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Totally Geodesic and Parallel Hypersurfaces of Cahen-Wallach Spacetimes

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Abstract. We completely classify and describe totally geodesic hypersurfaces of Cahen-Wallach spacetimes. We also describe parallel hypersurfaces and investigate their geometric properties.

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1. Introduction

An irreducible Lorentzian symmetric space has necessarily constant sectional curvature. For this reason, in pseudo-Riemannian settings, the weaker notion of indecomposability has been considered. A pseudo-Riemannian manifold (M, g) is said to be *indecomposable* if the holonomy group at each point $p \in M$ stabilizes only non-trivial subspaces $V \subset T_pM$ which are degenerate (that is, such that the restriction of the metric g_p on $V \times V$ is degenerate). Indecomposable symmetric Lorentzian manifolds of non-constant sectional curvature are known as *Cahen-Wallach spaces*. Since their introduction in [6], Cahen-Wallach spaces have been extensively studied in literature (see for example [1]–[3], [7], [8], [17], [23], [24], [25] and references therein).

The aim of this paper is to study some natural hypersurfaces in Cahen-Wallach spacetimes, namely, totally geodesic and parallel hypersurfaces. Observe that differently from the Riemannian case, a submanifold of a pseudo-Riemannian manifold can be degenerate. However, throughout this work, we shall always consider *nondegenerate* submanifolds and hypersurfaces.

A submanifold M of a pseudo-Riemannian manifold is *totally geodesic* if its second fundamental form h vanishes. Consequently, the geodesics of a totally geodesic submanifold are also geodesics of the ambient space. More in general, a submanifold is said to be *parallel* if h is covariantly constant. The parallelism of the second fundamental form may be seen as the extrinsic analogue of local symmetry. Indeed, parallel hypersurfaces of a locally symmetric ambient space are again locally symmetric. Thus, it is a natural and interesting problem to investigate totally geodesic and parallel hypersurfaces of a given pseudo-Riemannian manifold, in order to improve our knowledge and understanding of its geometric properties.

Parallel surfaces of three-dimensional manifolds, both Riemannian and Lorentzian, have been extensively studied. Some examples may be found in [11]– [15], [21], [22]. In particular, the classification of parallel surfaces in three-dimensional Cahen-Wallach spaces was obtained in [14]. It is clear that the difficulty of classifying and describing totally geodesic and parallel hypersurfaces increases with the dimension of the given ambient space. On the other hand, the case of a four-dimensional ambient space is particularly relevant, because of the possible physical interpretation of some Lorentzian four-manifolds as solutions to Einstein's field equations. Some results concerning the study of totally geodesic and parallel hypersurfaces in some four-dimensional pseudo-Riemannian ambient spaces may be found in [9]– [11], [16], [19], [20].

We shall investigate and describe totally geodesic and parallel hypersurfaces in Cahen-Wallach spacetimes, also emphasizing the description of such hypersurfaces as three-dimensional Brinkmann manifolds. Moreover, we determine minimal and CMC examples in the parallel case. The paper is organized in the following way. In Sect. 2 we report the description of the Levi-Civita connection and curvature of Cahen-Wallach spacetimes and some basic information on totally geodesic hypersurfaces and their generalizations and on Brinkmann manifolds. In Sect. 3 we consider hypersurfaces of Cahen-Wallach spacetimes and provide the complete classification and explicit description of totally geodesic hypersurfaces of these spaces. Parallel hypersurfaces are investigated in Sect. 4, obtaining a complete description of such hypersurfaces for the general Cahen-Wallach spacetimes and of further examples in the special case of an ε-space.

2. Preliminaries

2.1. On Totally Geodesic and Parallel Hypersurfaces

Let M be a hypersurface immersed into a $(n+1)$-dimensional pseudo-Riemannian manifold $\bar{M}$ with the isometric immersion $F : M \to \bar{M}$ and ξ a unit normal vector field on M with $g(\xi, \xi) = \varepsilon \in \{-1, 1\}$. Let ∇^M and ∇ denote the Levi-Civita connections of M and $\bar{M}$ respectively. For any tangent vector fields X, Y to M, the *formula of Gauss* gives

$$\nabla_X Y = \nabla^M_X Y + h(X, Y)\xi, \tag{2.1}$$

where h is called the *second fundamental form* of the immersion.

We recall that M is a *totally geodesic hypersurface* if $h = 0$. This is equivalent to requiring that every geodesic of M is also a geodesic of the ambient space $\bar{M}$.

Moreover, considering the covariant derivative $\nabla^M h$, defined by

$$(\nabla^M h)(X, Y, Z) = X(h(Y, Z)) - h(\nabla^M_X Y, Z) - h(Y, \nabla^M_X Z),$$

for all vector fields X, Y, Z tangent to M, the hypersurface is said to be *parallel* (or having a parallel second fundamental form) if

$$\nabla^M h = 0.$$

Equivalently, the parallelism of M can be expressed by $\nabla^M S = 0$, where $S(X) = -\nabla_X \xi$ denotes the Weingarten (or shape) operator of M. For a parallel hypersurface, all the extrinsic invariants derived from h are covariantly constant. Clearly, totally geodesic hypersurfaces are parallel.

Throughout the paper, we will follow the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for the curvature. Denoting by R^M and R the Riemann-Christoffel curvature tensors of M and $\bar{M}$ respectively, we can express the *equations of Gauss and Codazzi*, given respectively by

$$g(R(X, Y)Z, W) = g(R^M(X, Y)Z, W) + \varepsilon\,(h(X, Z)h(Y, W) - h(X, W)h(Y, Z))\,, \tag{2.2}$$

$$g(R(X, Y)Z, \xi) = \varepsilon\left((\nabla^M h)(X, Y, Z) - (\nabla^M h)(Y, X, Z)\right), \tag{2.3}$$

where X, Y, Z and W are tangent to M.

