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Bi-Hamiltonian structures of KdV type, cyclic Frobenius algebrae and Monge metrics

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Abstract

We study algebraic and projective geometric properties of Hamiltonian trios determined by a constant coefficient second-order operator and two first-order localizable operators of Ferapontov–Pavlov type. We show that first-order operators are determined by Monge metrics, and define a structure of cyclic Frobenius algebra. Examples include the AKNS system, a 2-component generalization of Camassa–Holm equation and the Kaup–Broer system. In dimension 2 the trio is completely determined by two conics of rank at least 2. We provide a partial classification in dimension 4.

Supplementary material for this article is available online

Keywords: Hamiltonian trios, projective geometry, Monge metrics

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1. Introduction

It was observed in [30] that many important bi-Hamiltonian structures of integrable (systems of) PDEs have the form

$$\left(P_1, Q_1 + \epsilon^k R_{k+1}\right), \tag{1.1}$$

(no sum over k) where P_1 and Q_1 are first-order compatible homogeneous Hamiltonian operators (Hamiltonian operators of hydrodynamic type) and R_{k+1} is a single $(k+1)$th-order homogeneous Hamiltonian operator compatible with P_1 and Q_1. Here, the homogeneity is defined with respect to the grading $\deg \partial_x = 1$.

Denoting with the square bracket the Schouten bracket we have

$$[P_1, Q_1] = [P_1, R_{k+1}] = [Q_1, R_{k+1}] = 0.$$

In other words the building blocks of the pair (1.1) (P_1, Q_1, R_{k+1}) define a *trio* of Hamiltonian structures. The above structure can be thought as a deformation of the bi-Hamiltonian structure of hydrodynamic type (P_1, Q_1). Due to the general theory of deformations the most interesting cases are $k = 1$ and $k = 2$ since for $k > 2$ the deformation R_{k+1} can be always eliminated by Miura type transformations [22]. The most famous example of such structures is the Hamiltonian trio

$$P = P_1 = \partial_x, \qquad Q = Q_1 + R_3, \quad Q_1 = 2u\partial_x + u_x, \quad R_3 = \partial_x^3. \tag{1.2}$$

Coupling Q_1 and R_3 one obtains the bi-Hamiltonian structure of the KdV hierarchy

$$\left(\partial_x, 2u\partial_x + u_x + \epsilon^2 \partial_x^3\right) \tag{1.3}$$

discovered by Magri in [29], while coupling P_1 and R_3 one obtains the bi-Hamiltonian structure of the Camassa–Holm hierarchy

$$\left(2u\partial_x + u_x, \partial_x + \epsilon^2 \partial_x^3\right). \tag{1.4}$$

Bi-Hamiltonian structures (1.1) obtained in this way have been introduced in [30] and have been called in [25] *bi-Hamiltonian structures of KdV type*. Another example (from [10, 22]) is the trio:

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \; Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix}, \; R_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix} \tag{1.5}$$

In this case one coupling yields the bi-Hamiltonian structure of the the so-called AKNS hierarchy, and the other one yields the bi-Hamiltonian structure of the two component Camassa–Holm hierarchy [10, 22].

In order to classify this kind of bi-Hamiltonian structures (with $k = 1, 2$) one can use the following strategy:

1. Use canonical forms of R_{k+1} under some natural groups of transformations preserving the form of (P_1, Q_1, R_{k+1}).
2. Compute compatibility conditions $[P_1, R_{k+1}] = 0$.
3. Use compatibility conditions to obtain trios of Hamiltonian operators.

The above strategy is motivated by the fact that there exist classifications of canonical forms of operators R_{k+1} under the action of various transformation groups, while trying to work with canonical forms of P_1 or Q_1 does not lead to manageable forms of the corresponding R_{k+1}, in view of the greater complexity of the latter.

There are two natural choices of the group of transformations to deal with: the group of diffeomorphisms of the dependent variables and the groups of reciprocal projective transformations of the independent variables. The latter group has been introduced in [14] as the group of projective transformations of the dependent variables coupled with a nonlocal transformation of the independent variable x of the type

$$d\tilde{x} = \Delta dx, \qquad \tilde{u}^i = \frac{T^i_j u^j + T^i_0}{\Delta}, \tag{1.6}$$

where $\Delta = T^0_i u^i + T^0_0$ and T's are constants. Depending on the choice of the group the problem admits a slightly different formulation. In the first case, since the group of diffeomorphisms preserves the locality of Hamiltonian operators it is possible to restrict the attention only to *local* first-order Hamiltonian operators (also known as Dubrovin–Novikov Hamiltonian operators)

$$P_1^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x. \tag{1.7}$$

In the second case, we can use the group of transformations of dependent variables to reduce the operator R_{k+1} to Doyle–Potemin canonical form: (see [26] and references therein)

$$R_{k+1} = \partial_x \circ R_{k-1} \circ \partial_x, \tag{1.8}$$

where R_{k-1} is a homogeneous operator of order $k-1$. Reciprocal projective transformations preserve this form [26]; however, they do not preserve locality of P_1, so that one is obliged to consider first-order Hamiltonian operators of localizable shape (or simply localizable), first introduced by Ferapontov and Pavlov in [13]:

$$P_1^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + w^i_k u^k_x \partial_x^{-1} u^j_x + u^i_x \partial_x^{-1} w^j_k u^k_x. \tag{1.9}$$

The first approach has been pursued in [25] using the results of [8, 16] for second-order operators R_2 and the results of [8, 14, 15, 17, 18] for third-order operators R_3. For instance, in the 2-component case the canonical forms are

$$R_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x^2, \tag{1.10}$$

$$R_3^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x^3, \tag{1.11}$$

$$R_3^{(2)} = \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{u^1} \\ \frac{1}{u^1}\partial_x & \frac{u^2}{(u^1)^2}\partial_x + \partial_x \frac{u^2}{(u^1)^2} \end{pmatrix} \partial_x, \tag{1.12}$$

$$R_3^{(3)} = \partial_x \begin{pmatrix} \partial_x & \partial_x \frac{u^2}{u^1} \\ \frac{u^2}{u^1}\partial_x & \frac{(u^2)^2+1}{2(u^1)^2}\partial_x + \partial_x \frac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} \partial_x. \tag{1.13}$$

And the corresponding compatible first-order operators are given in the following theorem.

Theorem: affine classification [25]. *P_1 is a Hamiltonian operator compatible with R_2 if and only if*

$$g^{11} = c_1 u^1 + c_2, \tag{1.14a}$$

$$g^{12} = \frac{1}{2} c_3 u^1 + \frac{1}{2} c_1 u^2 + c_5 \tag{1.14b}$$

$$g^{22} = c_3 u^2 + c_4. \tag{1.14c}$$

P_1 is a Hamiltonian operator compatible with $R_3^{(1)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2 + c_3, \tag{1.15a}$$

$$g^{12} = c_4 u^1 + c_1 u^2 + c_5 \tag{1.15b}$$

$$g^{22} = c_6 u^1 + c_4 u^2 + c_7 \tag{1.15c}$$

together with the algebraic conditions

$$c_1 c_4 - c_2 c_6 = 0, \quad c_3 c_4 - c_7 c_2 = 0, \quad c_3 c_6 - c_1 c_7 = 0. \tag{1.16}$$

P_1 is a Hamiltonian operator compatible with $R_3^{(2)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2, \tag{1.17a}$$

$$g^{12} = c_4 u^1 + \frac{c_3}{u^1} + \frac{c_2 \left(u^2\right)^2}{2u^1}, \tag{1.17b}$$

$$g^{22} = 2c_4 u^2 + \frac{c_6}{u^1} - \frac{c_1 \left(u^2\right)^2}{u^1} + c_5, \tag{1.17c}$$

together with the algebraic conditions

$$c_2 c_6 + 2c_1 c_3 = 0, \quad c_2 c_5 = 0, \quad c_1 c_5 = 0. \tag{1.18}$$

P_1 is a Hamiltonian operator compatible with $R_3^{(3)}$ if and only if

$$g^{11} = c_1 u^1 + c_2 u^2 + c_3, \tag{1.19a}$$

$$g^{12} = c_4 u^1 - \frac{c_2}{2u^1} + \frac{c_3 u^2}{u^1} + \frac{c_2 \left(u^2\right)^2}{2u^1}, \tag{1.19b}$$

$$g^{22} = 2c_4 u^2 + \frac{c_1}{u^1} + \frac{c_5 u^2}{u^1} - \frac{c_1 \left(u^2\right)^2}{u^1} + c_6, \tag{1.19c}$$

together with the algebraic conditions

$$c_2 c_5 + 2c_1 c_3 = 0, \quad c_2 c_6 - 2c_3 c_4 = 0, \quad c_1 c_6 + c_4 c_5 = 0. \tag{1.20}$$

The family of contravariant metrics (1.14) depends linearly on the parameters and thus any pair of metrics belonging to these families defines a bi-Hamiltonian structure of hydrodynamic type compatible with the second/third-order operator. Other families are defined by nonlinar constraints and the previous argument fail; see [25] for a complete list of compatible pairs within the families.

