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Set-Valued Bernstein Operators on the Hypercube: Asymptotics and Semigroups

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Abstract. Tensor-product Bernstein operators connect approximation of convex-valued maps on a hypercube with a degenerate diffusion in the space of support functions. We establish pointwise and uniform Voronovskaja formulas and quantitative Hausdorff estimates from exact centered moments, including the optimal fourth-moment constant. We also obtain simultaneous approximation of parameter derivatives. The saturation class consists of maps affine in each coordinate with respect to Minkowski addition; the complete approximation class of order n^{-1} follows from the weighted scalar inequalities of Ditzian and Zhou. We characterize first-order Minkowski corrections and express the best correction errors as distances to the support-function cone and its negative. Spherical Hessian inequalities give sufficient conditions for generalized Hukuhara differences at large degrees. The associated strongly continuous convex-valued semigroup admits an Aumann-integral representation, and its Bernstein product formula converges uniformly over all non-negative times for each initial map. Its stationary states coincide with the saturation class. An explicit smooth planar family shows that both generalized Hukuhara relations can fail at every degree and between every pair of distinct times, despite positive curvature and exact asymptotic formulas.

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1. Introduction

Positive approximation operators provide finite-dimensional representations of diffusion semigroups. For set-valued maps taking compact convex values, we have the problem of the existence of increments defined by Minkowski addition, which may fail even for smooth families of strictly convex bodies. We study this problem for tensor-product Bernstein operators and we extend

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the analysis from first-order asymptotics to the evolution generated by the limiting operator.

Let $Q_m = [0,1]^m$, $m \geq 1$, and let $\mathcal{K}(\mathbb{R}^d)$, $d \geq 1$, be the non-empty compact convex subsets of $\mathbb{R}^d$, with Minkowski addition and multiplication by non-negative scalars. The support function of $A \in \mathcal{K}(\mathbb{R}^d)$ is $h_A(u) = \max_{a\in A}\langle a, u\rangle$, $u \in \mathbb{S}^{d-1}$. Set $X_d := C(\mathbb{S}^{d-1})$ and $\mathcal{H}_d := \{h_A \mid A \in \mathcal{K}(\mathbb{R}^d)\}$. The standard support-function embedding has closed convex image $\mathcal{H}_d$ and satisfies

$$h_{\lambda A+\mu B} = \lambda h_A + \mu h_B, \qquad d_H(A,B) = \|h_A - h_B\|_{X_d} \quad (\lambda,\mu \geq 0); \tag{1.1}$$

see [16,17]. For a Hausdorff-continuous set-valued map $F \in C(Q_m, \mathcal{K}(\mathbb{R}^d))$ from Q_m to $\mathcal{K}(\mathbb{R}^d)$, write $H_F(x) = h_{F(x)}$ and $H_F(x,u) = h_{F(x)}(u)$. Thus $H_F \in C(Q_m; X_d)$, and the uniform Hausdorff metric equals the uniform norm distance between support-function maps. All differences below are taken in X_d; they need not be support functions. Unless specified otherwise, norms of maps on Q_m are uniform norms.

Let $n \in \mathbb{N}$, $n \geq 1$; for every $t \in [0,1]$ and $x = (x_1,\dots,x_m) \in [0,1]^m$ put

$$b_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}, \qquad p^Q_{n,\alpha}(x) := \prod_{i=1}^m b_{n,\alpha_i}(x_i),$$

where $\alpha = (\alpha_1,\dots,\alpha_m) \in \{0,\dots,n\}^m$ and endpoint values are defined by continuity. The scalar and convex-valued operators are defined by setting, for every $f \in C(Q_m)$, $F \in C(Q_m, \mathcal{K}(\mathbb{R}^d))$ and $x = (x_1,\dots,x_m) \in [0,1]^m$,

$$B_n f(x) := \sum_{\alpha\in\{0,\dots,n\}^m} p^Q_{n,\alpha}(x) f\left(\frac{\alpha}{n}\right),$$

$$\mathcal{B}_n F(x) := \sum_{\alpha\in\{0,\dots,n\}^m} p^Q_{n,\alpha}(x) F\left(\frac{\alpha}{n}\right).$$

The second sum is a weighted Minkowski sum. Here n is the degree in each coordinate, so the total degree is at most mn. The same finite sum defines B_n on X_d-valued maps, and (1.1) gives

$$H_{\mathcal{B}_n F} = B_n H_F. \tag{1.2}$$

The operator acts only on the cube variable. The differential operator governing the asymptotics is

$$\mathcal{A}_Q H(x) = \frac{1}{2}\sum_{i=1}^m q_i(x)\partial_{ii}H(x), \qquad q_i(x) := x_i(1-x_i). \tag{1.3}$$

Scalar Bernstein asymptotics, simultaneous approximation and semigroup approximation are classical topics in positive-operator theory [1–3,5,23]. For tensor-product Bernstein operators on a cube, Ditzian and Zhou [9, Theorem 2.2] established weighted direct and converse inequalities characterizing the complete approximation class of order n^{-1}. In the present paper, this scalar characterization is applied uniformly in the support direction; it is not presented as a new scalar approximation theorem.

Approximation of compact convex-valued functions by positive operators also has an established literature. Vitale [24] studied approximation of

convex set-valued functions, and Keimel and Roth [14] established a set-valued counterpart of Korovkin's theorem. Further results on Hausdorff-continuous set-valued maps and their relation to vector-valued approximation were developed in [6,7]. These results provide the background for the use of positive operators and support functions in the present work. Other constructions for compact-valued maps are discussed in [10,11], while related Korovkin results for non-convex values are considered in [8]. Here the values are convex, so their support functions determine both the sets and their Hausdorff distance exactly. Sendov's theory for augmented scalar graphs [[18], Sects. 4.2, 4.3] uses a different Hausdorff metric, allowing displacement of the parameter.

The tensor-product structure is reflected in the limiting operator and in its invariant maps. The generator (1.3) contains only coordinate second derivatives, and the invariant maps are determined by the 2^m vertex values through Minkowski interpolation. They are affine in each coordinate separately, rather than necessarily affine on the whole cube. For example, if $m \geq 2$ and $e \in \mathbb{R}^d$, the map $F(x) = \{x_1 x_2 e\}$ is reproduced by every $\mathcal{B}_n$.

Several familiar cases are included in the construction. The parameter dimension m and the value dimension d have different roles. When $m = 1$, the parameter operator is the classical univariate Bernstein operator. For singleton-valued maps $F(x) = \{f(x)\}$, with $f \in C(Q_m; \mathbb{R}^d)$, one has

$$\mathcal{B}_n F(x) = \{B_n f(x)\},$$

where B_n acts componentwise. Thus the scalar and vector-valued Bernstein operators are recovered exactly.

When $d = 1$, every value is an interval. If $F(x) = [\ell(x), u(x)]$, with continuous endpoints $\ell \leq u$, positivity gives

$$\mathcal{B}_n F(x) = [B_n \ell(x), B_n u(x)].$$

The corresponding Hausdorff error is the maximum of the two endpoint errors. The semigroup likewise evolves the endpoints through the scalar semigroup. These cases explain how the present results connect with ordinary approximation and interval-valued evolution.

The geometric issue becomes more substantial for values in dimension $d \geq 2$. At each fixed parameter, any two intervals admit one of the two generalized Hukuhara relations in (1.4); this need not hold for higher-dimensional convex sets. Example 5.4 gives a smooth planar family with strictly convex values and positive curvature for which both relations fail whenever $0 < x_1 < 1$. The example, therefore, distinguishes convergence and differentiability of support functions from the existence of an increment represented by a compact convex set.