Then, the second fundamental form h of the hypersurface is said to be *Codazzi* if $\nabla^M h$ is totally symmetric. By equation (2.3), it is easy to see that this is equivalent to requiring that $R(X, Y)\xi = 0$ for all vector fields X, Y tangent to M. Obviously, parallel hypersurfaces have a Codazzi second fundamental form.

Some other well known and relevant generalizations of totally geodesic hypersurfaces are minimal and CMC hypersurfaces. The *mean curvature* of a hypersurface M is given by

$$\frac{1}{n}\mathrm{tr}_{g_M} h = \frac{1}{n}\sum g^{ij}_M h_{ij},$$

where g_M is the pullback on M of the metric of the ambient space and g^{ij}_M are the components of $(g_M)^{-1}$ with respect to a given basis of vector fields tangent to M. The hypersurface is said to be *minimal* (respectively, *of constant mean curvature,* or *CMC)* if $\mathrm{tr}_{g_M} h = 0$ (respectively, $\mathrm{tr}_{g_M} h = \kappa$ for some real constant κ). Clearly, any totally geodesic hypersuface is minimal and every minimal hypersurface is CMC.

2.2. On Cahen-Wallach Spacetimes

Explicitly, an arbitrary four-dimensional Cahen-Wallach symmetric space is described as $\mathbb{R}^4$ equipped with the Lorentzian metric

$$g = \left(k_3 x_3^2 + k_4 x_4^2\right) dx_1^2 + 2dx_1 dx_2 + dx_3^2 + dx_4^2, \tag{2.4}$$

where $k_3, k_4 \neq 0$ are some real constants [6], [8]. In the special case where $k_3 = k_4 = k$, these spaces are also known as *ε-spaces*; they are locally conformally flat and admit a large group of isometries [25]. We shall denote by $\{\partial_i = \frac{\partial}{\partial x_i}\}$ the basis of coordinate vector fields. By (2.4), a Cahen-Wallach metric is completely determined by

$$g(\partial_1, \partial_1) = k_3 x_3^2 + k_4 x_4^2, \qquad g(\partial_1, \partial_2) = g(\partial_3, \partial_3) = g(\partial_4, \partial_4) = 1.$$

Using the *Koszul formula*, the Levi-Civita connection ∇ of g is explicitly described by

$$\nabla_{\partial_1} \partial_3 = k_3 x_3 \partial_2, \qquad \nabla_{\partial_1} \partial_4 = k_4 x_4 \partial_2, \qquad \nabla_{\partial_1} \partial_1 = -k_3 x_3 \partial_3 - k_4 x_4 \partial_4. \tag{2.5}$$

Starting from (2.5) and taking into account the symmetries of the curvature tensor, a direct calculation yields that with respect to the basis $\{\partial_i\}$ of coordinate vector fields, the curvature of a Cahen-Wallach metric is completely determined by the following non-vanishing components

$$\begin{aligned} R(\partial_1, \partial_3)\partial_1 &= -k_3 \partial_3, & R(\partial_1, \partial_3)\partial_3 &= k_3 \partial_2, \\ R(\partial_1, \partial_4)\partial_1 &= -k_4 \partial_4, & R(\partial_1, \partial_4)\partial_4 &= k_4 \partial_1. \end{aligned}$$

2.3. On Brinkmann Manifolds

Just like in the Riemannian case, a Lorentzian manifold admitting a non-null parallel vector field splits locally as a metric product. However, a different situation can occur in Lorentzian settings, where a null (lightlike) vector field can exist. A *Brinkmann manifold* is a Lorentzian manifold (M, g) admitting a parallel null vector field U, that is, such that $g(U, U) = \nabla U = 0$. Brinkmann manifolds have been extensively studied, both for their physical relevance and because they illustrate several geometric phenomena which do not have any Riemannian counterpart.

Cahen-Wallach symmetric spaces are examples of Brinkmann manifolds. In fact, from equations (2.4) and (2.5) it follows at once that ∂_2 is a parallel null vector field on any Cahen-Wallach space.

A Brinkmann manifold is a special kind of Walker manifold. A *Walker manifold* is a pseudo-Riemannian manifold which admits a non-trivial distribution $\mathcal{D}$ which is parallel (if $X \in \mathcal{D}$ then $\nabla X \in \mathcal{D}$) and null (the metric restricted to $\mathcal{D}$ vanishes identically). In dimension three, a Walker manifold admits local coordinates (t, x, y), with respect to which the Lorentzian metric tensor of M is expressed as follows:

$$\tilde{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{pmatrix}, \tag{2.6}$$

where $\varepsilon = \pm 1$, for some smooth function $f(t,x,y)$ [4], [18]. If ∂_t is a parallel null vector field and so, (M,g) is a Brinkmann manifold, then $f = f(x,y)$ and conversely.

The study of the geometry of three-dimensional Walker manifolds started with the pioneering paper [18] and continued in several different directions. In particular, parallel surfaces of three-dimensional Walker manifolds were completely classified in [14]. We may refer to [5] for an extensive survey on Walker manifolds.

As proved in [18], with respect to the above set of coordinates, three-dimensional locally symmetric Brinkmann manifolds are completely characterized by having a defining function f of the form

$$f(x,y) = \alpha x^2 + x\beta(y) + \gamma(y) \tag{2.7}$$

where α is a real constant and β, γ are arbitrary functions. In particular, if $\alpha = 0$, then the Brinkmann metric is flat [18].

