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Onesided Korovkin approximation

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Abstract

In this paper we study in detail some characterizations of Korovkin closures and we also introduce the notions of onesided upper and lower Korovkin closures. We provide some complete characterizations of these new closures which separate the roles of approximating functions in a Korovkin system. We also present some new characterizations of the classical Korovkin closure in spaces of integrable functions. Again we can introduce and characterize the upper and lower Korovkin closures. Finally, we provide some examples which justify the interest in these new closures.

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1. Introduction and notation

The aim of this paper is to analyze in detail some characterizations of Korovkin closures and to highlight the different roles of the functions involved in these characterizations.

This is an old problem in Korovkin approximation theory. Indeed, the classical Korovkin closure was first introduced in [3] and was studied in [4,5] (see also [2, Notes and References to Section 4.1,p. 209] and [12, Chapter 6]).

More recently, the Korovkin closure has been also studied in Banach algebras [1] and Lindenstrauss spaces [11].

However, until now only necessary and sufficient conditions have been obtained which ensure that the Korovkin closure coincides with the whole space. Here, we provide some

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characterizations which make it possible to determine exactly the subspace where the convergence of a net of operators can be assured even if this subspace does not coincide with the whole space. Examples showing the interest of these characterizations are also presented. In Proposition 2.4 and Corollary 2.5 we shall state some applications of particular interest.

Our analysis conducts us to introduce the notion of onesided Korovkin approximation which allows us to obtain some interesting consequences.

The introduction of onesided Korovkin approximation takes its origin from [9] where in the context of set-valued functions it is shown the possibility of studying the convergence of equicontinuous nets of linear monotone operators on set-valued functions having a convex graph using only affine set-valued functions (see also [6–8,10]).

We consider different cases of interest both in spaces of continuous than in spaces of integrable functions.

First, we recall the classical definition of Korovkin closure with respect to monotone (respectively, linear positive, linear contractive) operators for the identity operator.

Let X be a compact Hausdorff topological space and let $H \subset C(X,\mathbb{R})$.

The *Korovkin closure* $K(H)$ of H with respect to monotone (respectively, linear positive, linear contractive) operators for the identity operator is defined as follows

$$\begin{aligned} K(H) := \{ f \in C(X,\mathbb{R}) \mid\ & \lim_{i\in I^{\leq}} L_i(f) = f \textit{ for every equicontinuous net} \\ & (L_i)_{i\in I}^{\leq} \textit{ of monotone (respectively, linear positive,} \\ & \textit{linear contractive) operators from } C(X,\mathbb{R}) \textit{ into} \\ & \textit{itself satisfying } \lim_{i\in I^{\leq}} L_i(h) = h \textit{ uniformly on } X \\ & \textit{for every } h \in H \}\,. \end{aligned}$$

Moreover H is said to be a *Korovkin system* with respect to monotone (respectively, linear positive, linear contractive) operators for the identity operator if $K(H) = C(X,\mathbb{R})$.

Our analysis is inspired by a characterization of the Korovkin closure given in condition (v) in [2, Theorem 4.1.4, p. 199] (see the Notes to Section 4.1.4 of [2] for complete references), which states that $f \in K(H)$ if and only if

(KS) For every $x_0 \in X$ and $\varepsilon > 0$, there exist $h, k \in H$ such that

$$h \leq f \leq k\,, \qquad k(x_0) - \varepsilon < f(x_0) < h(x_0) + \varepsilon,$$

or equivalently, taking into account that X is compact,

(KS)$_1$ For every $\varepsilon > 0$, there exist $h_1, \dots, h_m, k_1, \dots, k_m \in H$ such that

$$h_j \leq f \leq k_j\,, \quad j = 1,\dots,m\,, \qquad \inf_{j=1,\dots,m} k_j - \varepsilon < f < \sup_{j=1,\dots,m} h_j + \varepsilon.$$

The concepts of upper and lower Korovkin closures introduced in the next section are obtained by considering separately the roles of the functions h_j and k_j, $j = 1,\dots,m$, involved in the above characterizations.

We shall consider these new closures both in spaces of continuous real functions than in L^p-spaces.

2. Onesided Korovkin approximation in spaces of continuous functions

We start with the following main definition.

Definition 2.1. Let X be a compact Hausdorff topological space and let $H \subset C(X, \mathbb{R})$.

Then, the *upper Korovkin closure* (respectively, the *lower Korovkin closure*) of H with respect to monotone operators for the identity operator is the subset $K(H)^+$ (respectively, $K(H)^-$) defined as follows

$$K(H)^+ := \{f \in C(X, \mathbb{R}) \mid \limsup_{i \in I^{\le}} L_i(f) \le f \textit{ for every equicontinuous net } (L_i)^{\le}_{i \in I} \textit{ of monotone operators from } C(X, \mathbb{R}) \textit{ into itself satisfying } \lim_{i \in I^{\le}} L_i(h) = h \textit{ uniformly on } X \textit{ for every } h \in H\}.$$

(respectively,

$$K(H)^- := \{f \in C(X, \mathbb{R}) \mid f \le \liminf_{i \in I^{\le}} L_i(f) \textit{ for every equicontinuous net } (L_i)^{\le}_{i \in I} \textit{ of monotone operators from } C(X, \mathbb{R}) \textit{ into itself satisfying } \lim_{i \in I^{\le}} L_i(h) = h \textit{ uniformly on } X \textit{ for every } h \in H\} .)$$

Moreover, H is said to be an *upper Korovkin system* (respectively, a *lower Korovkin system*) with respect to monotone operators for the identity operator if $K(H)^+ = C(X, \mathbb{R})$ (respectively, $K(H)^- = C(X, \mathbb{R})$).

The above definition is justified by the equality $K(H) = K(H)^+ \cap K(H)^-$.

Indeed, if $f \in K(H)^+ \cap K(H)^-$ and if $(L_i)^{\le}_{i \in I}$ is an equicontinuous net of monotone operators from $C(X, \mathbb{R})$ into itself satisfying $\lim_{i \in I^{\le}} L_i(h) = h$ uniformly on X for every $h \in H$, we have

$$f \le \liminf_{i \in I^{\le}} L_i(f) \le \limsup_{i \in I^{\le}} L_i(f) \le f$$

and this ensures that the limit $\lim_{i \in I^{\le}} L_i(f)$ exists and is equal to f uniformly on X. The converse inclusion is trivial.

In general, if f is only in the upper (or lower) Korovkin closure, we may not expect that $\lim_{i \in I^{\le}} L_i(f) = f$.

However, distinguishing between the upper and lower Korovkin closures, we have the possibility to construct some new interesting examples of Korovkin closures, as we shall see in the sequel.