The second approach has been pursued in [27] in the case $R_3^{ij} = \eta^{ij}\partial_x^3$ where (η^{ij}) is a symmetric constant non-degenerate matrix ($\det(\eta^{ij}) \neq 0$). The operator R_3 generates the simplest orbit of third-order operators under the action of the projective reciprocal transformation group. The study of compatibility conditions leads to the following results:

- the Christoffel symbols Γ^{ij}_k define a Frobenius algebra structure on the cotangent bundle of the manifold of dependent variables (u^i);
- the operator $P = L + N$ splits into its local part L and its nonlocal part N, and they are independently Hamiltonian operators;
- for $n \geq 3$, $N = 0$, and the operator becomes purely local.

The local trios that we got for $n > 2$ are known in the literature. They can be obtained as special cases of the results of [34] and also were studied in [2] in terms of Frobenius pencils. However we point out that the locality (for $n > 2$) is not an *a priori* assumption but the result of non trivial computations.

The aim of the present paper is to study the case $R_2^{ij} = \eta^{ij}\partial_x^2$ where (η^{ij}) is a skew-symmetric constant non-degenerate matrix ($\det(\eta^{ij}) \neq 0$). The operator R_2 generates the simplest orbit of second-order operators under the action of the projective reciprocal transformation group. The role of Frobenius algebra in this setting is played by a new type of algebra recently introduced by Buchstaber and Mikhailov and called *cyclic Frobenius algebra*.

Definition 1.1. [3]. Let $\mathcal{V}$ be some $\mathbb{C}$-linear space ($\dim(\mathcal{V} \geqslant 1$). A cyclic Frobenius algebra (CF-algebra) $\mathcal{A}$ is an associative algebra $\mathcal{A}$ with unity 1 equipped with a $\mathbb{C}$-bilinear skew-symmetric form $\eta(\cdot,\cdot) : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{V}$ such that

$$\eta(A, B \circ C) + \eta(B, C \circ A) + \eta(C, A \circ B) = 0 \tag{1.21}$$

where $A, B, C \in \mathcal{A}$ and $\circ$ is the product in the algebra.

Let $\mathcal{V} = \mathbb{C}$. Denoting by Γ^{ij}_k the structure constants of the product we have

$$\begin{aligned}\eta^{ij} A_i (B \circ C)_j + \eta^{ij} B_i (C \circ A)_j + \eta^{ij} C_i (A \circ B)_j \\ = \left(\eta^{ij}\Gamma^{lk}_j + \eta^{lj}\Gamma^{ki}_j + \eta^{kj}\Gamma^{il}_j\right) A_i B_l C_k = 0\end{aligned}$$

or, taking into account that A, B, C are arbitrary,

$$\Gamma^{ki}_j \eta^{jl} + \Gamma^{il}_j \eta^{jk} + \Gamma^{lk}_j \eta^{ji} = 0. \tag{1.22}$$

The main results of the paper concerning the compatibility of Hamiltonian operators can be summarized as follows.

Compatibility Theorem. *The Hamiltonian operators P, R are compatible if and only if*

- $w^i_j = W^i_j$ *where W is a constant matrix that is symmetric with respect to* η*:*

$$\eta(AW, B) = \eta(A, BW); \tag{1.23}$$

- *the contravariant Christoffel symbols are linear functions of the form*

$$\Gamma^{ij}_k = \partial_k\left(-W^j_s u^s u^i + b^{ij}_s u^s\right) = -W^j_k u^i - W^j_s u^s \delta^i_k + b^{ij}_k \tag{1.24}$$

for arbitrary constants b^{ij}_k.
- *the product with structure constants* Γ^{ij}_k

$$(A \circ B)_k = \Gamma^{ij}_k A_i B_j$$

endows the cotangent space T^*M of the manifold M of dependent variables with a structure of cyclic Frobenius algebra (without unity) and satisfy the conditions:

$$\eta(A\circ B,C)=\eta(A,C\circ B)\,. \tag{1.25}$$

The Theorem is stated and proved as theorem 3.1.

Notice that the condition (1.25) can be also written as

$$\Gamma^{ij}_{l}\eta^{lk}+\Gamma^{kj}_{l}\eta^{li}=0. \tag{1.26}$$

Indeed, relabelling the indices the condition

$$\eta^{ij}(A\circ B)_i=\eta^{ij}A_i(C\circ B)_j$$

reads

$$\left(\Gamma^{lk}_{i}\eta^{ij}+\Gamma^{jk}_{i}\eta^{il}\right)A_lB_kC_j=0.$$

Conditions (1.21) and (1.25) appear in the paper [34] as cocycle conditions arising from the compatibility between a local first-order Hamiltonian operator of hydrodynamic type defined by a flat linear metric and a second-order constant Hamiltonian operator defined by a skew-symmetric matrix. In this setting the contravariant Christoffel symbols of the linear metric are constant and define the structure constants of a Balinsky–Novikov algebra.

A corollary of the above theorem is that the nondegenerate symmetric bilinear form obtained from the the contravariant metric defining P_1 'lowering' the indices with η:

$$\bar{g}_{ab}=\eta_{jb}\eta_{ia}g^{ij} \tag{1.27}$$

is the Monge metric of a *quadratic line complex*, an algebraic variety that is defined in the Plücker embedding of the projective space with homogeneous coordinates $[v^1,\dots,v^{n+1}]$, where $u^i=v^i/v^{n+1}$, $i=1,\dots,n$, $v^{n+1}\neq 0$. See [14, 15] for more details on this construction. The equation that characterizes Monge metrics is

$$\bar{g}_{ij,k}+\bar{g}_{ki,j}+\bar{g}_{jk,i}=0 \tag{1.28}$$

can be obtained from the cyclic Frobenius algebra condition. In 2-component case there are no additional conditions and the general solution of compatibility conditions can be obtained starting from arbitrary Monge metric

$$\begin{aligned}
\bar{g}_{11}&=c_0(u^2)^2+c_3u^2+c_4,\\
\bar{g}_{12}&=-c_0u^1u^2-\frac{1}{2}c_3u^1-\frac{1}{2}c_1u^2+c_5,\\
\bar{g}_{22}&=c_0(u^1)^2+c_1u^1+c_2.
\end{aligned}$$

The above metric yields a flat contravariant metric g^{ij}, by means of (1.27), if and only if the coefficient c_0 vanishes. Notice that in the flat case we recover the metric of the above affine classification theorem.

It is known [36] that Plücker embedding provides an identification of the leading coefficient matrix of a second-order homogeneous Hamiltonian operator with an algebraic variety, more precisely, a *linear line congruence*. Such a variety is defined by a system of $n+1$ linear equations in $\mathbb{P}(\wedge^2V)$ and its intersection with Plücker variety.

It is then clear that there is a correspondence between trios of Hamiltonian operators P_1, Q_1 of the form (1.9) and R_2 of the form (1.8) and trios of algebraic varieties. In the case $n=2$ that is summarized by the following theorem.

Projective Correspondence Theorem. *If $n=2$, then there is a bijective correspondence between trios of mutually compatible Hamiltonian operators P_1, Q_1 of the form (1.9) and $R_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\partial_x^2$ and pairs of conics $\mathcal{C}_1$, $\mathcal{C}_2$ of rank at least 2.*

Note that the linear line congruence corresponding to R_2 degenerates to 0 in this case. The theorem is stated and proved as theorem 4.1.

In higher dimension the compatibility conditions include Monge's condition on the metric, but are not reduced to that condition only. In particular, the geometric interpretation of the further conditions as conditions on the trio of algebraic varieties underlying the compatible trio of Hamiltonian operators is still missing. However, it easy to realize that there are plenty of Hamiltonian trios in any dimension.