The geometric question is whether the limit $V_Q F = \mathcal{A}_Q H_F$, initially defined for twice continuously differentiable support-function maps, represents a convex set. A first-order correction by addition exists precisely when $V_Q F(x) \in \mathcal{H}_d$; the reversed addition relation requires $-V_Q F(x) \in \mathcal{H}_d$. These statements concern the two classical relations defining $A \ominus_{gH} B = C$:

$$\text{(i)} \quad A = B + C, \qquad \text{(ii)} \quad B = A + (-C), \qquad C \in \mathcal{K}(\mathbb{R}^d). \tag{1.4}$$

Here $h_{-C}(u) = h_C(-u)$, which differs in general from $-h_C(u)$. We determine the best errors when a correction is unavailable and give strict spherical Hessian criteria for existence at large degrees. An exact planar example shows why smoothness and positive curvature do not suffice.

The same limit operator generates a semigroup preserving the support-function cone. The resulting convex-valued evolution is compatible with every face, fixes the vertex values, and converges to their separately affine interpolant. This common stationary projection strengthens the usual Bernstein product formula: convergence is uniform for all $t \geq 0$, for each initial map. The planar example also gives an exact orbit on which neither generalized Hukuhara relation holds between distinct times. The construction uses deterministic analytic arguments throughout.

Sections 2 and 3 give the moment estimates and the quantitative asymptotic formulas. Section 4 treats saturation and the complete $O(n^{-1})$ class. Sections 5 and 6 develop the geometric criteria and the semigroup, respectively.

2. Tensor-Product Bernstein Moments and Preliminary Estimates

For an integer $r \geq 1$, $C^r(Q_m; X_d)$ denotes restrictions of C^r maps on an open neighbourhood of Q_m. Derivatives are Fréchet derivatives, with the multilinear operator norms induced by the Euclidean norm, and $\|D^r H\|_\infty = \sup_{x \in Q_m} \|D^r H(x)\|$. Write $\mathrm{pr}_i(y) = y_i$, $\mathrm{pr} = (\mathrm{pr}_1, \ldots, \mathrm{pr}_m)$, and $a_{ij}^Q(x) = q_i(x)\delta_{ij}$. In this section, for fixed x, put $z_i := \mathrm{pr}_i - x_i$ and $z := \mathrm{pr} - x$. All coefficients q_i, a_{ij}^Q below are evaluated at x. We set $b_{0,0} := 1$ and interpret endpoint values of the Bernstein basis by continuity.

Lemma 2.1. (Tensor-product Bernstein moments) *For $x \in Q_m$, $n \geq 1$, and $1 \leq i, j, k, \ell \leq m$,*

$$B_n z_i(x) = 0, \tag{2.1}$$

$$n B_n(z_i z_j)(x) = a_{ij}^Q, \tag{2.2}$$

$$n^2 B_n(z_i z_j z_k)(x) = \delta_{ij}\delta_{ik} q_i (1 - 2x_i). \tag{2.3}$$

The fourth moments satisfy

$$B_n(z_i z_j z_k z_\ell)(x) = \frac{a_{ij}^Q a_{k\ell}^Q + a_{ik}^Q a_{j\ell}^Q + a_{i\ell}^Q a_{jk}^Q}{n^2} + \frac{\delta_{ij}\delta_{ik}\delta_{i\ell}}{n^3} q_i(1 - 6q_i). \tag{2.4}$$

In particular, with $S_1 = \sum_i q_i$ and $S_2 = \sum_i q_i^2$,

$$\begin{aligned} B_n(|z|^2)(x) &= S_1/n, \\ B_n(|z|^4)(x) &= (S_1^2 + 2S_2)/n^2 + (S_1 - 6S_2)/n^3. \end{aligned} \tag{2.5}$$

Consequently, for $\gamma_m = m/4$ and $\beta_m = m(m+2)/16$,

$$B_n(|z|^2)(x) \leq \frac{\gamma_m}{n}, \qquad B_n(|z|^4)(x) \leq \frac{\beta_m}{n^2}. \tag{2.6}$$

Both constants are optimal uniformly over $x \in Q_m$ and $n \geq 1$.

Proof. Let $(k)_{\underline{r}} = k(k-1)\cdots(k-r+1)$, with $(k)_{\underline{0}} = 1$. Reindexing gives $\sum_{k=0}^{n} b_{n,k}(t)(k)_{\underline{r}} = (n)_{\underline{r}} t^r$; both sides vanish for $r > n$. Expanding powers in falling factorials and centering yields, for $M_r = \sum_{k=0}^{n} b_{n,k}(t)(k/n - t)^r$ and $q = t(1-t)$,

$$M_1 = 0, \qquad M_2 = q/n,$$
$$M_3 = q(1-2t)/n^2, \quad M_4 = 3q^2/n^2 + q(1-6q)/n^3.$$

The tensor-product sum factors over the coordinates. Hence a mixed centered moment vanishes if an index occurs exactly once. At order four, the remaining cases are four equal indices and two distinct pairs, the latter giving $q_i q_j/n^2$. This proves (2.1)–(2.4). Summing the second moments and using $|z|^4 = \sum_i z_i^4 + 2\sum_{i<j} z_i^2 z_j^2$ gives (2.5).

Since $0 \le q_i \le 1/4$, $S_1 \le m/4$, with equality at the midpoint, proving optimality of γ_m. Put $\eta := S_1^2 + 2S_2$ and $\mu := S_1^2 + S_1 - 4S_2$. Then

$$n^2 B_n(|z|^4)(x) = (1-n^{-1})\eta + n^{-1}\mu, \qquad \eta \le \beta_m.$$

For $m \ge 2$, Cauchy–Schwarz gives $\mu \le S_1 + (1-4/m)S_1^2 \le m^2/16$. Indeed, $s + (1-4/m)s^2$ is non-decreasing on $[0, m/4]$; indeed, for $m = 2, 3$, its derivative has minimum $m/2 - 1 \ge 0$, and for $m \ge 4$ this is immediate. Thus the scaled fourth moment is at most $\beta_m - m/(8n)$, attained at the midpoint. For $m = 1$, with $q = q_1$, it equals $(3 - 6/n)q^2 + q/n$, whose maximum on $[0, 1/4]$ is $3/16 - 1/(8n)$ if $n \ge 2$ and $1/12$ if $n = 1$. This proves (2.6); the midpoint values tend to β_m as $n \to \infty$, proving its optimality. □

Cauchy–Schwarz for the finite Bernstein sum also gives

$$B_n(|z|^3)(x) \le \sqrt{B_n(|z|^2)(x) B_n(|z|^4)(x)} \le \frac{m\sqrt{m+2}}{8n^{3/2}}. \tag{2.7}$$

If $L : (\mathbb{R}^m)^3 \to X_d$ is a continuous map which is linear in each variable, from (2.3) we get

$$B_n\big(L[z,z,z]\big)(x) = \frac{1}{n^2}\sum_i q_i(1-2x_i)L[e_i, e_i, e_i],$$

and, therefore,

$$\big\|B_n\big(L[z,z,z]\big)(x)\big\|_{X_d} \le \frac{m}{6\sqrt{3}\,n^2}\|L\|. \tag{2.8}$$

Here $\max_{0\le t\le 1} t(1-t)|1-2t| = 1/(6\sqrt{3})$, obtained by maximizing $s(1-s^2)/4$ for $s = |1-2t| \in [0,1]$. No optimality is claimed in (2.7) or (2.8).

The triangle inequality for finite non-negative weighted sums gives

$$\|B_n H(x)\|_{X_d} \le B_n\big(\|H(\cdot)\|_{X_d}\big)(x) \qquad (H \in C(Q_m; X_d)). \tag{2.9}$$

In particular, B_n is a contraction. For $d \ge 2$, the same assertion holds for $C^2(\mathbb{S}^{d-1})$-valued maps with $\|g\|_{C^2} = \|g\|_\infty + \|\nabla g\|_\infty + \|\nabla^2 g\|_\infty$, where ∇ denotes spherical covariant differentiation and tensor norms are induced by the standard Riemannian metric.