3. Totally Geodesic Hypersurfaces of Cahen-Wallach Spacetimes

Let $F : M \to (\bar{M}, g)$ denote the immersion of a hypersurface M into a Cahen-Wallach spacetime and ξ the unit normal vector field to the hypersurface. We look for some necessary algebraic conditions on the components of ξ with respect to the frame $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ on $\bar{M}$, in order for M to have a Codazzi second fundamental form. We prove the following.

Theorem 3.1. *Let $F : M \to \bar{M}$ be a hypersurface with a Codazzi second fundamental form and ξ the unit normal vector field, with $g(\xi,\xi) = \varepsilon \in \{-1,1\}$. Consider the coordinate vector fields $\{\partial_i\}$ on $\bar{M}$ introduced above. Then, $g(\xi, \partial_2) = 0$ and M is a timelike hypersurface. Moreover, some of the following holds:*

(I) Every point of M admits an open neighbourhood in M, on which either $\xi = b\partial_2 + \partial_3$ or $\xi = b\partial_2 + \partial_4$;
(II) M is an ε-space, that is, $k_3 = k_4 = k$.

Proof. Consider $\xi = a\partial_1 + b\partial_2 + c\partial_3 + d\partial_4$, for some functions $a, b, c, d : U \to \mathbb{R}$ such that $g(\xi,\xi) = \varepsilon = \pm 1 \neq 0$. Then, the following vector fields are tangent to the hypersurface:

$$X_1 = a\partial_1 - \rho\partial_2, \quad X_2 = c\partial_1 - \rho\partial_3, \quad X_3 = d\partial_1 - \rho\partial_4,$$
$$X_4 = c\partial_2 - a\partial_3, \quad X_5 = d\partial_2 - a\partial_4, \quad X_6 = d\partial_3 - c\partial_4,$$

where we put $\rho = a\left(k_3x_3^2 + k_4x_4^2\right) + b$. If h is Codazzi, then equation (2.3) yields that $R(X_i, X_j)\xi = 0$ for every $i, j \in \{1, \dots, 6\}$. In particular, we have

$$0 = R(X_1, X_4)\xi = -a^2k_3X_4,$$

which gives necessarily $a = 0$. Therefore, $g(\xi, \xi) = c^2 + d^2$ and so, M is necessarily timelike and c and d cannot both vanish. Moreover, from

$$0 = R(X_2, X_6)\xi = cd(k_3 - k_4)X_4 \tag{3.1}$$

we deduce that we have to consider separately two cases, depending on whether $k_3 = k_4$. If $k_3 = k_4$, then $\bar{M}$ is an ε-space and we get case (II) of the statement.

Assume now that $k_3 \neq k_4$. Then, by (3.1), either $c = 0$ or $d = 0$ at any point of M. On the other hand, c and d cannot vanish at the same point. Hence, each point $p \in M$ either admits a neighbourhood where $c \neq 0 = d$ (and so, $||\xi||^2 = c^2 + d^2 = 1$ yields $c = \pm 1$) or it admits a neighbourhood where $c = 0 \neq d$ (whence, $d = \pm 1$).

As the maps $\Lambda_3 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, x_4)$ and $\Lambda_4 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4)$ are isometries of the ambient space $\bar{M}$, we can fix $\xi = b\partial_2 + \partial_3$ and $\xi = b\partial_2 + \partial_4$ without loss of generality. □

Remark 3.2. We explicitly observe that in the proof above, when referring to the isometric immersion F, coordinates x_i stand for $x_i \circ F$. Throughout the paper we shall use this simplified notation but always taking into account their precise meaning.

We shall now proceed with the classification of totally geodesic hypersurfaces in the different cases listed in Theorem 3.1. Suppose that M is a hypersurface with a Codazzi second fundamental form.

Case (I): $\boldsymbol{\xi = b\partial_2 + \partial_3}$ (or $\boldsymbol{\xi = b\partial_2 + \partial_4}$).

If $\xi = b\partial_2 + \partial_3$, for some smooth function b on M, the vector fields

$$Y_1 = \partial_2, \qquad Y_2 = \partial_1 - b\partial_3, \qquad Y_3 = \partial_4 \tag{3.2}$$

span the tangent space to M at every point.

A direct calculation, using (3.2) and (2.5), gives

$$\begin{aligned}
&\nabla_{Y_1}Y_1 = 0, \quad \nabla_{Y_2}Y_1 = 0, \quad \nabla_{Y_3}Y_1 = 0,\\
&\nabla_{Y_1}Y_2 = -Y_1(b)\xi + bY_1(b)Y_1, \quad \nabla_{Y_3}Y_2 = -Y_3(b)\xi + (k_4x_4 + bY_3(b))Y_1,\\
&\nabla_{Y_2}Y_2 = -(Y_2(b) + k_3x_3)\xi + b(Y_2(b) - k_3x_3)Y_1 - k_4x_4Y_3,\\
&\nabla_{Y_1}Y_3 = 0, \quad \nabla_{Y_2}Y_3 = +k_4x_4Y_1, \quad \nabla_{Y_3}Y_3 = 0.
\end{aligned}$$

Then, since $h(Y_1, Y_2) = h(Y_2, Y_1)$ and $h(Y_2, Y_3) = h(Y_3, Y_2)$, we obtain

$$Y_1(b) = Y_3(b) = 0 \tag{3.3}$$

and so, using the Gauss formula (2.1), we get that the Levi-Civita connection on M is completely determined by the following possibly non-vanishing components:

$$\nabla^M_{Y_2}Y_2 = b(Y_2(b) - k_3x_3)Y_1 - k_4x_4Y_3, \qquad \nabla^M_{Y_2}Y_3 = \nabla^M_{Y_3}Y_2 = k_4x_4Y_1. \tag{3.4}$$