In particular, using the solver CRACK [39, 40], a package working within the computer algebra system Reduce [19], we obtain the general solution P_1 of the compatibility conditions $[P_1, R_2] = 0$ (with $R_2^{ij} = \eta^{ij}\partial_x^2$) for $n=4$. It turns out that there are 288 subcases, each depending on several parameters. This calculation has been performed on a compute server of the INFN, and it took 33GB of RAM and about 15 min of time.

Then, for each subcase one should find all compatible Q_1 in the same list. We did this computation for one selected P_1, generalizing the Kaup–Broer and AKNS bi-Hamiltonian pairs, and obtained a list of 64 subcases, again each of them depending on several parameters.

Both lists of solutions are available at the webpage of one of us [28] and also in the article's supplementary information. Here, we just wrote two examples of bi-Hamiltonian trios, a local one and a nonlocal one.

The paper is organized as follows: in section 2 we briefly recall the canonical form of second and third-order homogeneous operators under projective reciprocal transformations; in section 3 we compute the compatibility conditions between the simplest canonical form of a second-order homogeneous operator and a first-order operator of localizable shape; using these results in section 4 we study trios of such operators and we provide the classification in the case $n=2$ and the computational scheme by which we computed the classification in the case $n=4$, as well as some examples.

2. Hamiltonian operators and projective reciprocal transformations

First-order Hamiltonian operators are operators of the form

$$P_1^{ij} = g^{ij}(u)\,\partial_x + \Gamma^{ij}_k(u)\,u^k_x, \tag{2.1}$$

formally skew-adjoint and satisfying the Schouten bracket condition $[P_1, P_1] = 0$. In the non-degenerate case ($\det(g^{ij}) \neq 0$) Dubrovin and Novikov proved that P_1 is Hamiltonian if and only if g^{ij} is a flat contravariant pseudo-Riemannian metric and $\Gamma^j_{hk} = -g_{hi}\Gamma^{ij}_k$ are the Christoffel symbols of the associated Levi–Civita connection.

Higher-order Dubrovin–Novikov operators have a much more complicated form, see [9] for details. However, it was proved [1, 8, 18, 31] that if the order is 2 or 3, they admit, respectively, the canonical forms

$$R_2^{ij} = \partial_x \circ f^{ij} \circ \partial_x, \qquad R_3^{ij} = \partial_x \circ \left(\ell^{ij}\partial_x + c^{ij}_k u^k_x\right) \circ \partial_x. \tag{2.2}$$

The above canonical forms are invariant with respect to projective reciprocal transformations of the type (1.6) (see [14, 26, 36]). The result of such transformations on a first-order Dubrovin–Novikov Hamiltonian operator is a nonlocal Hamiltonian operator of localizable shape

$$P_1^{ij} = g^{ij}(u)\,\partial_x + \Gamma^{ij}_k(u)\,u^k_x + w^i_k(u)\,u^k_x\partial_x^{-1}u^j_x + u^i_x\partial_x^{-1}w^j_k(u)\,u^k_x. \tag{2.3}$$

Operators of this form have been studied in [11, 13] and naturally appear in the study and classification of integrable systems of PDEs (see for instance [23, 26]). Skew-adjointness and vanishing of the Schouten bracket in this case lead to the following list of conditions:

1. g^{ij} is a contravariant pseudo-Riemannian metric and Γ^{ij}_k are the contravariant Christoffel symbols of its Levi–Civita connection; equivalently, the following conditions hold:

$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s, \tag{2.4}$$

$$\partial_k g^{ij} = \Gamma^{ij}_k + \Gamma^{ji}_k; \tag{2.5}$$

2. the following equations hold:

$$g^{is}w^j_s = g^{js}w^i_s, \tag{2.6}$$

$$\nabla_i w^j_k = \nabla_k w^j_i, \tag{2.7}$$

$$R^{ij}_{kh} = w^i_k\delta^j_h - w^j_k\delta^i_h - w^i_h\delta^j_k + w^j_h\delta^i_k, \tag{2.8}$$

where ∇ is the Levi–Civita connection of g^{ij} and

$$R^{jk}_{sl} = g^{jp}R^k_{psl} = \frac{\partial\Gamma^{jk}_s}{\partial u^l} - \frac{\partial\Gamma^{jk}_l}{\partial u^s} + g_{st}\left(\Gamma^{tj}_m\Gamma^{mk}_l - \Gamma^{tk}_m\Gamma^{mj}_l\right) \tag{2.9}$$

is the Riemannian curvature tensor of g_{ij}.

Canonical forms of operators (2.2) under the action of projective reciprocal transformations have been found in [36] in the case of second-order operators and in [14] and [15] for third-order operators. The simplest canonical form of second-order operators (2.2) is $R_2^{ij} = \eta^{ij}\partial_x^2$ where η^{ij} are the entries of a constant skew-symmetric matrix.

3. Conditions of compatibility

In this section we calculate the conditions that are equivalent to the compatibility of P and R, i.e. the vanishing of the Schouten bracket $[P,R] = 0$, for a pair of Hamiltonian operators, where $P = P_1$ is a nonlocal localizable first-order homogeneous Hamiltonian operator as in (2.3) and $R^{ij} = R_2^{ij} = \eta^{ij}\partial_x^2$, with (η^{ij}) a constant skew-symmetric non-degenerate matrix.

Theorem 3.1. *The Hamiltonian operators P, R are compatible if and only if*

- *the functions w^i_j are constant and satisfy the condition*

$$w^i_l\eta^{lk} + w^k_l\eta^{li} = 0; \tag{3.1}$$

- *the contravariant Christoffel symbols* Γ^{ij}_k *satisfy the conditions:*

$$\Gamma^{ij}_l\eta^{lk} + \Gamma^{kj}_l\eta^{li} = 0, \tag{3.2}$$

$$\Gamma^{ki}_l\eta^{lj} + \Gamma^{ij}_l\eta^{lk} + \Gamma^{jk}_l\eta^{li} = 0, \tag{3.3}$$

$$\Gamma^{sj}_p\Gamma^{ir}_s - \Gamma^{sr}_p\Gamma^{ij}_s = 0, \tag{3.4}$$

$$\frac{\partial\Gamma^{kj}_l}{\partial u^s} = -\delta^j_s w^k_l - w^j_s\delta^k_l. \tag{3.5}$$

Proof. We will write differential operators by means of distributions as

$$P^{ij}_{xy} = g^{ij}\delta'(x-y) + \Gamma^{ij}_s u^s_x\delta(x-y) + u^i_x\nu(x-y)\,w^j_s u^s_y + w^i_s u^s_x\nu(x-y)\,u^j_y \tag{3.6}$$

and

$$R^{ij}_{xy} = \eta^{ij}\delta''(x-y). \tag{3.7}$$

We use Dubrovin–Zhang formula for the Schouten bracket:

$$\begin{aligned}[P,R]^{ijk}_{x,y,z} =& \frac{\partial P^{ij}_{x,y}}{\partial u^l(x)}R^{lk}_{x,z} + \frac{\partial P^{ij}_{x,y}}{\partial u^l(y)}R^{lk}_{y,z} + \frac{\partial P^{ki}_{z,x}}{\partial u^l(z)}R^{lj}_{z,y} + \frac{\partial P^{ki}_{z,x}}{\partial u^l(x)}R^{lj}_{x,y}\\ &+ \frac{\partial P^{jk}_{y,z}}{\partial u^l(y)}R^{li}_{y,x} + \frac{\partial P^{jk}_{y,z}}{\partial u^l(z)}R^{li}_{z,x} + \frac{\partial P^{ij}_{x,y}}{\partial u^l_x}\partial_x R^{lk}_{x,z} + \frac{\partial P^{ij}_{x,y}}{\partial u^l_y}\partial_y R^{lk}_{y,z}\\ &+ \frac{\partial P^{ki}_{z,x}}{\partial u^l_z}\partial_z R^{lj}_{z,y} + \frac{\partial P^{ki}_{z,x}}{\partial u^l_x}\partial_x R^{lj}_{x,y} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_y}\partial_y R^{li}_{y,x} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_z}\partial_z R^{li}_{z,x}.\end{aligned}$$

The vanishing of the distribution $[P,R]^{ijk}_{x,y,z}$ means that for any choice of the test functions $p_i(x), q_j(y), r_k(z)$ the triple integral

$$\iiint [P,R]^{ijk}_{x,y,z}\, p_i(x)\, q_j(y)\, r_k(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{3.8}$$

should vanish.