Let ω_F be the modulus of continuity of $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$, namely $\omega_F(\delta) = \sup_{x,y\in Q_m,\,|x-y|\le\delta} d_H(F(x), F(y))$, and let ω_F^c be its least concave majorant on $[0, \sqrt{m}]$. For $H \in C(Q_m; X_d)$, put

$$K_2(H,t) := \inf_{G\in C^2(Q_m;X_d)} \{\|H - G\|_\infty + t\|D^2 G\|_\infty\} \qquad (t > 0).$$

The comparison map G need not be a support-function map.

Proposition 2.2. (Direct approximation estimates) *For $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$, $n \ge 1$, and $x \in Q_m$,*

$$d_H(\mathcal{B}_n F(x), F(x)) \le \omega_F^c\left(\sqrt{S_1(x)/n}\right) \le \omega_F^c\left(\sqrt{m/(4n)}\right).$$

In particular, $\sup_x d_H(\mathcal{B}_n F(x), F(x)) \to 0$. *For $G \in C^2(Q_m; X_d)$,*

$$\begin{aligned} \|B_n G(x) - G(x)\|_{X_d} &\le \frac{S_1(x)}{2n}\|D^2 G\|_\infty, \\ \|B_n G - G\|_\infty &\le \frac{m}{8n}\|D^2 G\|_\infty. \end{aligned} \tag{2.10}$$

For arbitrary $H \in C(Q_m; X_d)$,

$$\|B_n H - H\|_\infty \le 2K_2\left(H, \frac{m}{16n}\right). \tag{2.11}$$

For $H = H_F$, this norm equals $\sup_x d_H(\mathcal{B}_n F(x), F(x))$.

Proof. By (1.2) and (2.9), the error is at most $B_n(\omega_F^c(|z|))(x)$. Jensen's inequality and Cauchy–Schwarz bound this by $\omega_F^c(\sqrt{B_n(|z|^2)(x)})$. Use (2.5) and $\omega_F^c(\delta) \to 0$ as $\delta \downarrow 0$. The remainder after the affine Taylor polynomial of G at x is bounded by $\frac{1}{2}\|D^2 G\|_\infty |z|^2$; reproduction of affine functions proves (2.10). Finally, contractivity gives $\|B_n H - H\|_\infty \le 2\|H - G\|_\infty + \frac{m}{8n}\|D^2 G\|_\infty$; taking the infimum over G proves (2.11). □

Proposition 2.3. (Simultaneous approximation of parameter derivatives) *Let $H \in C^1(Q_m; X_d)$. For $1 \le i \le m$, let ω_i^c be the least concave majorant on $[0, \sqrt{m}]$ of the modulus of continuity of $\partial_i H$, $1 \le i \le m$. Then*

$$\|\partial_i B_n H - \partial_i H\|_\infty \le \omega_i^c(\sigma_{m,n}), \qquad \sigma_{m,n}^2 = \begin{cases} (m-1)/4 + 1/3, & n = 1, \\ m/(4n) - 1/(6n^2), & n \ge 2. \end{cases}$$

Thus $\partial_i B_n H \to \partial_i H$ in $C(Q_m; X_d)$. In particular, if $H_F \in C^1(Q_m; X_d)$, then $D_x H_{\mathcal{B}_n F} \to D_x H_F$ in $C(Q_m; \mathcal{L}(\mathbb{R}^m, X_d))$.

Proof. Fix $i = 1, \dots, m$, and let $I_{n,i}$ consist of indices β with $0 \le \beta_i \le n-1$ and $0 \le \beta_j \le n$ for $j \ne i$. The weights $w_{n,\beta}^{(i)}(x) = b_{n-1,\beta_i}(x_i) \prod_{j\ne i} b_{n,\beta_j}(x_j)$ sum to one. Using $b'_{n,k} = n(b_{n-1,k-1} - b_{n-1,k})$, with out-of-range terms zero, and the fundamental theorem of calculus gives

$$\partial_i B_n H(x) = \sum_{\beta\in I_{n,i}} w_{n,\beta}^{(i)}(x) \int_0^1 \partial_i H((\beta + te_i)/n)\,\mathrm{d}t.$$

The first two binomial moments yield

$$\sum_{\beta\in I_{n,i}} w_{n,\beta}^{(i)}(x)\int_0^1 |(\beta+te_i)/n-x|^2\,\mathrm{d}t = \frac{1}{n}\sum_{j\neq i} q_j(x)+\frac{n-2}{n^2}q_i(x)+\frac{1}{3n^2}.$$

Indeed, the i-th contribution is $((n-1)q_i+\int_0^1(t-x_i)^2\,\mathrm{d}t)/n^2$, and the integral is $1/3-q_i$. Maximization over $0\le q_j\le 1/4$ gives $\sigma_{m,n}^2$, also when $n=1$. The norm inequality, Jensen's inequality and Cauchy–Schwarz now prove the modulus estimate. Since $\sigma_{m,n}\to 0$, uniform convergence follows. Finally, $\|L\|_{\mathcal{L}(\mathbb{R}^m,X_d)}\le\sqrt{m}\max_i\|L(e_i)\|_{X_d}$ and (1.2) give convergence of the full derivative. □

These derivatives describe variation in the cube variable in X_d; they need not be support functions or define Hukuhara derivatives of the convex-valued map.

3. Voronovskaja Formulas and Hausdorff Asymptotics

Definition 3.1. *(Second-order Taylor expansion relative to the cube)* A map $H\in C(Q_m;X_d)$ has a second-order Taylor expansion with Peano remainder relative to Q_m at $x\in Q_m$ if there are a linear map $L_x:\mathbb{R}^m\to X_d$ and a symmetric bilinear map $M_x:\mathbb{R}^m\times\mathbb{R}^m\to X_d$ such that

$$H(y)=H(x)+L_x(y-x)+\tfrac{1}{2}M_x[y-x,y-x]+r_x(y),\quad y\in Q_m\setminus\{x\},\tag{3.1}$$

and $\lim_{y\to x}\frac{\|r_x(y)\|_{X_d}}{|y-x|^2}=0$. The expansions are uniform if L_x and M_x are uniformly bounded and

$$\rho(\delta):=\sup_{\substack{x,y\in Q_m\\ 0<|y-x|\le\delta}}\frac{\|r_x(y)\|_{X_d}}{|y-x|^2}\longrightarrow 0\qquad(\delta\downarrow 0).\tag{3.2}$$

The same terminology will be used with $C^2(\mathbb{S}^{d-1})$ in place of X_d when $d\ge 2$.

If H_F has the expansion in Definition 3.1 at x, we define

$$V_QF(x,\cdot):=\frac{1}{2}\sum_{i=1}^m q_i(x)M_x[e_i,e_i].$$

For $H_F\in C^2(Q_m;X_d)$, this gives $V_QF=\mathcal{A}_QH_F$, with $\mathcal{A}_Q$ as in (1.3).

For $F\in C(Q_m;\mathcal{K}(\mathbb{R}^d))$, write

$$r_n(x)=n\big(H_{\mathcal{B}_nF}(x)-H_F(x)\big).$$

The following result makes no assumption about the existence of a Hukuhara difference. Its proof uses Taylor's formula and the finite Bernstein sums.

Theorem 3.2 (Voronovskaja formula). *Let* $F\in C(Q_m;\mathcal{K}(\mathbb{R}^d))$.

(i) *If H_F has a second-order Taylor expansion relative to Q_m at x, in the sense of Definition 3.1, then*

$$\|r_n(x) - V_Q F(x,\cdot)\|_{X_d} \longrightarrow 0.$$

(ii) *If these expansions are uniform, then*

$$\mathcal{R}_n(F) := \sup_{x\in Q_m} \|r_n(x) - V_Q F(x,\cdot)\|_{X_d} \longrightarrow 0. \tag{3.3}$$

In particular, $V_Q F \in C(Q_m; X_d)$.

