



Totally Umbilical Hypersurfaces of Lorentzian Reducible Spaces

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Abstract

We investigate totally umbilical hypersurfaces in Lorentzian reducible spaces, focusing on their classification and geometric properties. We explore conditions for the existence of proper totally umbilical hypersurfaces in a $(n + 1)$ -dimensional Lorentzian reducible manifold and provide explicit examples, especially in the three-dimensional case. Moreover, in this special case, we determine a general condition characterizing minimal and CMC surfaces and describe some examples.

Keywords Lorentzian spaces · Lorentzian products · Totally umbilical surfaces · Minimal surfaces

Mathematics Subject Classification 53B25 · 53C50

1 Introduction

A hypersurface M of a pseudo-Riemannian ambient space is said to be *totally umbilical* if its shape operator is proportional to the identity (equivalently, its second fundamental form h is a multiple of the metric on M). The class of totally umbilical submanifolds includes the totally geodesic ones, where h vanishes identically.

In recent years, totally umbilical hypersurfaces have been investigated in several Riemannian ambient spaces. In particular, totally umbilical hypersurfaces in Riemannian products of type $M^n \times \mathbb{R}$ were described in full generality in [14], showing how their existence determines the structure of the ambient space. The classification of totally umbilical surfaces in homogeneous Riemannian three-manifolds was obtained in [12] and [13]: proper examples exist in \mathbb{R}^3 , \mathbb{S}^3 , \mathbb{H}^3 , Sol_3 , $\mathbb{S}^2(c) \times \mathbb{R}$, $\mathbb{H}^2(c) \times \mathbb{R}$

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and in some non-unimodular metric Lie groups. Moreover, parallel and totally umbilical hypersurfaces were recently classified in the four-dimensional Thurston geometry Sol_0^4 [7].

Usually, when investigating some geometric property in the pseudo-Riemannian case, there are some similarities together with some interesting differences with respect to the Riemannian settings. In [2], the authors and M. Castrillón-Lopez investigated totally umbilical surfaces in the three-dimensional Lorentzian Heisenberg group, showing that only the flat left-invariant Lorentzian metric (which does not have a Riemannian counterpart) allows for totally umbilical surfaces.

Taking into account the cases where totally umbilical surfaces occur in homogeneous Riemannian three-manifolds, it is natural to investigate the Lorentzian ambient spaces $S_1^2(c) \times \mathbb{R}$, $\mathbb{H}_1^2(c) \times \mathbb{R}$ and $S^2(c) \times \mathbb{R}_1$, $\mathbb{H}^2(c) \times \mathbb{R}_1$. More in general, taking into account the Riemannian analogue studied in [14], one can consider reducible $(n + 1)$ -spaces and investigate conditions for the existence of their totally umbilical hypersurfaces.

Some examples of the study of (hyper)surfaces in Riemannian product ambient spaces may be found in [3], [5], [6], [15]. With regard to the geometry of surfaces in Lorentzian products ambient spaces, constant angle surfaces [8] and minimal surfaces [1] have been studied. Moreover, in [5], the first author and J. Van der Veken classified parallel surfaces in both Lorentzian and Riemannian three-dimensional reducible spaces. It is also worthwhile to mention the ever growing interest toward minimal and CMC surfaces, defined respectively by the vanishing and the constancy of the mean curvature.

The above cited classes of surfaces are linked with the totally umbilical ones in different ways. In fact, totally umbilical, parallel and minimal surfaces all admit the totally geodesic examples as a special case. On the other hand, a parallel (more in general, a semi-parallel) surface in a three-dimensional ambient space is either flat or totally umbilical [4]. Moreover, both in a Riemannian [11] and in a Lorentzian [2] ambient three-space, a surface admits a conformal Gauss map if and only if it is either minimal or totally umbilical.

In this paper we shall investigate totally umbilical hypersurfaces in an arbitrary Lorentzian product of a real line by an n -dimensional manifold. As such, the ambient space is given by any $(n + 1)$ -dimensional Lorentzian manifold (locally) isometric to either $\mathbb{R}_1 \times \mathbb{M}^n$ or $\mathbb{R} \times \mathbb{M}_1^n$, where the pedix “1” refers to the Lorentzian part. In order to refer to these two cases together, we set

$$\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{n,\delta} = \begin{cases} \mathbb{R}_1 \times \mathbb{M}^n & \text{if } \delta = -1, \\ \mathbb{R} \times \mathbb{M}_1^n & \text{if } \delta = 1. \end{cases}$$

The main result is the following characterization of Lorentzian reducible spaces admitting a proper (that is, not totally geodesic) totally umbilical hypersurface.

Theorem 1.1 $\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{n,\delta}$ admits a proper totally umbilical hypersurface if and only if the n -dimensional pseudo-Riemannian manifold $\mathbb{M}^{n,\delta}$ is (locally) isometric to a warped product of an interval of the Euclidean line or the Lorentzian line with some $(n - 1)$ -dimensional manifold, either Lorentzian or Riemannian.

As the existence of proper totally umbilical hypersurfaces requires the ambient space to be (locally) a warped product, we shall then focus on the case of three-dimensional Lorentzian warped products, providing several explicit examples of totally umbilical surfaces. Finally, since both totally umbilical and minimal surfaces are linked to the conformality of the Gauss map, we also investigate minimal surfaces in these ambient spaces.

The paper is organized in the following way. In Sect. 2 we report the needed information concerning the geometry of (hyper)surfaces. In Sect. 3 we study totally umbilical hypersurfaces of Lorentzian product spaces, firstly proving the above characterization and then focusing on the three-dimensional case. In Sect. 4 we deduce the equations which characterize minimal and CMC surfaces in three-dimensional Lorentzian warped products and describe some explicit examples.

2 Preliminaries

Let $F : \Sigma \rightarrow \tilde{M}$ be an isometric immersion of a pseudo-Riemannian hypersurface Σ into a pseudo-Riemannian space \tilde{M} of dimension $n + 1$. We shall denote by \tilde{g} and g the metric on \tilde{M} and its pullback to Σ by F , respectively.

Let N be a unit normal vector field on the hypersurface, with $\tilde{g}(N, N) = \varepsilon \in \{-1, 1\}$; if $\varepsilon = 1$ (respectively, $\varepsilon = -1$), then N is spacelike (respectively, timelike) and we call Σ a timelike (respectively, spacelike) hypersurface, meaning that the immersion is Lorentzian (respectively, Riemannian).

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of Σ and \tilde{M} respectively. For any vector fields X, Y tangent to Σ , the well known *formula of Gauss* reads

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N \tag{2.1}$$

and determines the *second fundamental form* h of the immersion as a symmetric bilinear form defined on $T\Sigma$.

As we mentioned in the Introduction, several significant properties of h define corresponding relevant classes of submanifolds. In particular, any point $p \in \Sigma$ admitting a real number $\lambda(p)$, which satisfies

$$h(X_p, Y_p) = \lambda(p)g_p(X_p, Y_p),$$

is called a *totally umbilical point*. The hypersurface Σ is called *totally umbilical* if every point $p \in M$ is totally umbilical, that is, for every point of Σ there is a neighbourhood $U \subseteq \Sigma$ and a smooth function $\lambda : U \rightarrow \mathbb{R}$, such that

$$h(X, Y) = \lambda g(X, Y), \tag{2.2}$$

for every $X, Y \in T\Sigma$. It is easy to observe that equation (2.2) can be expressed equivalently requiring that the shape operator S is a multiple of the identity, namely, $S = \lambda Id$.

Clearly, if (2.2) holds for $\lambda = 0$, then h vanishes identically and so, Σ is totally geodesic. A different generalization of totally geodesic hypersurfaces is given by the parallelism of the second fundamental form h . In fact, a hypersurface is said to be *parallel* (or having a parallel second fundamental form) if

$$\nabla h = 0,$$

which means that a parallel hypersurface is characterized by the covariant constancy of all the extrinsic invariants derived from h .

Next, using R and \tilde{R} to denote the Riemann–Christoffel curvature tensors of Σ and \tilde{M} respectively, we can express the definition of a *semi-parallel* immersion, that is, $R \cdot h = 0$, where

$$(R \cdot h)(X, Y, Z, W) = -h(R(X, Y)Z, W) - h(Z, R(X, Y)W)$$

for every $X, Y, Z, W \in T\Sigma$.