Moreover, the only possibly non-vanishing component of the second fundamental form h with respect to $\{Y_1, Y_2, Y_3\}$ is given by

$$h(Y_2, Y_2) = -Y_2(b) - k_3 x_3. \tag{3.5}$$

In particular, M is totally geodesic if and only if

$$Y_2(b) = -k_3 x_3. \tag{3.6}$$

It is easy to check that $[Y_i, Y_j] = 0$ for all indices i, j. So, the vector fields

$$Y_1 = \partial_t, \quad Y_2 = \partial_y, \quad Y_3 = \partial_x \tag{3.7}$$

may be taken as coordinate vector fields on M. Since $Y_1(b) = Y_3(b) = 0$ we then get that $b = b(y)$ and condition (3.6) can be rewritten in the form

$$b'(y) = -k_3 x_3. \tag{3.8}$$

Denote now by $F : M \to \bar{M},\ (t, x, y) \mapsto (F_1(t, x, y), \ldots, F_4(t, x, y))$ the immersion of the hypersurface in the local coordinates introduced above. By (3.2) and (3.7), we obtain

$$\begin{aligned} (\partial_t F_1, \partial_t F_2, \partial_t F_3, \partial_t F_4) &= \ (0, 1, 0, 0), \\ (\partial_x F_1, \partial_x F_2, \partial_x F_3, \partial_x F_4) &= (0, 0, 0, 1), \\ (\partial_y F_1, \partial_y F_2, \partial_y F_3, \partial_y F_4) &= \ (1, 0, -b, 0). \end{aligned} \tag{3.9}$$

From (3.9) we find that $F_3 = F_3(y)$ satisfies $\partial_y F_3 = -b(y)$. Then, by (3.8) we get $F_3''(y) = k_3 F_3$. Integrating we obtain $F_3(y) = \phi(y)$, where

$$\phi(y) = \begin{cases} A\cosh(\sqrt{k_3}\, y) + B\sinh(\sqrt{k_3}\, y) & \text{if } k_3 > 0, \\ A\cos(\sqrt{-k_3}\, y) + B\sin(\sqrt{-k_3}\, y) & \text{if } k_3 < 0, \end{cases} \tag{3.10}$$

for some real constants A and B. Therefore, as $b(y) = -\partial_y F_3 = -\phi'(y)$, we find

$$b(y) = \begin{cases} -A\sqrt{k_3}\sinh(\sqrt{k_3}\, y) - B\sqrt{k_3}\cosh(\sqrt{k_3}\, y) & \text{if } k_3 > 0, \\ -A\sqrt{-k_3}\sin(\sqrt{-k_3}\, y) + B\sqrt{-k_3}\cos(\sqrt{-k_3}\, y) & \text{if } k_3 < 0. \end{cases}$$

Moreover, again by integration of (3.9), we get

$$F_1 = y + c_1, \quad F_2 = t + c_2, \quad F_4 = x + c_4,$$

for some real constants c_1, c_2 and c_4. After a reparametrization, we obtain the immersion

$$F(t, x, y) = (y, t, \phi(y), x). \tag{3.11}$$

The case where $\xi = b\partial_2 + \partial_4$ can be treated exactly in the same way. It leads to the following explicit expression of the immersion:

$$F(t, x, y) = (y, t, x, \psi(y)), \tag{3.12}$$

where

$$\psi(y)=\begin{cases} A\cosh(\sqrt{k_4}\,y)+B\sinh(\sqrt{k_4}\,y) & \text{if } k_4>0,\\ A\cos(\sqrt{-k_4}\,y)+B\sin(\sqrt{-k_4}\,y) & \text{if } k_4<0, \end{cases} \tag{3.13}$$

for some real constants A, B and $b(y)=-\psi'(y)$. Observe that for an ε-space, $\phi(y)=\psi(y)$.

Remark 3.3. A hypersurface M as described in (3.11) is determined with respect to coordinates (x_1,x_2,x_3,x_4) of the ambient space by equation $x_3=\phi(x_1)$. Hence, it is an open part of the cylindrical hypersurface spanned by the curve of equation $x_3=\phi(x_1)$ in the (x_1,x_3)–plane. Correspondingly, a hypersurface M as described in (3.12) is an open part of the cylindrical hypersurface of equation $x_4=\psi(x_1)$.

Finally, we observe that in the special case where $b=g(\xi,\partial_1)=0$, we easily conclude that $\phi=0$ (respectively, $\psi=0$), so that the totally geodesic hypersurface is an open part of the hyperplane $x_3=0$ (respectively, $x_4=0$).

We shall now treat the case (II) in Theorem 3.1. As we shall see, the totally geodesic solutions are again of the forms we determined in case (I). We shall make use of the following well-known general result.

Lemma 3.4. *Let $\ell: M\to\mathbb{R}$ denote an arbitrary continuous function. Then,*

$$\begin{aligned}\Omega&=\Omega_1\cup\Omega_2\\ &=\{p\in M:\ell\neq 0 \textit{ in a neighbourhood of } p\}\\ &\quad\cup\{p\in M:\ell=0 \textit{ in a neighbourhood of } p\}\end{aligned}$$

is a dense open subset of M.

Case (II): $a=0$ and $k_3=k_4=k$. In this case, $\xi=b\partial_2+c\partial_3+d\partial_4$, where b,c,d are smooth functions on M. As $||\xi||^2=c^2+d^2=1$, there exists a smooth function θ on M such that $c=\cos\theta$ and $d=\sin\theta$.

Applying Lemma 3.4 to $\ell=b$, we have that there exists a dense open subset Ω of M, such that each point $p\in\Omega$ either admits a neighbourhood where $b=0$ or a neighbourhood where $b\neq 0$ at any point. Thus, we consider separately the following cases.