(iii) *If $H_F \in C^2(Q_m; X_d)$, then the convergence in (3.3) holds with $V_Q F = \mathcal{A}_Q H_F$. For $d \geq 2$, the same assertions hold in the norm of $C^2(\mathbb{S}^{d-1})$, provided the continuity, Taylor expansion, and regularity assumptions are made with values in that space.*

Proof. Write $H = H_F$. Reproduction of affine functions and (2.2), applied to (3.1), give

$$n(B_n H(x) - H(x)) - V_Q F(x) = n(B_n r_x)(x). \tag{3.4}$$

For fixed x, set $C_x := \sup_y \|r_x(y)\|_{X_d} < \infty$ and

$$\rho_x(\delta) := \sup_{\substack{y\in Q_m \\ 0<|y-x|\leq\delta}} \frac{\|r_x(y)\|_{X_d}}{|y-x|^2}.$$

Splitting into $|y-x| \leq \delta$ and its complement gives

$$\|r_x(y)\|_{X_d} \leq \rho_x(\delta)|y-x|^2 + C_x\delta^{-4}|y-x|^4.$$

The moment bounds imply $n\|(B_n r_x)(x)\|_{X_d} \leq \gamma_m \rho_x(\delta) + C_x\beta_m/(n\delta^4)$. Let $n \to \infty$ and then $\delta \downarrow 0$. For uniform expansions, the same proof uses ρ from (3.2) and $C_* = \sup_{x,y} \|r_x(y)\|_{X_d} < \infty$, which follows from boundedness of H, L_x, M_x on the bounded cube. Continuity of $V_Q F$ follows from uniform convergence.

For a C^2 extension, Taylor's formula gives, with $h = y - x$,

$$r_x(y) = \int_0^1 (1-t)\big(D^2H(x+th) - D^2H(x)\big)[h,h]\,\mathrm{d}t. \tag{3.5}$$

Uniform continuity of $D^2 H$ gives uniform expansions. The proof uses only norm inequalities and, therefore, applies unchanged in $C^2(\mathbb{S}^{d-1})$. □

We next make the uniform remainder estimates explicit. For a continuous map J on Q_m with values in a normed space, write

$$\omega(J;\delta) = \sup\{\|J(x) - J(y)\| \;\mid\; x, y \in Q_m,\ |x-y| \leq \delta\}, \qquad \delta \geq 0.$$

This convention defines the modulus also when $\delta > \sqrt{m}$.

Theorem 3.3. (Quantitative remainder estimates)
Let $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$.
If $H_F \in C^2(Q_m; X_d)$, then

$$\mathcal{R}_n(F) \leq \frac{m}{6}\,\omega\left(D^2 H_F; \sqrt{\frac{m+2}{4n}}\right). \tag{3.6}$$

If $H_F \in C^3(Q_m; X_d)$, then

$$\mathcal{R}_n(F) \le \frac{m\sqrt{m+2}}{48\sqrt{n}} \|D^3 H_F\|_\infty. \tag{3.7}$$

If $H_F \in C^4(Q_m; X_d)$, then

$$\mathcal{R}_n(F) \le \frac{1}{n}\left(\frac{m}{36\sqrt{3}}\|D^3 H_F\|_\infty + \frac{m(m+2)}{384}\|D^4 H_F\|_\infty\right). \tag{3.8}$$

For $d \ge 2$, the estimates remain valid with the $C^2(\mathbb{S}^{d-1})$ norm whenever the corresponding regularity holds with values in that space.

Proof. Let $H = H_F$. Subdivision of a segment in the convex set Q_m shows, for $a, \delta > 0$, that $\omega(D^2 H; a) \le (1 + a/\delta)\omega(D^2 H; \delta)$. Consequently, (3.5) gives

$$\|r_x(y)\|_{X_d} \le \left(\frac{|y-x|^2}{2} + \frac{|y-x|^3}{6\delta}\right)\omega(D^2 H; \delta).$$

By (3.4), (2.6), and (2.7),

$$\mathcal{R}_n(F) \le \left(\frac{\gamma_m}{2} + \frac{\sqrt{\gamma_m \beta_m}}{6\delta\sqrt{n}}\right)\omega(D^2 H; \delta).$$

Taking $\delta = \sqrt{\beta_m/(\gamma_m n)} = \sqrt{(m+2)/(4n)}$ proves (3.6).

If $H \in C^3$, the integral remainder after the quadratic Taylor polynomial satisfies

$$\|r_x(y)\|_{X_d} \le \tfrac{1}{6}\|D^3 H\|_\infty |y-x|^3.$$

Thus (2.7) proves (3.7). If $H \in C^4$, Taylor's formula to degree three gives

$$r_x(y) = \tfrac{1}{6} D^3 H(x)[y-x, y-x, y-x] + s_x(y),$$
$$\|s_x(y)\|_{X_d} \le \tfrac{1}{24}\|D^4 H\|_\infty |y-x|^4.$$

The signed third-moment estimate (2.8) and the fourth-moment bound yield

$$n\|(B_n r_x)(x)\|_{X_d} \le \frac{m}{36\sqrt{3}\,n}\|D^3 H\|_\infty + \frac{\beta_m}{24n}\|D^4 H\|_\infty.$$

This is (3.8). The final assertion follows by the same norm estimates. □

Remark 3.4 *(Boundary points and faces).* The coefficients in Definition 3.1 are unique also at boundary points. Subtracting two expansions along strictly inward rays and dividing successively by t and t^2 identifies their linear and quadratic terms on an open set of directions. The polynomial identity theorem, applied through continuous linear functionals, gives equality everywhere. Derivatives of different extensions, therefore, agree on Q_m.

Let Γ be a face of the cube, obtained by fixing some coordinates at 0 or 1. For $x \in \Gamma$, every Bernstein coefficient associated with a grid point outside Γ is zero. Thus restriction to Γ commutes with the corresponding lower-dimensional Bernstein operator. In (1.3), the coefficients q_i of the fixed coordinates vanish. The restricted Voronovskaja formula, therefore, involves precisely the free coordinates of that face. At a vertex, all q_i vanish and $\mathcal{B}_n F(x) = F(x)$ for every degree.

The support-function isometry and the reverse triangle inequality give the corresponding Hausdorff asymptotics

$$n\, d_H(\mathcal{B}_n F(x), F(x)) \longrightarrow \|V_Q F(x)\|_{X_d}$$

under the pointwise hypotheses, and

$$\sup_x |n\, d_H(\mathcal{B}_n F(x), F(x)) - \|V_Q F(x)\|_{X_d}| \le \mathcal{R}_n(F) \longrightarrow 0$$

under the uniform hypotheses. In the latter case, $n \sup_x d_H(\mathcal{B}_n F(x), F(x)) \to \|V_Q F\|_\infty$, with the same error bound. Hence the quantitative estimates above apply to both scalar errors. Under these uniform hypotheses, the Hausdorff error is $o(n^{-1})$ if and only if $V_Q F = 0$.

4. Saturation and the Optimal Approximation Class

For $\nu \in \{0,1\}^m$, put

$$w_\nu(x) := \prod_{i=1}^m x_i^{\nu_i}(1-x_i)^{1-\nu_i}, \qquad (PH)(x) := \sum_{\nu\in\{0,1\}^m} w_\nu(x) H(\nu).$$

Here $H \in C(Q_m; X_d)$, and P is the interpolation operator at the vertices whose range consists of maps affine in each coordinate separately. The identities $\sum_\nu w_\nu = 1$ and $B_n w_\nu = w_\nu$ give $B_n PH = PH$ for every $n \ge 1$.

The following estimate will also determine the long-time behaviour of the semigroup.

Lemma 4.1. *For every $H \in C(Q_m; X_d)$ and $\eta > 0$, there is a constant $C_\eta \ge 0$, independent of n and k, such that*

$$\|B_n^k H - PH\|_\infty \le \eta + \frac{mC_\eta}{4}\left(1 - \frac{1}{n}\right)^k \qquad (n \ge 1,\ k \ge 1). \tag{4.1}$$

In particular, for each fixed $n \ge 1$, the iterates $B_n^k H$ converge uniformly to PH as $k \to \infty$.