As we mentioned in the Introduction, in the case of surfaces in three-dimensional ambient spaces, a special relationship arises between the semi-parallel surfaces and the totally umbilical ones, as shown by the following result.

Lemma 2.1 [4] *Any parallel surface in a Lorentzian three-manifold (\tilde{M}, \tilde{g}) is semi-parallel. Moreover, a surface Σ in a Lorentzian manifold is semi-parallel if and only if it is either flat or totally umbilical.*

Next, the *equations of Gauss and Codazzi* are given respectively by

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + \varepsilon (h(X, Z)h(Y, W) - h(X, W)h(Y, Z)), \tag{2.3}$$

$$\tilde{g}(\tilde{R}(X, Y)Z, N) = \varepsilon ((\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z)), \tag{2.4}$$

where X, Y, Z, W are tangent to Σ .

We now recall that the *mean curvature* H of a surface Σ in a three-dimensional ambient space \tilde{M} is given by

$$H = \frac{1}{2} \text{tr}_g h = \frac{1}{2} \sum g^{ij} h_{ij},$$

where g^{ij} are the components of g^{-1} with respect to a given basis of vector fields tangent to Σ . The surface Σ is said to be *minimal* if $H = 0$ and *of constant mean curvature (CMC)* if $H \neq 0$ is a constant. Obviously, totally geodesic surfaces are minimal and minimal surfaces are CMC.

We now briefly report the description of the Gauss map for a submanifold of a Lorentzian ambient space and its characterization when the ambient space is three-dimensional, referring to [2] for more details.

Let $O\tilde{M}$ denote the $O(n, 1)$ -principal bundle of orthonormal frames on a $(n + 1)$ -dimensional Lorentzian manifold (\tilde{M}, \tilde{g}) . Starting from an isometric immersion $F :$

$(M, g) \rightarrow (\tilde{M}, \tilde{g})$ of an m -dimensional spacelike submanifold, one considers the Grassmannian

$$\mu^+ : O\tilde{M} \rightarrow G_m^+(\tilde{M}) = \frac{O\tilde{M}}{O(m) \times O(n-m, 1)},$$

Then, the Gauss map γ^+ of (M, g) is defined by

$$\begin{aligned} \gamma^+ : M &\rightarrow G_m^+(\tilde{M}) \\ p &\mapsto F_*(T_p M). \end{aligned}$$

Correspondingly, if $F : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ is an isometric immersion of an m -dimensional timelike manifold, we consider the Grassmannian

$$\mu^- : O\tilde{M} \rightarrow G_m^-(\tilde{M}) = \frac{O\tilde{M}}{O(n+1-m) \times O(m-1, 1)}$$

and the Gauss map γ^- of (M, g) is defined by

$$\begin{aligned} \gamma^- : M &\rightarrow G_m^-(\tilde{M}) \\ p &\mapsto F_*(T_p M). \end{aligned}$$

Then, the following result holds.

Proposition 2.2 [2] *Let (M, g) be a Riemannian (respectively, Lorentzian) surface in a three-dimensional Lorentzian ambient space (\tilde{M}, \tilde{g}) . Then, whatever the curvature of the ambient space, the Gauss map γ^+ (respectively, γ^-) is conformal if and only if (M, g) is pseudo-umbilical, that is, either minimal or totally umbilical.*

We end this Section recalling the definition of conformal vector fields, which will play an important role in the next Section. A vector field X on a pseudo-Riemannian manifold (\tilde{M}, \tilde{g}) is conformal if and only if $\mathcal{L}_X \tilde{g} = 2\phi \tilde{g}$ for some function ϕ . In this case, the flow of ξ consists of conformal maps. In particular, a vector field X is said to be closed conformal if and only if it is conformal and its dual one-form is closed. It is easy to check by direct calculation that X is closed conformal if and only if $\tilde{\nabla}_Y X = \phi Y$, for all $p \in \tilde{M}$ and $Y \in T_p \tilde{M}$.

3 Totally Umbilical Hypersurfaces in a Lorentzian Reducible Space

Following the notation we introduced in Sect. 1, we shall denote by $\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{n,\delta}$, $\delta = \pm 1$, an arbitrary Lorentzian product of a real line by an n -dimensional manifold. We then consider the canonical projection

$$\pi_\delta : \mathcal{M}_\delta \rightarrow \mathbb{M}^{n,\delta}$$

of \mathcal{M}_δ on $\mathbb{M}^{n,\delta}$. Let ξ denote a vector field on \mathcal{M}_δ , tangent to the fibers of π_δ and such that $\tilde{g}(\xi, \xi) = \delta$. We remark that ξ is a δ -unit Killing vector field.

Next, if we set $\partial_{x_0} := \xi$ and choose suitable coordinates (x_1, \dots, x_n) on $\mathbb{M}^{n,\delta}$, a general Lorentzian metric on \mathcal{M}_δ can be described by

$$\tilde{g} =: g_\delta = \delta dx_0^2 + g_{\mathbb{M}^{n,\delta}}(x_1, \dots, x_n). \tag{3.1}$$

As we mentioned in Sect. 2, totally geodesic hypersurfaces are totally umbilical. We then start from totally geodesic hypersurfaces of \mathcal{M}_δ . We have the following.

Remark 3.1 There are two natural families of examples of totally geodesic hypersurfaces of \mathcal{M}_δ , namely,

- (a) the slices $\{t_0\} \times \mathbb{M}^{n,\delta}$, which are spacelike for $\delta = -1$ and timelike for $\delta = 1$;
- (b) $\mathbb{R}_\delta \times \Sigma^{n-1} = \pi_\delta^{-1}(\Sigma^{n-1})$, where Σ^{n-1} (if it exists) is a totally geodesic hypersurface of $\mathbb{M}^{n,\delta}$. These hypersurfaces are spacelike when $\delta = 1$ and Σ^{n-1} is spacelike, otherwise they are timelike.

We may observe that in the above case (a) (respectively, (b)), vector field ξ is orthogonal (respectively, tangent) to the hypersurface. From now on, we shall focus our attention on the case of proper (i.e., not totally geodesic) totally umbilical hypersurfaces, so that at some point they are neither orthogonal nor tangent to ξ . Let Σ be such a hypersurface of \mathcal{M}_δ with unit normal N such that $g_\delta(N, N) = \varepsilon$. Then, the vector field ξ decomposes as

$$\xi = T + \nu N,$$

where T is tangent to Σ . Therefore,

$$g_\delta(T, T) = \delta - \varepsilon \nu^2$$

and $\nu = g_\delta(N, N)g_\delta(N, \xi) = \varepsilon g_\delta(N, \xi)$ is a smooth function called the *angle function* between N and ξ .

Since Σ is neither orthogonal nor tangent to ξ , the above decomposition is not trivial, in the sense that both ν and T do not vanish.

Next, let X be any tangent vector to Σ . Using (2.3) and the definition of the shape operator S , a straightforward calculation yields

$$\nabla_X T = \nu SX, \quad h(X, T) = -X(\nu). \tag{3.2}$$

If Σ is totally umbilical, then $SX = \lambda X$ for any vector field X tangent to Σ and some smooth function λ on Σ . Therefore, from (3.2) we get

$$\nabla_X T = \lambda \nu X,$$

which means that T is a closed conformal vector field on ξ .

Remark 3.2 Suppose that T is either lightlike or zero, that is, $g_\delta(T, T) = \delta - \varepsilon \nu^2 = 0$. Then, $\nu^2 = \varepsilon \delta$ and so, ν is a real constant. Moreover, from (3.2) we get $0 = h(X, T) = \lambda g(X, T)$ for all X . Thus, either $\lambda = 0$ or $T = 0$, so that Σ is totally geodesic. Therefore, we exclude this case.

Because of the above Remark, from now on we suppose that T is nondegenerate and we shall use the direction T to construct a local non-vanishing closed conformal field on $\mathbb{M}^{n,\delta}$. In fact, we prove the following.