We explicitly observe that the onesided definition is meaningful only with respect to equicontinuous net of monotone operators. Otherwise, a net of linear operators which converges on $H \subset C(X, \mathbb{R})$ converges also on $-H := \{-h \mid h \in H\}$ and therefore $K(H)^+ = K(H)^-$.

Since $K(H)^+$ and $K(H)^-$ are not in general subspaces of $C(X, \mathbb{R})$, it may be also useful to introduce the following notion.

We define a subset H of $C(X, \mathbb{R})$ to be *upward cofinal* (respectively, *downward cofinal*) in $C(X, \mathbb{R})$ if, for every $f \in C(X, \mathbb{R})$, there exists $k \in H$ such that $f \le k$ (respectively, for every $f \in C(X, \mathbb{R})$, there exists $h \in H$ such that $h \le f$).

At this point, we can state the following main characterization.

For the sake of brevity, we shall explicitly state only the results for the upper Korovkin closure. With the appropriate changes, similar results can be also stated for the lower Korovkin closure. In general, in this case, functions $k \in H$ satisfying $f \le k$ and $k(x) - \varepsilon < f(x)$ at some $x \in X$ should be replaced with functions $h \in H$ such that $h \le f$ and $f(x) < h(x) + \varepsilon$.

Theorem 2.2. *Let X be a compact Hausdorff topological space and let H be an upward cofinal subset of $C(X,\mathbb{R})$.*

If $f \in C(X,\mathbb{R})$, the following statements are equivalent:

(a) $f \in K(H)^+$;

(b) For every $x_0 \in X$ and $\varepsilon > 0$, there exists $k \in H$ such that

$$f \le k\,, \qquad k(x_0) - \varepsilon < f(x_0).$$

(c) For every $\varepsilon > 0$, there exist $k_1, \dots, k_m \in H$ such that

$$f \le k_j\,, \quad j = 1, \dots, m\,, \qquad \inf_{j=1,\dots,m} k_j - \varepsilon < f.$$

Proof. (a) $\Rightarrow$ (b) Assume that condition (a) holds and by contradiction that there exist $x_0 \in X$ and $\varepsilon_0 > 0$ such that $k(x_0) \ge f(x_0) + \varepsilon$ for every $k \in H$ satisfying $f \le k$. Consider the operator $L : C(X,\mathbb{R}) \to C(X,\mathbb{R})$ defined by setting, for every $g \in C(X,\mathbb{R})$ and $x \in X$,

$$L(g)(x) = \inf_{k \in H, g \le k} k(x)\,.$$

Since H is upward cofinal, the operator L is well-defined and it is obviously monotone. Moreover, $L(h) = h$ for every $h \in H$, but

$$L(f)(x_0) = \inf_{k \in H, f \le k} k(x) \ge f(x_0) + \varepsilon_0\,.$$

Hence, the net $(L_i)^{\le}_{i \in I}$ defined by $L_i = L$ for every $i \in I$ cannot satisfy the condition $\limsup_{i \in I^{\le}} L_i(f)(x_0) \le f(x_0)$ and this contradicts condition (a).

(b) $\Rightarrow$ (c) It follows using a straightforward argument based on the compactness of X.

(c) $\Rightarrow$ (a) Let $(L_i)^{\le}_{i \in I}$ be an equicontinuous net of monotone operators from $C(X,\mathbb{R})$ into itself satisfying

$$\lim_{i \in I^{\le}} L_i(h) = h$$

uniformly on X for every $h \in H$.

Let $\varepsilon > 0$ and from (c) consider $k_1, \dots, k_m \in H$ such that

$$f \le k_j\,, \quad j = 1, \dots, m\,, \qquad \inf_{j=1,\dots,m} k_j - \frac{\varepsilon}{2} < f\,. \tag{2.1}$$

Since every L_i is monotone, we have, for every $i \in I$,

$$L_i(f) \le L_i(k_j)\,, \quad j = 1, \dots, m.$$

Moreover, the net $(L_i(k_j))^{\le}_{i \in I}$ converges to k_j for every $j = 1, \dots, m$ and therefore we can find $\alpha \in I$ such that, for every $i \in I$, $i \ge \alpha$, and $j = 1, \dots, m$,

$$k_j \le L_i(k_j) + \frac{\varepsilon}{2}\,, \qquad L_i(k_j) \le k_j + \frac{\varepsilon}{2}\,. \tag{2.2}$$

From (2.1) and the second inequality in (2.2) we obtain, for every $i \ge \alpha$,

$$L_i(f) \le \inf_{j=1,\dots,m} L_i(k_j) \le \inf_{j=1,\dots,m} k_j + \frac{\varepsilon}{2} < f + \varepsilon$$

and consequently

$$\sup_{i \ge \alpha} L_i(f) \le f + \varepsilon$$

which finally yields

$$\inf_{\alpha\in I}\sup_{i\geq\alpha} L_i(f) \leq f+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\limsup_{i\in I^{\leq}} L_i(f) \leq f$. ■

Now, we give some examples of upper and lower Korovkin systems which can be obtained from classical well-known properties.

Proposition 2.3. *Let (X, σ) be a compact metric space and, for every $\lambda, \mu \in \mathbb{R}$ and $x_0 \in X$, consider the functions $r_{x_0,\lambda,\mu}, s_{x_0,\lambda,\mu} : X \to \mathbb{R}$ defined by setting*

$$r_{x_0,\lambda,\mu}(x) := \lambda + \mu\sigma(x, x_0)\ , \quad s_{x_0,\lambda,\mu}(x) := \lambda + \mu\sigma(x, x_0)^2\ , \qquad x \in X.$$

Then, the following subsets of $C(X, \mathbb{R})$:

$$H_1^+(X) := \{r_{x_0,\lambda,\mu} \mid x_0 \in X, \lambda \in \mathbb{R}, \mu \geq 0, \}$$

$$H_2^+(X) := \{s_{x_0,\lambda,\mu} \mid x_0 \in X, \lambda \in \mathbb{R}, \mu \geq 0, \}$$

are upper Korovkin systems in $C(X, \mathbb{R})$, while the subsets

$$H_1^-(X) := \{r_{x_0,\lambda,\mu} \mid x_0 \in X, \lambda \in \mathbb{R}, \mu \leq 0, \}$$

$$H_2^-(X) := \{s_{x_0,\lambda,\mu} \mid x_0 \in X, \lambda \in \mathbb{R}, \mu \leq 0, \}$$

are lower Korovkin systems in $C(X, \mathbb{R})$.

Proof. We show the property only for $H_1^+(X)$ since the same arguments can be applied in all other cases.

