^{-\frac{u^2}{4s}}\int_{n\beta}^{u} dD\frac{\sinh D}{\sqrt{\cosh D-\cosh(n\beta)}\sqrt{\cosh u-\cosh D}} \\ &= \pi\int_{n\beta}^{\infty} du u\, e^{-\frac{u^2}{4s}} = 2\pi s e^{-\frac{n^2\beta^2}{4s}}. \end{aligned} \tag{A.10}$$

So, considering a fixed value of $n \neq 0$ (one then sums over n treating separately $n=0$, see [77])

$$\begin{aligned} \log\det(-\nabla^2+m^2)\Big|_n &= -2\int_1^{q^{-1}}\frac{dr}{r}\int_0^{\infty}\frac{ds}{s}\frac{\sqrt{2}\, e^{-(m^2+\frac{1}{4})s}}{(4\pi s)^{3/2}}\frac{1}{\sqrt{2}q^{n/2}(-1+q^{-n})}2\pi s e^{-\frac{n^2\beta^2}{4s}} \\ &= -\frac{q^{n/2}e^{-n\beta\sqrt{\frac{1}{4}+m^2}}}{n(1-q^n)} = -\frac{1}{n}\frac{q^{nh}}{1-q^n}, \qquad h = \frac{1}{2}+\sqrt{\frac{1}{4}+m^2}, \end{aligned} \tag{A.11}$$

which confirms (4.52) in $\kappa=0$ case by a direct calculation.

B Casimir energy and $\beta \ll 1$ expansion of partition function

Given the single particle partition function $Z_{\text{s.p.}}(\beta)$,[22] we can determine the Casimir contribution from (4.54)

$$E_c = \frac{1}{2}\zeta_E(-1), \qquad \zeta_E(z) = \frac{1}{\Gamma(z)}\int_0^{\infty} d\beta\beta^{z-1}Z_{\text{s.p.}}(\beta). \tag{B.1}$$

This is a Mellin transform with inverse formula

$$Z_{\text{s.p.}}(\beta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, \beta^{-z}\,\Gamma(z)\zeta_E(z), \tag{B.2}$$

where c in the analyticity strip of $\zeta_E(z)$. As shown in [78] one can consider the full partition function

$$Z(\beta) = \text{PE}[Z_{\text{s.p.}}] = \exp\sum_{n=1}^{\infty}\frac{1}{n}Z_{\text{s.p.}}(n\beta), \tag{B.3}$$

[22]The present discussion holds for the standard Casimir energy and also for the supersymmetric Casimir energy if $Z_{\text{s.p.}}$ is the single particle superconformal index.

and (B.2) gives the following integral representation

$$\log Z(\beta) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz\, (n\beta)^{-z}\, \Gamma(z)\zeta_E(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \beta^{-z}\, \zeta(z+1)\Gamma(z)\zeta_E(z). \tag{B.4}$$

From the pole at $z = -1$ we find that the Laurent expansion of $\log Z(\beta)$ at small β has then a linear in β term of the form

$$\log Z(\beta) = \text{singular terms} + \#\beta^0 + E_c\beta + \cdots . \tag{B.5}$$

Actually, E_c can be extracted in an easier way by noting that $Z_{\text{s.p.}}(\beta)$ has also a Laurent expansion around $\beta = 0$ and the coefficient of the linear in β term should be $-2E_c$. In fact, one has then

$$\log Z(\beta) = \sum_{n=1}^{\infty} \frac{1}{n} Z_{\text{s.p.}}(n\beta) = \cdots + \zeta(0)(-2E_c)\beta + \cdots , \tag{B.6}$$

in agreement with (B.5). We have thus the simple relation

$$E_c = -\frac{1}{2} Z_{\text{s.p.}}(\beta)\Big|_{\beta \text{ term}} , \tag{B.7}$$

which is quite efficient for calculations. For instance, for a scalar field in AdS_2

$$Z_{\text{s.p.}}(\beta) = \frac{e^{-\beta h}}{1-e^{-\beta}} = \frac{1}{\beta} + (\tfrac{1}{2} - h) + \tfrac{1}{12}(1 - 6h + 6h^2)\beta + \cdots , \tag{B.8}$$

consistently with (4.57). As another example, for a conformally coupled scalar on $S^1 \times S^3$ one has

$$Z_{\text{s.p.}}(\beta) = \frac{e^{-\beta}(1-e^{-2\beta})}{(1-e^{-\beta})^4} = \frac{2}{\beta^3} - \frac{1}{120}\beta + \cdots , \tag{B.9}$$

reproducing the known value $E_c = \frac{1}{240}$.

