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Non-planar corrections to ABJM Bremsstrahlung function from quantum M2 brane

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Abstract
As was shown in Giombi and Tseytlin (2023 *Phys. Rev. Lett.* **130** 201601), the leading large N, fixed k correction in the localization result for the expectation value of the $\frac{1}{2}$ BPS circular Wilson loop in $U(N)_k \times U(N)_{-k}$ ABJM theory given by the $(\sin\frac{2\pi}{k})^{-1}$ factor can be reproduced on the dual M-theory side as the one-loop correction in the partition function of M2 brane in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with $\mathrm{AdS}_2 \times S^1$ world volume. Here we prove, following the suggestion in Giombi *et al* (2024 *J. High Energy Phys.* JHEP11(2024)056), that the analogous fact is true also for the corresponding correction $B_1 = -\frac{1}{2\pi k}\cot\frac{2\pi}{k}$ in the localization result for the Bremsstrahlung function associated with the Wilson loop with a small cusp in either AdS_4 or CP^3. The corresponding M2 brane is wrapped on the 11d circle and generalizes the type IIA string solution in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ ending on the cusped line. We show that the one-loop term in the M2 brane partition function reproduces the localization expression for B_1 as the coefficient of the leading term in its small cusp expansion.

Keywords: AdS/CFT, M-theory, ABJM theory

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1. Introduction

A series of recent papers have demonstrated how non-planar corrections in the ABJM theory [1] can be found by semiclassically quantizing the M2 brane in $AdS_4 \times S^7/\mathbb{Z}_k$ [2–4] (see also [5–8]). Here we shall follow [2, 4] and focus on the Bremsstrahlung function $B(\lambda,N)$ associated with the $\frac{1}{2}$-BPS Wilson loop with the aim to demonstrate how the result found from localization can be reproduced on the dual M-theory side by quantizing the M2 brane near the classical solution representing the Wilson line with a small cusp.

Let us start with a brief review of some relevant facts about the Bremsstrahlung function B. It determines the energy emitted by a moving quark given by $\Delta E = 2\pi B \int dt\, \dot{v}^2$ in the small velocity limit. In the $\mathcal{N}=4$ $SU(N)$ SYM theory it may be found from the exact localization result for the expectation value of the $\frac{1}{2}$-BPS circular Wilson loop as [9]

$$B_{\rm SYM} = \frac{1}{2\pi^2}\lambda\frac{\partial}{\partial\lambda}\log\langle W\rangle_{\rm SYM}\ , \tag{1.1}$$

where [10, 11]

$$\begin{aligned}\langle W\rangle_{\rm SYM} &= e^{\frac{\lambda}{8N^2}(N-1)}L^{(1)}_{N-1}\left(-\frac{\lambda}{4N}\right)\\ &= \frac{2N}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right)\left[1+\frac{1}{N^2}\left(\frac{\lambda^{3/2}}{96}\frac{I_2\left(\sqrt{\lambda}\right)}{I_1\left(\sqrt{\lambda}\right)}-\frac{\lambda}{8}\right)+\cdots\right].\end{aligned} \tag{1.2}$$

Expanded in large N and then also in large λ this gives

$$\begin{aligned}B_{\rm SYM}(\lambda,N) &= B_{\rm SYM}(\lambda)+\frac{1}{128\pi^2}\frac{\lambda^{3/2}}{N^2}+\cdots,\\ B_{\rm SYM}(\lambda) &= \frac{\sqrt{\lambda}}{4\pi^2}\frac{I_2\left(\sqrt{\lambda}\right)}{I_1\left(\sqrt{\lambda}\right)} = \frac{\sqrt{\lambda}}{4\pi^2}-\frac{3}{8\pi^2}+\cdots.\end{aligned} \tag{1.3}$$

One may also get B from the expression for the $\frac{1}{2}$-BPS Wilson loop wrapped w times on the circle as [12]

$$B(\lambda,N) = \frac{1}{4\pi^2}\frac{\partial}{\partial {\rm w}}\log\langle W\rangle\Big|_{{\rm w}=1}\ . \tag{1.4}$$

In the $\mathcal{N}=4$ SYM case this leads to the same expression as in (1.1) since the dependence on w can be incorporated into $\langle W\rangle$ by $\sqrt{\lambda}\to {\rm w}\sqrt{\lambda}$ [10]. Lewkowycz and Maldacena [12] has shown that (1.4) applies also in the ABJM case for the Bremsstrahlung function given in terms of the $\frac{1}{6}$-BPS Wilson loop.

The Bremsstrahlung function may also be related to the anomalous dimension $\Gamma_{\rm cusp}$ governing the logarithmic divergence of a Wilson loop with a cusp $\langle W\rangle \sim \exp[-\Gamma_{\rm cusp}\log(\Lambda_{\rm IR}/\Lambda_{\rm UV})]$. In the case of locally supersymmetric Wilson loops, the cusp anomaly $\Gamma_{\rm cusp} = \Gamma_{\rm cusp}(\lambda,N;\alpha,\beta)$ depends on the geometrical angle α between the two lines defining the cusp, and an internal angle β describing the change in the 'internal' orientation described by the scalar coupling.

In a small angle expansion around the BPS limit $\alpha = \pm\beta$ one can determine $B(\lambda,N)$ from the small α,β expansion of $\Gamma_{\rm cusp}$ as [13, 14]

$$\Gamma_{\rm cusp}(\lambda,N;\alpha,\beta) = -\left(\alpha^2-\beta^2\right)B(k,N)+\cdots. \tag{1.5}$$

One may also compute the Bremsstrahlung function corresponding to either $\frac{1}{2}$- or $\frac{1}{6}$-BPS Wilson loops in the ABJM theory by using a generalization of the identity [9] that expresses B as a derivative of the logarithm of the latitude Wilson loop with respect to the small latitude angle[4]. In the planar limit of $U(N)_k\times U(N)_{-k}$ ABJM theory one finds the following strong coupling expansion for the Bremsstrahlung function corresponding to the $\frac{1}{2}$-BPS Wilson loop [18]

$$B_{\rm ABJM}(\lambda)\Big|_{\lambda\gg1} = \frac{1}{2\pi}\sqrt{\frac{\lambda}{2}}-\frac{1}{4\pi^2}-\frac{1}{96\pi}\frac{1}{\sqrt{2\lambda}}+\cdots, \qquad \lambda=\frac{N}{k}=\text{fixed},\ N\to\infty\,. \tag{1.6}$$

This matches the prediction from string theory in $AdS_4\times CP^3$ at the two leading orders [21, 22].[5] Finding non-planar corrections in this approach is hard as the exact localization result is not known in the ABJM theory for a non-zero cusp angle.

An alternative approach based on mass-deformed localization matrix model was suggested in [23, 24]. The resulting exact expression for Bremsstrahlung function found in [24] reads (for $k>2$)

$$B_{\rm ABJM}(k,N) = -\frac{1}{(4\pi^2k)^{2/3}}\frac{{\rm Ai}'\left[\left(\frac{\pi^2}{2}k\right)^{1/3}\left(N-\frac{k}{24}-\frac{1}{3k}\right)\right]}{{\rm Ai}\left[\left(\frac{\pi^2}{2}k\right)^{1/3}\left(N-\frac{k}{24}-\frac{1}{3k}\right)\right]}-\frac{1}{2\pi k}\cot\frac{2\pi}{k}\,, \tag{1.7}$$

where prime indicates the derivative of the Airy function over its argument. As was pointed out in [4] the expression (1.7) can be reproduced in a simple way by applying (1.4) to the result for the localization result for the expectation value of the $\frac{1}{2}$-BPS circular Wilson loop in the w-fundamental representation (presumably equivalent to the result for the w-wrapped circular loop) [25]

$$\langle W\rangle_{\rm ABJM} = \frac{1}{2\sin\frac{2\pi{\rm w}}{k}}\frac{{\rm Ai}\left[\left(\frac{\pi^2}{2}k\right)^{1/3}\left(N-\frac{k}{24}-\frac{1}{3k}-\frac{2{\rm w}}{k}\right)\right]}{{\rm Ai}\left[\left(\frac{\pi^2}{2}k\right)^{1/3}\left(N-\frac{k}{24}-\frac{1}{3k}\right)\right]}\,. \tag{1.8}$$

Expanding in large N at fixed k one gets

$$\langle W\rangle_{\rm ABJM} = \frac{1}{2\sin\frac{2\pi{\rm w}}{k}}e^{\pi{\rm w}\sqrt{\frac{2N}{k}}}\left[1-\frac{\pi{\rm w}\left(k^2+24{\rm w}+8\right)}{24\sqrt{2}k^{3/2}}\frac{1}{\sqrt{N}}+\cdots\right], \tag{1.9}$$

[4] For $\frac{1}{2}$-BPS Wilson loop this identity was proposed and proved perturbatively in [15], and for the corresponding Bremsstrahlung function it was first introduced and then proved exactly in [16]. In the $\frac{1}{6}$-BPS Wilson loop case a similar identity for the Brehmstrahlung function was proved in [17] and further elaborated on in [18]. For a review of the Brehmstrahlung function in the ABJM theory see the contribution of L. Bianchi in [19] and also [20].

[5] This was partly shown in [21] by computing the one-loop contribution to $\Gamma_{\rm cusp}$ at $\beta=0$ and small α. They also computed $\Gamma_{\rm cusp}$ at $\alpha=0$ and small β with a result not consistent with the expected expression in (1.5). This was later corrected in [22].

$$B_{\rm ABJM}(k,N) = \frac{1}{4\pi^2}\frac{\partial}{\partial {\rm w}}\log\langle W\rangle_{\rm ABJM}\Big|_{{\rm w}=1} = \frac{1}{4\pi}\sqrt{\frac{2N}{k}} - \frac{1}{2\pi k}\cot\frac{2\pi}{k} - \frac{56+k^2}{96\pi\sqrt{2}k^{3/2}}\frac{1}{\sqrt{N}} + \cdots . \tag{1.10}$$

The first term in (1.10) is same as in the planar limit in (1.6). The cot term in (1.7) and (1.10) originates from the derivative of the logarithm of the 1/sin prefactor in (1.8). Expanded in large k it gives an infinite series of terms $1/k^{2p} = (\lambda/N)^{2p}$ that represent the leading large λ corrections at each order in $1/N$, with the first $p=0$ one reproducing the $-\frac{1}{4\pi^2}$ correction in (1.6).

Let us now recall some basic M-theory relations used in [2]. The M2 brane is placed into $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ background with the metric

$$ds^2 = \tfrac{1}{4}R^2 ds^2_{\rm AdS_4} + R^2 ds^2_{S^7/\mathbb{Z}_k} , \tag{1.11}$$

$$ds^2_{\rm AdS_4} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,\left(dx^2 + \cos^2 x\, d\theta^2\right) , \tag{1.12}$$

$$ds^2_{S^7/\mathbb{Z}_k} = ds^2_{\rm CP^3} + \frac{1}{k^2}(d\phi + kA)^2 ,\ \ \phi \equiv \phi + 2\pi , \tag{1.13}$$

$$ds^2_{\rm CP^3} = \frac{dw^s d\bar w^s}{1+|w|^2} - \frac{w_r\bar w_s dw^s d\bar w^r}{(1+|w|^2)^2} ,\ A = \frac{i}{2}\frac{w^s d\bar w^s - \bar w^s dw^s}{1+|w|^2} ,\ \ s,r = 1,2,3 , \tag{1.14}$$

and the 4-form field strength

$$F_4 = dC_3 = -\tfrac{3}{8}R^3 {\rm vol}_{\rm AdS_4} . \tag{1.15}$$

The leading order relation between the radius (in 11d Planck length units) and the parameters N,k of the dual ABJM theory is[6]

$$\left(\frac{R}{\ell_{\rm P}}\right)^6 = 2^5\pi^2 Nk . \tag{1.16}$$

The world-volume action for a probe M2 brane in this background is [27, 28][7]

$$S_{\rm M2} = T_2\int d^3\sigma\sqrt{-g} + T_2\int C_3 + \text{fermionic terms},\qquad T_2 = \frac{1}{(2\pi)^2\ell_{\rm P}^3} . \tag{1.17}$$

The resulting dimensionless effective M2 brane tension is

$$\mathrm{T}_2 \equiv R^3 T_2 = \tfrac{1}{\pi}\sqrt{2kN} . \tag{1.18}$$

This suggests that the expansions in (1.9) and (1.10) should be matched with the semiclassical (large T_2 at fixed k) expansions of the corresponding M2 brane expressions.

Note that expressed in terms of the type IIA string effective tension T and coupling $g_{\rm s}$

$$T = \frac{R^2}{8\pi\alpha'} = \sqrt{\frac{\lambda}{2}} ,\qquad g_{\rm s} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N} ,\qquad \lambda = \frac{N}{k} ,\qquad \frac{1}{k^2} = \frac{g_{\rm s}^2}{8\pi T} , \tag{1.19}$$

the subleading cot term in (1.10) represents the sum of the leading large tension corrections at each order in the string coupling (genus) expansion

$$-\frac{1}{2\pi k}\cot\frac{2\pi}{k} = -\frac{1}{4\pi^2} + \frac{g_{\rm s}^2}{24\pi T} + \frac{g_{\rm s}^4}{720T^2} + \frac{\pi g_{\rm s}^6}{15120T^3} + \cdots . \tag{1.20}$$

[6] As in [2, 4] we shall ignore the shift [26] $N\to N-\frac{1}{24}(k-k^{-1})$ as it will not be relevant to the order of the large N expansion that we will consider.

[7] We choose the sign of the action as appropriate for a Euclidean continuation that will be implicitly assumed below.

Subleading in large T terms at each order in g_s^2 should come from the next $1/\sqrt{N}$ term in (1.10) or the 2-loop M2 brane correction.

As was shown in [2] for $w = 1$ the exponent and the 1/sin prefactor in (1.9) can be reproduced on the dual M-theory side as, respectively, the classical and the one-loop corrections in the partition function for the M2 brane in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ wrapped on 11d circle and $\mathrm{AdS}_2 \subset \mathrm{AdS}_4$ (ending on a circle at the boundary).

The aim of the present paper will be to show that one can similarly reproduce the first two terms in B in (1.10) from the classical and one-loop corrections to the M2 brane partition function computing the cusp anomaly (1.5). The corresponding M2 brane solution will be a straightforward generalization of the type IIA string solution for the line with a cusp [22, 29] wrapped also on the 11d circle.