Case (II.a): $a=b=0$ and $k_3=k_4=k$. In this case, $\xi=\cos\theta\partial_3+\sin\theta\partial_4$ and the vector fields

$$Y_1=\partial_1,\qquad Y_2=\partial_2,\qquad Y_3=\sin\theta\partial_3-\cos\theta\partial_4 \tag{3.14}$$

span the tangent space to M at every point. A direct calculation, using (3.14) and (2.5), gives

$$\nabla_{Y_1}Y_1 = -k(x_3\cos\theta + x_4\sin\theta)\xi - k(x_3\sin\theta - x_4\cos\theta)Y_3, \quad \nabla_{Y_2}Y_1 = 0,$$
$$\nabla_{Y_3}Y_1 = k(x_3\sin\theta - x_4\cos\theta)Y_2, \quad \nabla_{Y_1}Y_2 = 0,$$
$$\nabla_{Y_2}Y_2 = 0, \quad \nabla_{Y_3}Y_2 = 0,$$
$$\nabla_{Y_1}Y_3 = Y_1(\theta)\xi + k(x_3\sin\theta - x_4\cos\theta)Y_2, \quad \nabla_{Y_2}Y_3 = Y_2(\theta)\xi,$$
$$\nabla_{Y_3}Y_3 = Y_3(\theta)\xi.$$

Then, since $h(Y_1, Y_3) = h(Y_3, Y_1)$ and $h(Y_2, Y_3) = h(Y_3, Y_2)$, we get

$$Y_1(\theta) = Y_2(\theta) = 0$$

and so, applying the Gauss formula (2.1), we find that the Levi-Civita connection on M is completely determined by the following possibly non-vanishing components:

$$\begin{aligned} \nabla^M_{Y_1}Y_1 &= -k(x_3\sin\theta - x_4\cos\theta)Y_3, \\ \nabla^M_{Y_1}Y_3 = \nabla^M_{Y_3}Y_1 &= k(x_3\sin\theta - x_4\cos\theta)Y_2. \end{aligned} \tag{3.15}$$

Moreover, the second fundamental form is determined by

$$h(Y_1, Y_1) = -k(x_3\cos\theta + x_4\sin\theta), \quad h(Y_3, Y_3) = Y_3(\theta). \tag{3.16}$$

By (3.16), M is totally geodesic if and only if

$$x_3\cos\theta + x_4\sin\theta = 0, \qquad Y_3(\theta) = 0. \tag{3.17}$$

In this case, as $Y_i(\theta) = 0$ for all indices $i = 1, 2, 3$, we have that θ is a real constant. Observe that the map

$$\begin{aligned} \Lambda : \bar{M} &\to \bar{M}, \\ (x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, \cos\theta x_3 + \sin\theta x_4, -\sin\theta x_3 + \cos\theta x_4) \end{aligned} \tag{3.18}$$

is an isometry of the ambient ε-space $\bar{M}$, for every real constant θ. Therefore, as the unit normal vector field to M is given by $\xi = \cos\theta\partial_3 + \sin\theta\partial_4$, without loss of generality it suffices to consider the case where $\theta = 0$ and so, $\xi = \partial_3$ (equivalently, we can set $\theta = \frac{\pi}{2}$ and get $\xi = \partial_4$).

Hence, this case corresponds to the special solution we found in Case (I) when $b = g(\xi, \partial_1) = 0$ (see Remark 3.3). Correspondingly, the totally geodesic condition written in (3.17), namely, $x_3\cos\theta + x_4\sin\theta = 0$, applying the above isometry reads $x_3 = 0$ (equivalently, $x_4 = 0$).

Case (II.b): $\boldsymbol{a = 0 \neq b}$ and $\boldsymbol{k_3 = k_4 = k}$. We now have $\xi = b\partial_2 + \cos\theta\partial_3 + \sin\theta\partial_4$ and the vector fields

$$Y_1 = \partial_2, \qquad Y_2 = \cos\theta\partial_1 - b\partial_3, \qquad Y_3 = \sin\theta\partial_1 - b\partial_4 \tag{3.19}$$

span the tangent space to M at every point.

Using (2.5), (3.19) and the formula $h(Y_i, Y_j) = -g(\nabla_{Y_i}\xi, Y_j)$, a long but straightforward computation yields

$$\begin{aligned}
&h(Y_1, Y_1) = h(Y_2, Y_1) = h(Y_3, Y_1) = 0,\\
&h(Y_1, Y_2) = Y_1(\cos\theta)b - \cos\theta Y_1(b),\\
&h(Y_2, Y_2) = Y_2(\cos\theta)b - \cos\theta Y_2(b) - k\cos^2\theta(x_3\cos\theta + x_4\sin\theta),\\
&h(Y_3, Y_2) = Y_3(\cos\theta)b - \cos\theta Y_3(b) - k\cos\theta\sin\theta(x_3\cos\theta + x_4\sin\theta),\\
&h(Y_1, Y_3) = Y_1(\sin\theta)b - \sin\theta Y_1(b),\\
&h(Y_2, Y_3) = Y_2(\sin\theta)b - \sin\theta Y_2(b) - k\cos\theta\sin\theta(x_3\cos\theta + x_4\sin\theta),\\
&h(Y_3, Y_3) = Y_3(\sin\theta)b - \sin\theta Y_3(b) - k\sin^2\theta(x_3\cos\theta + x_4\sin\theta).
\end{aligned} \tag{3.20}$$