Following [6, 24], we apply a procedure to collect together all terms which are related by a distributional identity. The procedure is the following

1. Using identities like

$$\nu(z-y)\,\delta(z-x) = \nu(x-y)\delta(x-z) \tag{3.9}$$

together with their differential consequences, we can eliminate all terms containing $\nu(z-y)\delta^{(n)}(z-x)$, $\nu(y-x)\delta^{(n)}(y-z)$, $\nu(x-z)\delta^{(n)}(x-y)$ producing nonlocal terms containing $\nu(x-y)\delta^{(n)}(x-z)$, $\nu(z-x)\delta^{(n)}(z-y)$, $\nu(y-z)\delta^{(n)}(y-x)$ and additional local terms.

2. Using the identity

$$f(z)\,\delta^{(n)}\,(x-z) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}\,(x)\,\delta^{(n-k)}\,(x-z)\,, \tag{3.10}$$

we can eliminate the dependence on z in the coefficients of $\nu(x-y)\delta^{(n)}(x-z)$, the dependence on y in the coefficients of $\nu(z-x)\delta^{(n)}(z-y)$ and the dependence on x in the coefficients of $\nu(y-z)\delta^{(n)}(y-x)$. After the first two steps the nonlocal part of $[P,R]^{ijk}_{x,y,z}$ has the form

$$a_1\,(x,y,z)\,\nu\,(x-y)\,\nu\,(x-z) + \text{cyclic}\,(x,y,z) \\ + \sum_{n\geqslant 0} b_n\,(x,y)\,\nu\,(x-y)\,\delta^{(n)}\,(x-z) + \text{cyclic}\,(x,y,z)\,. \tag{3.11}$$

3. The local part of $[P,R]^{ijk}_{x,y,z}$ can be reduced to the form

$$\sum_{m,n} e_{mn}\,(x)\,\delta^{(m)}\,(x-y)\,\delta^{(n)}\,(x-z) \tag{3.12}$$

using the identities (and their differential consequences)

$$\delta\,(z-x)\,\delta\,(z-y) = \delta\,(y-x)\,\delta\,(y-z) = \delta(x-y)\delta(x-z) \tag{3.13}$$

and the identities (3.10).

The fulfillment of the Jacobi identity turns out to be equivalent to the vanishing of each coefficient in the reduced form. Below a list of relevant coefficients in our case.

The vanishing of the coefficient of $\delta(x-y)\delta'''(x-z)$ provides the condition (3.2):

$$\frac{\partial g^{jk}}{\partial u^l}\eta^{li} + \Gamma^{ij}_l\eta^{lk} - \Gamma^{jk}_l\eta^{li} = \Gamma^{ij}_l\eta^{lk} + \Gamma^{kj}_l\eta^{li} = 0. \tag{3.14}$$

The same condition is provided by the vanishing of the coefficient $\delta'''(x-y)\delta(x-z)$.

The vanishing of coefficient of $\delta'(x-y)\delta''(x-z)$ provides the condition

$$\frac{\partial g^{ij}}{\partial u^l}\eta^{lk} + 2\frac{\partial g^{jk}}{\partial u^l}\eta^{li} - 3\Gamma^{jk}_l\eta^{li} = \Gamma^{ji}_l\eta^{lk} + \Gamma^{kj}_l\eta^{li} - \Gamma^{jk}_l\eta^{li} = 0, \tag{3.15}$$

and the vanishing of coefficient of $\delta''(x-y)\delta'(x-z)$ provides the condition

$$-\frac{\partial g^{ki}}{\partial u^l}\eta^{lj} + \frac{\partial g^{jk}}{\partial u^l}\eta^{li} - 3\Gamma^{jk}_l\eta^{li} = -\Gamma^{ki}_l\eta^{lj} + \Gamma^{kj}_l\eta^{li} - \Gamma^{jk}_l\eta^{li} = 0. \tag{3.16}$$

The difference between (3.15) and (3.16) is equivalent to condition (3.2), while their sum provides (3.3):

$$\Gamma^{ki}_l\eta^{lj} + \Gamma^{ij}_l\eta^{lk} + \Gamma^{jk}_l\eta^{li} = 0. \tag{3.17}$$

The coefficient of $\nu(x-y)\delta'''(x-z)$ is

$$w^j_s u^s_y \eta^{ik} + w^i_l(x) u^j_y \eta^{lk} + u^j_y w^k_l(x)\eta^{li} + w^j_s u^s_y \eta^{ki} = \left(w^{i_l}(x)\eta^{lk} + w^{k_l}(x)\eta^{li}\right) u^j_y; \tag{3.18}$$

its vanishing is (3.1). The same condition is obtained by the coefficients of $\nu(z-x)\delta'''(z-y)$ and $\nu(y-z)\delta'''(y-x)$.

The coefficient of $u^s_{xxx}\delta(x-y)\delta(x-z)$ is

$$\left(\frac{\partial\Gamma^{jk}_s}{\partial u^l}-\frac{\partial\Gamma^{jk}_l}{\partial u^s}\right)\eta^{li}+w^i_l\eta^{lj}\delta^k_s+w^k_s\eta^{ij}+w^j_s\eta^{ki}+\delta^j_s w^k_l\eta^{li}. \tag{3.19}$$

Replacing the condition (2.8) in the previous expression and requiring its vanishing we get condition (3.4):

$$\Gamma^{tj}_m\Gamma^{mk}_l-\Gamma^{tk}_m\Gamma^{mj}_l=0, \tag{3.20}$$

and the equivalent condition

$$\frac{\partial\Gamma^{jk}_s}{\partial u^l}-\frac{\partial\Gamma^{jk}_l}{\partial u^s}=w^j_l\delta^k_s+\delta^j_l w^k_s-w^j_s\delta^k_l-\delta^j_s w^k_l. \tag{3.21}$$

The coefficient of $\delta(x-y)\delta''(x-z)$ is

$$\frac{\partial\Gamma^{ij}_s}{\partial u^l}u^s_x\eta^{lk}+2\partial_x\left(\frac{\partial g^{jk}}{\partial u^l}\right)\eta^{li}+\frac{\partial\Gamma^{jk}_s}{\partial u^l}u^s_x\eta^{li}+u^i_x w^j_l\eta^{lk}+w^i_s u^s_x\eta^{jk}+u^k_x w^i_l\eta^{lj}$$
$$+w^k_s u^s_x\eta^{ij}-3\partial_x\left(\Gamma^{jk}_l\right)\eta^{li}+3u^j_x w^k_l\eta^{li}+3w^j_s u^s_x\eta^{ki}. \tag{3.22}$$

Using (2.5) in order to eliminate the derivative of g^{jk}, the above coefficient can be rewritten as

$$\left(\frac{\partial\Gamma^{kj}_l}{\partial u^s}+\delta^j_s w^k_l+w^j_s\delta^k_l\right)u^s_x\eta^{li}. \tag{3.23}$$

Thus the vanishing of this coefficient provides condition (3.5). The same condition is provided by the vanishing of the coefficient of $\delta''_{xy}\delta_{xz}$.

The coefficient of $\delta'(x-y)\delta'(x-z)$ is

$$2\partial_x\left(\frac{\partial g^{jk}}{\partial u^l}\right)\eta^{li}+2\frac{\partial\Gamma^{jk}_s}{\partial u^l}u^s_x\eta^{li}-u^i_x w^j_l\eta^{lk}-w^i_s u^s_x\eta^{jk}$$
$$+3u^k_x w^i_l\eta^{lj}+3w^k_s u^s_x\eta^{ij}-6\partial_x\left(\Gamma^{jk}_l\right)\eta^{li}+3u^j_x w^k_l\eta^{li}+3w^j_s u^s_x\eta^{ki}. \tag{3.24}$$

It can be proved that the above expression is equal to

$$\frac{\partial}{\partial u^s}\left(\Gamma^{kj}_l\eta^{li}+\Gamma^{ik}_l\eta^{lj}+\Gamma^{ji}_l\eta^{lk}\right), \tag{3.25}$$

and thus vanishes due to condition (3.3).

The coefficient of $\nu(x-y)\delta''(x-z)$ is

$$u^j_y u^s_x\left(\frac{\partial w^i_s}{\partial u^l}\eta^{lk}-\frac{\partial w^k_s}{\partial u^l}\eta^{li}+3\frac{\partial w^k_l}{\partial u^s}\eta^{li}\right), \tag{3.26}$$

which is the same as the coefficients of $\nu_{zx}\delta''_{zy}$ and $\nu_{yz}\delta''_{yx}$ up to renaming indices and variables.