The straight-line Wilson loop is described, like in [2], by the $\mathrm{AdS}_2 \times S^1$ M2 brane solution. In Poincare coordinates for the AdS_4 the AdS_2 metric is $z^{-2}(-\mathrm{d}t^2 + \mathrm{d}z^2)$ with t parametrizing the line. Here we will consider the M2 brane solution representing two lines with a relative angle α in AdS_4 and angle β in CP^3 and wrapped also on the angle ϕ in (1.13). It is given by the straightforward uplift of the IIA string solution in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ ending on a cusped line [29]. The IIA string world sheet is embedded into a subspace $\mathrm{AdS}_3 \times S^1$ with an angle of AdS_3 spanning the range $[\frac{\alpha}{2}, \pi - \frac{\alpha}{2}]$ corresponding to the directions of the two half-lines representing the cusp with a non-zero angle α. The coordinate of $S^1 \subset \mathrm{CP}^3$ belongs to the interval $[-\frac{\beta}{2}, \frac{\beta}{2}]$ where β is the 'internal' cusp angle.

The corresponding classical action of the M2 brane will be proportional to $\frac{1}{k}T_2$ and will match the first term in (1.10). Quantum M2 brane fluctuations around this classical solution will reproduce the $\mathcal{O}(T_2^0)$ term $\cot\frac{2\pi}{k}$ in (1.10). The computation of this one-loop M2 brane correction will be the aim of this paper.

In general, the M2 brane partition function will have the form

$$Z = \int [\mathrm{d}X\mathrm{d}\vartheta]\ \mathrm{e}^{-S[X,\vartheta]} = \mathcal{Z}_1\, \mathrm{e}^{-T_2 \bar{S}_{\mathrm{cl}}} \left[1 + \mathcal{O}\left(T_2^{-1}\right)\right] = \mathrm{e}^{-\mathcal{T}\,\Gamma_{\mathrm{cusp}}}\,, \tag{1.21}$$

$$\mathcal{Z}_1 = \mathrm{e}^{-\mathcal{T}\,\Gamma^{(1)}_{\mathrm{cusp}}}\,, \tag{1.22}$$

where (X, ϑ) are the bosonic and fermionic coordinates. $\mathcal{T} \to \infty$ is the range of the time direction t parametrizing the cusped line (it plays the role of an infrared cut off, cf the discussion above (1.5)). The one-loop $\Gamma^{(1)}_{\mathrm{cusp}}$ term should match the N^0 term in (1.10) while the 2-loop T_2^{-1} term should reproduce the subleading $N^{-1/2}$ term in (1.10).

The one-loop correction $\mathcal{Z}_1$ is given by the usual combination of determinants of the 2nd order fluctuation operators. Using static gauge and expanding the M2 brane 3d fluctuation fields in Fourier modes in S^1 we get as in [2] a tower of 2d massive fluctuation fields with the lowest $n = 0$ level corresponding to the IIA string fluctuations.

A complication compared to the circular or straight Wilson loop case in [2] is that here the induced metric is not just of a homogeneous $\mathrm{AdS}_2 \times S^1$ space as in the absence of the cusp and thus the computation of the fluctuation determinants is, in general, non-trivial. Because of the translation invariance in t the result for $\Gamma^{(1)}_{\mathrm{cusp}}$ can be represented in terms of the vacuum energy of the quadratic fluctuations around the classical solution (here I stands for the mode number labels)

$$\Gamma^{(1)}_{\mathrm{cusp}} \equiv E = \tfrac{1}{2} \sum_I (-1)^{\mathrm{F}_I}\, \omega_I\,. \tag{1.23}$$

The fluctuation energies $\omega_I(\alpha, \beta)$ may be evaluated in perturbation theory near the BPS limit, i.e. expanding in the small cusp angles like in [22]. The 'unperturbed' configuration corresponds to the straight BPS Wilson line in AdS_4 (and point-like in CP^3) for which the M2 geometry is $AdS_2 \times S^1$ (in this case $E=0$ due to supersymmetry which is explicit in the spectrum of fluctuations as in the string case in [22, 30]).[8]

As a result, as we will show below, $\Gamma^{(1)}_{\rm cusp}$ takes the form of (1.5) with the one-loop correction to the Bremsstrahlung function reproducing the subleading term in (1.10)

$$B^{(1)} = -\frac{1}{4\pi^2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{k^2n^2-2} = -\frac{1}{2\pi k}\cot\frac{2\pi}{k}\,, \qquad k>2\,, \tag{1.24}$$

$$B^{(1)} = \frac{1}{4\pi^2}\,, \quad k=1,2\,. \tag{1.25}$$

The $k=1,2$ values represent the predictions as, like for the BPS Wilson loop, the corresponding localization results for the Bremsstrahlung function are not currently available.

Let us note that the result for $B^{(1)}$ in (1.24) and (1.25) is manifestly finite. This is to be compared with the circular Wilson loop computation [2] of the one-loop $(\sin\frac{2\pi}{k})^{-1}$ prefactor in the $AdS_2 \times S^1$ M2 brane partition function where the sum over S^1 mode number n was linearly divergent and thus required the ζ-function regularization.

The plan of the rest of this paper is as follows. In section 2 we briefly review the classical solution for the IIA string in $AdS_4 \times CP^3$ with a world sheet ending on a cusped Wilson line with the geometrical angle α in AdS_4 and the internal angle β in CP^3. We then present its 11d uplift as an M2 brane embedded in $AdS_4 \times S^7/\mathbb{Z}_k$.

In section 3 we expand the M2 brane action to quadratic level and determine the spectrum of bosonic and fermionic fluctuations.

In section 4 we consider the case of $\beta=0$ and show how to reproduce the one-loop term in (1.10) from the cusp anomaly expanded in small angle α. This is achieved by using (1.23) and doing quantum-mechanical perturbation theory for the fluctuation energies in small α.

Similar analysis is repeated in section 5 for the case of a cusped Wilson line with $\alpha=0$ and small β demonstrating that this leads to the same expression (1.24) and (1.25) for the Bremsstrahlung function, in agreement with the expected BPS structure in (1.5).

2. String in $AdS_4 \times CP^3$ and M2 brane in $AdS_4 \times S^7/\mathbb{Z}_k$ solutions representing cusped Wilson line

2.1. String solution

We shall follow [13, 21] (see also [29]) and use global coordinates in AdS_4 as in (1.11). The CP^3 metric and A in (1.14) can be expressed in terms of 6 real angles as (see e.g. [35])

$$\begin{aligned} ds^2_{CP^3} &= d\gamma^2 + \cos^2\gamma\sin^2\gamma\left(d\psi + \tfrac{1}{2}\cos\theta_1\, d\varphi_1 - \tfrac{1}{2}\cos\theta_2\, d\varphi_2\right)^2 \\ &\quad + \tfrac{1}{4}\cos^2\gamma\left(d\theta_1^2 + \sin^2\theta_1\, d\varphi_1^2\right) + \tfrac{1}{4}\sin^2\gamma\left(d\theta_2^2 + \sin^2\theta_2\, d\varphi_2^2\right)\,, \end{aligned} \tag{2.1}$$

$$A = \tfrac{1}{2}\left(\cos 2\gamma\, d\psi + \cos^2\gamma\cos\theta_1\, d\varphi_1 + \sin^2\gamma\cos\theta_2\, d\varphi_2\right)\,, \tag{2.2}$$

[8] The resulting procedure of computing the quadratic in small angle terms in the one-loop determinants is closely related to the alternative interpretation of the Bremsstrahlung function as a coefficient in the 2-point function of the excitations on the Wilson line defect represented by string or M2 brane fluctuations in transverse directions (see [31, 32]). It is also somewhat similar to the one in the near short-string expansions discussed in [33, 34].

where $0 \leqslant \gamma < \frac{\pi}{2}, 0 \leqslant \psi < 2\pi, 0 \leqslant \theta_i \leqslant \pi, 0 \leqslant \varphi_i < 2\pi$. We will consider the configuration localised at

$$x = 0\,, \qquad \gamma = \tfrac{\pi}{4}\,, \qquad \theta_{1,2} = \tfrac{\pi}{2}\,, \qquad \varphi_{1,2} = 0\,, \tag{2.3}$$

and embedded into the subspace $\mathrm{AdS}_3 \times S^1 \subset \mathrm{AdS}_4 \times \mathrm{CP}^3$ with the metric

$$\mathrm{d}s^2 = \tfrac{1}{4}R^2\left(-\cosh^2\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\theta^2\right) + \tfrac{1}{4}R^2\mathrm{d}\psi^2\,. \tag{2.4}$$

The IIA string solution corresponding to the cusped Wilson line is described by

$$\rho = \rho(\theta)\,, \qquad \psi = \psi(\theta)\,, \tag{2.5}$$

with t and θ identified with the world-sheet coordinates. We will also use the coordinates (τ,σ), with τ proportional to t and σ being a particular function of θ.

The solution will have the following range of $\theta(\sigma)$ and $\psi(\sigma)$

$$\theta(\sigma) \in \left[\tfrac{\alpha}{2}, \pi - \tfrac{\alpha}{2}\right]\,, \quad \psi(\sigma) \in \left[-\tfrac{\beta}{2}, \tfrac{\beta}{2}\right]\,, \tag{2.6}$$

with the parameters α and β being the spatial cusp angle and the internal cusp angle discussed in the Introduction. The coordinate $\rho(\sigma)$ will cover the range $[\rho_{\min},\infty]$ twice.

The resulting induced string metric is

$$\mathrm{d}s^2 = \tfrac{1}{4}R^2\left[-\cosh^2\rho\,\mathrm{d}t^2 + \left(\rho'^2 + \sinh^2\rho + \psi'^2\right)\mathrm{d}\theta^2\right]\,. \tag{2.7}$$

The classical Nambu–Goto string action is then given by

$$S = T\int \mathrm{d}t\mathrm{d}\theta\, L\,, \quad L = \cosh\rho\sqrt{\rho'^2 + \sinh^2\rho + \psi'^2}\,, \tag{2.8}$$

where the effective string tension T was defined in (1.19). From (2.8) we get the conserved quantities: $\mathcal{E}$ (the 'energy' conjugate to θ) and J (the momentum corresponding to ψ).[9] The BPS limit is $\mathcal{E} = \pm J$ [29].[10] It is convenient to define the following constant parameters

$$p = \frac{1}{\mathcal{E}}\,, \quad q = \frac{J}{\mathcal{E}} = \frac{\psi'}{\sinh^2\rho}\,. \tag{2.9}$$

Let us introduce the function $\xi(\theta)$ defined in terms of $\rho(\theta)$ by [13]

$$\xi(\theta) = \frac{1}{b}\sqrt{\frac{p^2+b^4}{p^2\sinh^2\rho + b^2}}\,, \quad b^2 = \frac{p^2-q^2}{2} + \sqrt{p^2 + \frac{(p^2-q^2)^2}{4}}\,, \tag{2.10}$$

and also the parameter ε obeying

$$\varepsilon^2 = \frac{b^2\left(b^2-p^2\right)}{p^2+b^4}\,, \quad p^2 = \frac{b^4\left(1-\varepsilon^2\right)}{b^2+\varepsilon^2}\,, \quad q^2 = \frac{b^2\left(1-2\varepsilon^2-\varepsilon^2 b^2\right)}{b^2+\varepsilon^2}\,. \tag{2.11}$$

[9] Sign of $\mathcal{E}$ is conventional and we follow [29]. The explicit expressions are $-\mathcal{E} = \rho'\frac{\partial L}{\partial\rho'} + \psi'\frac{\partial L}{\partial\psi'} - L = -\frac{\sinh^2\rho\cosh\rho}{\sqrt{\rho'^2+\psi'+\sinh^2\rho}}$ and $J = \frac{\partial L}{\partial\psi'} = \frac{\cosh\rho\ \psi'}{\sqrt{\rho'^2+\psi'+\sinh^2\rho}}$.

[10] At these special points the classical action vanishes after adding a suitable boundary action required to have the correct Neumann boundary conditions for some of the string coordinates [36].

Solving the equations of motion for $\theta(\xi)$ one finds that the cusp angles are given in terms of the elliptic functions (with modulus ε^2)[11]

$$\alpha = \pi - \frac{2p^2}{b\sqrt{p^2+b^4}}\left[\Pi\left(\frac{b^4}{p^2+b^4}\middle|\varepsilon^2\right) - \mathbb{K}\left(\varepsilon^2\right)\right] , \quad \beta = \frac{2bq}{\sqrt{p^2+b^4}}\mathbb{K}\left(\varepsilon^2\right) . \tag{2.12}$$

The small angle limit is $p \gg 1$ with fixed q when

$$b = p + \frac{1-q^2}{2p} + \cdots , \qquad \varepsilon^2 = \frac{1-q^2}{p^2} + \cdots , \quad \alpha = \frac{\pi}{p} + \cdots , \quad \beta = \frac{\pi q}{p} + \cdots . \tag{2.13}$$

The (regularized) value of the classical action is

$$S_{\rm cl} = \mathcal{T}T\frac{2\sqrt{b^4+p^2}}{bp}\left[\frac{\left(b^2+1\right)p^2}{p^2+b^4}\mathbb{K}\left(\varepsilon^2\right) - \mathbb{E}\left(\varepsilon^2\right)\right] = \mathcal{T}T\left[\frac{\pi\left(q^2-1\right)}{2p^2} + \mathcal{O}\left(p^{-3}\right)\right]$$
$$= \frac{1}{2\pi}\mathcal{T}T\left(\beta^2-\alpha^2\right) + \cdots , \qquad \mathcal{T} \equiv \int \mathrm{d}t . \tag{2.14}$$

Comparing to (1.5) and (1.21) we conclude that we reproduce the leading planar strong-coupling (string tree-level) term in the ABJM Bremsstrahlung function as given in (1.6)

$$B^{(0)} = \frac{1}{2\pi}T = \frac{1}{2\pi}\sqrt{\frac{\lambda}{2}} . \tag{2.15}$$

Elliptic parametrisation

Replacing ρ by ξ introduced in (2.10) the induced metric (2.7) can be written as

$$\mathrm{d}s^2 = \frac{R^2}{4\xi^2}\left[-\frac{1+b^2}{b^2}\frac{1-\varepsilon^2\xi^2}{1-\varepsilon^2}\mathrm{d}t^2 + \frac{\mathrm{d}\xi^2}{1-\xi^2}\right] . \tag{2.16}$$

Let us now define the world-sheet coordinates (σ, τ) as

$$\sigma = F\left(\arcsin\xi|\varepsilon^2\right) - \mathbb{K} \qquad \in [-\mathbb{K}, \mathbb{K}] , \quad \tau = \frac{1}{b}\sqrt{\frac{1+b^2}{1-\varepsilon^2}}\, t \in \mathbb{R} , \tag{2.17}$$

$$\xi = \mathrm{sn}\left(\sigma + \mathbb{K}\right) = \frac{\mathrm{cn}\,\sigma}{\mathrm{dn}\,\sigma} , \tag{2.18}$$

where $\mathbb{K} \equiv \mathbb{K}(\varepsilon^2)$, F is the elliptic integral of the first kind and $\mathrm{cn}(\sigma) \equiv \mathrm{cn}(\sigma|\varepsilon^2)$ is a Jacobi elliptic function. This allows to write the metric (2.16) in the conformally flat form

$$\mathrm{d}s^2 = \tfrac{1}{4}R^2\frac{1-\varepsilon^2}{\mathrm{cn}^2\,\sigma}\left(-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2\right) . \tag{2.19}$$

Note that $\mathrm{cn}\,\sigma$ function here has an implicit dependence on ε. For zero cusp angle, i.e. $\varepsilon = 0$, we get $\mathrm{cn}\,\sigma \to \cos\sigma$ so that (2.19) reduces to the AdS_2 metric

$$\mathrm{d}s^2 = \tfrac{1}{4}R^2\frac{1}{\cos^2\sigma}\left(-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2\right) . \tag{2.20}$$

Expressed in terms of σ the string solution has the following explicit form

$$\theta(\sigma) = \frac{\pi}{2} + \frac{p^2}{b\sqrt{p^2+b^4}}\left[\sigma - \Pi\left(\tfrac{b^4}{p^2+b^4}, \mathrm{am}\left(\sigma+\mathbb{K}\right)|\varepsilon^2\right) + \Pi\left(\tfrac{b^4}{p^2+b^4}|\varepsilon^2\right)\right] \quad \psi(\sigma) = \frac{bq}{\sqrt{p^2+b^4}}\sigma . \tag{2.21}$$

[11] One finds two branches for the functions $\theta(\rho)$ and $\psi(\rho)$: one with ρ ranging from ∞ (AdS boundary) to a minimal value, and another with ρ growing from a minimal value to infinity.