Next, from $h(Y_1, Y_2) = h(Y_2, Y_1)$ and $h(Y_1, Y_3) = h(Y_3, Y_1)$ we get

$$Y_1(\theta) = 0, \quad Y_1(b) = 0.$$

We now set $H = x_3\cos\theta + x_4\sin\theta$ and assume that M is totally geodesic. Then, from (3.20) we have

$$-\sin\theta Y_2(\theta)b - \cos\theta Y_2(b) = kH\cos^2\theta, \tag{3.21}$$

$$\cos\theta Y_3(\theta)b - \sin\theta Y_3(b) = kH\sin^2\theta, \tag{3.22}$$

$$\cos\theta Y_2(\theta)b - \sin\theta Y_2(b) = kH\cos\theta\sin\theta, \tag{3.23}$$

$$-\sin\theta Y_3(\theta)b - \cos\theta Y_3(b) = kH\cos\theta\sin\theta, \tag{3.24}$$

We multiply equation (3.21) by $-\sin\theta$ and (3.23) by $\cos\theta$ and we sum them, obtaining $Y_2(\theta)b = 0$, that is, $Y_2(\theta) = 0$, as $b \neq 0$. In the same way, from (3.22) and (3.24) we get $Y_3(\theta) = 0$. So, θ is constant.

As we already observed in Case (II.a), since θ is a real constant, the map Λ described in (3.18) is an isometry of the ambient ε-space $\bar{M}$. Consequently, without loss of generality, it suffices to consider the case where $\theta = 0$ and so, $\xi = b\partial_2 + \partial_3$ (or, equivalently, $\theta = \frac{\pi}{2}$, whence, $\xi = b\partial_2 + \partial_4$). Thus, we obtain again the totally geodesic hypersurfaces described in Case (I).

The above calculations and conclusions are summarized in the following complete classification of totally geodesic hypersurfaces of Cahen-Wallach spacetimes.

Theorem 3.5. *Let M denote a totally geodesic hypersurface of a Cahen-Wallach spacetime $\bar{M}$. If $\bar{M}$ is not an ε-space, then one of the following holds:*

(a) M is an open part of the cylindrical hypersurface of equation $x_3 = \phi(x_1)$, where ϕ is given by (3.10).

(b) M is an open part of the cylindrical hypersurface of equation $x_4 = \psi(x_1)$, where ψ is given by (3.13).

Hyperplanes $x_3 = 0$ and $x_4 = 0$ occur as special solutions of case (a) and (b) respectively.

If $\bar{M}$ is an ε-space, then there exists an open dense subset Ω of M, such that any point $p \in \Omega$ admits a neighbourhood as described in one of above cases (a) and (b).

Remark 3.6. If $\bar{M}$ is an ε-space, then $\phi = \psi$. Moreover, interchanging x_3 and x_4 is an isometry. Therefore, the two cases listed in Theorem 3.5 are congruent in an ε-space.

Before proceeding with the investigation of parallel hypersurfaces of Cahen-Wallach spaces, we shall emphasize some relevant geometric properties which hold for hypersurfaces with a Codazzi second fundamental form. We prove the following.

Theorem 3.7. *Any hypersurface $F : M \to \bar{M}$ with a Codazzi second fundamental form in a Cahen-Wallach spacetime is a Brinkmann manifold. Moreover, if $\bar{M}$ is not an ε-space, then M is minimal.*

Proof. From the description of the Levi-Civita connection of a Cahen-Wallach spacetime given in (2.5), we see at once that ∂_2 is a parallel vector field. On the other hand, Theorem 3.1 yields that ∂_2 is tangent to M. Therefore, all hypersurfaces with a Codazzi second fundamental form admit a parallel null vector field, namely, the projection on M of the coordinate vector field ∂_2 of the ambient space. So, they are Brinkmann manifolds.

Next, suppose now that $\bar{M}$ is not an ε-space, so that we are in Case (I) of Theorem 3.1. Consider the case where the unit normal vector field is given by $\xi = b\partial_2 + \partial_3$ (the same argument holds when $\xi = b\partial_2 + \partial_4$).

Denoted by g_M the metric induced on M by the Cahen-Wallach metric g described by (2.4), from equations (3.2) we easily find that $g_M^{-1}(Y_2, Y_2) = 0$. On the other hand, by (3.5), $h(Y_2, Y_2)$ is the only possibly nonvanishing component of h with respect to $\{Y_i\}$. Therefore, we have

$$\mathrm{tr}_{g_M} h = \frac{1}{3} g_M^{-1}(Y_2, Y_2) h(Y_2, Y_2) = 0,$$

so that all such hypersurfaces with Codazzi second fundamental form are minimal. We may observe that the same arguments and the conclusion about minimality hold in the class of ε-spaces, provided one assumes that either $g(\xi, \partial_3)$ or $g(\xi, \partial_4)$ is a constant. □

4. Parallel Hypersurfaces in Cahen-Wallach Spacetimes

Clearly, totally geodesic hypersurfaces are also parallel. In this section we investigate *proper* parallel hypersurfaces, that is, the ones which are parallel but not totally geodesic. We start with the following.

Theorem 4.1. *Let $F : M \to \bar{M}$ be a proper parallel hypersurface of a Cahen-Wallach spacetime. If $\bar{M}$ is not an ε-space, then there exist local coordinates (t, x, y) on M such that up to isometries, the immersion is one of the following:*

$$(a) \qquad F(t, x, y) = (y, t, \phi(y) + C, x),$$

where ϕ is given by (3.10) and $C \neq 0$ is a real constant. So, M is an open part of the cylindrical hypersurface of equation $x_3 = \phi(x_1) + C$.

$$(b) \qquad F(t, x, y) = (y, t, x, \psi(y) + C),$$

where ψ is given by (3.13) and $C \neq 0$ is a real constant. So, M is an open part of the cylindrical hypersurface of equation $x_4 = \psi(x_1) + C$.