Theorem 3.3 *Let Σ be a proper totally umbilical hypersurface of \mathcal{M}_δ , with N its unit normal such that $g_\delta(N, N) = \varepsilon$. Then, the canonical projection π_δ is locally a diffeomorphism between an open neighborhood U of this point in Σ and the open subset $\pi_\delta(U)$ of $\mathbb{M}^{n,\delta}$.*

Let T be the tangent projection of ξ and denote by T_0 the projection of T to $\pi_\delta(U)$, rescaled so that

$$g_\delta(T_0, T_0) = \varepsilon \delta g_\delta(T, T).$$

Then, T_0 is a closed conformal field on $\pi_\delta(U)$.

Proof We denote by Σ a proper totally umbilical hypersurface of \mathcal{M}_δ and consider ξ, N, T and ν be as before. Assume that the shape operator associated to N is $S = \lambda Id$ with $\lambda \neq 0$, so that Σ is non-vertical and non-horizontal at some point. As a consequence, we can consider an open neighborhood U of this point in Σ where $\nu \neq 0$ and such that

$$\pi_{\delta|U} : U \rightarrow \pi_\delta(U) \subset \mathbb{M}^{n,\delta}$$

is a local diffeomorphism.

The, using the one-parameter group of translations defined by the Killing field ξ , we extend T, N, ν and λ to the whole of $\mathbb{R}_\delta \times \pi_\delta(U)$ and we denote these again by the same symbols.

Moreover, one can observe that ν and λ are constant on the fibers of π_δ and so they can be considered as functions on $\pi_\delta(U)$. Therefore, the vector field T_0 on $\pi_\delta(U)$ is given by

$$T_0 = \frac{1}{\nu}(d\pi)(T)$$

and its horizontal lift to $\pi_\delta(U) \times \mathbb{R}_\delta$ is

$$\tilde{T}_0 = \frac{T}{\nu} - \frac{\delta}{\nu} \tilde{g}(T, \xi)\xi = \varepsilon \nu \xi - N.$$

As in the Riemannian case, T_0 corresponds to the projection of N to $\pi_\delta(U)$ up to the sign.

Next, we consider a vector field X on $\pi_\delta(U)$ and its horizontal lift \tilde{X} . We get

$$\nabla_{\tilde{X}}^{\mathbb{M}^{n,\delta}} T_0 = (d\pi_\delta) \left(\tilde{\nabla}_{\tilde{X}} \tilde{T}_0 \right) = (d\pi_\delta) \left(\tilde{\nabla}_{\tilde{X}} \varepsilon \nu \xi - N \right) = -(d\pi_\delta) \left(\tilde{\nabla}_{\tilde{X}} N \right). \tag{3.3}$$

We observe that as ν is constant on the fiber of π_δ , $[\xi, N] = 0$ and so, $\tilde{\nabla}_\xi N = \tilde{\nabla}_N \xi = 0$. Moreover, let Y be a local vector field defined on Σ and such that $(d\pi_\delta)(Y) = X$.

We denote again by Y its extension to $\mathbb{R}_\delta \times \pi_\delta(U)$ using the flow of ξ . Then, $\tilde{X} = Y - \delta g_\delta(\tilde{Y}, \xi)\xi$ and we obtain

$$\tilde{\nabla}_{\tilde{X}}N = \tilde{\nabla}_Y N - \delta g_\delta(\tilde{Y}, \xi)\tilde{\nabla}_\xi N = \tilde{\nabla}_Y N = -SY = -\lambda Y. \tag{3.4}$$

Therefore, using (3.3) and (3.4), we get

$$\nabla_X^{\mathbb{M}^{n,\delta}} T_0 = (d\pi_\delta)(\lambda Y) = \lambda X,$$

that is, T_0 is a closed conformal vector field on $\mathbb{M}^{n,\delta}$. □

It is well known that the existence of a closed conformal vector field on an either Riemannian [14] or Lorentzian [9] manifold, determines the structure of the manifold itself, which is locally a warped product. So, because of the above Theorem 3.3, the n -dimensional pseudo-Riemannian manifold $\mathbb{M}^{n,\delta}$ is locally isometric to a warped product of an interval of the Euclidean line (if T_0 is spacelike) or the Lorentzian line (if T_0 is timelike) with some $(n - 1)$ -dimensional manifold, either Lorentzian or Riemannian. This proves the “if” part of Theorem 1.1.

In order to prove the “only if” part of Theorem 1.1, we shall now investigate in detail the possible cases where the reducible ambient space $\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{n,\delta}$ is Lorentzian, $\mathbb{M}^{n,\delta}$ is locally isometric to a warped product and the totally umbilical hypersurface is either spacelike or timelike. So, we have to consider three different cases, depending on whether $\delta = -1$ or $\delta = 1$ and with $\partial_{x_1} = T_0$ either spacelike or timelike.

Case (A): $\delta = 1$ and ∂_{x_1} spacelike. Since $\delta = 1$ and $T_0 = \partial_{x_1}$ is spacelike, we have that $\mathbb{M}^{n,\delta} = \mathbb{M}_1^n$ splits (locally) as the warped product $\mathbb{R} \times_r \mathbb{M}_1^{n-1}$, for some warping function r , and the metric given in (3.1) rewrites as

$$g_\delta = dx_0^2 + dx_1^2 + r(x_1)^2 g_{\mathbb{M}_1^{n-1}}(x_2, \dots, x_n). \tag{3.5}$$

Case (B): $\delta = -1$. Here, $\mathbb{M}^{n,\delta} = \mathbb{M}^n$ is Riemannian and it splits (locally) as the Riemannian warped product $\mathbb{R} \times_r \mathbb{M}^{n-1}$. In this case, the metric given in (3.1) rewrites as

$$g_\delta = -dx_0^2 + dx_1^2 + r(x_1)^2 g_{\mathbb{M}^{n-1}}(x_2, \dots, x_n). \tag{3.6}$$

We observe that in this case T_0 is necessarily spacelike. In fact, N and T are orthogonal to each other, non-vanishing and non degenerate; moreover, as $\delta = -1$ (that is, ξ is timelike), they cannot have the same causal character and so,

$$g_\delta(T_0, T_0) = -\varepsilon g_\delta(T, T) = -g_\delta(N, N)g_\delta(T, T) > 0.$$

Case (C): $\delta = 1$ and ∂_{x_1} timelike. Here, $\mathbb{M}^{n,\delta} = \mathbb{M}_1^n$ is locally isometric to the warped product $\mathbb{R}_1 \times_r \mathbb{M}^{n-1}$. Thus, the metric (3.1) rewrites as

$$g_\delta = dx_0^2 - dx_1^2 + r(x_1)^2 g_{\mathbb{M}^{n-1}}(x_2, \dots, x_n). \tag{3.7}$$

We now prove the existence of proper totally umbilical hypersurfaces Σ in a reducible ambient space \mathcal{M}_δ , as described above in Cases (A), (B) and (C). Such hypersurface Σ is necessarily tangent to the distribution orthogonal to the vector fields ∂_{x_0} and ∂_{x_1} at any point. So, Σ is a cylindrical hypersurface over a curve γ in the (x_0, x_1) -plane, namely,

$$\gamma(s) = (x_0(s), x_1(s)).$$

We now consider separately two cases, depending on whether the distribution spanned by ∂_{x_0} and ∂_{x_1} is Riemannian (Case (A)) or Lorentzian (Cases (B) and (C)).

Case (A). If ∂_{x_0} and ∂_{x_1} span a Riemannian distribution, then both the curve γ and the associated unit normal N are spacelike ($\varepsilon = 1$). This means that we can assume that γ is parametrized by arc length and so, there exists a smooth function θ such that

$$x'_0(s) = \sin \theta, \quad x'_1(s) = \cos \theta.$$

In this case, the vector fields

$$X_1 = \sin \theta \partial_{x_0} + \cos \theta \partial_{x_1}, \quad X_2 = \partial_{x_2}, \dots, X_n = \partial_{x_n},$$

span the tangent space to Σ .