C OSp(4*|4) ultra-short multiplet in the presence of the D3 brane

Unitary representations of OSp($2n^*|2m$) were discussed in [79] using an oscillator representation. The specific case OSp(4*|4) was summarized in [60]. Here we present in full details the case of its ultra-short representation and its reduction to the smaller representation in table 2 in the presence of the D3 giant.

C.1 Structure of OSp(4*|4) and oscillator representation

The group OSp(4*|4) ⊃ SO(4*) × USp(4) has a Jordan structure with respect to the maximal subgroup G^0 = U(2|2) ⊃ U(2) × U(2). At Lie superalgebra level

$$\mathfrak{osp}(4^*|4) = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1}, \qquad [\mathfrak{g}^a, \mathfrak{g}^b] \subseteq \mathfrak{g}^{a+b}, \qquad \mathfrak{g}^a = 0 \text{ for } |a| > 1. \tag{C.1}$$

Generators are split into

$$M_{AB} \in \mathfrak{g}^{-1}, \qquad M^A{}_B \in \mathfrak{g}^0, \qquad M^{AB} \in \mathfrak{g}^{+1}, \tag{C.2}$$

where the index $A = (i, \mu)$ with $i = 1, 2$, $\mu = 1, 2$ is in the fundamental of U(2|2) and we assign $\deg i = 0$, $\deg \mu = 1$. The Lie superalgebra is realized by introducing a pair of super-oscillators[23]

$$\xi_A = \begin{pmatrix} a_i \\ \alpha_\mu \end{pmatrix}, \qquad \eta_A = \begin{pmatrix} b_i \\ \beta_\mu \end{pmatrix}, \tag{C.3}$$

where the non-vanishing (anti) commutators are

$$[a_i, a^j] = [b_i, b^j] = \delta_i^j, \qquad \{\alpha_\mu, \alpha^\nu\} = \{\beta_\mu, \beta^\nu\} = \delta_\mu^\nu. \tag{C.4}$$

Generators in (C.2) can be written

$$M_{AB} = \xi_A \eta_B - \eta_A \xi_B, \quad M^{AB} = \eta^B \xi^A - \xi^B \eta^A, \quad M^A{}_B = \xi^A \xi_B + (-1)^{\deg A \deg B} \eta_B \eta^A. \tag{C.5}$$

The bosonic bilinears

$$M_{ij} = a_i b_j - a_j b_i, \quad M^{ij} = a^i b^j - a^j b^i, \quad M^i{}_j = a^i a_j + b_j b^i, \tag{C.6}$$

generate (at group level) $SO(4^*) = SL(2, \mathbb{R}) \times SU(2)$. In particular,

$$B^- = M_{12}, \qquad B^+ = M^{12}, \qquad B^0 = \frac{1}{2} M^i{}_i = \frac{1}{2}(a^i a_i + b_i b^i), \tag{C.7}$$

are generators of $SL(2, \mathbb{R})$ obeying

$$[B^+, B^-] = 2B^0, \qquad [B^0, B^\pm] = \pm B^\pm. \tag{C.8}$$

The bilinears

$$I^i{}_j = M^i{}_j - \frac{1}{2} \delta^i_j M^k{}_k, \tag{C.9}$$

generate SU(2). The trace condition is $I^2{}_2 = -I^1{}_1$ and one has standard algebra

$$[I^1{}_1, I^1{}_2] = I^1{}_2, \qquad [I^1{}_1, I^2{}_1] = -I^2{}_1, \qquad [I^1{}_2, I^2{}_1] = 2I^1{}_1. \tag{C.10}$$

Bilinears in the fermionic oscillators are[24]

$$\mathsf{M}_{\mu\nu} = \alpha_\mu \beta_\nu - \beta_\mu \alpha_\nu, \qquad \mathsf{M}^{\mu\nu} = \beta^\nu \alpha^\mu - \alpha^\nu \beta^\mu, \qquad \mathsf{M}^\mu{}_\nu = \alpha^\mu \alpha_\nu - \beta_\nu \beta^\mu. \tag{C.11}$$