Proof. Set $G := H - PH$ and $r := \sum_i q_i$. The function G vanishes at all vertices. Since the zero set of r is precisely the set of vertices, continuity and compactness give a constant C_η such that $\|G(x)\|_{X_d} \le \eta + C_\eta r(x)$ on Q_m.

The scalar second-moment identity gives $B_n r = (1 - 1/n) r$. The triangle inequality for finite sums with non-negative coefficients implies $\|B_n^k G(x)\|_{X_d} \le B_n^k(\|G(\cdot)\|_{X_d})(x)$. Consequently,

$$\|B_n^k G(x)\|_{X_d} \le \eta + C_\eta \left(1 - \frac{1}{n}\right)^k r(x).$$

Now use $\|r\|_\infty = m/4$ and $B_n^k PH = PH$ to obtain (4.1). For fixed n, let $k \to \infty$ and then $\eta \downarrow 0$. □

For $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$, write

$$\varepsilon_n(F) := \sup_{x \in Q_m} d_H(\mathcal{B}_n F(x), F(x)) = \|B_n H_F - H_F\|_\infty$$
$$= \sup_{u \in \mathbb{S}^{d-1}} \|B_n H_F(\cdot, u) - H_F(\cdot, u)\|_{C(Q_m)}.$$

Theorem 4.2. (Saturation theorem) *For $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$,*

$$\varepsilon_n(F) = o(n^{-1}) \quad \Longleftrightarrow \quad F(x) = \sum_{\nu \in \{0,1\}^m} w_\nu(x) F(\nu) \qquad (x \in Q_m),$$

where the right-hand side is a weighted Minkowski sum. In this case $\mathcal{B}_n F = F$ for every $n \geq 1$. Conversely, equality $\mathcal{B}_n F = F$ for even one $n \geq 1$ implies the displayed interpolation formula.

Proof. Put $H := H_F$. Telescoping and contractivity give $\|H - B_n^k H\|_\infty \leq k\varepsilon_n(F)$. If $n\varepsilon_n(F) \to 0$, use Lemma 4.1 with $k = \lfloor nt \rfloor$ and let $n \to \infty$, for fixed $t > 0$, to obtain

$$\|H - PH\|_\infty \leq \eta + \frac{mC_\eta}{4} e^{-t}.$$

Let $t \to \infty$ and then $\eta \downarrow 0$. Thus $H = PH$, which is the asserted Minkowski interpolation identity. Its converse follows from $B_n P = P$. If $\mathcal{B}_n F = F$ for one fixed n, then $B_n^k H = H$ for all k; the lemma again gives $H = PH$. □

The rate n^{-1} is attained outside this invariant class. If $e \in \mathbb{R}^d$ is a unit vector and $F(x) = \{x_1^2 e\}$, then

$$\mathcal{B}_n F(x) = \left\{ \left(x_1^2 + \frac{x_1(1 - x_1)}{n} \right) e \right\}, \qquad \varepsilon_n(F) = \frac{1}{4n}.$$

Thus n^{-1} is the saturation order in the uniform Hausdorff metric. The invariant maps are separately affine with respect to Minkowski addition, not necessarily affine on Q_m. For example, when $m \geq 2$, the map $F(x) = \{x_1 x_2 e\}$ is fixed by every $\mathcal{B}_n$ although its support-function map is not affine.

To describe the complete $O(n^{-1})$ class, put

$$\varphi_i(x) := \sqrt{q_i(x)}, \qquad P_Q(D) := \sum_{i=1}^m \varphi_i^2 \partial_i^2 = 2\mathcal{A}_Q.$$

For a scalar $f \in C(Q_m)$ and $t > 0$, define the weighted K-functional

$$K_Q(f, t) := \inf_{g \in C^3(Q_m)} \left\{ \|f - g\|_\infty + t^2 \|P_Q(D) g\|_\infty + t^3 \max_{1 \leq i \leq m} \|\varphi_i^3 \partial_i^3 g\|_\infty \right\}. \tag{4.2}$$

This is the cube functional of Ditzian and Zhou [9, (2.5)]. For convex-valued F, set

$$\widetilde{K}_Q(F, t) := \sup_{u \in \mathbb{S}^{d-1}} K_Q(H_F(\cdot, u), t).$$

Theorem 4.3. (The complete $O(n^{-1})$ approximation class)

There is a constant $M_m > 0$, depending only on m, such that for every $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$ and $n \geq 1$,

$$\varepsilon_n(F) \leq M_m \left[\widetilde{K}_Q(F, n^{-1/2}) + n^{-3/2} \|H_F\|_\infty \right], \tag{4.3}$$

$$\widetilde{K}_Q(F, n^{-1/2}) \leq M_m n^{-3/2} \sum_{k=1}^{n} k^{1/2} \varepsilon_k(F). \tag{4.4}$$

Consequently,

$$\sup_{x \in Q_m} d_H(\mathcal{B}_n F(x), F(x)) = O(n^{-1}) \quad \Longleftrightarrow \quad \widetilde{K}_Q(F, t) = O(t^2) \quad (t \downarrow 0).$$

Proof. Apply the scalar inequalities in [9, Theorem 2.2 and Section 7] to $f_u = H_F(\cdot, u)$, and take the supremum over u. Their constants depend only on m, while

$$\|f_u\|_\infty \leq \|H_F\|_\infty, \qquad \|B_k f_u - f_u\|_\infty \leq \varepsilon_k(F).$$

This proves (4.3) and (4.4). The direct inequality gives one implication. Conversely, if $\varepsilon_k(F) = O(k^{-1})$, then

$$\widetilde{K}_Q(F, n^{-1/2}) \leq C n^{-3/2} \sum_{k=1}^{n} k^{-1/2} = O(n^{-1}).$$

Monotonicity of $\widetilde{K}_Q(F, \cdot)$, together with $n^{-1} \leq 2t^2$ when $(n+1)^{-1/2} < t \leq n^{-1/2}$, gives the assertion for arbitrary $t \downarrow 0$. □

The infimum in (4.2) is taken separately for each u; the scalar approximants need not form the support function of a convex-valued map. We retain the third-order term appearing in the scalar endpoint theorem: the cited result does not justify its removal.

5. Minkowski Corrections and Generalized Hukuhara Differences

The support-function criterion for (1.4) is standard; see [19–21]. For $A, B \in \mathcal{K}(\mathbb{R}^d)$, put $\delta := h_A - h_B$. Case (i), the Hukuhara difference $A \ominus_H B$, holds precisely when $\delta = h_Q \in \mathcal{H}_d$, and its value is Q. Case (ii) holds precisely when $-\delta = h_Q \in \mathcal{H}_d$, and its value is $-Q$, with support function $-\delta(-u)$. Both hold precisely when $\delta(u) = \langle a, u \rangle$, with common value $\{a\}$. These assertions follow by taking support functions; injectivity gives uniqueness. If $\delta = h_Q = -h_P$, then $Q + P = \{0\}$, forcing opposite singletons. The everywhere defined total generalized Hukuhara difference of [21, Definition 9] is a different construction and is not covered by the non-existence statements below.

Theorem 5.1 (First-order Minkowski corrections). *Assume the pointwise hypotheses of Theorem 3.2 at x, and put $V := V_Q F(x, \cdot)$. The limits*

$$n\, d_H\left(\mathcal{B}_n F(x), F(x) + n^{-1} Q^+\right) \to 0, \quad n\, d_H\left(\mathcal{B}_n F(x) + n^{-1} Q^-, F(x)\right) \to 0$$

admit fixed correction sets $Q^+, Q^- \in \mathcal{K}(\mathbb{R}^d)$, independent of n, precisely when $V \in \mathcal{H}_d$ and $-V \in \mathcal{H}_d$, respectively. Each correction is unique, with $h_{Q^+} = V$ or $h_{Q^-} = -V$. Both exist precisely when $V(u) = \langle a, u\rangle$, and are then $\{a\}$ and $\{-a\}$.