We observe that in this case Σ is always timelike, since one of the tangent directions X_j for $j \geq 2$ is timelike. Moreover, the spacelike unit normal to Σ is given by

$$N = \cos \theta \partial_{x_0} - \sin \theta \partial_{x_1}.$$

Next, if we express the shape operator, following the argument used in [14] for the Riemannian case, we find the same equations (and so, the same solutions), concluding that Σ is totally umbilical if and only if

$$\theta'(s) = \sin \theta(s) \frac{r'}{r} \tag{3.8}$$

whence, $x_1(s)$ is determined by the equation

$$x'_1(s) = \pm \sqrt{1 - c^2 r^2(x_1(s))}, \tag{3.9}$$

for some real constant $c \neq 0$. Consequently, one obtains

$$x_0(s) = \pm \int_0^s \sqrt{1 - (x'_1(t))^2} dt = \pm c \int_0^s r(x_1(t)) dt \tag{3.10}$$

and we have the following.

Theorem 3.4 *Suppose that \mathbb{M}_1^n is locally isometric to the warped product $\mathbb{R} \times_r \mathbb{M}_1^{n-1}$, so that the metric g_δ of the ambient space \mathcal{M}_δ is given by (3.5) in local coordinates (x_0, \dots, x_n) . Then, there always exists a proper totally umbilical timelike hypersurface*

Σ of \mathcal{M}_δ , neither vertical nor horizontal. An explicit parametrization for Σ is given by

$$F(s, u_1, \dots, u_{n-1}) = (x_0(s), x_1(s), u_1, \dots, u_{n-1}),$$

where $x_0(s), x_1(s)$ are determined by (3.10) and (3.9) respectively.

While Case (A) can be considered as the Lorentzian equivalent of the Riemannian case investigated in [14], the remaining cases (B) and (C) do not have a Riemannian counterpart, because they occur when the distribution spanned by ∂_{x_0} and ∂_{x_1} is Lorentzian. We shall treat these two cases together.

Cases (B) and (C). We first observe that the metric tensors given in (3.6) and (3.7) can be rewritten in the following unified way:

$$g_\delta = \delta(dx_0^2 - dx_1^2) + r(x_1)^2 g_{\mathbb{M}^{n-1}}(x_2, \dots, x_n), \tag{3.11}$$

where $\delta = -1$ in Case (B) and $\delta = 1$ in case (C).

Starting from (3.11) we calculated the Christoffel symbols for $(\mathcal{M}_\delta, g_\delta)$ and determined the following components of the Levi-Civita connection of \mathcal{M}_δ :

$$\tilde{\nabla}_{\partial_{x_1}} \partial_{x_1} = 0, \quad \tilde{\nabla}_{\partial_{x_1}} \partial_{x_j} = \tilde{\nabla}_{\partial_{x_j}} \partial_{x_1} = \frac{r'}{r} \partial_{x_j}, \quad j \geq 2. \tag{3.12}$$

As ∂_{x_0} and ∂_{x_1} span a Lorentzian distribution, the curve γ and the associated unit normal N have opposite causal characters. This means that we can assume that γ is parametrized by $(-\varepsilon)$ -arc length, that is,

$$(x'_0(s))^2 - (x'_1(s))^2 = -\varepsilon\delta$$

and so, there exists a smooth function $\theta(s)$ such that

$$(x'_0(s), x'_1(s)) = \begin{cases} (\cosh \theta(s), \sinh \theta(s)) & \text{if } \varepsilon\delta = -1, \\ (\sinh \theta(s), \cosh \theta(s)) & \text{if } \varepsilon\delta = 1. \end{cases} \tag{3.13}$$

In this case, the vector fields

$$X_1 = x'_0(s)\partial_{x_0} + x'_1(s)\partial_{x_1}, \quad X_2 = \partial_{x_2}, \dots, X_n = \partial_{x_n}$$

span the tangent space to Σ ; moreover, the spacelike unit normal to Σ is given by

$$N = x'_1(s)\partial_{x_0} + x'_0(s)\partial_{x_1}.$$

Next, using (3.12), we obtain

$$\tilde{\nabla}_{X_1} N = \theta'(s)X_1, \quad \tilde{\nabla}_{X_j} N = \frac{r'}{r} x'_0(s)X_j,$$

for every $j \geq 2$. Hence, Σ is totally umbilical if and only if

$$\theta'(s) = \frac{r'}{r}x'_0(s). \tag{3.14}$$

Thus, we have

$$x'_1(s) = \theta'(s)x'_0(s) = \frac{r'}{r}(x'_0(s))^2 = \frac{r'}{r} \left[(x'_1(s))^2 - \varepsilon\delta \right]$$

which gives, by integration,

$$x'_1(s) = \pm\sqrt{c^2r^2(x_1) + \varepsilon\delta} \tag{3.15}$$

for some real constant $c \neq 0$. Therefore, one obtains

$$x_0(s) = \pm \int_0^s \sqrt{(x'_1(t))^2 - \varepsilon\delta} dt = \pm c \int_0^s r(x_1(t)) dt \tag{3.16}$$

and we have the following.

Theorem 3.5 *Suppose that $\mathbb{M}^{n,\delta}$ is locally isometric either to the Riemannian warped product $\mathbb{R} \times_r \mathbb{M}^{n-1}$ (if $\delta = -1$) or to the Lorentzian warped product $\mathbb{R}_1 \times_r \mathbb{M}^{n-1}$ (if $\delta = 1$), so that the metric g_δ of the ambient space \mathcal{M}_δ is given by (3.11) in local coordinates (x_0, \dots, x_n) . Then, there always exists a proper totally umbilical hypersurface Σ of \mathcal{M}_δ , neither vertical nor horizontal. An explicit parametrization for Σ is given by*

$$F(s, u_1, \dots, u_{n-1}) = (x_0(s), x_1(s), u_1, \dots, u_{n-1}),$$

where $x_0(s), x_1(s)$ are determined by (3.16) and (3.15) respectively.

We shall now focus on the case $n = 2$.

Remark 3.6 For $n = 2$, the direction $X_2 = \partial_{x_2} = \partial_{u_1}$ is fully tangent to Σ . Moreover, with respect to the tangent space spanned by $X_1 = \partial_s$ and X_2 , the metric on Σ , denoted by g^Σ , can be described as follows:

$$g^\Sigma = \begin{cases} ds^2 - r^2(x_1(s))du_1^2 & \text{in case (A),} \\ -\varepsilon ds^2 + r^2(x_1(s))du_1^2 & \text{in case (B),} \\ -\varepsilon ds^2 + r^2(x_1(s))du_1^2 & \text{in case (C).} \end{cases}$$

When $\delta = 1 = \varepsilon$, let us denote by Σ' the totally umbilical surface obtained in the Case (A) and by Σ'' the one obtained in (C), respectively. Then, these distinct surfaces show the same geometry. In fact, they are anti-isometric, that is, $g^{\Sigma'} = -g^{\Sigma''}$. For this reason, we can ignore the Case (A) from now on, since it is the anti-isometric analogue of the timelike surface (i.e., the one obtained for $\varepsilon = 1$) in the Case (C).

As a consequence, changing the variable u_1 with v , and x_0, x_1 with α, β , a totally umbilical surface of \mathcal{M}_δ can be parametrized by the function

$$F(s, v) = (\alpha(s), \beta(s), v),$$

which shows that it is a cylinder over the curve $(\alpha(s), \beta(s))$, where

$$\alpha(s) = \pm c \int_0^s r(\beta(t)) dt \tag{3.17}$$

and $\beta(s)$ is a solution of

$$\beta'(s) = \pm \sqrt{c^2 r^2(\beta(s)) + \varepsilon \delta}. \tag{3.18}$$

for some real constant $c \neq 0$.

Remark 3.7 When $n = 2$, a reducible space \mathcal{M}_δ admitting some proper totally umbilical surfaces is the Lorentzian product $\mathbb{R}_\delta \times \mathbb{M}^{2,\delta}$, where $\mathbb{M}^{2,\delta}$ is a warped product with some warping function r . We observe that $\mathbb{M}^{2,\delta}$ (and so, \mathcal{M}_δ) is flat if and only if $r'' = 0$. Moreover, $\mathbb{M}^{2,\delta}$ has constant Gaussian curvature $\mathcal{K} = \pm \frac{1}{R^2} \neq 0$ if and only if $r'' = -\mathcal{K}r$. In particular:

- if $r(x) = R \sin\left(\frac{x}{R}\right)$, then $\mathbb{M}^{2,\delta}$ has constant Gaussian curvature $\mathcal{K} = \frac{1}{R^2}$;
- if $r(x) = R \sinh\left(\frac{x}{R}\right)$, then $\mathbb{M}^{2,\delta}$ has constant Gaussian curvature $\mathcal{K} = -\frac{1}{R^2}$.