Small angle expansions

Starting with the relations in (2.10)–(2.12) and expanding at large p for fixed q gives

$$b^2 = p^2+1-q^2+\frac{q^2-1}{p^2}+\frac{\left(q^2-1\right)\left(q^2-2\right)}{p^4}+\cdots, \quad \varepsilon^2=\frac{1-q^2}{p^2}-\frac{\left(q^2-1\right)\left(q^2-3\right)}{p^4}+\cdots,$$
$$\alpha=\frac{\pi}{p}+\frac{\pi\left(3q^2-5\right)}{4p^3}+\cdots, \quad \beta=\frac{\pi q}{p}+\frac{\pi q\left(q^2-3\right)}{4p^3}+\cdots. \tag{2.22}$$

For $\beta=0$ (that corresponds to $q=0$) the small ε limit is the same as the small α limit, i.e. we get

$$\varepsilon^2=\frac{1}{p^2}-\frac{3}{p^4}+\cdots, \quad \alpha=\frac{\pi}{p}-\frac{5\pi}{4p^3}+\cdots, \quad \frac{\alpha^2}{\pi^2}=\varepsilon^2+\frac{1}{2}\varepsilon^4+\cdots. \tag{2.23}$$

Another useful limit is large p at fixed imaginary ε. From the relations in (2.11) and (2.22) we find

$$b^2=\frac{p^2}{1-\varepsilon^2}+\varepsilon^2-\frac{\left(1-\varepsilon^2\right)\varepsilon^4}{p^2}+\cdots, \quad q^2=-\frac{\varepsilon^2p^2}{1-\varepsilon^2}+1-2\varepsilon^2-\frac{\varepsilon^2\left(1-2\varepsilon^2+2\varepsilon^4\right)}{p^2}+\cdots,$$
$$\alpha=\mathcal{O}\left(p^{-1}\right), \quad \frac{\beta^2}{\pi^2}=-\frac{\left(4-5\varepsilon^2\right)^2\varepsilon^2}{16\left(1-\varepsilon^2\right)^3}+\mathcal{O}\left(p^{-2}\right). \tag{2.24}$$

Thus the case of $\alpha=0$ is obtained by taking $p\to\infty$. Then further expansion in small ε corresponds to the small β expansion (cf (2.23))

$$\frac{\beta^2}{\pi^2}=-\varepsilon^2-\frac{1}{2}\varepsilon^4+\cdots. \tag{2.25}$$

Case of $\beta=0$

Let us record the explicit form of the solution in the special case of the vanishing internal angle $\beta=0$ which is obtained for $q=0$. Then the above functions can be expressed in terms of the parameter ε, i.e.

$$b^2=\frac{1-2\varepsilon^2}{\varepsilon^2}, \qquad p^2=\frac{\left(1-2\varepsilon^2\right)^2}{\varepsilon^2\left(1-\varepsilon^2\right)}, \qquad \tau=\frac{1}{\sqrt{1-2\varepsilon^2}}t, \qquad \cosh^2\rho=\frac{1-\varepsilon^2}{1-2\varepsilon^2}\frac{1}{\mathrm{cn}^2\sigma}, \tag{2.26}$$

$$\theta(\sigma)=\frac{\pi}{2}+\varepsilon\sqrt{\frac{1-2\varepsilon^2}{1-\varepsilon^2}}\left[\sigma-\Pi\left(1-\varepsilon^2,\mathrm{am}\left(\sigma+\mathbb{K}\right)|\varepsilon^2\right)+\Pi\left(1-\varepsilon^2|\varepsilon^2\right)\right], \quad \psi(\sigma)=0. \tag{2.27}$$

In this case the relation between the small cusp angle α and $\varepsilon\ll 1$ is given in (2.23)

$$\alpha=\pi\left(\varepsilon+\tfrac{1}{4}\varepsilon^3+\ldots\right). \tag{2.28}$$

For the subsequent analysis of fluctuations it is useful to record the values of the following derivatives with respect to σ

$$\theta'^2(\sigma)=\frac{\varepsilon^2\left(1-\varepsilon^2\right)}{1-2\varepsilon^2}\frac{1}{\sinh^4\rho}, \quad \rho'^2(\sigma)=\frac{-1+\left(1-2\varepsilon^2\right)^2\cosh^2(2\rho)}{4\left(1-2\varepsilon^2\right)\sinh^2\rho}. \tag{2.29}$$

2.2. M2 brane solution

It is straightforward to uplift of the above 10d string cusp solution to 11d M2 brane solution wrapped also on 11d circle (cf (1.11)–(1.14)) so that the latter reduces to the former upon the 'double dimensional reduction' [37]. Let us consider, for example, the special case of $\beta = 0$ when the solution is localized at a point in CP^3 (in addition to (2.3)). Then the analog of the induced metric (2.4) written in the world-volume coordinates (t, θ, ϕ) is

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}\sigma^i\mathrm{d}\sigma^j = \tfrac{1}{4}R^2\left[-\cosh^2\rho\,\mathrm{d}t^2 + \left(\rho'^2 + \sinh^2\rho\right)\mathrm{d}\theta^2\right] + \frac{1}{k^2}R^2\,\mathrm{d}\phi^2\,. \tag{2.30}$$

Note that as follows from (1.15)

$$F_4 = \mathrm{d}C_3 = -\tfrac{3}{8}R^3\cosh\rho\,\sinh^2\rho\,\cos x\;\mathrm{d}t\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}x\,, \tag{2.31}$$

$$C_3 = \tfrac{3}{8}R^3\cosh\rho\,\sinh^2\rho\,\sin x\,\mathrm{d}t\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta\,. \tag{2.32}$$

Since on the solution $x = 0$ we have vanishing C_3 contribution to the M2 brane action in (1.17).

The volume part of the M2 brane action is

$$S_{\rm V} = \frac{1}{4k}T_2\int \mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\phi\,\cosh\rho\sqrt{\rho'^2 + \sinh^2\rho}\,, \tag{2.33}$$

which is the same as in the type IIA string case, i.e. we again reproduce (2.15).[12]

3. M2 brane fluctuations near AdS_4 cusp solution

Let us now derive the spectrum of masses of quadratic fluctuations around the M2 brane analog of the $\beta = 0$ solution (2.27). We shall use a static gauge where, in particular, the 11d angle ϕ is identified with the second spatial world-volume coordinate, i.e. is not fluctuating.

3.1. Bosonic fluctuations

$S^7/\mathbb{Z}_k$ scalars

As the $\beta = 0$ solution is trivial in CP^3, the $S^7/\mathbb{Z}_k$ fluctuations are completely decoupled from the AdS_4 fluctuations. This is similar to the case of the $AdS_2 \times S^1$ M2 brane solution corresponding to the circular Wilson loop case in [2, 38]. To quadratic order in the remaining 3 complex fluctuations of w^s in (1.14) (that have trivial classical values) we have

$$\mathrm{d}s^2_{S^7/\mathbb{Z}_k} = \mathrm{d}w^s\mathrm{d}\bar{w}^s + \frac{1}{k^2}\mathrm{d}\phi^2 + \frac{i}{k}\left(w^s\mathrm{d}\bar{w}^s - \bar{w}^s\mathrm{d}w^s\right)\mathrm{d}\phi + \cdots\,. \tag{3.1}$$

Using the indices $a, b = 0, 1$ for (τ, σ) and $i, j = 0, 1, 2$ for (τ, σ, ϕ) and denoting by prime the derivative with respect to ϕ, we have (cf (2.19))

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}\sigma^i\mathrm{d}\sigma^j\,, \quad g_{ij} = g^{(0)}_{ij} + \delta g_{ij}\,, \tag{3.2}$$

$$g^{(0)}_{ij} = R^2\begin{pmatrix}\frac{1}{4}\hat{g}_{ab} & 0\\ 0 & \frac{1}{k^2}\end{pmatrix}, \quad \hat{g}_{ab}\mathrm{d}\sigma^a\mathrm{d}\sigma^b = \frac{1-\varepsilon^2}{\mathrm{cn}^2\,\sigma}\left(-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2\right)\,, \tag{3.3}$$

[12] Since $\int \mathrm{d}\phi = 2\pi$ we get the prefactor of the integral over t and θ as $\frac{2\pi R}{k}\frac{R^2}{4}T_2 = \frac{\pi}{2k}T_2 = \frac{\sqrt{2kN}}{\pi}\frac{\pi}{2k} = \sqrt{\frac{\lambda}{2}} = T$.

$$\delta g_{ij} = R^2 \begin{pmatrix} \partial_a w^s \partial_b \bar{w}^s & h_{a3} \\ h_{b3} & w'^s \bar{w}'^s + \frac{i}{k}\left(w^s \bar{w}'^s - \bar{w}'^s w^s\right) \end{pmatrix} , \tag{3.4}$$

$$h_{a3} = \tfrac{1}{2} \bar{w}'^s \partial_a w^s + \tfrac{1}{2} w'^s \partial_a \bar{w}^s + \tfrac{i}{2k}\left(w^s \partial_a \bar{w}^s - \bar{w}^s \partial_a w^s\right) . \tag{3.5}$$

The WZ part of the M2 brane action does not contribute in this sector (cf (2.31)) while the quadratic fluctuation term in the volume part of the M2 brane action is found to be

$$T_2 \int \mathrm{d}^3\sigma\, \delta\left(\sqrt{-g}\right) = \frac{1}{2} T_2 \int \mathrm{d}^3\sigma \sqrt{-g^{(0)}}\, g^{(0)ij} \delta g_{ij}$$
$$= \frac{1}{8k} T_2 \int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\phi \,\sqrt{-\hat{g}}\, \left[4 \hat{g}^{ab} \partial_a w^s \partial_b \bar{w}^s + k^2 w'^s \bar{w}'^s + ik\left(w^s \bar{w}'^s - \bar{w}'^s w^s\right)\right] . \tag{3.6}$$

Expanding in Fourier modes in the ϕ coordinate, $w^s(\tau,\sigma,\phi) = \sum_n \mathrm{e}^{in\phi} w^s_n(\tau,\sigma)$, we get for the corresponding fluctuation Lagrangian

$$L_2 = 4\sqrt{-\hat{g}} \sum_n \left[\hat{g}^{ab} \partial_a w^s_{-n} \partial_b \bar{w}^s_n + \tfrac{1}{4}\left(k^2 n^2 + 2kn\right) w^s_{-n} \bar{w}^s_n\right] . \tag{3.7}$$

It describes three towers of complex scalar 2d fluctuations propagating in the metric $\hat{g}_{ab}$ in (3.3) with the masses (cf (2.11) in [2])

$$S^7/\mathbb{Z}_k: \quad m_n^2 = \tfrac{1}{4} kn\,(kn+2) , \quad n = 0, \pm 1, \pm 2, \ldots . \tag{3.8}$$

The case of $n=0$ corresponds to the massless 2d fields found in the string theory case. For $k \geqslant 2$ the values of m_n^2 are positive[13].

AdS₄ scalars

Using the AdS_4 metric in (1.12) and (1.13) with the coordinates (t,ρ,x,θ) let us fix the static gauge as[14]

$$\delta t = 0 , \quad \delta\theta = 0 , \quad \delta\phi = 0 , \tag{3.9}$$

and define

$$\delta\rho = -\sqrt{1 + \frac{\rho'^2}{\sinh^2\rho\, \theta'^2}}\, Z(\tau,\sigma,\phi) , \quad \delta x = \frac{1}{\sinh\rho} X(\tau,\sigma,\phi) . \tag{3.10}$$

Here $\rho(\sigma)$ and $\theta(\sigma)$ are the classical solution functions in (2.26) and (2.27) (see also (2.29)) and Z and X represent the two non-trivial AdS_4 3d fluctuation functions. Then the quadratic term in the expansion of the volume term in the M2 action may be written as

$$S_{2,\mathrm{V}} = \tfrac{1}{8} T_2 \int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\phi\, \sqrt{-g^{(0)}} \left[R^2 \left(g^{(0)}\right)^{ij} \left(\partial_i Z \partial_j Z + \partial_i X \partial_j X\right) + 4\left(4 + R^{(2)}\right) Z^2 + 8X^2\right] . \tag{3.11}$$

Here $g^{(0)}_{ij}$ was defined in (3.3) and $R^{(2)}$ is the scalar curvature of the 2d metric $\hat{g}_{ab}$ in (3.3)

$$R^{(2)} = -2\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2} \mathrm{cn}^4\sigma\right) . \tag{3.12}$$

[13] Note that for $k=1$ and $n=-1$ we get $m^2_{-1} = -\frac{1}{4}$ which saturates the stability bound in AdS_2 (to which the metric $\hat{g}_{ab}$ reduces in the zero cusp limit) as the corresponding conformal dimension given by $h = \frac{1}{2}(1+\sqrt{1+4m^2}) = \frac{1}{2}$.
[14] In the string case an alternative approach is discussed in [13].