Proof. A parallel hypersurface M has a Codazzi second fundamental form. As $\bar{M}$ is not an ε-space, from Case (I) of Theorem 3.1 we have that the normal unit vector field of M is given by either $\xi = b\partial_2 + \partial_3$ or $\xi = b\partial_2 + \partial_4$. Suppose first that $\xi = b\partial_2 + \partial_3$. Then, equations (3.2), (3.4), (3.5) and (3.7) of Case (I) of the previous Section hold. In particular, the second fundamental form is completely determined by

$$h(Y_2, Y_2) = -(Y_2(b) + k_3 x_3).$$

Taking into account (3.4) and (3.5), we have that $\nabla^M h = 0$ if and only if $Y_i(h(Y_2, Y_2)) = 0$ for all indices $i = 1, 2, 3$. But equation (3.3) yields

$$Y_1(h(Y_2, Y_2)) = -Y_1(Y_2(b)) = -Y_2(Y_1(b)) = 0$$

and, by the same argument, $Y_3(h(Y_2, Y_2)) = 0$. On the other hand,

$$Y_2(h(Y_2, Y_2)) = -(Y_2(Y_2(b)) - k_3 b).$$

Thus, requiring M to be parallel is equivalent to $Y_2(Y_2(b)) = k_3 b$, that is, with respect to the local coordinates (t, x, y) introduced in (3.7), $b''(y) = k_3 b$.

By integration, we then have

$$b(y) = \begin{cases} P\cosh(\sqrt{k_3}\, y) + Q\sinh(\sqrt{k_3}\, y) & \text{if } k_3 > 0 \\ P\cos(\sqrt{-k_3}\, y) + Q\sin(\sqrt{-k_3}\, y) & \text{if } k_3 < 0 \end{cases}$$

for some real constants P, Q.

Denote now by $F : M \to \bar{M}$, $(t, x, y) \mapsto (F_1(t, x, y), \ldots, F_4(t, x, y))$ the immersion of the hypersurface in the local coordinates introduced above. By (3.2) and (3.7) we obtain

$$\begin{aligned} (\partial_t F_1, \partial_t F_2, \partial_t F_3, \partial_t F_4) &= (0, 1, 0, 0), \\ (\partial_x F_1, \partial_x F_2, \partial_x F_3, \partial_x F_4) &= (0, 0, 0, 1), \\ (\partial_y F_1, \partial_y F_2, \partial_y F_3, \partial_y F_4) &= (1, 0, -b, 0). \end{aligned} \tag{4.1}$$

Integrating (4.1) we find

$$F_1 = y + c_1, \quad F_2 = t + c_2, \quad F_4 = x + c_4,$$

and

$$F_3(y) = \begin{cases} -\dfrac{P}{\sqrt{k_3}}\sinh(\sqrt{k_3}\, y) - \dfrac{Q}{\sqrt{k_3}}\cosh(\sqrt{k_3}\, y) + C & \text{if } k_3 > 0, \\ -\dfrac{P}{\sqrt{-k_3}}\sin(\sqrt{-k_3}\, y) + \dfrac{Q}{\sqrt{-k_3}}\cos(\sqrt{-k_3}\, y) + C & \text{if } k_3 < 0, \end{cases}$$

for some real constants c_1, c_2, c_4 and C. Observe that setting $(P,Q) = (-\sqrt{k_3}B, -\sqrt{k_3}A)$ (respectively, $(P,Q) = (-\sqrt{k_3}B, \sqrt{k_3}A)$) in the first (respectively, second) of the above cases, by (3.10) we conclude that $F_3(y) = \phi(y) + C$. After a reparametrization, we then get the immersion

$$F(t,x,y) = (y, t, \phi(y) + C, x).$$

By Theorem 3.5, this immersion is totally geodesic if and only if $C = 0$.

The case $\xi = b\partial_2 + \partial_4$ can be treated exactly in the same way, leading to case (b) in the statement. □

Remark 4.2. When $g(\xi, \partial_1) = 0$, as $\phi = \psi = 0$, from Theorem 4.1 we get as special cases of proper parallel hypersurfaces the hyperplanes of equation $x_3 = C$ and $x_4 = C$, where $C(\neq 0$, because of Remark 3.3) is a real constant.

Remark 4.3. We already know from Theorem 3.7 that parallel hypersurfaces M, as described in Theorem 4.1, are Brinkmann manifolds and minimal. Moreover, being parallel hypersurfaces in a locally symmetric space, they must be locally symmetric.

In fact, with respect to local coordinates (t,x,y), using (2.4), (3.2) and (3.7), a straightforward calculation yields that the metric g_M takes exactly the form (2.6), with

$$f(x,y) = \begin{cases} k_4x^2 + \left[k_3\phi(y)^2 + 2k_3C\phi(y) + C^2 + (\phi'(y))^2\right] \text{ in case (a)}, \\ k_3x^2 + \left[k_4\psi(y)^2 + 2k_4C\psi(y) + C^2 + (\psi'(y))^2\right] \text{ in case (b)}, \end{cases}$$

so that they are locally symmetric Brinkmann manifolds.

Parallel hypersurfaces described in the above Theorem 4.1 also exist in ε-spaces, but in this case they do not provide a full classification as for general Cahen-Wallach spacetimes. As it may be seen in the previous Section, equations are remarkably more complicated for hypersurfaces of an ε-space. The following result provides the full classification of proper parallel hypersurfaces in ε-spaces under the assumption that $b = g(\xi, \partial_1) = 0$.