We shall now provide some examples of totally umbilical surfaces in three-dimensional Lorentzian warped products, starting with the cases where $\mathbb{M}^{2,\delta}$ has constant Gaussian curvature $\mathcal{K} \neq 0$ (we may refer to [10] for more general information concerning totally umbilical surfaces in a flat Lorentzian ambient space.)

Case 1: $\mathcal{K} = \frac{1}{R^2} > 0$. Here, $r(x) = R \sin\left(\frac{x}{R}\right)$. Then, (3.18) rewrites as

$$\beta'(s) = \pm \sqrt{c^2 R^2 \sin^2\left(\frac{\beta(s)}{R}\right) + \varepsilon \delta}.$$

In both cases, that is, for $\varepsilon \delta = \pm 1$, one can give implicit solutions for $\beta(s)$ using elliptic integrals of the first kind, defined as

$$\bar{F}(u, k) = \int_0^u \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

As a consequence, for specific values of c and R , the solutions can be numerically evaluated or further simplified.

Case 2: $\mathcal{K} = -\frac{1}{R^2} < 0$. Here $r(x) = R \sinh\left(\frac{x}{R}\right)$. Then, (3.18) yields

$$\beta'(s) = \pm \sqrt{c^2 R^2 \sinh^2\left(\frac{\beta(s)}{R}\right) + \varepsilon \delta}.$$

Again, for $\varepsilon \delta = \pm 1$, implicit solutions for $\beta(s)$ can be given by elliptic integrals of the first kind and for specific values of c and R , such solutions can be numerically evaluated.

We already emphasized the fact that totally umbilical hypersurfaces do exist in all Lorentzian warped products, without assumptions on the curvature of \mathcal{M}_δ . We shall now describe some explicit examples of totally umbilical surfaces in a three-dimensional Lorentzian product $\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{2,\delta}$ with $\mathbb{M}^{2,\delta}$ a warped product of non-constant curvature.

Example 3.8 If we take $r^2(x) = \frac{x^{-2\mu} - \varepsilon \delta}{c^2}$ for $\mu \neq -1$ we can easily see that $\mathbb{M}^{2,\delta}$ is neither flat nor of constant curvature. Moreover, β satisfies

$$\beta'(s) = \pm \beta(s)^{-\mu},$$

that is, $\beta(s)^{\mu+1} = (\mu + 1)(s + c_1)$ for some real constant c_1 . After a reparametrization we can take $\beta(s) = As^{\frac{1}{\mu+1}}$, whence,

$$\gamma(s) = (\alpha(s), \beta(s)) = \left(\int_0^s \sqrt{A^{-2\mu} t^{\frac{-2\mu}{\mu+1}} - \varepsilon \delta} dt, As^{\frac{1}{\mu+1}} \right).$$

Example 3.9 We set $r(x) = \sqrt{x}$. Then, (3.18) rewrites as follows:

$$\beta'(s) = \pm \sqrt{c^2 \beta(s) + \varepsilon \delta},$$

which gives $\beta(s) = \frac{c^2}{4}(\pm s + C_1)^2 - \frac{\varepsilon \delta}{c^2}$, for some real constant C_1 . After a reparametrization we obtain

$$\beta(u) = \frac{c^2}{4}u^2 - \frac{\varepsilon \delta}{c^2}$$

and, so, (3.17) becomes

$$\begin{aligned} \alpha(u) &= c \int_0^u \sqrt{\frac{c^2}{4}t^2 - \frac{\varepsilon \delta}{c^2}} dt = \int_0^s \sqrt{\frac{c^4}{4}t^2 - \varepsilon \delta} dt \\ &= \frac{u}{2} \sqrt{\frac{c^4}{4}u^2 - \varepsilon \delta} - \frac{\varepsilon \delta}{c^2} \ln \left(\sqrt{\frac{c^4}{4}u^2 - \varepsilon \delta} + \frac{c^2}{2}u \right) + C_2, \end{aligned}$$

Fig. 1 Graph of the curve $\alpha(\beta)$ for $\varepsilon\delta = 1$, $c = \frac{1}{2}$ and $C_2 = 0$

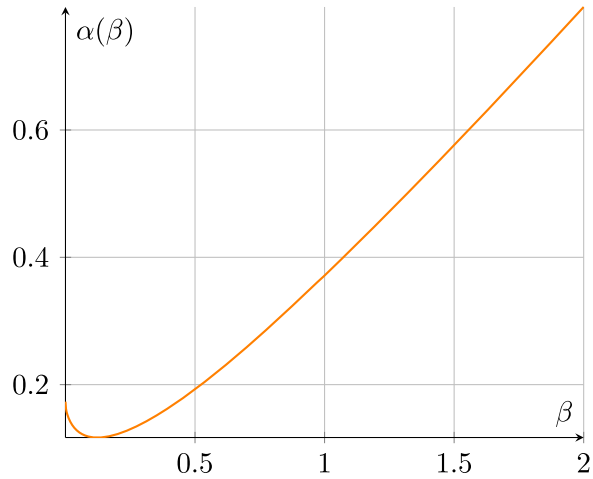
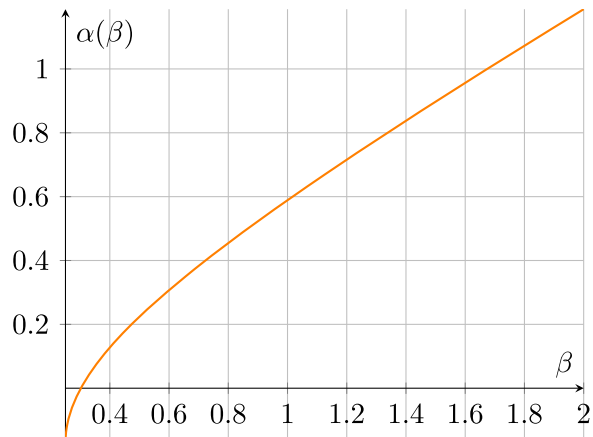


Fig. 2 Graph of the curve $\alpha(\beta)$ for $\varepsilon\delta = -1$, $c = \frac{1}{2}$ and $C_2 = 0$



for some real constant C_2 . Therefore, we can also express the equation of the curve $\gamma(u) = (\alpha(u), \beta(u))$ as follows:

$$\gamma : \alpha(\beta) = \frac{1}{c} \sqrt{\beta + \frac{\varepsilon\delta}{c^2}} \sqrt{\beta} - \frac{\varepsilon\delta}{c^2} \ln \left(\sqrt{\beta} + c \sqrt{\beta + \frac{\varepsilon\delta}{c^2}} \right) + C_2.$$

The plots in *Figure 1* and *Figure 2* show the curve $(\alpha(u), \beta(u))$ in the plane (α, β) for cases $\varepsilon\delta = 1$ and $\varepsilon\delta = -1$ respectively, choosing some particular values of c and C_2 .

4 Minimal Surfaces in Three-Dimensional Warped Products

In the previous section we showed that if proper totally umbilical hypersurfaces exist in a Lorentzian reducible space $\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{n,\delta}$, then $\mathbb{M}^{n,\delta}$ is (locally) a warped product. In the three-dimensional case we obtained a general description for totally

umbilical surfaces. Because of Proposition 2.2 we now consider minimal surfaces of these three-dimensional Lorentzian ambient spaces. In particular, we shall give a general condition to characterize their minimal and CMC immersions.