Let $f \in C(X, \mathbb{R})$, $x_0 \in X$ and $\varepsilon > 0$. Let $M > 0$ be such that $|f(x) - f(x_0)| \leq M$ for every $x \in X$ and let $\delta > 0$ be such that $|f(x) - f(x_0)| \leq \varepsilon/2$ whenever $x \in X$ satisfies $\sigma(x, x_0) \leq \delta$. Now consider the function $r_{x_0,\lambda,\mu}$ with $\lambda := f(x_0)+\varepsilon/2$ and $\mu := M/\delta$. Then $r_{x_0,\lambda,\mu} \in H_1^+(X)$.

We show that $f \leq r_{x_0,\lambda,\mu}$. Indeed, if $x \in X$ and $\sigma(x, x_0) \leq \delta$ we have $f(x) \leq f(x_0)+\varepsilon/2 \leq r_{x_0,\lambda,\mu}(x)$. If $\sigma(x_0, x) > \delta$ we have $f(x) \leq f(x_0) + M \leq r_{x_0,\lambda,\mu}(x)$. Therefore $f \leq r_{x_0,\lambda,\mu}(x)$.

Finally, we obviously have $r_{x_0,\lambda,\mu}(x_0) = f(x_0) + \varepsilon/2 < f(x_0) + \varepsilon$.

Therefore condition (b) of Theorem 2.2 is satisfied and from Theorem 2.2, (a), the proof is complete. ■

Observe that all the subsets $H_1^+(X)$, $H_2^+(X)$, $H_1^-(X)$ and $H_2^-(X)$ are both upward and downward cofinal subsets of $C(X, \mathbb{R})$ since each one of them contains the constant functions.

As a particular case, we can consider $X = [a, b]$. In this case the functions $r_{x_0,\lambda,\mu}, s_{x_0,\lambda,\mu} : [a, b] \to \mathbb{R}$ are defined by setting

$$r_{x_0,\lambda,\mu}(x) := \lambda + \mu|x - x_0|\ , \quad s_{x_0,\lambda,\mu}(x) := \lambda + \mu(x - x_0)^2\ , \qquad x \in [a, b]$$

and the corresponding subsets $H_1^+([a, b])$ and $H_2^+([a, b])$ of $C([a, b], \mathbb{R})$ are upper Korovkin systems in $C([a, b], \mathbb{R})$, while the subsets $H_1^-([a, b])$ and $H_2^-([a, b])$ are lower Korovkin systems in $C([a, b], \mathbb{R})$.

Now, we can state some properties of convex and concave functions in connections with the closure of the above subsets and the subspace of affine functions.

We recall that if $f : [a, b] \to \mathbb{R}$ is a continuous concave function, there exists $c \in]a, b[$ such that

$$f'_+(c) \le \frac{f(b) - f(a)}{b - a} \le f'_-(c)\,. \tag{2.3}$$

Indeed, the function $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$ is continuous and concave and takes the same value at the endpoints. Hence g attains its maximum at an internal point $c \in]a, b[$, where $g'_-(c) \cdot g'_+(c) \le 0$. The conditions

$$\left(f'_-(c) - \frac{f(b) - f(a)}{b - a}\right) \cdot \left(f'_+(c) - \frac{f(b) - f(a)}{b - a}\right) \le 0\,, \qquad f'_+(c) \le f'_-(c)$$

yield (2.3).

We shall denote by $A([a, b], \mathbb{R})$ the subspace of $C([a, b], \mathbb{R})$ consisting of all affine functions on $[a, b]$, i.e.,

$$A([a, b], \mathbb{R}) := \{h \in C([a, b], \mathbb{R}) \mid \exists\ p, q \in \mathbb{R}\ \forall\ x \in [a, b] : h(x) := px + q\}.$$

Since $A([a, b], \mathbb{R})$ contains the constant functions, it is both an upward and downward cofinal subset of $C([a, b], \mathbb{R})$.

Proposition 2.4. *If $f \in C([a, b], \mathbb{R})$ is concave (respectively, convex) we have $f \in K(A([a, b], \mathbb{R}))^+$ (respectively, $f \in K(A([a, b], \mathbb{R}))^-$).*

Proof. Let $f \in C([a, b], \mathbb{R})$ be a concave function and let $x_0 \in [a, b]$ and $\varepsilon > 0$.

If $x_0 \in]a, b[$, then f has finite left and right derivatives at x_0 and we can consider $p \in [f'_+(x_0), f'_-(x_0)]$ and the function $k(x) := p(x - x_0) + f(x_0)$, $x \in [a, b]$. Then k satisfies condition (b) in Theorem 2.2 and therefore we obtain $f \in K(A([a, b], \mathbb{R}))^+$.

Now, assume that $x_0 = a$. If $f'(a)$ is finite we can consider the tangent $k(x) := f'(a)(x - a) + f(a)$ to the graph of f at a and we have again that k satisfies condition (b) in Theorem 2.2 and therefore $f \in K(A([a, b], \mathbb{R}))^+$.

Finally, assume $f'(a) = +\infty$. Since f is continuous at a and $f'(a) = +\infty$ we can find $\delta > 0$ such that

$$|f(x) - f(a)| \le \varepsilon\,, \qquad \frac{f(x) - f(a)}{x - a} \ge 0$$

for every $x \in [a, a+\delta]$; the first inequality also yields $f(x) \le f(a)+\varepsilon$ for every $x \in [a, a+\delta]$.

From (2.3), there exists $c \in]a, a + \delta[$ such that

$$f'_+(c) \le \frac{f(a + \delta) - f(a)}{\delta} \le f'_-(c)$$

and, in particular, $f'_-(c) \ge 0$. Now, consider the left tangent to the graph of f at the point $(c, f(c))$, i.e. the function $k(x) := f(c) + f'_-(c)(x - c)$, $x \in [a, b]$. Since f is concave we have $f \le k$. Moreover $k(a) = f(c) + f'_-(c)(a - c) \le f(c) \le f(a) + \varepsilon$ and therefore also in this case condition (b) of Theorem 2.2 is satisfied. Consequently $f \in K(A([a, b], \mathbb{R}))^+$ and this completes the proof.

Obviously the same reasoning can be applied to the point b.

Then f satisfies condition (b) in Theorem 2.2 and we can conclude that $f \in K(A([a, b], \mathbb{R}))^+$.

If f is convex, the reasoning is at all similar. ■

At this point, we can state a further consequence of Propositions 2.3 and 2.4.

We denote by $C_{\text{conv}}([a,b],\mathbb{R})$ (respectively, by $C_{\text{conc}}([a,b],\mathbb{R})$) the subset of $C([a,b],\mathbb{R})$ consisting of all continuous convex (respectively, concave) functions on $[a,b]$ and as before by $A([a,b],\mathbb{R}) := C_{\text{conv}}([a,b],\mathbb{R}) \cap C_{\text{conc}}([a,b],\mathbb{R})$ the subspace of all affine functions on $[a,b]$.