The coefficient of $\nu(x-y)\delta'(x-z)$ is

$$-2u^j_y\partial_x\left(\frac{\partial w^k_s}{\partial u^l}u^s_x\right)\eta^{li}+3u^j_y\partial^2_x\left(w^k_l\right)\eta^{li}, \tag{3.27}$$

and the same expression, up to renaming indices and variables, holds for the coefficients of $\nu_{zx}\delta'_{zy}$ and $\nu_{yz}\delta'_{yx}$. In the expression (3.27), the coefficient of $u^s_{xx}u^i_y$ is

$$-2\frac{\partial w^k_s}{\partial u^l}\eta^{li}+3\frac{\partial w^k_l}{\partial u^s}\eta^{li}. \tag{3.28}$$

The coefficient of $\nu(x-y)\delta(x-z)$ is

$$-u^j_y\partial^2_x\left(\frac{\partial w^k_s}{\partial u^l}u^s_x\right)\eta^{li}+u^j_y\partial^3_x\left(w^k_l\right)\eta^{li}, \tag{3.29}$$

and the same expression holds for the coefficients of $\nu_{zx}\delta_{zy}$ and $\nu_{yz}\delta_{yx}$ up to renaming indices and variables. In the expression (3.29) the vanishing of the coefficient of $u^j_ju^s_{xxx}$ provides the closure condition

$$\left(-\frac{\partial w^k_s}{\partial u^l}+\frac{\partial w^k_l}{\partial u^s}\right)\eta^{li}=0.$$

Replacing this condition in (3.28) we obtain that the functions w^i_j are constant. In particular this tells us that

$$\Gamma^{kj}_l=-w^k_lu^j-w^j_su^s\delta^k_l+b^{kj}_l \tag{3.30}$$

where b^{kj}_l are constant.

Taking into account this fact the coefficient of $\delta(x-y)\delta'(x-z)$ is

$$\begin{aligned}&\partial^2_x\left(\frac{\partial g^{jk}}{\partial u^l}\right)\eta^{li}+2\partial_x\left(\frac{\partial\Gamma^{jk}_s}{\partial u^l}u^s_x\right)\eta^{li}+2u^k_{xx}w^i_l\eta^{lj}\\&\quad+2\partial_x\left(w^k_su^s_x\right)\eta^{ij}-3\partial^2_x\left(\Gamma^{jk}_l\right)\eta^{li}+3u^j_{xx}w^k_l\eta^{li}+3\partial_x\left(w^j_su^s_x\right)\eta^{ki}.\end{aligned} \tag{3.31}$$

This coefficient vanishes due to previous conditions. Indeed:

$$\begin{aligned}&\frac{\partial\Gamma^{kj}_l}{\partial u^s}\eta^{li}+2\left(\frac{\partial\Gamma^{jk}_s}{\partial u^l}-\frac{\partial\Gamma^{jk}_l}{\partial u^s}\right)\eta^{li}+2\delta^k_sw^i_l\eta^{lj}+2w^k_s\eta^{ij}+3\delta^j_sw^k_l\eta^{li}+3w^j_s\eta^{ki}\\&\quad=\left(-\delta^j_sw^k_l-w^j_s\delta^k_l\right)\eta^{li}+2\left(w^j_l\delta^k_s+\delta^j_lw^k_s-w^j_s\delta^k_l-\delta^j_sw^k_l\right)\eta^{li}\\&\qquad+2\delta^k_sw^i_l\eta^{lj}+2w^k_s\eta^{ij}+3\delta^j_sw^k_l\eta^{li}+3w^j_s\eta^{ki}=0.\end{aligned} \tag{3.32}$$

The coefficient of $\delta'(x-y)\delta(x-z)$ is

$$\begin{aligned}&2\partial_x\left(\frac{\partial\Gamma^{jk}_s}{\partial u^l}u^s_x\right)\eta^{li}+2u^i_x\partial_x\left(w^j_l\right)\eta^{lk}+3u^k_{xx}w^i_l\eta^{lj}\\&\quad+3\partial_x\left(w^k_su^s_x\right)\eta^{ij}-3\partial^2_x\left(\Gamma^{jk}_l\right)\eta^{li}+2u^j_{xx}w^k_l\eta^{li}+2\partial_x\left(w^j_su^s_x\right)\eta^{ki}.\end{aligned} \tag{3.33}$$

It vanishes due to previous conditions; the calculation is similar to that of (3.32).

Finally, the coefficient of $\delta(x-y)\delta(x-z)$ is

$$\partial_x^2\left(\frac{\partial\Gamma^{jk}_s}{\partial u^l}u^s_x\right)\eta^{li}+u^k_{xxx}w^i_l\eta^{lj}+\partial_x^2\left(w^k_s u^s_x\right)\eta^{ij}-\partial_x^3\left(\Gamma^{jk}_l\right)\eta^{li} \\ +u^j_{xxx}w^k_l\eta^{li}+\partial_x^2\left(w^j_s u^s_x\right)\eta^{ki}. \tag{3.34}$$

Again, this coefficient vanishes due to previous conditions. □

There are very interesting geometric and algebraic consequences of theorem 1. First of all, very recently a new algebraic structure has been introduced in the theory of Integrable Systems, namely *cyclic Frobenius algebra* [3], in a framework which is different from ours. It turns out that it also arises in our context.

Corollary 3.2. *The Christoffel symbols Γ^{ij}_k endow the cotangent space T^*M of the manifold M of dependent variables (u^i) with a structure of cyclic Frobenius algebra.*

Proof. The conditions that should be satisfied are exactly (3.2)–(3.4). □

An even more surprising fact is the interpretation as an algebraic variety of the leading coefficient of any first-order nonlocal homogeneous Hamiltonian operator P that is compatible with our constant-coefficient second-order Hamiltonian operator R (3.7).

Corollary 3.3. *Let us introduce the nondegenerate symmetric bilinear form*

$$\bar{g}_{ab}=\eta_{jb}\eta_{ia}g^{ij}. \tag{3.35}$$

Then, $\bar{g}_{ab}$ is the Monge metric of a quadratic line complex.

Proof. Summing the condition (3.3) with the same condition with the indices i,k swapped we obtain the condition

$$g^{ki}_{,l}\eta^{lj}+g^{ij}_{,l}\eta^{lk}+g^{jk}_{,l}\eta^{li}=0, \tag{3.36}$$

where $g^{ki}_{,l}=\partial g^{ki}/\partial u^l$. The above condition can be rewritten in lower indices by multiplication by $\eta_{kb}\eta_{ic}\eta_{ja}$, yielding

$$\bar{g}_{bc,a}+\bar{g}_{ca,b}+\bar{g}_{ab,c}=0. \tag{3.37}$$

The above condition is equivalent to the fact that $\bar{g}_{ab}$ is a Monge metric, which is S. Lie's representation of quadratic line complexes (see [14, 15]). This proves the Corollary. □

Remark 3.4. It is known [14] that under projective reciprocal transformations (1.6) a Monge metric transforms as $(\bar{g}_{ij}(\tilde{u}))=(\bar{g}_{hk}(u))/\Delta^4$. Moreover, it has been proved in [36] that the leading coefficient matrix of a second-order homogeneous Hamiltonian operator in Doyle–Potëmin canonical form transforms as $(\eta_{ij}(\tilde{u}))=(\eta_{hk}(u))/\Delta^3$.

That implies that the leading coefficient matrix (g^{ij}) of a first-order operator (1.9) that is compatible with a second-order operator $R^{ij}_2=\eta^{ij}\partial_x^2$ transforms as $(g^{ij}(\tilde{u}))=(g^{hk}(u))\Delta^2$, which is how the metric of the first-order operator transforms under a generic reciprocal transformations according with [13].

4. Classification of bi-Hamiltonian trios

The general problem of the classification of local bi-Hamiltonian trios can be formulated as follows: classify the bi-Hamiltonian trios of operators of the form

$$A_1 = P_1 + R_2, \qquad A_2 = Q_1, \tag{4.1}$$

where

- P_1, Q_1 are local homogeneous first-order Hamiltonian operators;
- R_2 is a local homogeneous second-order Hamiltonian operator;
- the three operators are mutually compatible:

$$[P_1, Q_1] = [R_2, P_1] = [R_2, Q_2] = 0. \tag{4.2}$$

Of course, in view of the complexity of the general form of R_2, the problem can be reformulated when R_2 is written in the canonical form (2.2). This can always be done by means of a point transformation of the dependent variables, without changing the shape of the three operators.