Using the notation introduced in Sect. 3, any three-dimensional Lorentzian reducible manifold \mathcal{M}_δ admitting totally umbilical surfaces is (locally) isometric to

$$\mathcal{M}_\delta = \mathbb{R}_\delta \times \mathbb{M}^{2,\delta} = \begin{cases} \mathbb{R}_1 \times \mathbb{M}^2 & \text{if } \delta = -1, \\ \mathbb{R} \times \mathbb{M}_1^2 & \text{if } \delta = 1, \end{cases}$$

where $\mathbb{M}^{2,\delta}$ is locally a warped product of two real lines, namely,

$$\mathbb{M}^{2,\delta} = \begin{cases} \mathbb{R} \times_r \mathbb{R} & \text{if } \delta = -1, \\ \text{either } \mathbb{R} \times_r \mathbb{R}_1 \text{ or } \mathbb{R}_1 \times_r \mathbb{R} & \text{if } \delta = 1, \end{cases}$$

for some real smooth function r .

As a consequence, in suitable coordinates (x_0, x_1, x_2) , the Lorentzian metric of \mathcal{M}_δ can be described by

$$g_\delta = \delta dx_0^2 + \tau(dx_1^2 - \delta r^2(x_1)dx_2^2), \quad \delta = \pm 1, \quad \tau = \pm 1, \tag{4.1}$$

for some smooth function $r \neq 0$.

Remark 4.1 The correspondence between the cases (A), (B) and (C) introduced in Sect. 3 and the possible values of τ and δ is the following:

- Case (A) corresponds to $\delta = \tau = 1$;
- Case (B) corresponds to $\delta = -\tau = -1$;
- Case (C) corresponds to $\delta = -\tau = 1$.

We explicitly remark that if $\delta = -1$ then necessarily $\tau = 1$.

By (4.1), vector fields

$$e_1 = \partial_{x_1}, \quad e_2 = \frac{\delta}{r(x_1)}\partial_{x_2}, \quad e_3 = \partial_{x_0} \tag{4.2}$$

form an orthonormal basis, with $g_\delta(e_1, e_1) = \tau$, $g_\delta(e_2, e_2) = -\tau\delta$ and $g_\delta(e_3, e_3) = \delta$. Applying the *Koszul formula* we find that with respect to this basis, the Levi-Civita connection is completely determined by the following non-vanishing components:

$$\nabla_{e_2}e_2 = \delta \frac{r'}{r}e_1, \quad \nabla_{e_2}e_1 = \frac{r'}{r}e_2. \tag{4.3}$$

Consequently, the curvature tensor is determined, up to symmetries, by

$$R(e_1, e_2)e_1 = -\frac{r''}{r}e_2.$$

Let now $F : \Sigma \rightarrow \mathcal{M}_\delta$ denote the isometric immersion of a surface Σ into \mathcal{M}_δ . We look for some conditions on the components of the unit normal vector field N with respect to the orthonormal frame $\{e_1, e_2, e_3\}$ in order for Σ to be minimal or, more in general, CMC. We prove the following characterization.

Theorem 4.2 *Let $F : \Sigma \rightarrow \mathcal{M}_\delta$ denote a CMC surface and N the unit normal vector field, with $g_\delta(N, N) = \varepsilon \in \{-1, 1\}$. Consider the orthonormal vector field $\{e_i\}$ on \mathcal{M}_δ introduced in (4.2). Then, every point of Σ admits a neighbourhood $U \subseteq \Sigma$, on which $N = ae_1 + be_2 + ce_3$, for some functions $a, b, c : U \rightarrow \mathbb{R}$ which satisfy $(a, b) \neq (0, 0)$, $c \neq 0$ and*

$$Y_1(a) + Y_2(b) + \tau ac \frac{r'}{r} = -2\tau c\rho, \tag{4.4}$$

where vector fields

$$Y_1 = \tau ce_1 - \delta ae_3, \quad Y_2 = \tau ce_2 + be_3 \tag{4.5}$$

span $T\Sigma$ in U and ρ is a real constant. In particular, Σ is proper minimal (that is, minimal but not totally geodesic) if and only if $\rho = 0$ in (4.4), namely,

$$Y_1(a) + Y_2(b) + \tau ac \frac{r'}{r} = 0, \tag{4.6}$$

while Σ is proper CMC (that is, not minimal) if and only if $\rho \neq 0$ in (4.4).

Proof Consider the normal vector field $N = ae_1 + be_2 + ce_3$, for some functions $a, b, c : U \rightarrow \mathbb{R}$ such that $g_\delta(N, N) = \tau(a^2 - \delta b^2) + \delta c^2 = \varepsilon = \pm 1$. We set

$$\begin{aligned} \Omega_1 &= \{p \in \Sigma : c \neq 0 \text{ in a neighbourhood of } p\}, \\ \Omega_2 &= \{p \in \Sigma : c = 0 \text{ in a neighbourhood of } p\}. \end{aligned}$$

Then, $\Omega = \Omega_1 \cup \Omega_2$ is a dense open subset of Σ . Clearly, each point $p \in \Omega$ admits a neighbourhood where either $c = 0$ or $c \neq 0$ everywhere. Working on the dense open subset Ω , we shall consider separately two cases, depending on whether $c = 0$.

If $c = 0$, then the ε -unit vector field normal to Σ is $N = ae_1 + be_2$, with $\tau(a^2 - \delta b^2) = \varepsilon$. It is easy to observe that such a surface is isometric to a totally geodesic surface as described (for $n = 2$) in the case (b) of the Remark 3.1. Thus, they are neither proper minimal nor proper CMC and we exclude this case.

So, we are left with the case where $c \neq 0$. In this case, the ε -unit normal vector field is given by $N = ae_1 + be_2 + ce_3$ and $\tau(a^2 - \delta b^2) + \delta c^2 = \varepsilon$. We observe that for $a = b = 0 \neq c$, we get a totally geodesic surface (see also Remark 3.1, (a)), namely,

$$\Sigma = \{t_0\} \times \mathbb{M}^{2,\delta}$$

for some $t_0 \in \mathbb{R}$. Thus, we need to assume $(a, b) \neq (0, 0)$ in order to get proper minimal and CMC surfaces.

Since $c \neq 0$, the vector fields Y_1, Y_2 described in (4.5) span the tangent plane to Σ at every point. Moreover, with respect to the basis $\{Y_1, Y_2\}$, the metric on Σ is determined by

$$g^\Sigma = \begin{pmatrix} \delta a^2 + \tau c^2 & -ab \\ -ab & \delta b^2 - \delta \tau c^2 \end{pmatrix}. \tag{4.7}$$

Using (4.5), (4.3) and applying the Gauss formula (2.1), we find that the Levi-Civita connection on Σ is completely determined by the following components:

$$\begin{aligned} \nabla_{Y_1}^\Sigma Y_1 &= \delta \varepsilon \left(Y_1(c) \frac{c^2 - \tau b^2}{c} + \delta \tau a Y_1(a) \right) Y_1 - \varepsilon \tau \left(Y_1(c) \frac{ab}{c} - b Y_1(a) \right) Y_2, \\ \nabla_{Y_1}^\Sigma Y_2 &= \delta \varepsilon \tau \left(Y_1(c) \frac{ab}{c} - a Y_1(b) \right) Y_1 + \varepsilon \delta \left(\delta Y_1(c) \frac{\tau a^2 + \delta c^2}{c} - \tau b Y_1(b) \right) Y_2, \\ \nabla_{Y_2}^\Sigma Y_1 &= \delta \varepsilon \left(Y_2(c) \frac{c^2 - \tau b^2}{c} + \delta \tau a Y_2(a) + abc \frac{r'}{r} \right) Y_1 \\ &\quad - \varepsilon \tau \left(Y_2(c) \frac{ab}{c} - b Y_2(a) - c(\tau a^2 + \delta c^2) \frac{r'}{r} \right) Y_2, \\ \nabla_{Y_2}^\Sigma Y_2 &= \delta \varepsilon \tau \left(Y_2(c) \frac{ab}{c} - a Y_2(b) + \delta c(c^2 - \tau b^2) \frac{r'}{r} \right) Y_1 \\ &\quad + \varepsilon \delta \left(\delta Y_2(c) \frac{\tau a^2 + \delta c^2}{c} - \tau b Y_2(b) - abc \frac{r'}{r} \right) Y_2 \end{aligned}$$

and the second fundamental form h is determined by

$$\begin{aligned} h(Y_1, Y_1) &= a Y_1(c) - c Y_1(a), \\ h(Y_1, Y_2) &= \delta c Y_1(b) - \delta b Y_1(c), \\ h(Y_2, Y_1) &= a Y_2(c) - c Y_2(a) - \delta \tau b c^2 \frac{r'}{r}, \\ h(Y_2, Y_2) &= \delta c Y_2(b) - \delta b Y_2(c) + \delta \tau a c^2 \frac{r'}{r}. \end{aligned} \tag{4.8}$$