Splitting the derivatives into the τ, σ and ϕ ones we may write (3.11) as

$$S_{2,\mathrm{V}} = \tfrac{1}{8k}\mathrm{T}_2 \int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\phi \, \sqrt{-\hat{g}} \, \Big[\hat{g}^{ab}\left(\partial_a Z \partial_b Z + \partial_a X \partial_b X\right) + \left(4 + R^{(2)}\right) Z^2 + 2X^2 + \tfrac{1}{4}k^2 \left(Z'^2 + X'^2\right)\Big] \ . \tag{3.13}$$

To find the fluctuation part of the WZ term $S_{\mathrm{WZ}} = T_2 \int C_3$ in the M2 brane action in (1.17) we note that according to (2.31)

$$\begin{aligned} C_3 &= \tfrac{3}{8} R^3 \cosh(\rho + \delta\rho) \sinh^2(\rho + \delta\rho) \, \frac{1}{\sinh \rho} \, (X + \cdots) \, \sqrt{1 - 2\varepsilon^2} \, \mathrm{d}\tau \wedge (\rho' \mathrm{d}\sigma + \mathrm{d}\delta\rho) \wedge \theta' \mathrm{d}\sigma \\ &= -\tfrac{3}{8} R^3 \sqrt{1 - 2\varepsilon^2} \cosh\rho \sinh\rho \, \theta' X \delta\rho' \, \mathrm{d}\tau \wedge \mathrm{d}\sigma \wedge \mathrm{d}\phi + \cdots . \end{aligned} \tag{3.14}$$

Note that in our notation $\rho' \equiv \partial_\sigma \rho(\sigma)$, $\theta' \equiv \partial_\sigma \theta'(\sigma)$, while for the fluctuations $\delta\rho'(\tau, \sigma, \phi) \equiv \partial_\phi \delta\rho$, etc. Expressing $\delta\rho$ in terms of Z in (3.10) and using the explicit form of the solution functions $\rho(\sigma), \theta(\sigma)$ we get

$$S_{2,\mathrm{WZ}} = \tfrac{3}{8} T_2 \left(1 - 2\varepsilon^2\right) \int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\phi \, \cosh^2\rho \, X Z' = \tfrac{3}{8} \mathrm{T}_2 \int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}\phi \, \sqrt{-\hat{g}} X Z' \ . \tag{3.15}$$

Combining (3.13) and (3.15) we get for the corresponding quadratic fluctuation Lagrangian (factoring out $\frac{1}{2}\sqrt{-\hat{g}}$)

$$L_{2,\mathrm{AdS}} = \hat{g}^{ab}\left(\partial_a Z \partial_b Z + \partial_a X \partial_b X\right) + \left(4 + R^{(2)}\right) Z^2 + 2X^2 + \tfrac{1}{4}k^2 \left(Z'^2 + X'^2\right) + \tfrac{3}{2}k \left(X Z' - X' Z\right) \ . \tag{3.16}$$

Note that for zero cusp angle, i.e. $\varepsilon = 0$, when $\hat{g}_{ab}$ in (3.3) reduces to the AdS_2 metric (cf (2.20)) and $R^{(2)}$ in (3.12) takes the -2 value, the mass terms of Z and X become the same. Then after the Fourier expansion in ϕ we get the same two towers of massive 2d fields in AdS_2 as found in [2]

$$\varepsilon = 0: \qquad m_n^2 = \tfrac{1}{4}(kn - 2)(kn - 4) \ . \tag{3.17}$$

For generic ε expanding in Fourier modes one finds the following 4×4 mass-squared matrix for the $\sin(n\phi)$, $\cos(n\phi)$ modes of Z and X

$$\mathcal{M}_n^2 = \begin{pmatrix} 2 + \frac{R^{(2)}}{2} + \frac{k^2 n^2}{8} & 0 & 0 & \frac{3kn}{4} \\ 0 & 2 + \frac{R^{(2)}}{2} + \frac{k^2 n^2}{8} & -\frac{3kn}{4} & 0 \\ 0 & -\frac{3kn}{4} & 1 + \frac{k^2 n^2}{8} & 0 \\ \frac{3kn}{4} & 0 & 0 & 1 + \frac{k^2 n^2}{8} \end{pmatrix} . \tag{3.18}$$

The corresponding eigenvalues are (each with multiplicity 2)

$$m_{n,\pm}^2 = 3 + \tfrac{1}{2} R^{(2)} + \tfrac{1}{4} k^2 n^2 \pm \tfrac{1}{2} \sqrt{\left(2 + R^{(2)}\right)^2 + 9k^2 n^2} \ . \tag{3.19}$$

Considering the limit when ε is small (and thus $2 + R^{(2)} < 0$ according to (3.12)) we get[15]

$$m_{0,\pm}^2 = 3 + \tfrac{1}{2} R^{(2)} \pm \tfrac{1}{2} |2 + R^{(2)}| = \begin{cases} 2 \\ 4 + R^{(2)} \end{cases} = \begin{cases} 2 \\ 2 - 2\cos^4\sigma \, \varepsilon^2 + \cdots \end{cases} , \tag{3.20}$$

[15] The $\pm$ signs in (3.19) are taken into account in (3.21) by the fact that $n \in \mathrm{Z}$.

$$m_{n,\pm}^2 = \tfrac{1}{4}(kn-2)(kn-4) - \cos^4\sigma\,\varepsilon^2 + \cdots, \qquad n = \pm 1, \pm 2, \ldots \,. \tag{3.21}$$

Note that the coefficient of the $\cos^4\sigma$ term is different in the cases $n \neq 0$ and $n = 0$.

3.2. Fermionic fluctuations

The general structure of the fermionic part of the M2 action in 11d background in (1.17) is

$$S_{\rm F} = T_2 \int d^3\sigma \left[\sqrt{-g}\, g^{ij}\partial_i X^{\rm M}\bar\vartheta\Gamma_{\rm M}\hat D_j\vartheta - \tfrac{1}{2}\varepsilon^{ijk}\partial_i X^{\rm M}\partial_j X^{\rm N}\bar\vartheta\Gamma_{\rm MN}\hat D_k\vartheta + \cdots\right], \tag{3.22}$$

$$g_{ij} = \partial_i X^{\rm M}\partial_j X^{\rm N} G_{\rm MN}(X), \qquad G_{\rm MN} = E^A_{\rm M}E^A_{\rm N}, \qquad \Gamma_{\rm M} = E^A_{\rm M}(X)\,\Gamma_A, \tag{3.23}$$

$$\hat D_i = \partial_i X^{\rm M}\hat D_{\rm M}, \qquad \hat D_{\rm M} = \partial_{\rm M} + \tfrac{1}{4}\Gamma_{\rm AB}\Omega^{AB}_{\rm M} - \tfrac{1}{288}\left(\Gamma^{\rm PNKL}_{\rm M} - 8\,\Gamma^{\rm PNK}\delta^L_{\rm M}\right)F_{\rm PNKL}\,. \tag{3.24}$$

In the type IIA GS string limit of the cusp solution one finds the Dirac action for the 3+3 2d fermions with masses ± 1 and 2 fermions with mass 0 [21]. Being independent of ε these are the same values as in the case of the BPS Wilson loop.

In the case of the $AdS_2 \times S^1$ M2 brane solution without cusp one finds as in [2, 38] 8 towers of 2d fermionic modes with masses

$$m_n = \tfrac{1}{2}kn \pm 1 \quad (3+3\ \text{ modes } \vartheta_\pm);\ m_n = \tfrac{1}{2}kn \quad (2\ \text{ modes } \vartheta'), \qquad n = 0, \pm 1, \pm 2, \ldots \,. \tag{3.25}$$

These reduce to the IIA string values for $n = 0$.

In the presence of the cusp, i.e. for $\varepsilon \neq 0$, the structure of the M2 brane action and the factorized form of the M2 brane solution in $AdS_4 \times S^7/\mathbb{Z}_k$ implies that one can get the M2 brane fermionic masses from their IIA string values by the same $\frac{1}{2}kn$ shift. The reason is that the fermion operator depends just on the induced metric and the F_4 background and thus should be universal. A similar pattern in the structure of the 11d fermion spectrum was observed in [3, 6, 39]. Detailed form of the Dirac operator in the induced metric (2.19) will be given in the next section.

4. One-loop M2 brane correction for small cusp in AdS_4

The log of the resulting one-loop M2 brane partition function (1.21) may be written as the sum of the contributions of 8+8 bosonic and fermionic towers of 2d fields, i.e. symbolically,

$$-\log \mathcal{Z}_1 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{p=1}^{8}\left(\log\det\Delta_{B_p,n} - \log\det\Delta_{F_p,n}\right). \tag{4.1}$$

Here the massive scalar Laplacian Δ_B and the fermionic operator Δ_F are defined using the 2d metric $\hat g_{ab}$ in (3.3)

$$ds^2 = \frac{1-\varepsilon^2}{\mathrm{cn}^2\sigma}\left(-d\tau^2 + d\sigma^2\right), \qquad -\mathbb{K} < \sigma < \mathbb{K}, \tag{4.2}$$

i.e. Δ_B has the following structure

$$\Delta_B = -\frac{1}{\sqrt{-\hat g}}\partial_a\left(\sqrt{-\hat g}\,\hat g^{ab}\partial_b\right) + m^2\,. \tag{4.3}$$

Rescaling all the operators by $\sqrt{-\hat g}$ and using that (4.2) is conformally flat we get

$$\Delta'_B = \partial_\tau^2 - \partial_\sigma^2 + \frac{1-\varepsilon^2}{\mathrm{cn}^2\sigma}m^2\,. \tag{4.4}$$

The effect of such rescaling should be trivial after summing over n as there is no Weyl anomaly in 3 dimensions[16]. Due to the translational invariance in τ, integrating over the corresponding momentum component one can, by the standard argument, express (4.1) in terms of the sum over the characteristic frequencies or eigenvalues of the spatial 1d operator in (4.4) (cf (1.22) and (1.23))

$$-\log \mathcal{Z}_1 = \mathcal{T} E\,, \quad E = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{p=1}^{8} \sum_{\ell} \left[\omega_{B_p,\ell}(n) - \omega_{F_p,\ell}(n)\right]\,, \quad \mathcal{T} = \int \mathrm{d}\tau\,, \tag{4.5}$$

where the bosonic frequencies ω_ℓ are defined by

$$\left(-\partial_\sigma^2 + \frac{1-\varepsilon^2}{\mathrm{cn}^2\,\sigma} m^2\right) \Phi_\ell = \omega_\ell^2 \Phi_\ell\,. \tag{4.6}$$

The fermionic ω_ℓ are the eigenvalues of the corresponding Dirac operator

$$\left[-i\gamma^1 \left(\partial_\sigma + \frac{\mathrm{sn}\,\sigma\,\mathrm{dn}\,\sigma}{2\,\mathrm{cn}\,\sigma}\right) + \frac{\sqrt{1-\varepsilon^2}}{\mathrm{cn}\,\sigma} m\right] \Psi_\ell = \gamma^0 \omega_\ell \Psi_\ell\,, \tag{4.7}$$

where γ^a are the 'flat' 2d gamma matrices defined in terms of the Pauli matrices $\hat{\sigma}_i$ as

$$\gamma^a = (\hat{\sigma}_1,\, i\hat{\sigma}_3)\,. \tag{4.8}$$

The explicit values of the masses were given in (3.8) and (3.19) for the bosons and in (3.25) for the fermions.

Given a complicated σ-dependent form of the operators in (4.6) and (4.7) determining their spectrum is, in general, a non-trivial problem. As we are interested in the correction to the Bremsstrahlung function, it is sufficient to find the leading terms in ω_ℓ in the small cusp angle α (or small ε, cf (2.13)) expansion. To do this we shall follow the same approach as was used in [22] in the corresponding IIA string case[17].

For $\varepsilon = 0$ we have $\mathrm{cn}\,\sigma \to \cos\sigma$ so that the operators in (4.6) and (4.7) reduce to the corresponding (rescaled) ones in the AdS_2 case. The associated eigen-functions Φ_ℓ, Ψ_ℓ were discussed, e.g. in [43]. For the scalars with the Dirichlet boundary conditions one finds

$$\Phi_{h,\ell}(\sigma) = \frac{\sqrt{\ell!\,\Gamma(\ell+2h)\,(\ell+h)}}{2^{h-\frac{1}{2}}\,\Gamma\left(\ell+h+\frac{1}{2}\right)} \cos^h\sigma\, P_\ell^{\left(h-\frac{1}{2},h-\frac{1}{2}\right)}(\sin\sigma)\,, \tag{4.9}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi_{h,\ell}(\sigma)\, \Phi_{h,\ell'}(\sigma) = \delta_{\ell,\ell'}\,, \tag{4.10}$$

$$\omega_\ell = \ell + h\,, \quad h = \tfrac{1}{2}\left(1+\sqrt{1+4m^2}\right)\,, \quad \ell = 0,1,2,\dots\,, \tag{4.11}$$

[16] The non-trivial part of the total Weyl anomaly cancels already in the GS string case, i.e. at the $n = 0$ level [30]. There is formally a residual UV divergence proportional to the Euler number that should cancel against the path integral measure [40]. This divergence and thus the Weyl anomaly cancels automatically in the M2 brane case as there are no log UV divergences in the 3d case [2]. Possible subtleties related to a Weyl rescaling of the 2d metric were discussed in the string context in [41].

[17] This perturbative approach does not require to write the second order fluctuation operators in the explicit Lamé form (for a similar near-AdS_2 expansion see [42]). Also, as discussed in [22], in this approach it is straightforward to implement the fermionic boundary conditions compatible with the $\mathcal{N} = 6$ supersymmetry in the zero cusp limit.

where h denotes the corresponding AdS_2 conformal dimension (i.e. $m^2 = h(h-1)$). For the fermions in AdS_2 with mass $m < \frac{1}{2}$ we have

$$\Psi_{h,\ell} = \left(\psi^1_{h,\ell}, \psi^2_{h,\ell}\right) , \quad \psi^1_{h,\ell}(\sigma) = \frac{\sqrt{\ell!\,\Gamma(\ell+2h)}}{2^{h-\frac{1}{2}}\,\Gamma(\ell+h)} \cos^h\sigma \cos\left(\tfrac{\sigma}{2} + \tfrac{\pi}{4}\right) P^{(h,h-1)}_\ell(\sin\sigma) , \tag{4.12}$$

$$\begin{aligned} \psi^2_{h,\ell}(\sigma) &= -\frac{\sqrt{\ell!\,\Gamma(\ell+2h)}}{2^{h-\frac{1}{2}}\,\Gamma(\ell+h)} \cos^h\sigma \sin\left(\tfrac{\sigma}{2} + \tfrac{\pi}{4}\right) P^{(h-1,h)}_\ell(\sin\sigma) , \\ \omega_\ell &= \ell + h , \qquad h = \tfrac{1}{2} - m . \end{aligned} \tag{4.13}$$

For the fermions with $m > -\frac{1}{2}$ we have instead

$$\Psi_{h,\ell} = \left(\chi^1_{h,\ell}, \chi^2_{h,\ell}\right) , \qquad \chi^1_{h,\ell}(\sigma) = \frac{\sqrt{\ell!\,\Gamma(\ell+2h)}}{2^{h-\frac{1}{2}}\,\Gamma(\ell+h)} \cos^h\sigma \cos\left(\tfrac{\sigma}{2} + \tfrac{\pi}{4}\right) P^{(h-1,h)}_\ell(\sin\sigma) , \tag{4.14}$$

$$\begin{aligned} \chi^2_{h,\ell}(\sigma) &= -\frac{\sqrt{\ell!\,\Gamma(\ell+2h)}}{2^{h-\frac{1}{2}}\,\Gamma(\ell+h)} \cos^h\sigma \sin\left(\tfrac{\sigma}{2} + \tfrac{\pi}{4}\right) P^{(h,h-1)}_\ell(\sin\sigma) , \\ \omega_\ell &= \ell + h , \qquad h = \tfrac{1}{2} + m . \end{aligned} \tag{4.15}$$

In both cases[18]

$$\int_{-\pi/2}^{\pi/2} \frac{d\sigma}{\cos\sigma} \Psi^\dagger_{h,\ell} \Psi_{h,\ell'} = \delta_{\ell,\ell'} . \tag{4.16}$$

Let us now find the first non-trivial terms in the small ε expansion of the eigenvalues ω_ℓ for $n \neq 0$ first assuming $k > 2$ and then discussing the special cases of $k = 1, 2$.