Then, we can use the projective classification of (non-degenerate) second-order homogeneous operators [36] at the price of allowing P_1 and Q_1 to have localizable shape (see [26]). Indeed, the projective classification makes use of projective reciprocal transformations which transform local operators into nonlocal ones.

In this paper, we will just consider the orbit of R_2 under the action of projective reciprocal transformations that contains the constant operator $R_2^{ij} = \eta^{ij}\partial_x^2$, so to apply the results from the previous Section.

For this reason, we reformulate and restrict the above problem to: classify the bi-Hamiltonian trios of operators of the form

$$A_1 = P_1 + R_2, \qquad A_2 = Q_1, \tag{4.3}$$

where

- P_1, Q_1 are nonlocal homogeneous first-order Hamiltonian operators that are localizable (by means of the same projective reciprocal transformation);
- $R_2^{ij} = \eta^{ij}\partial_x^2$ is a constant-coefficient local homogeneous second-order Hamiltonian operator;
- the three operators are mutually compatible:

$$[P_1, Q_1] = [R_2, P_1] = [R_2, Q_2] = 0. \tag{4.4}$$

We will be able to give a complete answer in the case $n = 2$ and a partial answer in the case $n = 4$, due to the complex structure of the space of solutions.

We observe that solutions the above version of the problem contain trios of local operators as a particular case, but they also contain trios where the two first-order operators cannot be *simultaneously* localized; hence, we obtain solutions with non-removable nonlocal terms.

The Hamiltonian operators of our trios are uniquely identified by algebraic varieties. We now give a brief description of the procedure that allows us to make the above identification, which, in essence, boils down to Plücker embedding.

We assume that (u^i) are affine coordinates of an n-dimensional projective space $\mathbb{P}(V)$, where V is a vector space with $\dim V = n+1$ and coordinates (v^i). Homogeneous coordinates on $\mathbb{P}(V)$ are denoted by $[v^1, \ldots, v^{n+1}]$, in such a way that $u^i = v^i/v^{n+1}$. We recall that Plücker embedding (of lines) is the natural injective map $\mathrm{Gr}(2,V)\hookrightarrow\mathbb{P}(\wedge^2 V)$, where $\mathrm{Gr}(2,V)$ is the Grassmanniann of planes in V, which can be identified as the space of projective lines in $\mathbb{P}(V)$.

Elements of $\mathbb{P}(\wedge^2 V)$ can be represented as $[p^{ij}]$, where p^{ij} are coordinates with respect to the basis $e_i\wedge e_j$, $i<j$, of $\wedge^2 V$, (e_i) being a basis of V. The coordinates p^{ij} are Plücker coordinates.

The image of Plücker embedding can be characterized as the space of of decomposable forms in $\wedge^2 V$; it is an algebraic variety described by a system of homogeneous quadratic relations between Plücker coordinates: $p^{ij}p^{kh}-p^{ik}p^{jh}+p^{ih}p^{jk}=0$, where $i<j<k<h$. The system is empty if $n=2$, consists of one quadric only if $n=3$, 5 quadrics if $n=4$, etc.

A single, additional quadratic equation $X^{\mathrm{T}}\mathcal{Q}X=0$, where $X=(p^{ij})$ and $\mathcal{Q}$ is a symmetric matrix of order $\dim\wedge^2 V=\binom{n+1}{2}$, together with the equations that define Plücker variety is a quadratic line complex.

The lines of the quadratic line complex passing through a single point x in the projective space form a quadratic cone. This x-dependent family of cones endows the projective space with a conformal structure, the Monge metric. The Monge metric is obtained by considering lines through two infinitesimally close points P, with coordinates $[v^1,\ldots,v^{n+1}]$, and $P+\mathrm{d}P$, with coordinates $[v^1+\mathrm{d}v^1,\ldots,v^{n+1}+\mathrm{d}v^{n+1}]$. Then, the Plücker coordinates are the minors $p^{ij}=v^i\mathrm{d}v^j-v^j\mathrm{d}v^i$, with $i,j=1,\ldots,n+1$, $i<j$, of the matrix

$$\begin{pmatrix} v^1 & \cdots & v^{n+1}\\ v^1+\mathrm{d}v^1 & \cdots & v^{n+1}+\mathrm{d}v^{n+1}\end{pmatrix}. \tag{4.5}$$

In affine coordinates, upon substituting $v^{n+1}=1$, $\mathrm{d}v^{n+1}=0$, the Monge metric is a quadratic form with respect to the one-forms

$$u^i\mathrm{d}u^j-u^j\mathrm{d}u^i,\quad i<j,\qquad \mathrm{d}u^i \tag{4.6}$$

(modulo Plücker variety); its coefficients are quadratic polynomials (but such a condition is not enough to characterize Monge metrics). The above geometric construction has been exploited by many geometers in the past, like A. Clebsch, S. Lie and C. Segre, but has been forgotten until recently (see [14, 15] and references therein, and the history paper [33]).

From the above discussion, it is easy to generate an ansatz for a first-order operator P_1 that is compatible with a constant-coefficient second-order operator R_2, using the formula (3.35) and a generic Monge metric $\bar{g}_{ij}$.

We remark that also R_2 defines a projective variety in the same space as the above quadratic line complex, according with the identification in [36]. More precisely, the two-form $\eta_{ij}\mathrm{d}u^i\wedge\mathrm{d}u^j$ can be made into a three-form $\eta_{ij\,n+1}\mathrm{d}v^i\wedge\mathrm{d}v^j\wedge\mathrm{d}v^{n+1}$, where $\eta_{ij\,n+1}=\eta_{ij}$, and this yields an algebraic variety in $\mathbb{P}(\wedge^2 V)$ defined by the equations $\eta_{ijk}p^{jk}=0$ and Plücker's variety equations (here η_{ijk} is obtained from $\eta_{ij\,n+1}=\eta_{ij}$ by skew-symmetrization). Such a variety is a *linear line congruence*. We will discuss it in the case $n=4$.

4.1. Case $n=2$: classification

Theorem 4.1. *Let R_2 and P_1 be Hamiltonian operators of the following shape:*

$$R_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad P_1^{ij}=g^{ij}\partial_x+\Gamma^{ij}_k u^k_x+w^i_k u^k_x\partial_x^{-1}u^j_x+u^i_x\partial_x^{-1}w^j_h u^h_x. \tag{4.7}$$

Then, the following conditions are equivalent:

- $[R_2,P_1]=0$;

- *The local part of P_1 is determined by an arbitrary non-degenerate Monge metric $(\bar{g}_{ab})$ through the formula (3.35). More explicitely, we have*

$$\begin{aligned} g^{11} &= c_0(u^1)^2 + c_1u^1 + c_2, \\ g^{12} &= c_0u^1u^2 + \frac{1}{2}c_3u^1 + \frac{1}{2}c_1u^2 + c_5, \\ g^{22} &= c_0(u^2)^2 + c_3u^2 + c_4, \end{aligned} \tag{4.8}$$

where c_0, c_1, c_2, c_3, c_4, c_5 are arbitrary parameters. The nonlocal part of P_1 is given by $(w^i_j) = -1/2c_0\,\mathrm{Id}$, hence the operator is of Mokhov–Ferapontov type [12] and has the form

$$P_1^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x - c_0 u^i_x \partial_x^{-1} u^j_x. \tag{4.9}$$

Proof. The unknown metric g^{ij} can be reconstructed from a Monge metric using (3.35). In this way, (3.3) will be solved by construction. Then, a simple calculation proves that the equations (3.2) and (3.4) are verified.

From (3.1) we easily obtain $w^1_2 = w^2_1 = 0$ and $w^1_1 = w^2_2$. If we use such conditions, all other equations are identically verified, with the exception of (3.5) that yields the equation

$$c_0 = -2w^1_1 = -2w^2_2. \tag{4.10}$$

Concerning the Hamiltonian conditions on P_1, we see that (2.7) is verified by the contravariant metric (4.8) and $w^i_j = -1/2c_0\delta^i_j$. Moreover, it is easy to calculate that the only nonzero component of the curvature (g_{ij}) is $R^{12}_{12} = -c_0$; using the condition (4.10), we immediately see that (2.8) is verified.