Next, using (4.7) and (4.8), we obtain that the mean curvature of Σ is given by

$$\begin{aligned} H &= \frac{1}{2} \text{tr}_g \Sigma h = \frac{1}{2} \sum (g^\Sigma)^{-1}_{ij} h_{ij} \\ &= -\frac{\varepsilon}{2c^2} [(a Y_1(c) - c Y_1(a))(\delta b^2 - \delta \tau c^2) + (\delta c Y_2(b) - \delta b Y_2(c) + \delta \tau a c^2 \frac{r'}{r})(\delta a^2 + \tau c^2) \\ &\quad + (\delta c Y_1(b) - \delta b Y_1(c) + a Y_2(c) - c Y_2(a) - \delta \tau b c^2 \frac{r'}{r})ab] \\ &= -\frac{\tau}{2c} [Y_1(a) + Y_2(b) + \tau a c \frac{r'}{r}]. \end{aligned}$$

Consequently, the case where Σ is CMC is characterized by equation (4.4), which is obtained by the above expression of H requiring that $H = \rho$ is some real constant. In particular, Σ is minimal if and only if $\rho = 0$, leading to equation (4.6), while the case $\rho \neq 0$ corresponds to proper CMC surfaces. \square

We shall now provide some explicit examples of minimal and CMC surfaces.

Example 4.3 ($\tau = \delta = 1, a = 0$). Setting $a = 0$ and $\delta = \tau = 1$, the condition (4.6) simplifies remarkably and we can give explicit examples of minimal surfaces. In this case, $N = be_2 + ce_3$ and we must consider separately two cases, depending on the sign of $-b^2 + c^2 = g_\delta(N, N) = \varepsilon = \pm 1$.

If $\varepsilon = 1$ (respectively, $\varepsilon = -1$), then Σ is timelike (respectively, spacelike) and there exists a smooth function $\theta : U \rightarrow \mathbb{R}$ such that $c = \cosh \theta$ (respectively, $\sinh \theta$) and $b = \sinh \theta$ (respectively, $\cosh \theta$). In both cases, when $a = 0$, condition (4.6) becomes $Y_2(\theta) = 0$, where Y_2 is given in (4.5). Next, a direct calculation, using (4.5), $a = 0$ and the minimal condition, yields

$$\begin{aligned} \nabla_{Y_1}^\Sigma Y_1 &= \frac{b}{c} Y_1(\theta) Y_1, & \nabla_{Y_2}^\Sigma Y_1 &= \varepsilon c^3 \frac{r'}{r} Y_2, \\ \nabla_{Y_1}^\Sigma Y_2 &= 0, & \nabla_{Y_2}^\Sigma Y_2 &= c^2 \frac{r'}{r} Y_2 \end{aligned} \tag{4.9}$$

and, using (4.8), the second fundamental form h is given by

$$h(Y_1, Y_1) = h(Y_2, Y_2) = 0, \quad h(Y_1, Y_2) = \varepsilon Y_1(\theta), \quad h(Y_2, Y_1) = -bc^2 \frac{r'}{r}.$$

Consequently, by the symmetry of the second fundamental form h , we deduce that

$$\varepsilon Y_1(\theta) = -bc^2 \frac{r'}{r}. \tag{4.10}$$

Therefore, using (4.9) and the symmetry condition it is easy to check that vector fields

$$\partial_u = \frac{1}{c(u, v)} Y_1, \quad \partial_v = \frac{1}{b(u, v)} Y_2 \tag{4.11}$$

may be taken as coordinate vector fields on Σ . As $Y_2(\theta) = 0$, we can observe that b and c do not depend on v , namely $b(u, v) = b(u)$ and $c(u, v) = c(u)$.

We now denote by $F : \Sigma \rightarrow \mathbb{M}_1^2 \times \mathbb{R}$, $(u, v) \mapsto (F_0(u, v), F_1(u, v), F_2(u, v))$ the immersion of the minimal surface Σ in the local coordinates introduced above. By (4.2), (4.5) and (4.11) we obtain

$$\begin{aligned} (\partial_u F_0, \partial_u F_1, \partial_u F_2) &= (0, 1, 0), \\ (\partial_v F_0, \partial_v F_1, \partial_v F_2) &= \left(1, 0, \frac{1}{r(F_1)} \frac{c}{b} \right). \end{aligned} \tag{4.12}$$

Using (4.10), a direct calculation shows that $\partial_u \left(\frac{1}{r(F_1)} \frac{c}{b} \right) = \partial_v \left(\frac{1}{r(F_1)} \frac{c}{b} \right) = 0$. So, we get

$$\frac{1}{r(F_1)} \frac{c}{b} = \kappa$$

where κ is a real constant. Integrating (4.12) we obtain

$$F(u, v) = (v + c_3, u + c_1, \kappa v + c_2),$$

for some real constants c_i . So, after a suitable reparametrization, we get that Σ is described as follows:

$$F(u, v) = (v, u, \kappa v + \mu),$$

where κ and μ are some real constants.

Therefore, referring to the global coordinates (x_0, x_1, x_2) described at the beginning of this Section, $\mathbb{M}_1^2 \times \mathbb{R}$ admits as proper minimal surfaces (either timelike or spacelike) the “planes” of equation $x_2 = \kappa x_0 + \mu$.

Remark 4.4 Let us observe that the “planes” Σ described in the Example 4.3 are minimal but, in general, not totally geodesic. In particular, the minimal surfaces Σ are totally geodesic only in the special cases where either $\theta = 0$ or $r' = 0$ (which implies that the ambient space is flat and planes are obviously totally geodesic).

Example 4.5 ($\tau = -\delta = 1, a = 0$). We proceed similarly to the previous example. We now have $N = be_2 + ce_3$ and we shall consider again separately two cases, depending on the sign of $\varepsilon = b^2 - c^2$.

When $\varepsilon = 1$ (respectively, $\varepsilon = -1$), then Σ is timelike (respectively, spacelike) and there exists a smooth function $\theta : U \rightarrow \mathbb{R}$ such that $c = \cosh \theta$ (respectively, $\sinh \theta$) and $b = \sinh \theta$ (respectively, $\cosh \theta$).

In both cases, when $a = 0$, the condition (4.6) becomes $Y_2(\theta) = 0$, where Y_2 is given in (4.5). Next, following the argument described in Example 4.3, we conclude that, referring to the global coordinates (x_0, x_1, x_2) , $\mathbb{M}_1^2 \times \mathbb{R}$ admits as proper minimal surfaces the (either timelike or spacelike) “planes” of equation $x_2 = \kappa x_0 + \mu$, for some real constants $\kappa \neq 0$ and μ . Again, these planes are minimal but not totally geodesic, unless we consider the special cases described in Remark 4.4.

Example 4.6 ($\tau = -\delta = -1, a = 0$). We proceed as in the two previous examples. We now have $N = be_2 + ce_3$ with $\varepsilon = b^2 + c^2 = 1$, which means that Σ is timelike and that there exists a smooth function $\theta : U \rightarrow \mathbb{R}$ such that $(b, c) = (\cos \theta, \sin \theta)$.

Since $a = 0$, the condition (4.6) becomes $Y_2(\theta) = 0$, where Y_2 is given in (4.5). By the same argument described in Example 4.3, we find that, referring to the global coordinates (x_0, x_1, x_2) , $\mathbb{M}_1^2 \times \mathbb{R}$ admits as proper minimal surfaces the timelike “planes” of equation $x_2 = -\kappa x_0 + \mu$, for some real constants $\kappa \neq 0$ and μ .

Also in this case, such planes are minimal but not totally geodesic, unless we consider the special cases specified in Remark 4.4.