Corollary 2.5. *We have*

$$C_{\text{conv}}([a,b],\mathbb{R}) \subset K(A([a,b],\mathbb{R}) \cup H_1^+([a,b])),$$

$$C_{\text{conv}}([a,b],\mathbb{R}) \subset K(A([a,b],\mathbb{R}) \cup H_2^+([a,b]))$$

and further

$$C_{\text{conc}}([a,b],\mathbb{R}) \subset K(A([a,b],\mathbb{R}) \cup H_1^-([a,b])),$$

$$C_{\text{conc}}([a,b],\mathbb{R}) \subset K(A([a,b],\mathbb{R}) \cup H_2^-([a,b])).$$

Proof. We show only the inclusion $C_{\text{conv}}([a,b],\mathbb{R}) \subset K(A([a,b],\mathbb{R}) \cup H_1^+([a,b]))$ since all the other ones are similar.

Let $f \in C_{\text{conv}}([a,b],\mathbb{R})$ and let $x_0 \in [a,b]$ and $\varepsilon > 0$.

From Proposition 2.3, we know that $H_1^+([a,b])$ is an upper Korovkin system in $C([a,b],\mathbb{R})$. Moreover, we have already observed that $H_1^+([a,b])$ is an upward cofinal subset of $C([a,b],\mathbb{R})$. Hence, we can apply Theorem 2.2, (b), and obtain the existence of $k \in H_1^+([a,b]) \subset A([a,b],\mathbb{R}) \cup H_1^+([a,b])$ such that

$$f \le k\,, \qquad k(x_0) - \varepsilon < f(x_0).$$

Moreover, from Proposition 2.4, we have $f \in K(A([a,b],\mathbb{R}))^-$. Since $A([a,b],\mathbb{R})$ is a downward cofinal subset of $C([a,b],\mathbb{R})$, we can apply the analogous of Theorem 2.2, (b), for the lower closure. This yields the existence of $h \in A([a,b],\mathbb{R}) \subset A([a,b],\mathbb{R}) \cup H_1^+([a,b])$ such that

$$h \le f\,, \qquad f(x_0) < h(x_0) + \varepsilon.$$

Since $x_0 \in [a,b]$ and $\varepsilon > 0$ are arbitrarily chosen, from **(KS)** we conclude that $f \in K(A([a,b],\mathbb{R}) \cup H_1^+([a,b]))$. ■

As a consequence, if an equicontinuous net $(L_i)_{i\in I}^{\le}$ of monotone operators from $C([a,b],\mathbb{R})$ into itself satisfies

$$\lim_{i\in I^{\le}} L_i(h) = h$$

uniformly on $[a,b]$ for every affine function $h \in A([a,b],\mathbb{R})$ and for every function $h(x) := \lambda + \mu|x - x_0|$ (or alternatively $h(x) := \lambda + \mu(x - x_0)^2$) with $x_0 \in [a,b]$, $\lambda \in \mathbb{R}$ and $\mu \ge 0$, then it converges to f for every convex function $f \in C_{\text{conv}}([a,b],\mathbb{R})$.

Analogously, if $(L_i)_{i\in I}^{\le}$ converges to h for every affine function $h \in A([a,b],\mathbb{R})$ and for every function $h(x) := \lambda + \mu|x - x_0|$ (or alternatively $h(x) := \lambda + \mu(x - x_0)^2$) with $x_0 \in [a,b]$, $\lambda \in \mathbb{R}$ and $\mu \le 0$, then it converges to f for every concave function $f \in C_{\text{conc}}([a,b],\mathbb{R})$.

3. Onesided Korovkin approximation in L^p-spaces

Convergence properties of suitable sequences of operators in L^p-spaces are often obtained by using the same characterizations which hold in spaces of continuous functions and the universal Korovkin properties (see [2, Section 3.2 and Corollary 4.1.7]).

Our aim is to obtain an independent and general specific characterization of the Korovkin closures in L^p-spaces and also in this case to introduce and study upper and lower Korovkin closures.

First, we fix some notation.

Let X be a compact Hausdorff topological space and μ be a positive Radon measure on X.

It is well-known (see, e.g., [2, Theorem 1.2.4]) that there exists a unique regular finite Borel measure ν on X such that $\mu(f) = \int f\,d\nu$ for every function $f : X \to \mathbb{R}$. To avoid supplementary notation, we shall denote all integrals by $\int f\,d\mu$.

Let $1 \le p \le +\infty$ and consider the space $L^p(X,\mu)$ endowed with the usual norm $\|f\|_p := \left(\int_X |f(x)|^p\right)^{1/p}$ if $1 \le p < +\infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x\in X}|f(x)|$ if $p = +\infty$.

The *Korovkin closure* of a subset $H \subset L^p(X,\mu)$ with respect to equicontinuous nets of monotone operators is defined as follows

$$\begin{aligned} K(H)_p := \{f \in L^p(X,\mu) \mid\ & \lim_{i\in I^{\le}} \|L_i(f) - f\|_p = 0 \textit{ for every equicontinuous net} \\ & (L_i)_{i\in I}^{\le} \textit{ of monotone operators from } L^p(X,\mu) \\ & \textit{into itself satisfying } \lim_{i\in I^{\le}} \|L_i(h) - h\|_p = 0 \\ & \textit{for every } h \in H\}\,. \end{aligned}$$

In this general context it may be interesting to consider also nets of positive linear operators (respectively, of contractive positive linear operators). However, for the sake of brevity, we shall omit the discussion of the Korovkin closures corresponding to these nets of operators.

In L^p-spaces, a subset H of $L^p(X,\mu)$ is said to be *cofinal* in $L^p(X,\mu)$ if, for every $f \in L^p(X,\mu)$, there exist $h,k \in H$ such that $k \le f \le h$ almost everywhere in X.

At this point, we can state the following characterization of $K(H)_p$.

Theorem 3.1. *Let X be a compact Hausdorff topological space, μ be a positive Radon measure on X and $1 \le p \le +\infty$.*

Let H be a cofinal subset of $L^p(X,\mu)$.

If $f \in L^p(X,\mu)$, the following statements are equivalent:

(a) $f \in K(H)_p$.