The Monge metric of the operator P_1 is

$$\begin{aligned} \bar{g}_{11} &= c_0(u^2)^2 + c_3u^2 + c_4, \\ \bar{g}_{12} &= -c_0u^1u^2 - \frac{1}{2}c_3u^1 - \frac{1}{2}c_1u^2 + c_5, \\ \bar{g}_{22} &= c_0(u^1)^2 + c_1u^1 + c_2. \end{aligned}$$

It is easy to prove that the (symmetric) matrix $\mathcal{Q}$ of the corresponding quadratic line complex is generic (up to the non-degeneracy requirement): if we fix Lie's form of Plücker's coordinates

$$X^{\mathrm{T}} = \left(u^1\mathrm{d}u^2 - u^2\mathrm{d}u^1, \mathrm{d}u^1, \mathrm{d}u^2\right) \tag{4.11}$$

the Monge metric $\bar{g}$ is the quadratic expression $\bar{g} = X^{\mathrm{T}}\mathcal{Q}X$ where

$$\mathcal{Q} = \begin{pmatrix} c_0 & -\frac{1}{2}c_3 & \frac{1}{2}c_1 \\ -\frac{1}{2}c_3 & c_4 & c_5 \\ \frac{1}{2}c_1 & c_5 & c_2 \end{pmatrix} \tag{4.12}$$

This is a generic conic in $\mathbb{P}(V)$ (up to the non-degeneracy requirement on (g^{ij})).

□

Corollary 4.2. *The Hamiltonian operator P_1 is local if and only if $c_0 = 0$; in this case, the operator coincides with the class that has been found in [25].*

Note that locality is not preserved by projective reciprocal transformations.

We are ready to state the Projective Correspondence Theorem.

Theorem 4.3 (Projective Correspondence Theorem). *Let $n = 2$. Then, a trio of mutually compatible Hamiltonian operators P_1, Q_1, R_2 of the form (4.3) is equivalently given by any two conics $\mathcal{C}_1$, $\mathcal{C}_2$ in $\mathbb{P}(V)$, each of rank at least 2.*

Proof. We observe that the action of projective reciprocal transformations on R_2 yields R_2 multiplied by the determinant of the projective transformation, so R_2 is invariant under the action of $SL(V)$.

Then, the action of $SL(V)$ on V induces an action on $\wedge^2 V$ that, in the case $n = 2$, is bijective on $SL(\wedge^2 V)$. This means that conics in $\mathbb{P}(\wedge^2 V)$ can be classified by their rank (provided we regard V as a complex vector space).

The rank of the quadratic line complex corresponding to a non-degenerate Monge metric must be at least 2; a rank 1 quadratic line complex yields a degenerate Monge metric.

Finally, we observe that any pencil $P_1 + \lambda Q_1$ of operators of the type (4.8) is of operators of the same type, due to the linearity of the coefficients. That implies that any two operators whose metric is defined by (4.8) are compatible. □

Remark 4.4. When $n = 2$ the Plücker variety is empty. Moreover, it is immediate to prove that the algebraic variety defined by R_2, a linear line complex, degenerates to 0. So, no other algebraic variety else than the two conics of the above statements come into play when $n = 2$.

A first projective classification of bi-Hamiltonian trios of the shape of theorem 4.1 can be made in the following way.

Proposition 4.5. *With respect to the action of projective reciprocal transformations, there are two inequivalent classes of trios R_2, P_1, Q_1 that are mutually compatible and of the type (4.7). They are described by*

1. *R_2, $P_{1,2}$, Q_1, where the quadratic line complex corresponding to $P_{1,2}$ has rank 2 and Q_1 is arbitrary, and*
2. *R_2, $P_{1,3}$, Q_1, where the quadratic line complex corresponding to $P_{1,3}$ has rank 3 and Q_1 is arbitrary.*

The classification is far from being complete; indeed, finding the invariants of a pair of quadratic forms is a well-known problem. Let us consider the pencil of conics $\mathcal{C}_1 - \lambda\mathcal{C}_2$ in $\mathbb{P}^2(\mathbb{C})$. The group $\mathrm{SL}(3,\mathbb{C})$ acts on the pencil in a natural way. The characteristic polynomial of the pencil $\det(\mathcal{C}_1 - \lambda\mathcal{C}_2)$ is multiplied by a constant after the action of a group element, hence its roots are invariants of the pencil.

Proposition 4.6. *Let $\mathrm{rk}(\mathcal{C}_1) = 3$, and assume that the three roots of the characteristic polynomial of the pencil $\det(\mathcal{C}_1 - \lambda\mathcal{C}_2)$ are distinct; denote them by λ_i, $i = 1, 2, 3$. Then, there exists a basis of $\mathbb{C}^3$ such that $\mathcal{C}_1 = \mathrm{Id}$ and $\mathcal{C}_2 = \mathrm{diag}(\lambda_i)$.*

Proof. There exists a basis in which the pencil can be rewritten as $\mathrm{Id} - \lambda\mathcal{C}_2$. The group of stabilizers of Id is $\mathrm{SO}(3,\mathbb{C})$. It is easy to prove that the characteristic vectors are independent: indeed, they are eigenvectors of $\mathcal{C}_1^{-1}\mathcal{C}_2$. Such vectors provide the basis in which the canonical form of the statement is achieved. □

Historically, in the case when one of $\mathcal{C}_1$, $\mathcal{C}_2$ is non-degenerate the problem was solved by Weierstrass [37, 38], while in the degenerate case a solution was provided by Kronecker [20] and Dickson [7]. See [35] for a modern treatment of the problem and related references.

4.2. Case $n = 2$: examples

We will consider, as the simplest example in $n = 2$ components, the Poisson pencil of the Kaup–Broer system (first obtained in [21]). The trio is defined by

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2\partial_x & \partial_x u^1 - \partial_x^2 \\ u^1\partial_x + \partial_x^2 & u^2\partial_x + \partial_x u^2 \end{pmatrix}, \tag{4.13}$$

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_x^2. \tag{4.14}$$

The first-order operators have the leading coefficient matrices

$$\left(g_1^{ij}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \left(g_2^{ij}\right) = \begin{pmatrix} 2 & u^1 \\ u^1 & 2u^2 \end{pmatrix}. \tag{4.15}$$

The corresponding Monge metrics are

$$(\bar{g}_{1,ab}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{4.16}$$

and

$$(\bar{g}_{2,ab}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 2 & u^1 \\ u^1 & 2u^2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2u^2 & -u^1 \\ -u^1 & 2 \end{pmatrix}. \tag{4.17}$$

We recall that Plücker's coordinates in Monge form are

$$u^1\mathrm{d}u^2 - u^2\mathrm{d}u^1, \quad \mathrm{d}u^1, \quad \mathrm{d}u^2. \tag{4.18}$$

With respect to the above coordinates, the matrices of the quadratic line complexes take the form

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{4.19}$$

Indeed, it is easy to realize that

$$\bar{g}_{2,ab}\mathrm{d}u^a\mathrm{d}u^b = -2\left(u^1\mathrm{d}u^2 - u^2\mathrm{d}u^1\right)\mathrm{d}u^1 + 2\mathrm{d}u^2\mathrm{d}u^2, \tag{4.20}$$

and similarly for the other Monge metric. We observe that Plücker variety is empty for the Plücker embedding of $\mathbb{P}^2$, so the above quadratic forms provide the only defining equations for the corresponding quadratic line complexes.

Remark 4.7. Note that $rk(Q_1) = 2$ and $rk(Q_2) = 3$. That means that, while Q_1 defines a third-order homogeneous Hamiltonian operator according with [14], Q_2 *does not* define a local third-order HHO (but see [4], as it could be nonlocal!).

Remark 4.8. Another remarkable example is the AKNS Hamiltonian trio (see [25] and references therein); we will not calculate the corresponding quadratic line complexes here as they can be found as in the above Example; however, both first-order operators are defined by a Monge metric whose matrix Q has rank 2: Q_1 is the same as in the previous example and the other is

$$Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.21}$$

4.3. Case $n = 4$: classification

When $n = 4$, we have been able to find a complete solution of the problem. We used the following algorithm.

First of all, we fix a second-order operator, for example

$$R_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \partial_x^2. \tag{4.22}$$

We observe that the corresponding 3-form is

$$\eta = -2\mathrm{d}v^1 \wedge \mathrm{d}v^4 \wedge \mathrm{d}v^5 - 2\mathrm{d}v^2 \wedge \mathrm{d}v^3 \wedge \mathrm{d}v^5.$$

The equations of the corresponding linear line congruence are $\eta_{ijk}p^{jk} = 0$, which translate into the system

$$p^{14} + p^{23} = 0,\ p^{i5} = 0,\ i < 5, \tag{4.23}$$

which yield a linear line congruence upon intersection with $\mathrm{Gr}(2, V)$.