We shall now investigate totally umbilical surfaces that are CMC, that is, we specialize condition (4.4) for totally umbilical surfaces (considering for $n = 2$ the hypersurfaces obtained in the previous section). Let Σ denote the surface parametrized by

$$F(s, t) = (x_0(s), x_1(s), t),$$

that is, some cylindrical surface over the curve $(x_0(s), x_1(s))$. We shall consider again the possible cases of totally umbilical surfaces we investigated in Sect. 3, that is, Case (A) and Cases (B) and (C) (which we shall again treat together).

Case (A). In this case, the vector field normal to the totally umbilical surface Σ is given by

$$N = -\sin \theta(s)\partial_{x_1} + \cos \theta(s)\partial_{x_0} = -\sin \theta(s)e_1 + \cos \theta(s)e_3.$$

Correspondingly, we obtain

$$a = -\sin \theta(s), \quad b = 0, \quad c = \cos \theta(s)$$

and so condition (4.4) rewrites as

$$-\theta'(s) - \sin \theta(s)\frac{r'}{r} = \rho. \tag{4.13}$$

Then, from (3.8) and (4.13), we get that the surface Σ is CMC and totally umbilical if and only if

$$\begin{cases} \theta' = -\frac{\rho}{2}s + k, \\ \sin \theta(s)\frac{r'}{r} = -\frac{\rho}{2}, \end{cases} \tag{4.14}$$

for some real constant k . Moreover, since $\partial_s = \sin \theta e_3 + \cos \theta e_1$, we obtain

$$x'_0(s) = \sin\left(-\frac{\rho}{2}s + k\right), \quad x'_1(s) = \cos\left(-\frac{\rho}{2}s + k\right).$$

Integrating we get

$$x_0(s) = \frac{2}{\rho} \cos\left(-\frac{\rho}{2}s + k\right) + B, \quad x_1(s) = -\frac{2}{\rho} \sin\left(-\frac{\rho}{2}s + k\right) + A,$$

for some real constants A and B .

Next, using the second equation of (4.14) we conclude that, locally, we must have

$$\frac{r'}{r} = \frac{1}{x_1(s) - A},$$

that is, $r(x_1)|_\Sigma = \Lambda(x_1 - A)$, for some real constant Λ . This means that totally umbilical surfaces of constant mean curvature can only occur where the space \mathcal{M}_δ is locally flat. Moreover, observe that the surface Σ is then a round sphere of radius $\frac{2}{\rho}$, since it satisfies the equation

$$(x_0 - B)^2 + (x_1 - A)^2 = \frac{4}{\rho^2}.$$

In fact, round spheres are totally umbilical and CMC in a flat Lorentzian ambient space.

Cases (B) and (C). From Remark 4.1, in these cases we have $\tau = -\delta$. Following the same argument of the previous case, here the vector field normal to the totally umbilical surface Σ is given by

$$N = x'_0(s)\partial_{x_1} + x'_1(s)\partial_{x_0} = x'_0(s)e_1 + x'_1(s)e_3,$$

where $-\delta(x'_0(s))^2 + \delta(x'_1(s))^2 = \varepsilon$ and so, $(x'_0(s))^2 - (x'_1(s))^2 = -\varepsilon\delta$. Thus, with the notation used in Theorem 4.2, we have

$$a = x'_0(s), \quad b = 0, \quad c = x'_1(s).$$

Consequently, condition (4.4) now reads

$$\theta'(s) + x'_0(s)\frac{r'}{r} = \rho, \tag{4.15}$$

which, using (3.14), leads to conclude that the surface Σ is CMC and totally umbilical if and only if

$$\begin{cases} \theta(s) = \frac{\rho}{2}s + k, \\ x'_0(s)\frac{r'}{r} = \frac{\rho}{2}, \end{cases} \tag{4.16}$$

for some real constant k . Integrating the expressions of $x'_1(s)$ and $x'_0(s)$ as they appear in (3.13), we obtain

$$x_0(s) = \frac{2}{\rho}x'_1(s) + B, \quad x_1(s) = \frac{2}{\rho}x'_0(s) + A,$$

for some real constants A and B .

Next, the second equation of (4.16) yields, locally,

$$\frac{r'}{r} = \frac{1}{x_1(s) - A},$$

which implies again, $r(x_1)|_\Sigma = \Lambda(x_1 - A)$ for some real constant Λ . Also in this case, totally umbilical CMC surfaces only occur where the ambient space \mathcal{M}_δ is flat. Moreover, observe that the surface Σ can be described by

$$(x_0 - B)^2 - (x_1 - A)^2 = \varepsilon\delta\frac{4}{\rho^2},$$

so that Σ is a hyperboloid, which is indeed totally umbilical and CMC in a flat Lorentzian ambient space.

Remark 4.7 More in general, requiring that totally umbilical hypersurfaces we described in the previous Section are CMC is equivalent to the condition

$$\theta'(s) = \pm\frac{\rho}{2} = \text{constant}.$$

In fact, in this case we obtain

$$\tilde{\nabla}_{X_j}N = \pm\frac{\rho}{2}X_j,$$

for every $j = 1, \dots, n$. Therefore, the shape operator satisfies $S = \lambda Id$, where $\lambda = \mp\frac{\rho}{2} \neq 0$ is a real constant.

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Data availability This paper does not refer to any data outside the paper itself.

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References

1. Alarcón, E.M., Alfás, L.J., dos Santos, F.R.: A new approach to minimal and maximal hypersurfaces in product spaces. *Results Math.* **74**, 116 (2019)
2. Calvaruso, G., Castrillón-Lopez, M., Pellegrino, L.: *On totally umbilical and minimal surfaces of the Lorentzian Heisenberg groups*, submitted
3. Calvaruso, G., Kowalczyk, D., Van der Veken, J.: On extrinsic symmetries of hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$. *Bull. Austral. Math. Soc.* **82**, 390–400 (2010)
4. Calvaruso, G., Van der Veken, J.: Parallel surfaces in three-dimensional Lorentzian Lie groups. *Taiwanese J. Math.* **14**, 223–250 (2010)
5. Calvaruso, G., Van der Veken, J.: Parallel surfaces in three-dimensional reducible spaces. *Proc. Roy. Soc. Edinburgh Sect. A* **143**, 483–491 (2013)
6. de Lima, R.F., dos Santos, J.P.: Totally umbilical hypersurfaces of product spaces. *Manuscripta Math.* **169**, 649–666 (2022)
7. D'haene, M., Inoguchi, J.-I., Van der Veken, J.: Parallel and totally umbilical hypersurfaces of the four-dimensional Thurston geometry Sol_0^4 . *Math. Nachr.* **297**, 1879–1891 (2024)
8. Fu, Y., Nistor, A.I.: Constant angle property and canonical principal directions for surfaces in $\mathbb{M}^2(c) \times \mathbb{R}_1$. *Mediterr. J. Math.* **10**, 1035–1049 (2013)
9. Gutiérrez, M., Olea, B.: Global decomposition of a Lorentzian manifold as a Generalized Robertson-Walker space. *Differ. Geom. Appl.* **27**, 146–156 (2009)
10. Hong, S.-K.: Totally umbilic lorentzian surfaces embedded in L^n . *Bull. Kor. Math. Soc.* **34**, 9–17 (1997)
11. Jensen, G.R., Rigoli, M.: Harmonic Gauss maps. *Pacific J. Math.* **136**, 261–282 (1989)
12. Manzano, J.M., Souam, R.: The classification of totally umbilical surfaces in homogeneous 3-manifolds. *Math. Z.* **279**, 557–576 (2015)
13. Souam, R., Toubiana, E.: Totally umbilic surfaces in homogeneous 3-manifolds. *Comment. Math. Helv.* **84**, 673–704 (2009)
14. Souam, R., Van der Veken, J.: Totally umbilical hypersurfaces of manifolds admitting a unit Killing field. *Trans. Amer. Math. Soc.* **364**, 3609–3626 (2010)
15. Van der Veken, J., Vrancken, L.: Parallel and semi-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$. *Bull. Braz. Math. Soc. (N.S.)* **39**, 355–370 (2008)

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