(b) For almost all $x_0 \in X$ and for every $\varepsilon > 0$, there exist $h,k \in H$ and a neighborhood U of x_0 such that $h \le f \le k$ a.e. and, if $1 \le p < \infty$,

$$\left(\int_K |k(x) - f(x)|^p\,d\mu(x)\right)^{1/p} \le \frac{\mu(K)}{\mu(X)}\,\varepsilon,$$

$$\left(\int_K |f(x) - h(x)|^p\,d\mu(x)\right)^{1/p} \le \frac{\mu(K)}{\mu(X)}\,\varepsilon$$

whenever K is a measurable subset of U, while, if $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in U} |k(x) - f(x)| \le \varepsilon\,, \qquad \operatorname*{ess\,sup}_{x\in U} |f(x) - h(x)| \le \varepsilon.$$

(c) *For almost all* $x_0 \in X$ *and for every* $\varepsilon > 0$*, there exist* $h, k \in H$ *and a neighborhood* U *of* x_0 *such that* $h \le f \le k$ *a.e. and, if* $1 \le p < \infty$,

$$\left(\int_K |k(x) - h(x)|^p \, d\mu(x)\right)^{1/p} \le \frac{\mu(K)}{\mu(X)}\, \varepsilon$$

whenever K *is a measurable subset of* U*, while, if* $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in U} |k(x) - h(x)| \le \varepsilon.$$

(d) *For every* $\varepsilon > 0$*, there exist* $h_1, \ldots, h_m, k_1, \ldots, k_m \in H$ *such that*

$$h_j \le f \le k_j \text{ a.e.}, \qquad j = 1, \ldots, m,$$

and

$$\left\| \inf_{j=1,\ldots,m} k_j - \sup_{j=1,\ldots,m} h_j \right\|_p \le \varepsilon.$$

Proof. $(a) \Rightarrow (b)$ Let $f \in K(H)_p$. We show only the existence of the function $k \in H$ in (b), since the proof of the existence of the function h is at all similar.

By contradiction, assume that there exists a measurable subset $S \subset X$ such that $\mu(S) := s > 0$ and $\varepsilon_0 > 0$ such that, for every $x_0 \in S$, for every $k \in H$ such that $f \le k$ a.e. and for every neighborhood U of x_0, we have, for some measurable subset K of U,

$$\left(\int_K |k(x) - f(x)|^p \, d\mu(x)\right)^{1/p} \ge \frac{\mu(K)}{\mu(X)}\, \varepsilon_0$$

if $1 \le p < \infty$ or

$$\operatorname*{ess\,sup}_{x\in K} |k(x) - f(x)| \ge \frac{\mu(K)}{\mu(X)}\, \varepsilon_0$$

if $p = \infty$.

Consider the operator $L : L^p(X, \mu) \to L^p(X, \mu)$ defined by setting, for every $g \in L^p(X, \mu)$ and $x \in X$,

$$L(g)(x) := g(x) + \inf_{\substack{U \text{ neighborhood of } x\\ k\in H, g\le k \text{ a.e.}}} \sup_{\substack{K\subset U,\\ \mu(K)>0}} \frac{\mu(X)}{\mu(K)} \left(\int_K |k(t) - g(t)|^p \, d\mu(t)\right)^{1/p}$$

if $1 \le p < \infty$ or

$$L(g)(x) := g(x) + \inf_{\substack{U \text{ neighborhood of } x\\ k\in H, g\le k \text{ a.e.}}} \operatorname*{ess\,sup}_{t\in U} |k(t) - g(t)|$$

if $p = \infty$.

Since H is cofinal, L is well-defined and it is obviously monotone. Moreover, for every $h \in H$, we have $L(h) = h$ but

$$\|L(f) - f\|_p \ge \left(\int_S |L(f)(t) - f(t)|^p \, d\mu(t)\right)^{1/p} \ge \varepsilon_0\, s^{1/p}$$

if $1 \le p < \infty$ or

$$\|L(f) - f\|_p \ge \operatorname*{ess\,sup}_{x_0\in S} |L(f)(x_0) - f(x_0)| \ge \varepsilon_0$$

if $p = \infty$.

Hence a net $(L_i)_{i\in I}^{\leq}$ defined by $L_i := L$ for every $i \in I$ converges to h for every $h \in H$ but $(L_i(f))_{i\in I}^{\leq}$ does not converge to f in $L^p(X, \mu)$ and this contradicts condition (a).

$(b) \Rightarrow (c)$ It is obvious since

$$\left(\int_K |k(t) - h(t)|^p \, d\mu(t)\right)^{1/p} \leq \left(\int_K |k(t) - f(t)|^p \, d\mu(t)\right)^{1/p} + \left(\int_K |f(t) - h(t)|^p \, d\mu(t)\right)^{1/p}$$

if $1 \leq p < \infty$ or

$$\operatorname*{ess\,sup}_{t\in U} |k(t) - h(t)| \leq \operatorname*{ess\,sup}_{t\in U} |k(t) - f(t)| + \operatorname*{ess\,sup}_{t\in U} |f(t) - h(t)|$$

if $p = \infty$.

$(c) \Rightarrow (d)$ Let $\varepsilon > 0$. Since H is cofinal, we can consider $h_0, k_0 \in H$ such that $h_0 \leq f \leq k_0$ almost everywhere. Let N be the subset of X consisting of all $x_0 \in X$ such that condition (c) does not hold at x_0. Hence $\mu(N) = 0$ and since μ is regular there exists an open measurable subset S of X such that $N \subset S$ and

$$\int_S |k_0(x) - h_0(x)|^p \, d\mu(x) \leq \frac{1}{2}\varepsilon^p$$

if $1 \leq p < \infty$ or

$$\operatorname*{ess\,sup}_{x\in S} |k_0(x) - h_0(x)| \leq \varepsilon$$

if $p = \infty$.

For every $x_0 \in X \setminus S$, we can consider $h, k \in H$ and an open measurable neighborhood $U(x_0)$ of x_0 as provided in condition (c). The subset $X \setminus S$ is compact and hence we can extract a finite covering $(U_j)_{j=1\dots,m}$ of $X \setminus S$. Moreover, from condition (c), for every $j = 1, \dots, m$, there exist $h_j, k_j \in H$ such that $h_j \leq f \leq k_j$ a.e. and, for every measurable subset K of U_j,

$$\int_K |k_j(x) - h_j(x)|^p \, d\mu(x) \leq \frac{\mu(K)}{2\mu(X)}\, \varepsilon^p$$

if $1 \leq p < \infty$ or

$$\operatorname*{ess\,sup}_{x\in K} |k_j(x) - h_j(x)| \leq \varepsilon$$

if $p = \infty$.

Setting $U_0 := S$, we obtain a finite covering $(U_j)_{j=0,1,\dots,m}$ of X.

Now, set $K_0 := S$ and, for every $j = 1, \dots, m$, $K_j := U_j \setminus (U_0 \cup \dots \cup U_{j-1})$.