4.1. Expansion of bosonic fluctuation frequencies

CP^3 scalars

For the 6 CP^3 scalars with the mass in (3.8) the AdS_2 conformal dimension in (4.11) is (assuming $k > 2$)

$$h_n = \tfrac{1}{2} + \tfrac{1}{2}|kn+1| = \begin{cases} 1 + \tfrac{1}{2}kn , & n \geqslant 0 , \\ -\tfrac{1}{2}kn , & n < 0 . \end{cases} \tag{4.17}$$

As in [22] let us rescale $\sigma \to \tilde{\sigma}$ and $\omega_\ell \to \tilde{\omega}_\ell$ as

$$\tilde{\sigma} = \frac{\pi}{2\mathbb{K}}\sigma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] , \qquad \tilde{\omega}_\ell = \frac{2\mathbb{K}}{\pi}\omega_\ell , \qquad \frac{2\mathbb{K}}{\pi} = 1 + \tfrac{1}{4}\varepsilon^2 + \cdots . \tag{4.18}$$

Then (4.6) takes the form

$$\left[-\partial^2_{\tilde{\sigma}} + \left(\frac{2\mathbb{K}}{\pi}\right)^2 \frac{1-\varepsilon^2}{\mathrm{cn}^2\left(\frac{2\mathbb{K}}{\pi}\tilde{\sigma}\right)} m^2\right] \Phi_{h,\ell} = \tilde{\omega}^2_\ell \Phi_{h,\ell} . \tag{4.19}$$

[18] For fermions with $|m| \leq \frac{1}{2}$ it is possible to adopt alternative quantizations, i.e. choose one of the two possibilities above [44–46]. Here this will happen only when $m = 0$, which may occur (for generic k) only for $n = 0$.

As m in (3.8) is independent of ε, expanding in small ε gives

$$\left(-\partial_{\tilde{\sigma}}^2 + \frac{m^2}{\cos^2\tilde{\sigma}} - \tfrac{1}{2}\varepsilon^2 m^2 + \cdots\right)\Phi_{h,\ell} = \tilde{\omega}_\ell^2 \Phi_{h,\ell}\,, \tag{4.20}$$

and thus using (4.10) we get

$$\tilde{\omega}_\ell^2 = (\ell + h_n)^2 - \tfrac{1}{2}\varepsilon^2 m^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\, \Phi_{h_n,\ell}^2 + \cdots = (\ell+h_n)^2 - \tfrac{1}{2}\varepsilon^2 h_n(h_n-1) + \cdots\,. \tag{4.21}$$

As a result, for the original ω_ℓ in (4.18) we obtain

$$\omega_\ell = (\ell + h_n)\left(1 - \tfrac{1}{4}\varepsilon^2\right) - \frac{h_n(h_n-1)}{4(\ell+h_n)}\varepsilon^2 + \cdots\,. \tag{4.22}$$

AdS$_4$ scalars

For the $\varepsilon = 0$ value of masses in (3.17) $m^2 = \frac{1}{4}(kn-2)(kn-4)$ we find from (4.11) (assuming again that $k > 2$)

$$h_n = \tfrac{1}{2} + \tfrac{1}{2}|kn-3| = \begin{cases} -1 + \tfrac{1}{2}kn\,, & n > 0\,, \\ 2 - \tfrac{1}{2}kn\,, & n \le 0\,. \end{cases} \tag{4.23}$$

Taking into account (3.21), the expansion of (4.19) reads

$$\left(-\partial_{\tilde{\sigma}}^2 + \frac{h_n(h_n-1)}{\cos^2\tilde{\sigma}} - \left[\tfrac{1}{2}h_n(h_n-1) + c_n\cos^2\tilde{\sigma}\right]\varepsilon^2 + \cdots\right)\Phi_{h_n,\ell} = \tilde{\omega}_\ell^2\Phi_{h_n,\ell}\,, \tag{4.24}$$

where (cf a remark after (3.21))

$$c_0 = 2\,, \qquad\qquad c_{n\neq 0} = 1\,. \tag{4.25}$$

Using that

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\, \Phi_{h,\ell}^2 \cos^2\tilde{\sigma} = \frac{h(h-1) + (h+\ell)^2 - 1}{2(h+\ell+1)(h+\ell-1)}\,, \tag{4.26}$$

we obtain

$$\begin{aligned}\tilde{\omega}_\ell^2 &= (\ell+h_n)^2 - \varepsilon^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\, \Phi_{h_n,\ell}^2 \left[\tfrac{1}{2}h_n(h_n-1) + c_n\cos^2\tilde{\sigma}\right] + \cdots \\ &= (\ell+h_n)^2 - \left[\tfrac{1}{2}h_n(h_n-1) + c_n\frac{h_n(h_n-1) + (h_n+\ell)^2 - 1}{2(h_n+\ell+1)(h_n+\ell-1)}\right]\varepsilon^2 + \cdots\,.\end{aligned} \tag{4.27}$$

For $n \neq 0$ we then get for ω_ℓ in (4.18)

$$\omega_\ell = (\ell+h_n)\left(1 - \tfrac{1}{4}\varepsilon^2\right) - \left[\frac{h_n(h_n-1)+1}{4(h_n+\ell)} + \frac{h_n(h_n-1)}{4(h_n+\ell)(h_n+\ell+1)(h_n+\ell-1)}\right]\varepsilon^2 + \cdots\,. \tag{4.28}$$

Note that the sum over ℓ of the last term in square brackets is convergent (and independent of h_n)

$$\sum_{\ell=0}^{\infty} \frac{h_n(h_n-1)}{4(h_n+\ell)(h_n+\ell+1)(h_n+\ell-1)} = \frac{1}{8}\,. \tag{4.29}$$

4.2. Expansion of fermionic fluctuation frequencies

The values of the fermionic masses (that do not depend on ε) were given in (3.25). As was mentioned above, one should consider separately the cases of $m > -\frac{1}{2}$ and $m < \frac{1}{2}$. Assuming $k > 2$ we have

$$n > 0: \quad \tfrac{kn}{2} \pm 1 > -\tfrac{1}{2}, \quad \tfrac{kn}{2} > -\tfrac{1}{2}\,; \quad n < 0: \quad \tfrac{kn}{2} \pm 1 < \tfrac{1}{2}, \quad \tfrac{kn}{2} < \tfrac{1}{2}. \tag{4.30}$$

Hence the corresponding values of h_n and the spinor type (i.e. with components ψ as in (4.12) or χ as in (4.14)) for the $3+3$ fermions with mass $\frac{1}{2}kn \pm 1$ and $1+1$ fermions with mass $\frac{1}{2}kn$ are as in table 1.

Table 1. Mass, conformal dimension, and the spinor wave functions for the 8 towers of fermionic modes. $\vartheta_\pm = (\vartheta^r_\pm)$, $r = 1,2,3$, represents 3+3 states, and $\vartheta' = (\vartheta'^s)$, $s = 1,2$, represents 1+1 fermionic states.

	m	sign (n)	h_n	spinor
$\vartheta_\pm$	$\frac{kn}{2} \pm 1$	$n > 0$	$\frac{kn}{2} \pm 1 + \frac{1}{2}$	(χ^1, χ^2)
ϑ'	$\frac{kn}{2}$	$n > 0$	$\frac{kn}{2} + \frac{1}{2}$	(χ^1, χ^2)
$\vartheta_\pm$	$\frac{kn}{2} \pm 1$	$n < 0$	$-\frac{kn}{2} \mp 1 + \frac{1}{2}$	(ψ^1, ψ^2)
ϑ'	$\frac{kn}{2}$	$n < 0$	$-\frac{kn}{2} + \frac{1}{2}$	(ψ^1, ψ^2)

(4.31)

After the rescaling in (4.18) equation (4.7) reads

$$\left[-i\gamma^1\left(\partial_{\tilde\sigma} + \frac{2\mathbb{K}}{\pi}\frac{\mathrm{sn}\,\tilde\sigma\,\mathrm{dn}\,\tilde\sigma}{2\,\mathrm{cn}\,\tilde\sigma}\right) + \frac{2\mathbb{K}}{\pi}\frac{\sqrt{1-\varepsilon^2}}{\mathrm{cn}\left(\frac{2\mathbb{K}}{\pi}\tilde\sigma\right)}m\right]\Psi_\ell = \gamma^0\tilde\omega_\ell\Psi_\ell\,. \tag{4.32}$$

Expanding in small ε gives

$$\left[-i\gamma^1\left(\partial_{\tilde\sigma} + \tfrac{1}{2}\tan\tilde\sigma + \tfrac{1}{8}\sin(2\tilde\sigma)\,\varepsilon^2 + \cdots\right) + \frac{m}{\cos\tilde\sigma} - \tfrac{1}{4}m\cos\tilde\sigma\,\varepsilon^2 + \cdots\right]\Psi_\ell = \gamma^0\tilde\omega_\ell\Psi_\ell\,. \tag{4.33}$$

Next we are to apply the standard first-order perturbation theory using $\Psi_{h_n,\ell}$ corresponding to a particular value of m in (4.31). We find[19]

$$\begin{aligned}\tilde\omega_\ell &= \ell + h_n - \varepsilon^2\int_{-\pi/2}^{\pi/2}\frac{\mathrm{d}\tilde\sigma}{\cos\tilde\sigma}\Psi^\dagger_{h_n,\ell}\gamma^0\left[\tfrac{1}{8}\sin(2\tilde\sigma)\gamma^1 + \tfrac{1}{4}m\cos\tilde\sigma\right]\Psi_{h_n,\ell} \\ &= \ell + h_n - \tfrac{1}{4}\varepsilon^2 m\int_{-\pi/2}^{\pi/2}\frac{\mathrm{d}\tilde\sigma}{\cos\tilde\sigma}\;\Psi^\dagger_{h_n,\ell}\gamma^0\Psi_{h_n,\ell}\;\cos\tilde\sigma\,,\end{aligned} \tag{4.34}$$

where we used that $\Psi^\dagger\gamma^0\gamma^1\Psi = 0$ which is true for any $m > -\frac{1}{2}$ or $< \frac{1}{2}$. Here

[19] Starting with a generic expression $[i(\partial_\sigma + \frac{1}{2}\tan\sigma)\gamma^1 + (\omega + \varepsilon^2\delta\omega + \cdots)\gamma^0 + B(\sigma) + C(\sigma)\varepsilon^2 + \cdots](\Psi + \delta\Psi\varepsilon^2 + \cdots) = 0$, we get at first order $\int\frac{\mathrm{d}\sigma}{\cos\sigma}\Psi^\dagger\gamma^0\left[i(\partial_\sigma + \frac{1}{2}\tan\sigma)\gamma^1 + \omega\gamma^0 + B(\sigma)\right]\delta\Psi + \int\frac{\mathrm{d}\sigma}{\cos\sigma}\Psi^\dagger\left[\delta\omega + \gamma^0 C(\sigma)\right]\Psi$. Integrating by parts we find that $\delta\omega = -\int\mathrm{d}\sigma\Psi^\dagger\gamma^0 C(\sigma)\Psi$.

$$\int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\tilde{\sigma}}{\cos\tilde{\sigma}} \Psi^{\dagger}_{h,\ell}\gamma^0\Psi_{h,\ell}\ \cos\tilde{\sigma} = \mp\frac{2\,(2h-1)\,(h+\ell)}{(2h+2\ell-1)\,(2h+2\ell+1)}$$
$$= \mp\left[\frac{2h-1}{2\ell+2h+1} + \frac{2h-1}{(2\ell+2h-1)\,(2\ell+2h+1)}\right], \qquad (4.35)$$

where the upper sign applies to the (ψ^1,ψ^2) case and the lower sign to the (χ^1,χ^2) case. In (4.35) we separated a term which gives divergence when summed over ℓ and a term that gives a convergent h-independent sum

$$\sum_{\ell=0}^{\infty} \frac{2h-1}{(2\ell+2h-1)\,(2\ell+2h+1)} = \frac{1}{2}\,. \qquad (4.36)$$

Thus using (4.31) and (4.34) we get

$$\omega_\ell = (\ell+h_n)\left(1-\tfrac{1}{4}\varepsilon^2\right) \pm \tfrac{1}{4}m\left[\frac{2h_n-1}{2\ell+2h_n+1} + \frac{2h_n-1}{(2\ell+2h_n-1)\,(2\ell+2h_n+1)}\right]\varepsilon^2 + \cdots. \qquad (4.37)$$

Using that $h=\frac{1}{2}\mp m$ (assuming the same sign/spinor correspondence as in (4.35)) we end with

$$\omega_\ell = (\ell+h_n)\left(1-\tfrac{1}{4}\varepsilon^2\right) + \tfrac{1}{4}\left(\tfrac{1}{2}-h_n\right)\left[\frac{h_n-\frac{1}{2}}{\ell+h_n+\frac{1}{2}} + \frac{h_n-\frac{1}{2}}{\left(\ell+h_n-\frac{1}{2}\right)\left(\ell+h_n+\frac{1}{2}\right)}\right]\varepsilon^2 + \cdots. \qquad (4.38)$$

Notice that here there are no $\pm$ signs, i.e. the expression is the same for both types of the spinors once m is written in terms of h_n.