Reciprocal projective transformations act non-trivially on R_2, but we will know all canonical forms of the trios if we compute P_1 as the nonlocal and localizable first-order homogeneous operators that are compatible with R_2: $[R_2, P_2] = 0$. In this subspace of operators we must then compute all pairs P_1, Q_1 of operators that are compatible: $[P_1, Q_1] = 0$ to form trios (see (4.3) and thereafter).

We brought to an end the first part of the above programme: we computed all P_1 that are compatible with R_2 and have the above form. The results are available at the link [28] and also in the article's supplementary information. The calculation was nontrivial and was performed on a compute server of the Istituto Nazionale di Fisica Nucleare (INFN—Italian National Institute of Nuclear Physics), using Reduce [19, 32] and about 64GB of RAM for 1 h.

It is worth to describe the algorithm that we used.

1. First of all, since we know that the metric of the first-order operator is a Monge metric, we calculate the most general Monge metric in the case $n = 4$. It is parametrized by a finite number of constants.
2. We also know that w^i_j are constants, and we use this information in the setup of the computation.
3. Christoffel symbols Γ^{ij}_k are determined by the formula (3.30) in terms of w^i_j and of new unknown constants b^{ij}_k. Summarizing, the unknowns are constants, and are: the coefficients in the Monge metric, the coefficients in the 'tail' w^i_j and the coefficients b^{ij}_k that make up Γ^{ij}_k.
4. Then, compatibility equations are solved. There are 2 groups of linear equations in the above unknowns: (3.1) and (3.2). The conditions (3.3) and (3.5) are automatically satisfied. The nonlinear condition is the associativity condition (3.4).
   The Hamiltonian operator conditions on P_1 are (2.5) (which is linear with respect to the unknowns), (2.4), (2.6) and (2.7) (which are nonlinear). Note that the equations (2.8) are automatically fulfilled.
5. The overdetermined system solver CRACK [39, 40], a package working in Reduce, was used to solve the above nonlinear algebraic equations. The solution obtained in this way is too involved to be printed out here, since it consists of 288 subcases. The full list can be found in a compressed folder available at the link [28] and also in the article's supplementary information.

6. It is excessively complicated to write down all solutions of the compatibility conditions from $[P_1, Q_1] = 0$, where P_1 and Q_1 are two solutions of the above equations (think of each of the 288 subcases to be used in a compatibility computation with another operator from each of the 288 subcases). However, the solutions are computable in reasonable time with modern computers, see below.

We observe that the results obtained are not exactly a classification of the trios with the given R_2; indeed, the reciprocal transformations act on R_2 with a stabilizer, that might be used to reduce the number of constants in P_1. At the moment, we do not consider this problem.

4.4. Case $n = 4$: a subclass

In view of the complexity of the compatibility calculation of the operators in the full set of solutions of $[R_2, P_1] = 0$, we can present here the results for a subset of all possible trios: namely, those that are a direct generalization of the Kaup–Broer and the AKNS trios in section 4.2.

Indeed, we can observe that in those examples P_1 has always constant form (in particular, its matrix is the 'antidiagonal identity'). We can therefore postulate the form of P_1 (besides the form of R_2) as

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \partial_x \tag{4.24}$$

and then find, in the set of solutions of $[R_2, Q_1] = 0$ with the given ansatz of Q_1, those that are compatible with P_1: $[P_1, Q_1] = 0$.

We obtain 64 cases of first-order operators Q_1 as above. The computation is shorter than that of $[R_2, Q_2] = 0$ only, and can be done on a modern laptop. It makes use of the packages developed in [5] in order to calculate the conditions $[P_1, Q_1] = 0$. Here, we will just show two cases, one is local and the other is nonlocal.

4.4.1. Local case.

The metric of the first-order operator is

$$\left(g^{ij}\right) = \begin{pmatrix} 2b_2^{11}u^2 + c_{55} & c_{54} & b_2^{11}u^4 + b_1^{13}u^1 - c_{49} & b_1^{13}u^2 - c_{34} \\ c_{54} & 0 & b_1^{13}u^2 - c_{34} & 0 \\ b_2^{11}u^4 + b_1^{13}u^1 - c_{49} & b_1^{13}u^2 - c_{34} & 2b_1^{13}u^3 + c_{46} & 2b_1^{13}u^4 + c_{31} \\ b_1^{13}u^2 - c_{34} & 0 & 2b_1^{13}u^4 + c_{31} & 0 \end{pmatrix} \tag{4.25}$$

The free parameters are

$$b_2^{11}, b_1^{13}, c_{31}, c_{34}, c_{46}, c_{49}, c_{54}, c_{55}. \tag{4.26}$$

Nonzero coefficients in the Christoffel symbols are determined by the only nonzero constants b_k^{ij}, which are

$$b_2^{14} = b_1^{13}, \quad b_2^{23} = b_1^{13}, \quad b_4^{31} = b_2^{11},$$
$$b_3^{33} = b_1^{13}, \quad b_4^{34} = b_1^{13}, \quad b_4^{43} = b_1^{13}.$$

It turns out that nonzero Christoffel symbols (in upper indices) are

$$\Gamma_2^{11} = b_2^{11}, \quad \Gamma_1^{13} = b_1^{13}, \quad \Gamma_2^{14} = b_1^{13}, \quad \Gamma_2^{23} = b_1^{13},$$
$$\Gamma_4^{31} = b_2^{11}, \quad \Gamma_3^{33} = b_1^{13}, \quad \Gamma_4^{34} = b_1^{13}, \quad \Gamma_4^{43} = b_1^{13}.$$

4.4.2. Nonlocal case. The metric of the first-order operator is

$$
(g^{ij}) = \begin{pmatrix}
0 & c_{54}-(u^1)^2w_1^2 & 0 & -(c_{34}+u^1u^3w_1^2) \\
c_{54}-(u^1)^2w_1^2 & 2b_1^{22}u^1+c_{53}-2u^1u^2w_1^2 & -(c_{34}+u^1u^3w_1^2) & b_1^{22}u^3-c_{33}-u^1u^4w_1^2-u^2u^3w_1^2 \\
0 & -(c_{34}+u^1u^3w_1^2) & 0 & c_{31}-(u^3)^2w_1^2 \\
-(c_{34}+u^1u^3w_1^2) & b_1^{22}u^3-c_{33}-u^1u^4w_1^2-u^2u^3w_1^2 & c_{31}-(u^3)^2w_1^2 & c_{28}-2u^3u^4w_1^2
\end{pmatrix} \tag{4.27}
$$

The nonlocal part is defined by the free parameter w_1^2 (with the requirement $w_1^2 \neq 0$) and the equations

$$w_3^4 = w_1^2, \qquad w_j^i = 0 \quad \text{otherwise.} \tag{4.28}$$

The free parameters are

$$b_1^{22}, w_{21}, c_{28}, c_{31}, c_{33}, c_{34}, c_{53}, c_{54} \tag{4.29}$$

The only nonzero constants b_k^{ij} are

$$b_1^{22}, \quad b_3^{42} = b_1^{22}. \tag{4.30}$$

The nonzero Christoffel symbols are

$$
\begin{aligned}
&\Gamma_1^{12} = -u^1w_1^2, \quad \Gamma_1^{14} = -u^3w_1^2, \quad \Gamma_1^{21} = -u^1w_1^2, \quad \Gamma_1^{22} = b_1^{22} - u^2w_1^2, \\
&\Gamma_2^{22} = -u^1w_1^2, \quad \Gamma_1^{23} = -u^3w_1^2 \quad \Gamma_1^{24} = -u^4w_1^2, \quad \Gamma_2^{24} = -u^3w_1^2, \\
&\Gamma_3^{32} = -u^1w_1^2, \quad \Gamma_3^{34} = -u^3w_1^2, \quad \Gamma_3^{41} = -u^1w_1^2, \quad \Gamma_3^{42} = b_1^{22} - u^2w_1^2 \\
&\Gamma_4^{42} = -u^1w_1^2, \quad \Gamma_3^{43} = -u^3w_1^2, \quad \Gamma_3^{44} = -u^4w_1^2, \quad \Gamma_4^{44} = -u^3w_1^2.
\end{aligned}
$$

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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