Theorem 4.4. *Let $F : M \to \bar{M}$ be a proper parallel hypersurface of a four-dimensional ε-space. Assume that the normal unit vector field ξ of M satisfies $g(\xi, \partial_1) = 0$. If F is not included in one of the cases listed in the above Theorem 4.1, then there exist local coordinates (t,x,y) on M, such that up to isometries, the immersion is given by*

$$F(t,x,y) = \left(y, t, \frac{1}{\lambda}\cos(\lambda x + \mu), \frac{1}{\lambda}\sin(\lambda x + \mu)\right),$$

for some real constants $\lambda \neq 0$ and μ. So, M is the cylindrical hypersurface of equation $x_3^2 + x_4^2 = \frac{1}{\lambda^2}$.

Proof. As $g(\xi, \partial_1) = 0$, the unit vector field normal to M is given by $\xi = \cos\theta\partial_3 + \sin\theta\partial_4$ and the vector fields Y_i described in (3.14) span the tangent space to M at every point. Moreover, equations (3.15) and (3.16) of Case (II.a) hold. A direct calculation then yields that M is parallel if and only if

$$\begin{aligned} 0 &= \nabla^M_{Y_3} h(Y_1, Y_1) = k(x_3 \sin\theta - x_4 \cos\theta) Y_3(\theta), \\ 0 &= \nabla^M_{Y_3} h(Y_3, Y_3) = Y_3(Y_3(\theta)). \end{aligned} \tag{4.2}$$

Since $[Y_i, Y_j] = 0$ for all indices i, j, the vector fields

$$Y_1 = \partial_y, \quad Y_2 = \partial_t, \quad Y_3 = \partial_x \tag{4.3}$$

may be taken as coordinate vector fields on M. By (3.16), $Y_1(\theta) = Y_2(\theta) = 0$ and so, the function θ only depends on x. Moreover, by (4.2), $\theta''(x) = Y_3(Y_3(\theta)) = 0$, so that $\theta'(x) = Y_3(\theta)$ is constant everywhere. Depending on whether $\theta'(x) = 0$, we have the following cases.

Case (1): $\theta'(x) = 0$.

Then, θ is a real constant. By the isometry Λ of an ε-space described in (3.18), without loss of generality we reduce to the case where $\xi = \partial_3$ (equivalently, $\xi = \partial_4$). So, we obtain the special cases of proper parallel hypersurfaces listed in Theorem 4.1, which we already described in Remark 4.2.

Case (2): $\theta'(x) = \lambda \neq 0$.

Integrating, we then have

$$\theta(x) = \lambda x + \mu$$

for some real constants $\lambda \neq 0$ and μ. Moreover, from (4.2) we deduce

$$x_3 \sin\theta = x_4 \cos\theta. \tag{4.4}$$

Denote now by $F : M \to \bar{M}$, $(t, x, y) \mapsto (F_1(t, x, y), \ldots, F_4(t, x, y))$ the immersion of the hypersurface in the local coordinates introduced above. By (3.14) and (4.3), we obtain

$$\begin{aligned} (\partial_t F_1, \partial_t F_2, \partial_t F_3, \partial_t F_4) &= (0, 1, 0, 0), \\ (\partial_x F_1, \partial_x F_2, \partial_x F_3, \partial_x F_4) &= (0, 0, \sin\theta, -\cos\theta), \\ (\partial_y F_1, \partial_y F_2, \partial_y F_3, \partial_y F_4) &= (1, 0, 0, 0). \end{aligned}$$

By integration we find

$$\begin{aligned} &F_1 = y + c_1, \quad F_2 = t + c_2, \quad F_3 = -\frac{\cos(\lambda x + \mu)}{\lambda} + c_3, \\ &F_4 = -\frac{\sin(\lambda x + \mu)}{\lambda} + c_4, \end{aligned}$$

for some real constants c_i, $i = 1, .., 4$. Finally, since equation (4.4) must be satisfied, we get

$$c_3 \sin(\lambda x + \mu) = c_4 \cos(\lambda x + \mu),$$

for all values of x and so, $c_3 = c_4 = 0$. Applying isometries of the ambient space, we obtain the following parametrization:

$$F(t, x, y) = \left(y, t, \frac{\cos(\lambda x + \mu)}{\lambda}, \frac{\sin(\lambda x + \mu)}{\lambda}\right),$$

which is never totally geodesic. □

With regard to geometric properties, we have the following.

Proposition 4.5. *Proper parallel hypersurfaces M of an ε-space $\bar{M}$, as described in Theorem 4.4, are flat and CMC.*

Proof. Let M denote a proper parallel hypersurface of an ε-space as described in Theorem 4.4. With respect to local coordinates (t, x, y), by equations (2.4), (3.14) and (4.3) we find that the metric g_M takes the form (2.6), where

$$f(x, y) = \frac{k}{\lambda^2}. \tag{4.5}$$

Comparing (4.5) with (2.7) we conclude that g_M is a locally symmetric Brinkmann metric, coherently with the fact that M is a parallel hypersurface in a locally symmetric space. In particular, (M, g_M) is flat [18] (the same conclusion also follows calculating directly the curvature from (3.15), since now we have $x_3 \sin\theta - x_4 \cos\theta = 0$).

Since g_M is of the form (2.6) (with f given by (4.5)), a straightforward calculation yields that

$$g_M^{-1}(\partial_x, \partial_x) = 1, \qquad g_M^{-1}(\partial_y, \partial_y) = 0.$$

On the other hand, from (3.16) and (4.3) we have that h is completely determined by $h(\partial_y, \partial_y)$ and $h(\partial_x, \partial_x) = \theta'(x)$. So, we have

$$\frac{1}{3}\mathrm{tr}_{g_M} h = \frac{1}{3} g_M^{-1}(\partial_x, \partial_x) h(\partial_x, \partial_x) = \frac{1}{3}\theta'(x) = \frac{1}{3}\lambda,$$

whence we conclude that these proper parallel hypersurfaces are CMC. □

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