4.3. n = 0: string theory limit

A similar computation in the type IIA string theory case was previously done in [22]. The string case corresponds to keeping only the $n=0$ modes. To take this limit requires some care as some details of the general n formulae above depended on assumptions that $n\neq 0$. For the 6 CP^3 scalar fluctuations we can set $h_0=1$ (cf (4.17)) and then from (4.22) we get

$$\omega_\ell = (\ell+1)\left(1-\tfrac{1}{4}\varepsilon^2\right) + \cdots. \qquad (4.39)$$

For the two AdS_4 scalar fluctuations we have the corresponding AdS_2 dimensions $h_0=2$ in (4.23). One mode has no corrections to its mass in (3.20) and then

$$\omega_\ell = (\ell+2)\left(1-\tfrac{1}{4}\varepsilon^2\right) - \frac{1}{2\,(\ell+2)}\varepsilon^2 + \cdots. \qquad (4.40)$$

For the other mode we need to use $c_0=2$ in (4.27) and this gives

$$\tilde{\omega}_\ell^2 = (\ell+2)^2 - \varepsilon^2\int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\,\Phi^2_{2,\ell}\left(1+2\cos^2\tilde{\sigma}\right) + \cdots = (\ell+2)^2 - \frac{2\,(\ell+2)^2}{(\ell+1)\,(\ell+3)}\varepsilon^2 + \cdots, \qquad (4.41)$$

which leads to

$$\omega_\ell = (\ell+2)\left(1-\tfrac{1}{4}\varepsilon^2\right) - \left[\frac{1}{\ell+1} - \frac{1}{(\ell+1)\,(\ell+3)}\right]\varepsilon^2 + \cdots. \qquad (4.42)$$

For the fermions, we have 1+1 states with $m=0$ and $h_0=\frac{1}{2}$ and 3+3 states with $m=\pm1$ and $h_0=\frac{3}{2}$. Using (4.38) we get

$$\begin{aligned} m=\ 0: &\quad \omega_\ell=\left(\ell+\tfrac{1}{2}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)+\cdots, \\ m=\pm1: &\quad \omega_\ell=\left(\ell+\tfrac{3}{2}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)-\left[\frac{1}{4(\ell+1)}-\frac{1}{8(\ell+1)(\ell+2)}\right]\varepsilon^2+\cdots. \end{aligned} \tag{4.43}$$

4.4. One-loop vacuum energy

We are now ready to compute the total vacuum energy in (4.5) by summing the characteristic frequencies $\omega_\ell(n)$ over all modes. We shall first assume that $k>2$ and then consider the special cases of $k=1,2$.

$\boldsymbol{k>2}$: Let us start with the $n>0$ modes for which (cf (4.17), (4.23), (4.31))

$$h_{\rm CP}=1+\tfrac{kn}{2}\,,\quad h_{\rm AdS}=\tfrac{kn}{2}-1\,,\qquad h_{\vartheta_\pm}=\tfrac{kn}{2}\pm1+\tfrac{1}{2}\,,\quad h_{\vartheta'}=\tfrac{kn}{2}+\tfrac{1}{2}\,. \tag{4.44}$$

Doing the sum over ℓ for each n let us define the contribution of each of the 8+8 scalar and fermionic modes as

$$E_n=\tfrac{1}{2}\sum_{\ell=0}^{\infty}\omega_\ell(n)\ . \tag{4.45}$$

Expressing the result in terms of the Hurwitz zeta-function $\zeta(s,a)=\sum_{\ell=0}^{\infty}\frac{1}{(\ell+a)^s}$ we then get (including minus sign for the fermions)[20]

$$\begin{aligned} E_n^{\rm CP}&=6\times\left[\tfrac{1}{2}\zeta(-1,h_{\rm CP})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}h_{\rm CP}(h_{\rm CP}-1)\,\zeta(1,h_{\rm CP})\,\varepsilon^2+\cdots\right], \\ E_n^{\rm AdS}&=2\times\left[\tfrac{1}{2}\zeta(-1,h_{\rm AdS})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{\rm AdS}(h_{\rm AdS}-1)+1\right]\zeta(1,h_{\rm AdS})\,\varepsilon^2-\tfrac{1}{16}\varepsilon^2+\cdots\right], \\ E_n^{\vartheta\pm}&=-3\times\left[\tfrac{1}{2}\zeta(-1,h_{\vartheta_\pm})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta_\pm}\right)^2\zeta\left(1,h_{\vartheta_\pm}+\tfrac{1}{2}\right)\varepsilon^2+\tfrac{1}{16}\left(\tfrac{1}{2}-h_{\vartheta_\pm}\right)\varepsilon^2+\cdots\right], \\ E_n^{\vartheta'}&=-2\times\left[\tfrac{1}{2}\zeta(-1,h_{\vartheta'})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta'}\right)^2\zeta\left(1,h_{\vartheta'}+\tfrac{1}{2}\right)\varepsilon^2+\tfrac{1}{16}\left(\tfrac{1}{2}-h_{\vartheta'}\right)\varepsilon^2+\cdots\right]. \end{aligned} \tag{4.46}$$

Using the explicit values of dimensions h_n in (4.44) we get for the total

$$\begin{aligned} E_n&=E_n^{\rm CP}+E_n^{\rm AdS}+E_n^{\vartheta+}+E_n^{\vartheta-}+E_n^{\vartheta'} \\ &=\zeta\left(-1,-1+\tfrac{kn}{2}\right)-\tfrac{3}{2}\zeta\left(-1,-\tfrac{1}{2}+\tfrac{kn}{2}\right)-\zeta\left(-1,\tfrac{1}{2}+\tfrac{kn}{2}\right)+3\zeta\left(-1,1+\tfrac{kn}{2}\right) \\ &\quad-\tfrac{3}{2}\zeta\left(-1,\tfrac{3}{2}+\tfrac{kn}{2}\right)+\left[\tfrac{1}{8}(-1+2kn)+\tfrac{3}{32}(-2+kn)^2\zeta\left(1,\tfrac{kn}{2}\right)+\tfrac{1}{16}\left(-12+6kn-k^2n^2\right)\right. \\ &\quad\times\zeta\left(1,-1+\tfrac{kn}{2}\right)-\tfrac{1}{8}kn(3+kn)\,\zeta\left(1,1+\tfrac{kn}{2}\right)+\tfrac{3}{32}(2+kn)^2\zeta\left(1,2+\tfrac{kn}{2}\right) \\ &\quad-\tfrac{1}{4}\zeta\left(-1,-1+\tfrac{kn}{2}\right)+\tfrac{3}{8}\zeta\left(-1,-\tfrac{1}{2}+\tfrac{kn}{2}\right)+\tfrac{1}{4}\zeta\left(-1,\tfrac{1}{2}+\tfrac{kn}{2}\right)-\tfrac{3}{4}\zeta\left(-1,1+\tfrac{kn}{2}\right) \\ &\quad\left.+\tfrac{3}{8}\zeta\left(-1,\tfrac{3}{2}+\tfrac{kn}{2}\right)\right]\varepsilon^2+\cdots. \end{aligned} \tag{4.47}$$

This expression can be simplified using the relations

$$\zeta\left(1,-1+\tfrac{kn}{2}\right)=\zeta\left(1,\tfrac{kn}{2}\right)+\frac{1}{\frac{kn}{2}-1}\,,\qquad\qquad \zeta\left(1,1+\tfrac{kn}{2}\right)=\zeta\left(1,\tfrac{kn}{2}\right)-\tfrac{4}{kn}\,, \tag{4.48}$$

[20] Note that the structure of the expressions for $\vartheta_\pm$ and ϑ' fermions is the same due to the comment after (4.38).

$$\zeta\left(1,2+\tfrac{kn}{2}\right)=\zeta\left(1,\tfrac{kn}{2}\right)-\tfrac{4}{kn}-\tfrac{1}{1+\frac{kn}{2}}\,,\qquad \zeta\left(-1,a\right)=\tfrac{1}{12}\left(-1+6a-6a^2\right)\,. \tag{4.49}$$

One finds that all log divergent terms proportional to $\zeta(1,\frac{kn}{2})$ cancel out. Also, the ε-independent terms combine to zero (reflecting the vanishing of vacuum energy in the AdS_2 limit [30]) so that

$$E_n=\frac{6-5kn}{4kn\,(kn-2)}\,\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right)\,. \tag{4.50}$$

Let us note that in (4.47) we used the zeta-function regularization to subtract the linear divergences present in the sum over ℓ for the contributions of the individual fluctuations. In fact, this regularization is not necessary once all these contributions are combined together—the resulting sum over ℓ is manifestly finite. Indeed, the explicit form of the total E_n as the sum over ℓ is given by

$$E_n=\left[\tfrac{1}{8}\,(2kn-1)+\sum_{\ell=0}^{\infty}e_{n,\ell}\right]\,\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right)\,, \tag{4.51}$$

$$e_{n,\ell}=-\frac{1}{2}-\frac{1+l+l^2}{2\,(-2+2l+kn)}+\frac{3\,(1+l)^2}{4\,(2l+kn)}-\frac{(1+l)\,(-1+2l)}{2\,(2+2l+kn)}+\frac{3\,(1+l)^2}{4\,(4+2l+kn)}\,. \tag{4.52}$$

Here $e_{n,\ell}\big|_{\ell\gg1}=-\frac{1}{8}(12+k^2n^2)\ell^{-2}+\mathcal{O}(\ell^{-3})$ so that the sum over ℓ is convergent and is given by

$$\sum_{\ell=0}^{\infty}e_{n,\ell}=\frac{12-12kn+5k^2n^2-2k^3n^3}{8kn\,(kn-2)}\,. \tag{4.53}$$

As a result, (4.51) gives the same result as in (4.50) without using the zeta-function regularization.

One can check that the same expression is found also for $n<0$ when

$$h_{\rm CP}=-\tfrac{kn}{2}\,,\qquad h_{\rm AdS}=2-\tfrac{kn}{2}\,,\qquad h_{\vartheta_\pm}=-\tfrac{kn}{2}\mp1+\tfrac{1}{2}\,,\qquad h_{\vartheta'}=-\tfrac{kn}{2}+\tfrac{1}{2}\,. \tag{4.54}$$

Summing over $n\neq0$ then gives

$$\begin{aligned}\sum_{n\neq0}E_n&=\sum_{n=1}^{\infty}\left[\frac{6-5kn}{4kn\,(kn-2)}+\frac{6+5kn}{4kn\,(kn+2)}\right]\varepsilon^2+\cdots=-2\sum_{n=1}^{\infty}\frac{1}{k^2n^2-4}\,\varepsilon^2+\cdots\\&=\left(-\frac{1}{4}+\frac{\pi}{2k}\cot\frac{2\pi}{k}\right)\,\varepsilon^2+\cdots\,.\end{aligned} \tag{4.55}$$

Note that this sum over n is also manifestly convergent. This is to be compared to the Wilson loop case in [2] where the sum over the M2 brane modes in the partition function was linearly divergent and was defined using the Riemann zeta-function regularisation[21].

It remains to add the contribution of the string-level $n=0$ fluctuations that can be computed using (4.39)–(4.43) as was already done in [22]. Explicitly, we find

$$E_0=\sum_{\ell=0}^{\infty}\frac{1}{4\,(\ell+1)\,(\ell+2)}\,\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right)=\frac{1}{4}\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right)\,. \tag{4.56}$$

[21] Explicitly, in [2] one had for the one-loop M2 brane correction to $-\log\langle W\rangle$: $\Gamma_1=\sum_{n=1}^{\infty}\log\left(\frac{k^2n^2}{4}-1\right)$. This gave $\Gamma_1=\log\left[2\sin\frac{2\pi}{k}\right]$ after using that $\zeta(0)=-\frac{1}{2}$, $\zeta'(0)=-\frac{1}{2}\log(2\pi)$.

Here the sum over ℓ is also convergent, i.e. does not require to use the zeta-function regularization.

Combing (4.55) and (4.56) we conclude that

$$E=\sum_{n=-\infty}^{\infty} E_n=\frac{\pi}{2k}\cot\frac{2\pi}{k}\,\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right)\ . \tag{4.57}$$

Using that in the small cusp angle limit $\alpha=\pi\varepsilon+\ldots$ (see (2.28)) we observe that the one-loop correction $\Gamma^{(1)}_{\rm cusp}=E$ in (1.22) indeed scales as α^2. Then using (1.5) we conclude that the one-loop correction to the Bremsstrahlung function is given by (1.24).

$\boldsymbol{k=1}$: For $k=1$ the relation (4.17) for h_n for the CP^3 scalars still applies. Equation (4.23) for h_n for AdS_4 scalars applies for $n>3$ while for $n=1,2$ one has

$$h_{1,\rm AdS}=\tfrac{1}{2}+\tfrac{1}{2}|1-3|=\tfrac{3}{2}\ ,\qquad\qquad h_{2,\rm AdS}=\tfrac{1}{2}+\tfrac{1}{2}|2-3|=1\ . \tag{4.58}$$

For the fermions, the data in table (4.31) requires a modification for the mass $m_n=\frac{kn}{2}\mp 1$ for $n=\pm 1$. In these cases one has

$$m_1=-\tfrac{1}{2}\quad\rightarrow\quad h^{\vartheta-}_1=1\ ;\qquad\qquad m_{-1}=\tfrac{1}{2}\quad\rightarrow\quad h^{\vartheta+}_{-1}=1\ . \tag{4.59}$$

With these changes we find

$$E_1=-\tfrac{1}{4}\varepsilon^2+\cdots\ ,\qquad E_2=-\tfrac{5}{8}\varepsilon^2+\cdots\ ,\qquad E_{-1}=\tfrac{11}{12}\varepsilon^2+\cdots\ . \tag{4.60}$$

Using also that for $k=1$ we have $E_{-2}=\frac{1}{2}$ the analog of the sum in (4.55) is found to be

$$\begin{aligned}\sum_{n=-\infty}^{\infty} E_n&=\frac{1}{4}\varepsilon^2+\left(-\frac{1}{4}-\frac{5}{8}+\frac{11}{12}+\frac{1}{2}\right)\varepsilon^2+\sum_{n=3}^{\infty}\left[\frac{6-5n}{4n\,(n-2)}+\frac{6+5n}{4n\,(n+2)}\right]\varepsilon^2+\cdots\\&=-\frac{1}{4}\varepsilon^2+\cdots.\end{aligned} \tag{4.61}$$

$\boldsymbol{k=2}$: For $k=2$, (4.17) is still valid while in (4.23) for $n=1$ one should use $h_1=\frac{1}{2}+\frac{1}{2}|2-3|=1$. For the fermions, the values in table (4.31) apply also for $k=2$. As a result, $E_1=-\frac{5}{8}\varepsilon^2+\cdots$. Using also that $E_{-1}=\frac{1}{2}$ we find

$$\begin{aligned}\sum_{n=-\infty}^{\infty} E_n&=\frac{1}{4}\varepsilon^2+\left(-\frac{5}{8}+\frac{1}{2}\right)\varepsilon^2+\sum_{n=2}^{\infty}\left[\frac{6-10n}{8n\,(2n-2)}+\frac{6+10n}{8n\,(2n+2)}\right]\varepsilon^2+\cdots\\&=-\frac{1}{4}\varepsilon^2+\cdots.\end{aligned} \tag{4.62}$$

As as result, from (4.61) and (4.62) we get the corresponding values (1.25) for the one-loop correction to the Bremsstrahlung function.