If $r_n \to V_Q F$ uniformly on Q_m, the corresponding uniform limits, replacing the fixed sets by $Q^+(x)$ and $Q^-(x)$, hold for $Q^\pm \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$ precisely when the respective cone membership holds at every x. Here the supremum over x is taken before each limit. The correction maps are unique, with support functions $\pm V_Q F(x, \cdot)$. Both exist precisely when $V_Q F(x, u) = \langle a(x), u\rangle$ for some $a \in C(Q_m; \mathbb{R}^d)$.

Proof. For every $Q \in \mathcal{K}(\mathbb{R}^d)$, the isometry gives

$$n\, d_H(\mathcal{B}_n F(x), F(x) + n^{-1}Q) = \|r_n(x) - h_Q\|_{X_d}, \tag{5.1}$$

$$n\, d_H(\mathcal{B}_n F(x) + n^{-1}Q, F(x)) = \|r_n(x) + h_Q\|_{X_d}. \tag{5.2}$$

Pass to the limit and use injectivity and the preceding criterion. In the uniform case, $V_Q F$ is continuous, so the inverse support-function isometry makes $Q^\pm$ continuous; the same identities prove necessity and uniform convergence. For singleton corrections, continuity follows also from $\|\langle a - b, \cdot\rangle\|_{X_d} = |a - b|$. □

For $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$, define the best correction errors

$$\begin{aligned} E_n^+(x) &:= \inf_Q n\, d_H(\mathcal{B}_n F(x), F(x) + n^{-1}Q), \\ E_n^-(x) &:= \inf_Q n\, d_H(\mathcal{B}_n F(x) + n^{-1}Q, F(x)), \\ E_n^{\min}(x) &:= \min\{E_n^+(x), E_n^-(x)\}, \end{aligned} \tag{5.3}$$

where $Q \in \mathcal{K}(\mathbb{R}^d)$. We write $\operatorname{dist}(g, S) = \inf_{s \in S} \|g - s\|_{X_d}$.

Proposition 5.2. (Best-approximation errors) *Every $g \in X_d$ has a best approximation in $\mathcal{H}_d$. All minimizing bodies lie in the closed ball of radius $2\|g\|_{X_d}$ centered at zero and form a Hausdorff compact collection. If $g_j \to g$ and Q_j minimizes for g_j, some subsequence converges to a minimizing body for g. In particular, both infima in (5.3) are attained, and*

$$\begin{aligned} E_n^\pm(x) &= \operatorname{dist}(\pm r_n(x), \mathcal{H}_d) = \operatorname{dist}(r_n(x), \pm\mathcal{H}_d), \\ E_n^{\min}(x) &= \operatorname{dist}(r_n(x), \mathcal{H}_d \cup (-\mathcal{H}_d)). \end{aligned} \tag{5.4}$$

If $r_n(x) \to V = V_Q F(x, \cdot)$, then for $S = \mathcal{H}_d, -\mathcal{H}_d, \mathcal{H}_d \cup (-\mathcal{H}_d)$,

$$|\operatorname{dist}(r_n(x), S) - \operatorname{dist}(V, S)| \le \|r_n(x) - V\|_{X_d}. \tag{5.5}$$

Under uniform Voronovskaja convergence, these limits are uniform in x, and all minimizing bodies are uniformly bounded for sufficiently large degrees. Moreover, $E_n^+(x) = 0$ and $E_n^-(x) = 0$ characterize cases (i) and (ii), respectively; $E_n^{\min}(x) = 0$ characterizes existence of at least one case. If $V \notin \mathcal{H}_d \cup (-\mathcal{H}_d)$, neither case holds for any sufficiently large degree.

Proof. Since $0 \in \mathcal{H}_d$, the distance is at most $\|g\|_{X_d}$. A minimizing sequence satisfies $\|h_{Q_j}\|_{X_d} \le 2\|g\|_{X_d} + 1$ after finitely many terms. For $z \in Q_j \backslash \{0\}$, $|z| \le h_{Q_j}(z/|z|)$, so Blaschke's selection theorem gives a minimizing limit; see [17]. For minimizers the bound improves to $2\|g\|_{X_d}$. This common containing ball, the support-function isometry, and continuity of distance prove compactness and the subsequence assertion. Equations (5.1)–(5.2) give (5.4). Distance is 1-Lipschitz, giving (5.5) and its uniform version; the radius bound applies to $r_n(x)$. Finally, the cones and their union are closed. Thus zero distance is equivalent to membership, while the distance of V from the union is positive when V lies outside it. □

No continuous choice of minimizers is asserted, and uniqueness can fail; indeed, for $d \ge 2$, $g(u) = |u_2| - 1$ has distance one from $\mathcal{H}_d$, attained by every segment $[-ce_2, ce_2]$, $0 \le c \le 1$. The width inequality at $\pm e_1$ gives the lower bound; $h(u) = c|u_2|$ attains it.

For the differential criteria assume $d \ge 2$. Let σ_u be the Euclidean inner product on $T_u\mathbb{S}^{d-1}$, and set $\mathcal{W}[g] := \nabla^2_{\mathbb{S}^{d-1}} g + g\sigma$ for $g \in C^2(\mathbb{S}^{d-1})$. The classical criterion is $g \in \mathcal{H}_d$ if and only if $\mathcal{W}[g] \succeq 0$, where the inequality means non-negativity on every tangent vector; see [17, Sections 1.7 and 2.5] and [13]. Consequently, cases (i) and (ii) for smooth support functions are equivalent to $\mathcal{W}[h_A - h_B] \succeq 0$ and $\mathcal{W}[h_A - h_B] \preceq 0$, respectively. Also, $\mathcal{W}[g] = 0$ precisely for restrictions of linear functions, by applying the criterion to g and $-g$. In the plane the criterion is $g + g'' \ge 0$.

Theorem 5.3. (Strict differential criteria) *Let $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$, $d \ge 2$, and $x \in Q_m$. Suppose $H_F \in C(Q_m; C^2(\mathbb{S}^{d-1}))$ has a second-order Taylor expansion relative to Q_m at x, with Peano remainder measured in $C^2(\mathbb{S}^{d-1})$. Put $\delta_n := h_{\mathcal{B}_n F(x)} - h_{F(x)}$ and $V = V_Q F(x, \cdot)$.*

(i) If $\mathcal{W}[V] \succeq \eta\sigma$ for some $\eta > 0$, case (i) holds for all sufficiently large n, and $nC_n^+ \to Q^+$, where $C_n^+ = \mathcal{B}_n F(x) \ominus_H F(x)$ and $h_{Q^+} = V$.

(ii) If $\mathcal{W}[V] \preceq -\eta\sigma$ for some $\eta > 0$, case (ii) holds for all sufficiently large n. The body P_n^- satisfying $F(x) = \mathcal{B}_n F(x) + P_n^-$ and the generalized Hukuhara value $C_n^- = -P_n^-$ satisfy $nP_n^- \to Q^-$, $nC_n^- \to -Q^-$, with $h_{Q^-} = -V$.

(iii) If $\mathcal{W}[V]$ has a strictly positive value and a strictly negative value on unit tangent vectors, possibly at different points of the sphere, neither case holds for any sufficiently large n.

The inequalities in (i–ii) hold on the entire sphere, and the limits are in the Hausdorff metric. These conclusions are uniform on a compact $K \subset Q_m$ if $r_n \to V_Q F$ uniformly there in $C^2(\mathbb{S}^{d-1})$ and the strict bounds have a common positive margin. In (iii), the directions may depend on x, provided their values are at least η and at most $-\eta$, respectively.

Proof. Theorem 3.2 gives $n\delta_n \to V$ in C^2, hence

$$\sup_u \|\mathcal{W}[n\delta_n](u) - \mathcal{W}[V](u)\|_{\mathrm{op}} \longrightarrow 0.$$

All strict signs persist, uniformly under the stated assumptions. The support-function criteria give existence or non-existence. The limits follow from $h_{nC_n^+} =$

$n\delta_n \to V$, $h_{nP_n^-} = -n\delta_n \to -V$, and the isometries given by support functions and reflection. □

If $\mathcal{W}[V] \succeq 0$, is not zero, and vanishes on a non-zero tangent vector, then $V \in \mathcal{H}_d$ and a positive direction eventually excludes case (ii). On a null direction, however, $\mathcal{W}[n\delta_n - V]$ may have either sign, so convergence alone does not decide the case (i). The reversed statement holds for $\mathcal{W}[V] \preceq 0$. If $\mathcal{W}[V] = 0$, both limiting corrections are singletons, but neither finite-degree relation is determined by the limit alone.