They are $3 + 3 + 4 = 10$,[25] and generate $USp(4) = SO(5)_R$. Fermionic bilinears are the following $4 \times 4 = 16$ (real) supercharges

$$Q_{i\mu} = a_i \beta_\mu - b_i \alpha_\mu, \qquad Q^{i\mu} = a^i \beta^\mu - b^i \alpha^\mu, \qquad S^i{}_\mu = a^i \alpha_\mu + b^i \beta_\mu, \qquad S^\mu{}_i = a_i \alpha^\mu + b_i \beta^\mu. \tag{C.12}$$

[23] The general case requires several copies of these two super-oscillators [79], but for the discussion of the ultra-short representations a single pair is enough.

[24] We use a different font to make it clear whether a certain operator with explicit indices has them of Latin or Greek type.

[25] Oscillators are fermionic so for instance $\mathsf{M}_{11} \neq 0$.

C.2 The ultra short multiplet

The shortened multiplet in (3.7) contains only spin 0 and spin $\frac{1}{2}$ states. It is built starting from the oscillator vacuum $|0\rangle$ annihilated by $a_i, b_i, \alpha_\mu, \beta_\mu$. With respect to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$ it obeys

$$B^0|0\rangle = |0\rangle, \qquad I^i{}_j|0\rangle = 0. \tag{C.13}$$

Acting on it with the generators of SO(5) in (C.11) we get the additional $3+1$ states

$$(\alpha^\mu\beta^\nu + \alpha^\nu\beta^\mu)|0\rangle, \qquad \alpha^2\alpha^1\beta^2\beta^1|0\rangle, \tag{C.14}$$

that together with $|0\rangle$ give the **5** of $\mathrm{SO}(5)_R$. Thus we have a first lowest weight states in the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{SO}(5)$ representation, cf. (5.38),

$$\varphi(1,\mathbf{1},\mathbf{5}). \tag{C.15}$$

Acting with the supercharges $Q^{i\mu}$ on $|0\rangle$ we get 4 non vanishing states with dilatation eigenvalue $h=\frac{3}{2}$. Taking their orbit with SU(2) and $\mathrm{SO}(5)_R$ generators gives 8 states. These have $(J^1{}_2)^2$ and definite $J^1{}_1$ eigenvalue equal to $\pm\frac{1}{2}$ and belong to the **2** of SU(2). Let us list their explicit form

$$\begin{array}{rr} & J^1{}_1 \\ (-\alpha^1b^1+\beta^1a^1)\,|0\rangle & +\frac{1}{2} \\ (-\alpha^1b^2+\beta^1a^2)\,|0\rangle & +\frac{1}{2} \\ (-\alpha^2b^1+\beta^2a^1)\,|0\rangle & -\frac{1}{2} \\ (-\alpha^2b^2+\beta^2a^2)\,|0\rangle & -\frac{1}{2} \\ \hline (\alpha^1\beta^2\beta^1a^1+\alpha^2\alpha^1\beta^1b^1)\,|0\rangle & +\frac{1}{2} \\ (\alpha^1\beta^2\beta^1a^2+\alpha^2\alpha^1\beta^1b^2)\,|0\rangle & -\frac{1}{2} \\ (\alpha^2\alpha^1\beta^2b^1+\alpha^2\beta^2\beta^1a^1)\,|0\rangle & +\frac{1}{2} \\ (\alpha^2\alpha^1\beta^2b^2+\alpha^2\beta^2\beta^1a^2)\,|0\rangle & -\frac{1}{2} \end{array} \tag{C.16}$$