5. One-loop M2 brane correction for small cusp in CP^3

Let us now discuss the case of the 'internal' cusp, i.e. when $\alpha=0$ while β is non-zero. Considering the small β limit we will confirm that the coefficient of the leading β^2 in the corresponding one-loop correction (1.22) and (1.23) is indeed the same up to sign as of the α^2 term computed above, i.e. is given again by (1.24) and (1.25).[22]

The $\alpha=0$ case corresponds to $p\to\infty$ at fixed imaginary ε when (see (2.24), (2.25) and (2.17))

$$b=\frac{p}{\sqrt{1-\varepsilon^2}}\to\infty\,,\quad t=\sqrt{1-\varepsilon^2}\,\tau,\quad \theta'(\sigma)=0\,,\quad \varepsilon=i\epsilon\,, \tag{5.1}$$

$$\psi'^2=-\varepsilon^2,\quad \rho'^2=1+\left(1-\varepsilon^2\right)\sinh^2\rho\quad \cosh^2\rho=\frac{1}{\mathrm{cn}^2\,\sigma}\,, \tag{5.2}$$

$$\beta^2=-\pi^2\frac{\left(4-5\varepsilon^2\right)^2\varepsilon^2}{16\left(1-\varepsilon^2\right)^3}\,. \tag{5.3}$$

The small β limit corresponds to $\varepsilon\ll 1$ when $\beta=i\pi\varepsilon+\ldots$ which is consistent with the range of $\psi=i\varepsilon\sigma$ being $[-\frac{1}{2}\beta,\frac{1}{2}\beta]$ as in (2.6).

5.1. Fluctuation spectrum

Let us start with bosonic fluctuations and fix a static gauge similarly to (3.9) (see [13])

$$\delta t=0\,,\quad \delta\psi=0\,,\quad \delta\phi=0\,, \tag{5.4}$$

$$\delta x=\frac{1}{\sinh\rho}X(\tau,\sigma,\phi)\,,\quad \delta\rho=-\frac{i}{\varepsilon}\sqrt{\rho'^2-\varepsilon^2}\,W(\tau,\sigma,\phi)\,,\quad \delta\theta=\frac{1}{\sinh\rho}V(\tau,\sigma,\phi)\,. \tag{5.5}$$

The 'volume' part of the quadratic fluctuation action then takes the form (cf (3.6) and (3.11))[23]

$$\begin{aligned}S_{2,\mathrm{V}}=\tfrac{1}{4}R^2T_2\int \mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}\phi\,\sqrt{-g^{(0)}}\,\Big\{&\tfrac{1}{2}\Big[\left(g^{(0)}\right)^{ij}\left(\partial_iV\partial_jV+\partial_iW\partial_jW+\partial_iX\partial_jX\right)\\ &+m_V^2V^2+m_W^2W^2+m_X^2X^2\Big]-\frac{4k}{R^2}\sqrt{\frac{\varepsilon^2\,\mathrm{cn}^2\,\sigma}{1-\varepsilon^2}+1}\,\tilde\gamma\,\partial_\phi W+\left(g^{(0)}\right)^{ij}\\ &\times\left(2\,\partial_i\tilde\gamma\partial_j\tilde\gamma+\frac{1}{4}\,\partial_i\tilde\theta_r\partial_j\tilde\theta_r+\frac{1}{4}\,\partial_i\tilde\varphi_r\partial_j\tilde\varphi_r\right)+2m_\gamma^2\tilde\gamma^2-\frac{k}{R^2}\tilde\theta_r\partial_\phi\tilde\varphi_r-\frac{i}{2\varepsilon}m_\gamma^2\,\tilde\theta_r\partial_\sigma\tilde\varphi_r\Big\}\,,\end{aligned}$$

$$m_V^2=m_X^2=\frac{8}{R^2}+m_\gamma^2\,,\quad m_W^2=\frac{4}{R^2}\left(2+R^{(2)}\right)+m_\gamma^2\,,\quad m_\gamma^2=\tfrac{4}{R^2}\tfrac{\varepsilon^2}{1-\varepsilon^2}\,\mathrm{cn}^2\,\sigma\,. \tag{5.6}$$

Here the metric $g^{(0)}_{ij}$ is the same as in (3.3) and the $\mathrm{cn}\,\sigma$ factors may be written in terms of the conformal factor of the $\hat g_{ab}$ metric as $\sqrt{-\hat g}=\frac{1-\varepsilon^2}{\mathrm{cn}^2\,\sigma}$. The two pairs of fields $(\tilde\theta_r,\tilde\varphi_r)$ $(r=1,2)$ enter symmetrically and to diagonalize their Lagrangian we may perform the redefinition

$$\tilde\theta_r=\sqrt{2}\left(A_r\cosh\tfrac{k\sigma}{2}-iB_r\sinh\tfrac{k\sigma}{2}\right)\,,\quad \tilde\varphi_r=\sqrt{2}\left(B_r\cosh\tfrac{k\sigma}{2}+iA_r\sinh\tfrac{k\sigma}{2}\right)\,, \tag{5.7}$$

[22] Let us note that this case has a relation to the latitude Wilson loop for which there is no so far an exact localization result for the expectation value for a finite angle.

[23] We use tilde to denote fluctuations of the CP^3 angles that have fixed values in (2.3) and rescale $\tilde\gamma\to\frac{1}{2}\tilde\gamma$.

and use (5.2). As a result, we can put (5.6) into the form[24]

$$S_{2,\mathrm{V}} = \tfrac{1}{8}R^2T_2\int \mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}\phi\,\sqrt{-g^{(0)}}\Big[(g^{(0)})^{ij}\Big(\partial_iV\partial_jV+\partial_iW\partial_jW+\partial_iX\partial_jX$$
$$+\partial_i\tilde{\gamma}\partial_j\tilde{\gamma}+\partial_iA_r\partial_jA_r+\partial_iB_r\partial_jB_r\Big)+m_V^2V^2+m_W^2W^2+m_X^2X^2+m_\gamma^2\tilde{\gamma}^2+m_A^2(A_r^2+B_r^2)$$
$$-\tfrac{2k}{R^2}\sqrt{1+R^2m_A^2}\,(\tilde{\gamma}\,\partial_\phi W-W\partial_\phi\tilde{\gamma})-\tfrac{2k}{R^2}(A_r\partial_\phi B_r-B_r\partial_\phi A_r)\Big]\,,$$
$$\times\, m_A^2=\tfrac{1}{4}m_\gamma^2=\tfrac{1}{R^2}\tfrac{\varepsilon^2}{1-\varepsilon^2}\,\mathrm{cn}^2\,\sigma=\tfrac{\varepsilon^2}{R^2}\tfrac{1}{\sqrt{-\hat{g}}}\,, \tag{5.8}$$

where we used that $g^{(0)}_{ij}$ is given by (3.3). In the IIA string limit when terms with ∂_ϕ are absent this action for 8 bosonic fluctuations is equivalent the one found in [21].

The contribution of the WZ term is found using (2.31) as in (3.14). As here $\theta'=0$ (see (5.1)) we get

$$C_3=\tfrac{3}{8}R^3\cosh(\rho+\delta\rho)\sinh^2(\rho+\delta\rho)\,\frac{1}{\sinh\rho}\,(X+\cdots)\,\sqrt{1-\varepsilon^2}\mathrm{d}\tau\wedge(\rho'\mathrm{d}\sigma+\mathrm{d}\delta\rho)\wedge\mathrm{d}\delta\theta$$
$$=\tfrac{3}{8}R^3\sqrt{1-\varepsilon^2}\cosh\rho\,\sinh\rho\,X\rho'\,(\delta\theta)'\;\mathrm{d}\tau\wedge\mathrm{d}\sigma\wedge\mathrm{d}\phi+\cdots\,, \tag{5.9}$$

where again $\rho'\equiv\partial_\sigma\rho(\sigma)$ while the prime on all the 3d fluctuation fields is assumed to be the derivative over ϕ, i.e. $(\delta\rho)'\equiv\partial_\phi\delta\rho$, $(\delta\theta)'\equiv\partial_\phi\delta\theta$. We may now use (5.2) and (5.5) to get

$$S_{2,\mathrm{WZ}}=\tfrac{3}{8}T_2R^3\sqrt{1-\varepsilon^2}\int\mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}\phi\;\cosh\rho\,\rho'XV'=\tfrac{3}{8}\mathrm{T}_2\int\mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}\phi\;\sqrt{-\hat{g}}\;\sqrt{1+R^2m_A^2}\,XV'\,. \tag{5.10}$$

According to (3.3), $\sqrt{-g^{(0)}}=\frac{R}{k}\sqrt{-\hat{g}}$ so we get from (5.8) and (5.10)

$$S_{2,\mathrm{V}}+S_{2,\mathrm{WZ}}=\tfrac{1}{8k}T_2\int\mathrm{d}\tau\mathrm{d}\sigma\mathrm{d}\phi\;\sqrt{-\hat{g}}\;(L_{\mathrm{kin}}+L_{\mathrm{mass}}+L_{\mathrm{mix}})\,, \tag{5.11}$$

$$L_{\mathrm{kin}}=\hat{g}^{ab}\left(\partial_aV\partial_bV+\partial_aX\partial_bX+\partial_aW\partial_bW+\partial_a\tilde{\gamma}\partial_b\tilde{\gamma}+\partial_aA_r\partial_bA_r+\partial_aB_r\partial_bB_r\right)\,, \tag{5.12}$$

$$L_{\mathrm{mass}}=\tfrac{R^2}{4}\left[m_V^2V^2+m_X^2X^2+m_W^2W^2+m_\gamma^2\tilde{\gamma}^2+m_A^2\left(A_r^2+B_r^2\right)\right]$$
$$+\tfrac{k^2}{4}\left(V'^2+W'^2+X'^2+\tilde{\gamma}'^2+A_r'^2+B_r'^2\right)$$
$$-\tfrac{k}{2}\left(\sqrt{1+R^2m_A^2}\,[(\tilde{\gamma}\,W'-W\tilde{\gamma}')-3\,(XV'-VX')]+A_rB_r'-B_rA_r'\right)\,, \tag{5.13}$$

where we put the terms with $(\ldots)'=\partial_\phi(\ldots)$ into the mass part anticipating the expansion in Fourier modes in ϕ.

[24] The reason why the simple rotation in (5.7) works is due to the special dependence of the $(\tilde{\theta},\tilde{\varphi})$ mixing term in (5.6) on σ. Indeed, if we start with a model Lagrangian $L=\sqrt{-\hat{g}}\,\hat{g}^{ab}\left[\partial_a\Phi_1\partial_b\Phi_1+\partial_a\Phi_2\partial_b\Phi_2+\mu(\sigma)\Phi_1\partial_\sigma\Phi_2\right]$ and assume that $\mu(\sigma)=\frac{c}{\sqrt{-\hat{g}}}$ where c is a constant then for a conformally flat $\hat{g}_{ab}$ its conformal factor drops out, i.e. the corresponding action is the same as in flat space. Thus it can be diagonalized by a rotation with an angle $\frac{c}{2}\sigma$. The resulting mass term is then proportional to $\frac{1}{\sqrt{-\hat{g}}}$ like m_A^2 in (5.8).

In general, for two mixed 3d scalars Φ_1, Φ_2 with canonical kinetic terms and the mass terms like in (5.13)

$$L_{\text{mass}} = m_1^2\Phi_1^2 + m_2^2\Phi_2^2 + m_{12}^2\left(\Phi_1\Phi_2' - \Phi_2\Phi_1'\right) , \tag{5.14}$$

decomposing the fields in $\sin(n\phi)$, $\cos(n\phi)$ components the resulting mass matrix

$$\mathcal{M}^2 = \begin{pmatrix} m_1^2 & 0 & 0 & nm_{12}^2 \\ 0 & m_1^2 & -nm_{12}^2 & 0 \\ 0 & -nm_{12}^2 & m_2^2 & 0 \\ nm_{12}^2 & 0 & 0 & m_2^2 \end{pmatrix} , \tag{5.15}$$

has eigenvalues (cf (3.18) and (3.19))

$$m_\pm^2 = \tfrac{1}{2}\left[m_1^2 + m_2^2 \pm \sqrt{\left(m_1^2 - m_2^2\right)^2 + 4n^2 m_{12}^4}\,\right] . \tag{5.16}$$

Assuming $n \neq 0$, we then obtain the following expansions of masses for the above pairs of mixed bosonic fluctuations:

$$m_{\mathrm{V,X}}^2 = 2 - \tfrac{3}{2}kn + \tfrac{1}{4}k^2n^2 + \left(1 - \tfrac{3}{4}kn\right)\cos^2\sigma\,\varepsilon^2 + \cdots , \tag{5.17}$$

$$m_{\tilde{\gamma},\mathrm{W}}^2 = \tfrac{1}{2}kn + \tfrac{1}{4}k^2n^2 + \left[\left(1 + \tfrac{1}{4}kn\right)\cos^2\sigma - \cos^4\sigma\right]\varepsilon^2 + \cdots , \tag{5.18}$$

$$m_{\mathrm{A,B}}^2 = \tfrac{1}{2}kn + \tfrac{1}{4}k^2n^2 + \tfrac{1}{4}\cos^2\sigma\,\varepsilon^2 + \cdots . \tag{5.19}$$

The quadratic fermionic fluctuations in (3.22) are again governed by the the Dirac operator in (4.7). While in the $\beta = 0$ case in section 4 the mass parameter m was independent of σ, here one finds that in the IIA string limit [22]

$$m_0(\sigma) = \tfrac{1}{4}\left[s_1 - s_2 + \frac{\mathrm{d}n\,\sigma}{\sqrt{1-\varepsilon^2}}\left(s_3 + 3s_1s_2\right)\right] , \tag{5.20}$$

where s_1, s_2, s_3 take ± 1 values giving masses of 8 fermionic 2d modes. In the present M2 brane case where the Dirac operator contains also a ∂_ϕ term the fermion masses are again given by (5.20) with an extra Universal term $\frac{1}{2}kn$ as in (3.25), i.e.

$$m = m_0(\sigma) + \tfrac{1}{2}kn , \qquad n = \pm 1, \pm 2, \ldots . \tag{5.21}$$

5.2. Expansion of fluctuation frequencies

Like in section 4 we can now determine the leading terms in the expansion of the corresponding fluctuation frequencies ω_ℓ in small ε or, equivalently, in small β.