Then $(K_j)_{j=0,\dots,m}$ is a finite covering of pairwise disjoint measurable subsets of X and obviously

$$\begin{aligned}\left\| \inf_{j=0,\dots,m} k_j - \sup_{j=1,\dots,m} h_j \right\|_p^p &= \int_S |\inf_{j=0,1,\dots,m} k_j(x) - \sup_{j=1,\dots,m} h_j(x)|^p \, d\mu(x) \\ &\quad + \sum_{j=1}^{m} \int_{K_j} |\inf_{i=0,1,\dots,m} k_i(x) - \sup_{i=1,\dots,m} h_i(x)|^p \, d\mu(x) \\ &\leq \int_S |k_0(x) - h_0(x)|^p \, d\mu(x)\end{aligned}$$

$$+\sum_{j=1}^{m}\int_{K_j}|k_j(x)-h_j(x)|^p\,d\mu(x)$$
$$\le \frac{1}{2}\,\varepsilon^p+\frac{\varepsilon^p}{2\mu(X)}\sum_{j=1}^{m}\mu(K_j)\le \varepsilon^p$$

if $1\le p<\infty$ or

$$\left\|\inf_{j=0,\dots,m}k_j-\sup_{j=1,\dots,m}h_j\right\|_\infty=\max_{j=0,1,\dots,m}\operatorname*{ess\,sup}_{x\in K_j}|\inf_{i=0,1,\dots,m}k_i(x)-\sup_{i=1,\dots,m}h_i(x)|$$
$$\le \max_{j=0,1,\dots,m}\operatorname*{ess\,sup}_{x\in K_j}|k_j(x)-h_j(x)|\le\varepsilon$$

if $p=\infty$.

In any case, we have obtained $\left\|\inf_{j=0,\dots,m}k_j-\sup_{j=1,\dots,m}h_j\right\|_p\le\varepsilon$ and this completes the proof of d).

$(d)\ \Rightarrow\ (a)$ Let $(L_i)_{i\in I}^{\le}$ be an equicontinuous net of monotone operators from $L^p(X,\mu)$ into itself satisfying $\lim_{i\in I^{\le}}\|L_i(h)-h\|_p=0$ for every $h\in H$. In order to show that $\lim_{i\in I^{\le}}\|L_i(f)-f\|_p=0$ we fix $\varepsilon>0$ and consider $h_1,\dots,h_m,k_1,\dots,k_m\in H$ such that

$$h_j\le f\le k_j\quad a.e.,\qquad j=1,\dots,m\,,\qquad \|\inf_{j=1,\dots,m}k_j-\sup_{j=1,\dots,m}h_j\|_p\le\frac{\varepsilon}{2}.$$

It follows also

$$\sup_{j=1,\dots,m}h_j\le f\le\inf_{j=1,\dots,m}k_j\quad a.e.$$

and since every L_i, $i\in I$, is monotone, we have also, for every $i\in I$,

$$L_i(h_j)\le L_i(f)\le L_i(k_j)\quad a.e.$$

Moreover, since the nets $(L_i(h_j))_{i\in I}^{\le}$ and $(L_i(k_j))_{i\in I}^{\le}$ converge to h_j and respectively k_j for every $j=1,\dots,m$, we can find $\alpha\in I$ such that, for every $i\ge\alpha$ and $j=1,\dots,m$,

$$\|L_i(h_j)-h_j\|_p\le\frac{\varepsilon}{4m}\,,\qquad \|L_i(k_j)-k_j\|_p\le\frac{\varepsilon}{4m}.$$

Fix $i\ge\alpha$ and consider the sets

$$X_i^+:=\{x\in X\mid L_i(f)(x)-f(x)\ge 0\}\,,\qquad X_i^-:=\{x\in X\mid L_i(f)(x)-f(x)\le 0\}.$$

For every $j=1,\dots,m$ and almost all $x\in X_i^+$, we have

$$\begin{aligned}0&\le L_i(f)(x)-f(x)\le L_i(k_j)(x)-h_j(x)\\&=L_i(k_j)(x)-k_j(x)+k_j(x)-h_j(x)\\&\le|L_i(h_j)(x)-h_j(x)|+|L_i(k_j)(x)-k_j(x)|+(k_j(x)-h_j(x))\end{aligned}$$

and analogously, for almost all $x\in X_i^-$,

$$\begin{aligned}0&\le -L_i(f)(x)+f(x)\le -L_i(h_j)(x)+k_j(x)\\&=-L_i(h_j)(x)+h_j(x)+k_j(x)-h_j(x)\\&\le|L_i(h_j)(x)-h_j(x)|+|L_i(k_j)(x)-k_j(x)|+(k_j(x)-h_j(x))\,.\end{aligned}$$

Hence, for almost all $x\in X$,

$$|L_i(f)(x)-f(x)|\le|L_i(h_j)(x)-h_j(x)|+|L_i(k_j)(x)-k_j(x)|+(k_j(x)-h_j(x))$$

$$\le \sum_{s=1}^{m} |L_i(h_s)(x) - h_s(x)| + \sum_{s=1}^{m} |L_i(k_s)(x) - k_s(x)|$$
$$+ (k_j(x) - h_j(x)).$$

The preceding inequality holds for every $j = 1, \ldots, m$ and therefore we have also

$$|L_i(f)(x) - f(x)| \le \sum_{s=1}^{m} |L_i(h_s)(x) - h_s(x)| + \sum_{s=1}^{m} |L_i(k_s)(x) - k_s(x)|$$
$$+ \inf_{j=1,\ldots,m} k_j(x) - \sup_{j=1,\ldots,m} h_j(x).$$

Finally, we have

$$\|L_i(f) - f\|_p \le \sum_{j=1}^{m} \|L_i(h_j) - h_j\|_p + \sum_{j=1}^{m} \|L_i(k_j) - k_j\|_p$$
$$+ \left\| \inf_{j=1,\ldots,m} k_j - \sup_{j=1,\ldots,m} h_j \right\|_p$$
$$\le \sum_{j=1}^{m} \frac{\varepsilon}{4m} + \sum_{j=1}^{m} \frac{\varepsilon}{4m} + \frac{\varepsilon}{2} = \varepsilon$$

and this completes the proof. ■

Condition (b) of Theorem 3.1 suggests to distinguish also in the L^p-setting between the upper and lower Korovkin closures.

We start with the following main definition.

Definition 3.2. Let $1 \le p \le \infty$ and let $H \subset L^p(X, \mu)$.