These 8 states correspond to the lowest weight states, cf. (5.38),

$$\psi(\tfrac{3}{2},\mathbf{2},\mathbf{4}). \tag{C.17}$$

Next, we consider the states

$$Q^{i\mu}Q^{j\nu}|0\rangle. \tag{C.18}$$

These have $h=2$. Some of them are conformal descendant of $\varphi(1,\mathbf{1},\mathbf{5})$. Additional primary states are the three ones

$$\begin{array}{rr} & J^1{}_1 \\ (\alpha^1\beta^2a^1b^1+\alpha^2\alpha^1b^1b^1-\alpha^2\beta^1a^1b^1+\beta^2\beta^1a^1a^1)\,|0\rangle & 1 \\ (\alpha^1\beta^2a^1b^2+\alpha^2\alpha^1b^2b^1-\alpha^2\beta^1a^2b^1+\beta^2\beta^1a^2a^1)\,|0\rangle & 0 \\ (\alpha^1\beta^2a^2b^2+\alpha^2\alpha^1b^2b^2-\alpha^2\beta^1a^2b^2+\beta^2\beta^1a^2a^2)\,|0\rangle & -1 \end{array} \tag{C.19}$$

and build the last lowest weight representation in (5.38) (one can check that $A^{\mu\nu}$ give zero)

$$\phi(2,\mathbf{3},\mathbf{1}). \tag{C.20}$$

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C.3 Breaking R-symmetry $\mathbf{SO(5)_R \to SO(2)_R \times SO(3)_R}$

The $\mathrm{SO}(5)_R$ generators

$$\mathsf{L}^1 = -\frac{i}{2}(\mathsf{M}^1{}_2 - \mathsf{M}^2{}_1), \qquad \mathsf{L}^2 = -\frac{1}{2}(\mathsf{M}^2{}_1 + \mathsf{M}^1{}_2), \qquad \mathsf{L}^3 = \frac{1}{2}(\mathsf{M}^1{}_1 - \mathsf{M}^2{}_2), \tag{C.21}$$

obey

$$[\mathsf{L}^i, \mathsf{L}^j] = i\varepsilon^{ijk}\,\mathsf{L}^k, \tag{C.22}$$

and generate $\mathrm{SO}(3)_R$. Also

$$[\mathsf{L}^i, \mathsf{R}] = 0, \qquad \mathsf{R} = \frac{1}{2}(\mathsf{M}^1{}_1 + \mathsf{M}^2{}_2), \tag{C.23}$$

and R generates $\mathrm{SO}(2)_R$ (commuting with $\mathrm{SO}(3)_R$).

The state $|0\rangle$ is part of $\mathbf{5}$ of $\mathrm{SO}(5)_R$ and is a singlet of $\mathrm{SO}(3)_R$ with $\mathrm{SO}(2)_R$ quantum number $\mathsf{R} = -1$. It is the scalar φ_- in table 2. Same for the state $\alpha^2\alpha^1\beta^2\beta^1|0\rangle$ but with $\mathrm{SO}(2)_R$ number $\mathsf{R} = +1$. It is the scalar φ_- in table 2. The three states $(\alpha^\mu\beta^\nu + \alpha^\nu\beta^\mu)|0\rangle$ are in the $\mathbf{3}$ of SO(3) and have $\mathrm{SO}(2)_R$ charge $\mathsf{R} = 0$. We denote representations of states in the presence of the D3 brane by four labels

$$(B^0 \equiv h, \dim \mathrm{SU}(2), \mathsf{R}, \dim \mathrm{SO}(3)_R). \tag{C.24}$$

The five scalars corresponding to S^5 fluctuations are then

$$\begin{array}{rl} |0\rangle & \varphi_-(1,\mathbf{1},-1,\mathbf{1}) \\ (\alpha^\mu\beta^\nu + \alpha^\nu\beta^\mu)|0\rangle & \varphi_0(1,\mathbf{1},0,\mathbf{3}) \\ \alpha^2\alpha^1\beta^2\beta^1|0\rangle & \varphi_+(1,\mathbf{1},+1,\mathbf{1}) \end{array} \tag{C.25}$$

Let us look at the 16 supercharges. We have conformal weights

$$\begin{aligned} [B^0, Q^{i\mu}] &= \frac{1}{2}Q^{i\mu}, & [B^0, Q_{i\mu}] &= -\frac{1}{2}Q_{i\mu}, \\ [B^0, S^i{}_\mu] &= \frac{1}{2}S^i{}_\mu, & [B^0, S_i{}^\mu] &= -\frac{1}{2}S_i{}^\mu, \end{aligned} \tag{C.26}$$

and R charges

$$\begin{aligned} [\mathsf{R}, Q^{i\mu}] &= \frac{1}{2}Q^{i\mu}, & [\mathsf{R}, Q_{i\mu}] &= -\frac{1}{2}Q_{i\mu}, \\ [\mathsf{R}, S^i{}_\mu] &= -\frac{1}{2}S^i{}_\mu, & [\mathsf{R}, S_i{}^\mu] &= \frac{1}{2}S_i{}^\mu. \end{aligned} \tag{C.27}$$