V, X scalars

The AdS_2 conformal dimension h_n corresponding to the $\varepsilon = 0$ value of the mass here is the same as in (4.23). Then using (5.17) the analogs of (4.20)–(4.22) are

$$-\left[\partial_{\tilde{\sigma}}^2 - \frac{h_n(h_n-1)}{\cos^2\tilde{\sigma}} + \left[\tfrac{1}{2}h_n(h_n-1) - 1 + \tfrac{3}{4}kn\right]\varepsilon^2 + \cdots\right]\Phi_{h_n,\ell} = \tilde{\omega}_\ell^2\Phi_{h_n,\ell} , \tag{5.22}$$

$$
\begin{aligned}
\tilde{\omega}_\ell^2 &= (\ell+h_n)^2 - \varepsilon^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\, \Phi_{h_n,\ell}^2 \left[\tfrac{1}{2}h_n(h_n-1) - 1 + \tfrac{3}{4}kn\right] + \cdots \\
&= (\ell+h_n)^2 - \left[\tfrac{1}{2}h_n(h_n-1) - 1 + \tfrac{3}{4}kn\right]\varepsilon^2 + \cdots, \\
\omega_\ell &= (\ell+h_n)\left(1-\tfrac{1}{4}\varepsilon^2\right) - \frac{h_n(h_n-1) - 2 + \frac{3}{2}kn}{4(\ell+h_n)}\varepsilon^2 + \cdots. \qquad (5.23)
\end{aligned}
$$

$\tilde{\gamma}, W$ scalars

Here the conformal dimension h_n is the same as in (4.17) and using (5.18) we get

$$
-\left[\partial_{\tilde{\sigma}}^2 - \frac{h_n(h_n-1)}{\cos^2\tilde{\sigma}} + \left[\tfrac{1}{2}h_n(h_n-1) - 1 - \tfrac{1}{4}kn + \cos^2\tilde{\sigma} + \cdots\right]\varepsilon^2 + \cdots\right]\Phi_{h_n,\ell} = \tilde{\omega}_\ell^2 \Phi_{h_n,\ell}\,, \qquad (5.24)
$$

$$
\begin{aligned}
\tilde{\omega}_\ell^2 &= (\ell+h_n)^2 - \varepsilon^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\, \Phi_{h_n,\ell}^2 \left[\tfrac{1}{2}h_n(h_n-1) - 1 - \tfrac{1}{4}kn + \cos^2\tilde{\sigma}\right] + \cdots \\
&= (\ell+h_n)^2 - \left[\tfrac{1}{2}h_n(h_n-1) - 1 - \tfrac{1}{4}kn + \frac{-1+h_n(h_n-1)+(h_n+\ell)^2-1}{2(h_n+\ell+1)(h_n+\ell-1)}\right]\varepsilon^2 + \cdots, \\
\omega_\ell &= (\ell+h_n)\left(1-\tfrac{1}{4}\varepsilon^2\right) - \left[\frac{h_n(h_n-1) - 1 - \frac{1}{2}kn}{4(\ell+h_n)} + \frac{h_n(h_n-1)}{4(h_n+\ell)(h_n+\ell+1)(h_n+\ell-1)}\right]\varepsilon^2 + \cdots, \qquad (5.25)
\end{aligned}
$$

where the last term is the same as in (4.28).

A_r, B_r scalars

Again the conformal dimension h_n is the same as in (4.17) and using (5.19) we get

$$
-\left[\partial_{\tilde{\sigma}}^2 - \frac{h_n(h_n-1)}{\cos^2\tilde{\sigma}} + \left[\tfrac{1}{2}h_n(h_n-1) - \tfrac{1}{4}\right]\varepsilon^2 + \cdots\right]\Phi_{h_n,\ell} = \tilde{\omega}_\ell^2 \Phi_{h_n,\ell}\,, \qquad (5.26)
$$

$$
\begin{aligned}
\tilde{\omega}_\ell^2 &= (\ell+h_n)^2 - \varepsilon^2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\tilde{\sigma}\, \Phi_{h_n,\ell}^2 \left[\tfrac{1}{2}h_n(h_n-1) - \tfrac{1}{4}\right] + \cdots \\
&= (\ell+h_n)^2 - \left[\tfrac{1}{2}h_n(h_n-1) - \tfrac{1}{4}\right]\varepsilon^2 + \cdots, \\
\omega_\ell &= (\ell+h_n)\left(1-\tfrac{1}{4}\varepsilon^2\right) - \frac{h_n(h_n-1) - \frac{1}{2}}{4(\ell+h_n)}\varepsilon^2 + \cdots. \qquad (5.27)
\end{aligned}
$$

Fermions

Using the values of masses in (5.20) and (5.21) we find the following small ε expansions of the corresponding mass term in the equation (4.32) for the fermionic ω_ℓ

$$
-\frac{2\mathbb{K}}{\pi}\frac{\sqrt{1-\varepsilon^2}}{\mathrm{cn}\left(\frac{2\mathbb{K}}{\pi}\tilde{\sigma}\right)}\, m\left(\tfrac{2\mathbb{K}}{\pi}\tilde{\sigma}\right) = \begin{cases}
-\frac{\frac{1}{2}kn}{\cos\tilde{\sigma}} + \frac{1}{4}\left(\frac{kn}{2} \pm 1\right)\cos\tilde{\sigma}\,\varepsilon^2 + \cdots\,, & \\
-\frac{\frac{1}{2}kn+1}{\cos\tilde{\sigma}} + \frac{1}{4}\left(\frac{kn}{2} + 1 - 2\right)\cos\tilde{\sigma}\,\varepsilon^2 + \cdots\,, & \text{(2modes)} \\
-\frac{\frac{1}{2}kn-1}{\cos\tilde{\sigma}} + \frac{1}{4}\left(\frac{kn}{2} - 1 + 2\right)\cos\tilde{\sigma}\,\varepsilon^2 + \cdots\,, & \text{(2modes)} \\
-\frac{\frac{1}{2}kn+1}{\cos\tilde{\sigma}} + \frac{1}{4}\left(\frac{kn}{2} + 1 - 1\right)\cos\tilde{\sigma}\,\varepsilon^2 + \cdots\,, & \\
-\frac{\frac{1}{2}kn-1}{\cos\tilde{\sigma}} + \frac{1}{4}\left(\frac{kn}{2} - 1 + 1\right)\cos\tilde{\sigma}\,\varepsilon^2 + \cdots\,. &
\end{cases} \qquad (5.28)
$$

Comparing with (4.33) where we had the term $-\frac{m}{\cos\tilde{\sigma}}+\frac{1}{4}m\cos\tilde{\sigma}\,\varepsilon^2+\cdots$ (the γ^1 term from connection part was shown to give no contribution) we see that the ε^2 terms in (5.28) lead to

$$-\frac{m}{\cos\tilde{\sigma}}+\tfrac{1}{4}\,(m+\delta m)\cos\tilde{\sigma}\;\varepsilon^2+\cdots, \tag{5.29}$$

with δm shifts for the $(3+3)+(1+1)=8$ fermions given by

m	$\frac{kn}{2}+1$	$\frac{kn}{2}-1$	$\frac{kn}{2}$
δm	$-2,-2,-1$	$2,2,1$	$1,-1$

Equation (4.38) should then be modified by δm, giving

$$\begin{aligned}\omega_\ell &= (\ell+h_n)\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{4}\left(\tfrac{1}{2}-h_n\mp\delta m\right)\\ &\times\left[\frac{h_n-\frac{1}{2}}{\ell+h_n+\frac{1}{2}}+\frac{2h_n-1}{(2\ell+2h_n-1)(2\ell+2h_n+1)}\right]\varepsilon^2+\cdots.\end{aligned} \tag{5.30}$$

5.3. One-loop vacuum energy

Following the discussion in section 4.4 let us start with $n>0$ modes and recall the values of conformal dimensions in the zero-cusp AdS_2 limit (labelling fermions as in the $\beta=0$ case)

$$n>0:\quad h_{\gamma W}=h_{AB}=1+\tfrac{kn}{2}\,,\quad h_{VX}=\tfrac{kn}{2}-1\,,\quad h_{\vartheta_\pm}=\tfrac{kn}{2}\pm 1+\tfrac{1}{2}\,,\quad h_{\vartheta'}=\tfrac{kn}{2}+\tfrac{1}{2}\,. \tag{5.31}$$

We then doing the sum over ℓ in (4.45) we find

$$\begin{aligned}E_n^{\gamma W}&=2\times\left[\tfrac{1}{2}\zeta(-1,h_{\gamma W})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{\gamma W}(h_{\gamma W}-1)-1-\tfrac{kn}{2}\right]\zeta(1,h_{\gamma W})\,\varepsilon^2-\tfrac{1}{16}\varepsilon^2+\cdots\right],\\ E_n^{AB}&=4\times\left[\frac{1}{2}\zeta(-1,h_{AB})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{AB}(h_{AB}-1)-\tfrac{1}{2}\right]\zeta(1,h_{AB})\,\varepsilon^2+\cdots\right],\end{aligned} \tag{5.32}$$

$$\begin{aligned}E_n^{VX}&=2\times\left[\tfrac{1}{2}\zeta(-1,h_{VX})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{VX}(h_{VX}-1)-2+\tfrac{3}{2}kn\right]\zeta(1,h_{VX})\,\varepsilon^2+\cdots\right],\\ E_n^{\vartheta\pm}&=2\times\left[-\tfrac{1}{2}\zeta(-1,h_{\vartheta_\pm})\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta_\pm}\pm 2\right)\left[\left(\tfrac{1}{2}-h_{\vartheta_\pm}\right)\zeta\left(1,h_{\vartheta_\pm}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right]\\ &\quad+1\times\left[-\tfrac{1}{2}\zeta(-1,h_{\vartheta_\pm})\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta_\pm}\pm 1\right)\left[\left(\tfrac{1}{2}-h_{\vartheta_\pm}\right)\zeta\left(1,h_{\vartheta_\pm}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right]\\ E_n^{\vartheta'}&=1\times\left[-\tfrac{1}{2}\zeta(-1,h_{\vartheta'})\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta'}+1\right)\left[\left(\tfrac{1}{2}-h_{\vartheta'}\right)\zeta\left(1,h_{\vartheta'}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right]\\ &\quad+1\times\left[-\tfrac{1}{2}\zeta(-1,h_{\vartheta'})\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta'}-1\right)\left[\left(\tfrac{1}{2}-h_{\vartheta'}\right)\zeta\left(1,h_{\vartheta'}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right].\end{aligned}$$

Similarly, for $n<0$ we get

$$n<0:\quad h_{\gamma W}=h_{AB}=-\tfrac{kn}{2}\,,\qquad h_{VX}=2-\tfrac{kn}{2}\,,\quad h_{\vartheta_\pm}=-\tfrac{kn}{2}\mp 1+\tfrac{1}{2}\,,\quad h_{\vartheta'}=-\tfrac{kn}{2}+\tfrac{1}{2}\,, \tag{5.33}$$

$$\begin{aligned}E_n^{\gamma W}&=2\times\left[\tfrac{1}{2}\zeta(-1,h_{\gamma W})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{\gamma W}(h_{\gamma W}-1)-1-\tfrac{kn}{2}\right]\zeta(1,h_{\gamma W})\,\varepsilon^2-\tfrac{1}{16}\varepsilon^2+\cdots\right],\\ E_n^{AB}&=4\times\left[\frac{1}{2}\zeta(-1,h_{AB})\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{AB}(h_{AB}-1)-\tfrac{1}{2}\right]\zeta(1,h_{AB})\,\varepsilon^2+\cdots\right],\end{aligned} \tag{5.34}$$

$$
\begin{aligned}
E_n^{\mathrm{VX}} &= 2\times\left[\tfrac{1}{2}\zeta\left(-1,h_{\mathrm{VX}}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)-\tfrac{1}{8}\left[h_{\mathrm{VX}}\left(h_{\mathrm{VX}}-1\right)-2+\tfrac{3}{2}kn\right]\zeta\left(1,h_{\mathrm{VX}}\right)\varepsilon^2+\cdots\right]\\
E_n^{\vartheta\pm} &= 2\times\left[-\tfrac{1}{2}\zeta\left(-1,h_{\vartheta_\pm}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta_\pm}\mp 2\right)\left[\left(\tfrac{1}{2}-h_{\vartheta_\pm}\right)\zeta\left(1,h_{\vartheta_\pm}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right]\\
&\quad+1\times\left[-\tfrac{1}{2}\zeta\left(-1,h_{\vartheta_\pm}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta_\pm}\mp 1\right)\left[\left(\tfrac{1}{2}-h_{\vartheta_\pm}\right)\zeta\left(1,h_{\vartheta_\pm}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right]\\
E_n^{\vartheta\,\prime} &= 1\times\left[-\tfrac{1}{2}\zeta\left(-1,h_{\vartheta\,\prime}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta\,\prime}+1\right)\left[\left(\tfrac{1}{2}-h_{\vartheta\,\prime}\right)\zeta\left(1,h_{\vartheta\,\prime}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right]\\
&\quad+1\times\left[-\tfrac{1}{2}\zeta\left(-1,h_{\vartheta\,\prime}\right)\left(1-\tfrac{1}{4}\varepsilon^2\right)+\tfrac{1}{8}\left(\tfrac{1}{2}-h_{\vartheta\,\prime}-1\right)\left[\left(\tfrac{1}{2}-h_{\vartheta\,\prime}\right)\zeta\left(1,h_{\vartheta\,\prime}+\tfrac{1}{2}\right)-\tfrac{1}{2}\right]\varepsilon^2+\cdots\right].
\end{aligned}
$$

The difference between the expressions for $n>0$ and $n<0$ is only in the sign of δm in (5.30) which is due to the fact that we should use (ψ^1,ψ^2) spinors in the $n<0$ case, cf (4.31).

Note that for $n\neq 0$ we do not have massless fermions and thus there is no ambiguity in the choice of their quantization. This is to be compared with the string theory, i.e. $n=0$, case considered in [22] where for $\alpha=0,\ \beta\neq 0$ there were ε corrections to massless fermions and one had to choose a quantization consistent with $\mathcal{N}=6$ supersymmetry in the zero cusp limit. This subtlety is not present for $n\neq 0$.

Adding all mode contributions together for $n=0$ we find $E_0=\frac{1}{4}\varepsilon^2+\mathcal{O}(\varepsilon^2)$ as in [22] (cf (4.56)). The total sum over n is finite and is given by the same expression as in the $\beta=0$ case in (4.57)

$$
E=\sum_{n=-\infty}^{\infty}E_n=\frac{\pi}{2k}\cot\frac{2\pi}{k}\,\varepsilon^2+\mathcal{O}\left(\varepsilon^4\right)\ . \tag{5.35}
$$

Taking into account (2.25), i.e. that in the small β cusp limit $\varepsilon^2=-\frac{\beta^2}{\pi^2}+\ldots$ (while in $\beta=0$ case we had $\varepsilon^2=\frac{\alpha^2}{\pi^2}$, (cf (2.23)) we confirm that the α^2 and $-\beta^2$ terms have the coefficient (1.24), in agreement with (1.5). The same conclusion is reached also in the special $k=1,2$ cases.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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