Example 5.4. *(An exact counterexample on the hypercube)* Let $m \geq 1$, $d = 2$, $R > 1$, and $a(x) := q_1(x)$. Set $H_F(x, \theta) := R + a(x)\cos(2\theta)$. Since $0 \leq a \leq 1/4$,

$$H_F + \partial_{\theta\theta} H_F = R - 3a(x)\cos(2\theta) \geq R - \frac{3}{4} > 0.$$

Thus $F \in C(Q_m; \mathcal{K}(\mathbb{R}^2))$, and every value has a C^∞, strictly convex boundary with positive curvature. Indeed, the boundary parametrization $\gamma = hu + h'u'$ satisfies $\gamma' = (h + h'')u'$, with positive radius of curvature $h + h''$. Lemma 2.1 gives $B_n a = (1 - n^{-1})a$; hence $h_{\mathcal{B}_n F} = R + (1 - n^{-1})a\cos(2\theta)$, and the same curvature bound applies to every approximant. For $0 < x_1 < 1$, the expression $\delta_n + \delta_n'' = 3a(x)n^{-1}\cos(2\theta)$ changes sign, so neither generalized Hukuhara relation holds at any degree. Since $\mathcal{A}_Q a = -a$, one has exactly

$$r_n = V_Q F = -a\cos(2\theta), \qquad n\, d_H(\mathcal{B}_n F(x), F(x)) = a(x).$$

Thus $\sup_x n\, d_H(\mathcal{B}_n F(x), F(x)) = 1/4$. Both distances $\operatorname{dist}(\pm a\cos(2\theta), \mathcal{H}_2)$ equal a: either function takes the value $-a$ at antipodal directions, so an error smaller than a would force negative width; the zero support function attains the bound. Consequently $E_n^+(x) = E_n^-(x) = E_n^{\min}(x) = a(x)$. If $x_1 \in \{0, 1\}$, both relations hold with value $\{0\}$. All these identities hold for every $n \geq 1$.

6. Semigroup Generation and Bernstein Product Formula

The operator $\mathcal{A}_Q$ also determines a convex-valued evolution. Put

$$E := C(Q_m; X_d), \qquad \mathcal{C} := \{H \in E \mid H(x) \in \mathcal{H}_d \text{ for all } x \in Q_m\},$$

and let $\mathcal{P}_d$ be the space of X_d-valued polynomial maps on Q_m. For convex-valued maps we use $d_\infty(F, G) = \sup_{x \in Q_m} d_H(F(x), G(x))$. Generation follows from Trotter's approximation theorem [22, Theorem 5.3], in the formulation of [15, Theorem 1].

Theorem 6.1 (Generation and product formula). *The closure of $(\mathcal{A}_Q, \mathcal{P}_d)$ generates a strongly continuous contraction semigroup $T(t)$ on E. Its generator A satisfies $C^2(Q_m; X_d) \subset D(A)$ and $AH = \mathcal{A}_Q H$ on this space. Both $\mathcal{P}_d$ and $C^2(Q_m; X_d)$ are cores for A. For every $H \in E$ and $T_0 > 0$,*

$$\sup_{0 \leq t \leq T_0} \|B_n^{\lfloor nt \rfloor} H - T(t)H\|_E \longrightarrow 0. \tag{6.1}$$

The scalar semigroup $S_Q(t)$ obtained on $C(Q_m)$ is positive, fixes 1*, and is the tensor product of the m one-dimensional semigroups generated by $\frac{1}{2}x(1-x)D^2$. Moreover,*

$$(T(t)H)(x,u) = S_Q(t)\big(H(\cdot,u)\big)(x).$$

Proof. The norm proof of Theorem 3.2 gives $n(B_n - I)H \to \mathcal{A}_Q H$ for every $H \in C^2(Q_m; X_d)$. Also, $\|B_n^k\|_{E\to E} \le 1$, and $\mathcal{P}_d$ is dense in E, by finite-rank approximation and scalar polynomial approximation. For a multi-index α,

$$\mathcal{A}_Q x^\alpha = \frac{1}{2} \sum_{\substack{i=1,\dots,m \\ \alpha_i \ge 2}} \alpha_i(\alpha_i - 1)(x^{\alpha - e_i} - x^\alpha).$$

Thus $\lambda - \mathcal{A}_Q$, on polynomials of bounded total degree, has a triangular coefficient matrix with positive diagonal entries $\lambda + \frac{1}{2}\sum_i \alpha_i(\alpha_i - 1)$, for $\lambda > 0$. Successive solution for the X_d-valued coefficients gives $(\lambda - \mathcal{A}_Q)\mathcal{P}_d = \mathcal{P}_d$. Trotter's theorem, therefore, applies with either $\mathcal{P}_d$ or $C^2(Q_m; X_d)$ as the initial domain. The resulting semigroups coincide, being limits of the same iterates. Their generators coincide as well, proving the core and generator assertions.

For $H \in \mathcal{P}_d$, set $M_H := \sup_n \|n(B_n - I)H\|_E < \infty$. Telescoping gives

$$\|B_n^{\lfloor nt \rfloor} H - B_n^{\lfloor ns \rfloor} H\|_E \le M_H(|t - s| + n^{-1}).$$

A finite subdivision of $[0, T_0]$ and continuity of $T(\cdot)H$ upgrade pointwise convergence to uniform convergence there. Density and contractivity give (6.1). Scalar positivity, preservation of 1, and commutation with evaluation at u pass to the limit. Finally, the powers of B_n factor on products $\prod_i f_i(x_i)$ into the corresponding one-dimensional powers. Passage to the limit and density give the tensor-product assertion. □

These one-dimensional factors are the neutral binary Wright–Fisher semigroups; see [22, Section 6]. However, their tensor product on Q_m differs from the neutral $(m+1)$-allele operator on a simplex since it has no mixed derivatives and no constraint $\sum_i x_i \le 1$.

Theorem 6.2. (The convex-valued semigroup) *There is a unique semigroup $\mathcal{T}(t)$ on $C(Q_m; \mathcal{K}(\mathbb{R}^d))$ satisfying $H_{\mathcal{T}(t)F} = T(t)H_F$. It is strongly continuous, non-expansive in d_∞, additive with respect to Minkowski addition, positively homogeneous, and inclusion preserving. It fixes maps constant in x, commutes with restriction to every face, and preserves each vertex value. For every F,*

$$\mathcal{B}_n^{\lfloor nt \rfloor} F \longrightarrow \mathcal{T}(t)F \quad \text{locally uniformly in } t \ge 0 \tag{6.2}$$

with respect to d_∞. If $\mu_t(x, \cdot)$ is the Borel probability measure representing $f \mapsto (S_Q(t)f)(x)$, then

$$(\mathcal{T}(t)F)(x) = \int_{Q_m} F(y)\, \mu_t(x, \mathrm{d}y), \tag{6.3}$$

where the right-hand side is the Aumann integral.

Proof. Since $B_n\mathcal{C} \subset \mathcal{C}$ and $\mathcal{C}$ is closed, (6.1) gives $T(t)\mathcal{C} \subset \mathcal{C}$. The support-function isometry defines $\mathcal{T}(t)$ uniquely and transfers the semigroup law, continuity, contractivity, and algebraic properties. Order preservation follows from scalar positivity and $F \subset G$ if and only if $H_F \leq H_G$. Preservation of constant maps and commutation with restrictions to faces pass from the Bernstein operators to their product limit; vertices are the zero-dimensional faces. Intertwining gives (6.2).