The supercharges that are preserved in the presence of the wrapped D3 brane are those with $B^0 = \mathsf{R}$ [18] i.e. $Q^{i\mu}$ and $Q_{i\mu}$. Starting from $|0\rangle$ we consider $Q^{i\mu}|0\rangle$ and get four fermionic states with $\mathrm{SO}(2)_R \times \mathrm{SO}(3)_R$ quantum numbers

$$\begin{array}{lcc} & \mathsf{R} & \mathsf{L}^3 \\ (-\alpha^1 b^1 + \beta^1 a^1)\,|0\rangle & -\frac{1}{2} & +\frac{1}{2} \\ (-\alpha^1 b^2 + \beta^1 a^2)\,|0\rangle & -\frac{1}{2} & -\frac{1}{2} \\ (-\alpha^2 b^1 + \beta^2 a^1)\,|0\rangle & -\frac{1}{2} & +\frac{1}{2} \\ (-\alpha^2 b^2 + \beta^2 a^2)\,|0\rangle & -\frac{1}{2} & -\frac{1}{2} \end{array} \tag{C.28}$$

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This are the four states in second line of table 2

$$\psi_-(\tfrac{3}{2},\mathbf{2},-\tfrac{1}{2},\mathbf{2}), \tag{C.29}$$

and of course are the first four in (C.16) corresponding to $\psi \to (\psi_-,\psi_+)$ under R-symmetry breaking.

The same relation applies to descendants. Consider for instance the application of B^+ to the scalar φ_-. Since B^+ commutes with generators of SU(2) and also with R and generators of SO(3) we have

$$B^+(1,\mathbf{1},-1,\mathbf{1}) = (2,\mathbf{1},-1,\mathbf{1}), \tag{C.30}$$

which is the next mode of φ_-. We also have

$$[B^+, Q^{i\mu}] = 0. \tag{C.31}$$

Hence, supersymmetry relates the n-th mode of φ_-, with $h = 1+n$, to the n-th mode of ψ_-, with $h = \frac{3}{2}+n$.

Let us now apply two supersymmetries to φ_-. We get the states

$$Q^{i\mu}Q^{j\nu}|0\rangle. \tag{C.32}$$

As in the discussion after (C.18), we have 6 states. Three are level 1 conformal descendants $B^+\varphi_0$ and three are the states in $(2,\mathbf{3},\mathbf{1})$ in (C.20). Two examples illustrating the two cases are

$$\begin{aligned} &B^+\alpha^1\beta^1|0\rangle = -Q^{2,1}Q^{1,1}|0\rangle, \\ &(\alpha^1\beta^2a^1b^1 + \alpha^2\alpha^1b^1b^1 - \alpha^2\beta^1a^1b^1 + \beta^2\beta^1a^1a^1)\,|0\rangle = Q^{1,2}Q^{1,1}|0\rangle. \end{aligned} \tag{C.33}$$

At level 3, we find 4 independent states of the form

$$Q^{i\mu}Q^{j\nu}Q^{k\rho}|0\rangle, \tag{C.34}$$

and they are (linear combinations of) the B^+ descendents of the last four states in (C.16), i.e. of the spinor ψ_+. Finally, there is a single state of the form

$$Q^{i\mu}Q^{j\nu}Q^{k\rho}Q^{\ell\lambda}|0\rangle. \tag{C.35}$$

It is obtained for example with this product of 4 supercharges

$$Q^{1,1}Q^{2,2}Q^{1,2}Q^{2,1}|0\rangle = \alpha^2\alpha^1\beta^2\beta^1\Big((a^1)^2(b_2)^2 - 2a^2a^1b^2b^1 + (a^2)^2(b^1)^2\Big)|0\rangle = (B^+)^2\alpha^2\alpha^1\beta^2\beta^1|0\rangle, \tag{C.36}$$

and is the level 2 conformal descendent of φ_+, cf. last line in (C.25). Our analysis is summarized in (5.40).

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