The *upper Korovkin closure* (respectively, the *lower Korovkin closure*) of H in $L^p(X, \mu)$ with respect to monotone operators for the identity operator is the subset $K(H)_p^+$ (respectively, $K(H)_p^-$) defined as follows

$$K(H)_p^+ := \{f \in L^p(X, \mu) \mid \lim_{i \in I^{\le}} \|(L_i(f) - f)^+\|_p = 0 \textit{ for every equicontinuous net } (L_i)_{i \in I}^{\le} \textit{ of monotone operators from } L^p(X, \mu) \textit{ into itself satisfying } \lim_{i \in I^{\le}} \|L_i(h) - h\|_p = 0 \textit{ for every } h \in H\}.$$

(respectively,

$$K(H)_p^- := \{f \in L^p(X, \mu) \mid \lim_{i \in I^{\le}} \|(L_i(f) - f)^-\|_p = 0 \textit{ for every equicontinuous net } (L_i)_{i \in I}^{\le} \textit{ of monotone operators from } L^p(X, \mu) \textit{ into itself satisfying } \lim_{i \in I^{\le}} \|L_i(h) - h\|_p = 0 \textit{ for every } h \in H\}.)$$

As usual, we have denoted by $f^+ := \max\{f, 0\}$ (respectively, by $f^- := \max\{-f, 0\}$) the positive (respectively, negative) part of f.

Moreover, H is said to be an *upper Korovkin system* (respectively, a *lower Korovkin system*) with respect to monotone operators for the identity operator if $K(H)_p^+ = L^p(X,\mu)$ (respectively, $K(H)_p^- = L^p(X,\mu)$).

Also in this case, it may be useful to distinguish the upward and downward cofinal property.

Namely, a subset H of $L^p(X,\mu)$ is said to be *upward cofinal* (respectively, *downward cofinal*) in $L^p(X,\mu)$ if, for every $f \in L^p(X,\mu)$, there exists $h \in H$ such that $f \le h$ almost everywhere (respectively, for every $f \in L^p(X,\mu)$, there exists $k \in H$ such that $k \le f$ almost everywhere).

In the following result we state a characterization of the upper Korovkin closure. The corresponding characterization of the lower Korovkin closure can be obtained in a similar way and it is omitted for the sake of brevity.

Theorem 3.3. *Let X be a compact Hausdorff topological space, μ be a positive Radon measure on X and $1 \le p \le +\infty$.*

Let H be an upward cofinal subset of $L^p(X,\mu)$.

If $f \in L^p(X,\mu)$, the following statements are equivalent:

(a) $f \in K(H)_p^+$.

(b) For almost all $x_0 \in X$ and for every $\varepsilon > 0$, there exist $k \in H$ and a neighborhood U of x_0 such that $f \le k$ a.e. and, if $1 \le p < \infty$,

$$\left(\int_K |k(x) - f(x)|^p \, d\mu(x)\right)^{1/p} \le \frac{\mu(K)}{\mu(X)}\,\varepsilon,$$

whenever K is a measurable subset of U, while, if $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in U} |k(x) - f(x)| \le \varepsilon$$

(c) For every $\varepsilon > 0$, there exist $k_1, \dots, k_m \in H$ such that

$$f \le k_j \ a.e., \qquad j = 1, \dots, m,$$

and

$$\left\| \inf_{j=1,\dots,m} k_j - f \right\|_p \le \varepsilon.$$

Proof. $(a) \Rightarrow (b)$ The proof is at all similar to that of $(a) \Rightarrow (b)$ in Theorem 3.1, taking into account that the operator L considered there satisfies $(L(f) - f)^+ = L(f) - f$ a.e.
$(b) \Rightarrow (c)$ Let $\varepsilon > 0$. Since H is upward cofinal, there exists $k_0 \in H$ such that $f \le k_0$ almost everywhere. Let N be the subset of X consisting of all $x_0 \in X$ such that condition (b) does not hold at x_0. Hence $\mu(N) = 0$ and since μ is regular there exists an open measurable subset S of X such that $N \subset S$ and

$$\int_S |k_0(x) - h_0(x)|^p \, d\mu(x) \le \frac{1}{2}\varepsilon^p$$

if $1 \le p < \infty$ or

$$\operatorname*{ess\,sup}_{x\in S} |k_0(x) - h_0(x)| \le \varepsilon.$$

For every $x_0 \in X \setminus S$, we can consider $h, k \in H$ and an open measurable neighborhood $U(x_0)$ of x_0 as provided in condition (b). The subset $X \setminus S$ is compact and hence we can extract a finite covering $(U_j)_{j=1\dots,m}$ of $X \setminus S$. Moreover, from condition (b), for every $j = 1, \dots, m$, there exists $k_j \in H$ such that $f \le k_j$ a.e. and, for every measurable subset K of U_j,

$$\int_K |k_j(x) - f(x)|^p \, d\mu(x) \le \frac{\mu(K)}{2\mu(X)} \, \varepsilon^p$$

if $1 \le p < \infty$ or

$$\operatorname*{ess\,sup}_{x\in K} |k_j(x) - f(x)| \le \varepsilon$$

if $p = \infty$.

Setting $U_0 := S$, we obtain a finite covering $(U_j)_{j=0,1,\dots,m}$ of X.

Now, set $K_0 := S$ and, for every $j = 1, \dots, m$, $K_j := U_j \setminus (U_0 \cup \dots \cup U_{j-1})$.

Then $(K_j)_{j=0,\dots,m}$ is a finite covering of pairwise disjoint measurable subsets of X and obviously

$$\begin{aligned}
\left\| \inf_{j=0,\dots,m} k_j - f \right\|_p^p &= \int_S \Big| \inf_{j=0,1,\dots,m} k_j(x) - f(x) \Big|^p \, d\mu(x) \\
&\quad + \sum_{j=1}^m \int_{K_j} \Big| \inf_{i=0,1,\dots,m} k_i(x) - f(x) \Big|^p \, d\mu(x) \\
&\le \int_S |k_0(x) - f(x)|^p \, d\mu(x) + \sum_{j=1}^m \int_{K_j} |k_j(x) - f(x)|^p \, d\mu(x) \\
&\le \frac{1}{2} \varepsilon^p + \frac{\varepsilon^p}{2\mu(X)} \sum_{j=1}^m \mu(K_j) \le \varepsilon^p
\end{aligned}$$

if $1 \le p < \infty$ or

$$\begin{aligned}
\left\| \inf_{j=0,\dots,m} k_j - f \right\|_\infty &= \max_{j=0,1,\dots,m} \operatorname*{ess\,sup}_{x\in K_j} \Big| \inf_{i=0,1,\dots,m} k_i(x) - f(x) \Big| \\
&\le \max_{j=0,1,\dots,m} \operatorname*{ess\,sup}_{x\in K_j} |k_j(x) - f(x)| \le \varepsilon
\end{aligned}$$

if $p = \infty$.