The Riesz representation theorem gives $\mu_t(x, \cdot)$, and evaluation at u identifies $(T(t)H_F)(x)$ with the Bochner integral of H_F against this measure. The Aumann integral is the set of integrals of measurable selections belonging to $F(y)$ almost everywhere for this measure. Since F is continuous and compact convex-valued on a compact domain, this integral is non-empty, compact, and convex, with support function $\int h_{F(y)}\, \mu_t(x, \mathrm{d}y)$; see [4]. This proves (6.3). □

Corollary 6.3. (The Cauchy problem and the stationary limit) *For every $H_0 \in \mathcal{C}$, the constrained Cauchy problem*

$$H'(t) = AH(t), \qquad H(0) = H_0, \qquad H(t) \in \mathcal{C} \tag{6.4}$$

has the unique mild solution $H(t) = T(t)H_0$, which is classical in E if $H_0 \in D(A)$. If $H_0 \in C^2(Q_m; X_d)$, then $H \in C([0,\infty); C^2(Q_m; X_d)) \cap C^1([0,\infty); E)$, and

$$\partial_t H(t,x) = \frac{1}{2}\sum_{i=1}^{m} x_i(1-x_i)\partial_{ii}H(t,x)$$

in X_d, including at the boundary. No independent boundary data are prescribed since restrictions evolve on their respective faces and vertex values remain fixed. For every $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$,

$$\begin{aligned} \mathcal{T}(t)F &\longrightarrow \mathcal{P}F \quad \text{in } d_\infty \quad (t \to \infty), \\ (\mathcal{P}F)(x) &= \sum_{\nu \in \{0,1\}^m} w_\nu(x) F(\nu). \end{aligned} \tag{6.5}$$

The stationary maps are exactly the saturation class of Sect. 4.

Proof. The abstract Cauchy problem follows from semigroup theory [12, Chapter II, Section 6] and cone invariance. Here the mild formulation means

$$\int_0^t H(s)\,\mathrm{d}s \in D(A), \qquad H(t) - H_0 = A\int_0^t H(s)\,\mathrm{d}s.$$

For $|\alpha| \leq 2$, $n \geq 2$, differentiation of the Bernstein sum and the fundamental theorem of calculus, as in Proposition 2.3, give

$$\|\partial^\alpha B_n G\|_\infty \leq \prod_i \frac{(n)_{\underline{\alpha_i}}}{n^{\alpha_i}} \|\partial^\alpha G\|_\infty \leq \|\partial^\alpha G\|_\infty.$$

After the prefactor, the derivative is a non-negative normalized sum of integrals of $\partial^\alpha G$. The same formulas give simultaneous approximation in the C^2 norm; see [23], whose norm proof applies to X_d-valued maps. For polynomial data, convergence in (6.1) also holds in C^2, since the iterates lie in a fixed finite-dimensional space of polynomials with coefficients in the span of the

initial coefficients. The derivative bounds and polynomial approximation in C^2, therefore, show that $T(t)$ preserves C^2, is strongly continuous there, and satisfies $\|\partial^\alpha T(t)G\|_\infty \le \|\partial^\alpha G\|_\infty$. The generator identity in Theorem 6.1 gives the differential equation.

For arbitrary $H \in E$, choose C_η as in Lemma 4.1. Taking $n \to \infty$ in (4.1) with $k = \lfloor nt \rfloor$, for fixed $t > 0$, yields

$$\|T(t)H - PH\|_E \le \eta + \frac{mC_\eta}{4} e^{-t}. \tag{6.6}$$

Letting $t \to \infty$ and then $\eta \downarrow 0$ gives $T(t)H \to PH$. Since $T(t)P = P$, the stationary maps are precisely the range of P. The support-function isometry gives (6.5). □

Corollary 6.4. (Uniform approximation for all non-negative times) *For every* $H \in E$,

$$\lim_{n\to+\infty} \sup_{t\ge 0} \|B_n^{\lfloor nt \rfloor} H - T(t)H\|_E = 0.$$

Consequently, for every $F \in C(Q_m; \mathcal{K}(\mathbb{R}^d))$,

$$\lim_{n\to+\infty} \sup_{t\ge 0} d_\infty\big(\mathcal{B}_n^{\lfloor nt \rfloor} F, \mathcal{T}(t)F\big) = 0. \tag{6.7}$$

These assertions concern each fixed initial map; operator-norm convergence is not asserted.

Proof. Fix $H \in E$ and $\eta > 0$. For $t \ge T_0 \ge 1$, $k = \lfloor nt \rfloor \ge 1$ and $(1 - n^{-1})^k \le e^{-k/n} \le e^{1-t}$. Combining (4.1) and (6.6), with the same C_η, gives

$$\sup_{\substack{n\ge 1\\ t\ge T_0}} \|B_n^{\lfloor nt \rfloor} H - T(t)H\|_E \le 2\eta + \frac{mC_\eta}{4}(e+1)e^{-T_0}.$$

Choose η small and then T_0 large; on $[0, T_0]$ use (6.1). The isometry gives (6.7). □

The convex-valued evolution associated with (6.4) need not admit generalized Hukuhara differences, even when its support function is smooth in time.

Example 6.5. *(An exact orbit without generalized Hukuhara differences)* Take F_0 from Example 5.4, with $H_{F_0}(x, \theta) = R + q_1(x)\cos(2\theta)$. Writing $F_t = \mathcal{T}(t)F_0$, the identities $\mathcal{A}_Q q_1 = -q_1$ and $B_n q_1 = (1 - n^{-1})q_1$ give

$$\begin{aligned} H_{F_t}(x, \theta) &= R + e^{-t} q_1(x)\cos(2\theta), \\ H_{\mathcal{B}_n^k F_0}(x, \theta) &= R + (1 - n^{-1})^k q_1(x)\cos(2\theta). \end{aligned} \tag{6.8}$$

The factor $(1-n^{-1})^0$ is interpreted as 1. All sets have smooth strictly convex boundaries with positive curvature, as in that example. If $s \ne t$ and $0 < x_1 < 1$, the support-function difference is a non-zero multiple of $\cos(2\theta)$; neither it nor its negative is a support function. Both generalized Hukuhara relations, therefore, fail, although

$$\begin{aligned} \partial_t H_{F_t} &= -e^{-t} q_1 \cos(2\theta) = \mathcal{A}_Q H_{F_t}, \\ d_H(F_t(x), F_s(x)) &= q_1(x)|e^{-t} - e^{-s}|. \end{aligned}$$

The metric derivative at x is $e^{-t}q_1(x)$, with the right-hand interpretation at $t = 0$, whereas $\partial_t H_{F_t}(x, \cdot)$ belongs to neither $\mathcal{H}_2$ nor $-\mathcal{H}_2$ when $0 < x_1 < 1$. Moreover,

$$\sup_{t\geq 0} d_\infty\big(\mathcal{B}_n^{\lfloor nt\rfloor} F_0, F_t\big) = \frac{1}{4}(1 - e^{-1/n}) \leq \frac{1}{4n}. \tag{6.9}$$

To prove this, set $h := 1/n$, $a := 1 - h$, and $b := e^{-h}$. For $k \geq 1$,

$$\begin{aligned} 0 \leq b^k - a^k &= (b - a)\sum_{j=0}^{k-1} b^{k-1-j}a^j \\ &\leq \frac{b-a}{h} \leq \frac{h}{2} \leq 1 - b. \end{aligned}$$

The last two inequalities use $e^{-h} \leq 1 - h + h^2/2$ and $0 < h \leq 1$. If $kh \leq t < (k + 1)h$, then

$$|a^k - e^{-t}| \leq \max\{b^k - a^k, b^k(1 - b)\} \leq 1 - b,$$

also for $k = 0$. The supremum $1-b$ is approached as $t \uparrow h$. Since $\|q_1\|_\infty = 1/4$, (6.8) gives (6.9).

The product formula represents the evolution; numerical implementation is not considered here.

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Declarations

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