In any case, we have obtained $\left\| \inf_{j=0,\dots,m} k_j - f \right\|_p \le \varepsilon$ and this completes the proof of (c).

$(c) \Rightarrow (a)$ Let $(L_i)_{i\in I}^{\le}$ be an equicontinuous net of monotone operators from $L^p(X, \mu)$ into itself satisfying $\lim_{i\in I^{\le}} \|L_i(h) - h\|_p = 0$ for every $h \in H$. In order to show that $\lim_{i\in I^{\le}} \|(L_i(f) - f)^+\|_p = 0$ we fix $\varepsilon > 0$ and consider $k_1, \dots, k_m \in H$ such that

$$f \le k_j \quad a.e., \quad j = 1, \dots, m, \qquad \| \inf_{j=1,\dots,m} k_j - f \|_p \le \frac{\varepsilon}{2}.$$

It follows also, for every $j = 1, \dots, m$ and $i \in I$,

$$L_i(f) \le L_i(k_j) \quad a.e.$$

Moreover, there exists $\alpha \in I$ such that, for every $i \ge \alpha$ and $j = 1, \dots, m$, $\|L_i(k_j) - k_j\|_p \le \varepsilon/(2m)$.

For every $i \ge \alpha$, let $X_i^+ := \{x \in X \mid L_i(f)(x) - f(x) \ge 0\}$; hence, for almost all $x \in X_i^+$,

$$0 \le L_i(f)(x) - f(x) \le L_i(k_j)(x) - f(x)$$

$$
\begin{aligned}
&= L_i(k_j)(x) - k_j(x) + k_j(x) - f(x) \\
&\leq |L_i(k_j)(x) - k_j(x)| + (k_j(x) - f(x)) \\
&\leq \sum_{s=1}^{m} |L_i(k_s)(x) - k_s(x)| + (k_j(x) - f(x))
\end{aligned}
$$

and, since $j = 1, \ldots, m$ is arbitrary, we have also

$$0 \leq L_i(f)(x) - f(x) \leq \sum_{j=1}^{m} |L_i(k_j)(x) - k_j(x)| + \left(\inf_{j=1,\ldots,m} k_j(x) - f(x) \right) .$$

Finally, it follows

$$\|(L_i(f) - f(x))^+\|_p \leq \sum_{j=1}^{m} \|L_i(k_j) - k_j\|_p + \left(\inf_{j=1,\ldots,m} k_j(x) - f(x) \right) \leq \varepsilon$$

and this completes the proof. ■

Of course, we have $K(H)_p = K(H)_p^+ \cap K(H)_p^-$ and in general $K(H)_p$ does not coincide with $K(H)_p^+$ nor with $K(H)_p^-$.

However, if we limit ourselves to consider equilipschitzian nets of monotone operators, the Korovkin closure and the upper and lower Korovkin closures coincide, as stated in the following proposition.

Proposition 3.4. *Let X be a compact Hausdorff topological space, μ be a positive Radon measure on X and $1 \leq p \leq +\infty$.*

Let H be a cofinal subset of $L^p(X, \mu)$.

If $f \in K(H)_p^+$ or $f \in K(H)_p^-$ and if $(L_i)_{i \in I}^{\leq}$ is an equilipschitzian net of monotone operators from $L^p(X, \mu)$ into itself satisfying $\lim_{i \in I^{\leq}} \|L_i(h) - h\|_p = 0$ for every $h \in H$, then we have also

$$\lim_{i \in I^{\leq}} \|L_i(f) - f\|_p = 0.$$

Proof. We consider the case where $f \in K(H)_p^+$. Let $(L_i)_{i \in I}^{\leq}$ be an equilipschitzian net of monotone operators from $L^p(X, \mu)$ into itself satisfying $\lim_{i \in I^{\leq}} \|L_i(h) - h\|_p = 0$ for every $h \in H$ and let $M > 0$ such that $\|L_i(g_1) - L_i(g_2)\|_p \leq M \|g_1 - g_2\|_p$ for every $i \in I$ and $g_1, g_2 \in L^p(X, \mu)$.

Let $\varepsilon > 0$; with the same reasoning as in the proof of $(c) \Rightarrow (a)$ in Theorem 3.3, we can consider $k_1, \ldots, k_m \in H$ such that

$$f \leq k_j \quad a.e., \quad j = 1, \ldots, m, \qquad \| \inf_{j=1,\ldots,m} k_j - f \|_p \leq \frac{\varepsilon}{2}$$

and we can obtain $\alpha \in I$ such that, for every $i \geq \alpha$ and $j = 1, \ldots, m$, $\|L_i(k_j) - k_j\|_p \leq \varepsilon/(2m)$. Consequently, for every $i \geq \alpha$ and almost all $x \in X_i^+$ (see the proof of $(c) \Rightarrow (a)$ in Theorem 3.3),

$$0 \leq L_i(f)(x) - f(x) \leq \sum_{j=1}^{m} |L_i(k_j)(x) - k_j(x)| + \left(\inf_{j=1,\ldots,m} k_j(x) - f(x) \right) .$$

Moreover, for almost all $x \in X$ such that $L_i(f)(x) - f(x) \leq 0$ we have, for every $j = 1, \ldots, m$,

$$0 \leq f(x) - L_i(f)(x) \leq k_j(x) - L_i(k_j)(x) + L_i(k_j)(x) - L_i(f)(x)$$

and hence

$$|(L_i(f)-f)^-(x)| \le \sum_{j=1}^{m} |k_j(x)-L_i(k_j)(x)| + \sum_{j=1}^{m} |L_i(k_j)(x)-L_i(f)(x)| \,.$$

Finally, taking into account the equilipschitzian property,

$$\begin{aligned}\|(L_i(f)-f)^-\|_p &\le \sum_{j=1}^{m} \|L_i(k_j)-k_j\|_p + \sum_{j=1}^{m} \|L_i(k_j)-L_i(f)\|_p \\ &\le \sum_{j=1}^{m} \|L_i(k_j)-k_j\|_p + M\sum_{j=1}^{m} \|k_j-f\|_p \,.\end{aligned}$$

Putting together the above inequalities, we obtain

$$\|L_i(f)-f(x)\|_p \le \sum_{j=1}^{m} \|L_i(k_j)-k_j\|_p + (M+1)\left(\inf_{j=1,\dots,m} k_j(x)-f(x)\right) \le \varepsilon + \frac{M}{2}\,\varepsilon$$

and this shows that $\lim_{i\in I^{\le}} \|L_i(f)-f\|_p = 0$. ■

Data availability

No data was used for the research described in the article.

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