Nucl. Phys. B 1001 (2024) 116507

Contents lists available at ScienceDirect

Nuclear Physics, Section B

journal homepage: www.elsevier.com/locate/nuclphysb

High Energy Physics – Theory

Large N expansion of superconformal index of $k = 1$ ABJM theory and semiclassical M5 brane partition function

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ARTICLE INFO

Editor: Clay Córdova

ABSTRACT

It was shown in arXiv:2309.10786 that the leading non-perturbative contribution to the large N expansion of the superconformal index of the (2,0) 6d theory (which describes low-energy dynamics of N coincident M5 branes) is reproduced by the semiclassical partition function of quantum M2 brane wrapped on $S^1 \times S^2$ in a twisted version of $\mathrm{AdS}_7 \times S^4$ background. Here we demonstrate an analogous relation for the leading non-perturbative contribution to the large N expansion of the superconformal index of the $\mathcal{N} = 8$ supersymmetric level-one $U(N) \times U(N)$ ABJM theory (which describes low-energy dynamics of N coincident M2 branes). The roles of M2 and M5 branes get effectively interchanged. Namely, the large N correction to the ABJM index is found to be given by the semiclassical partition function of quantum M5 brane wrapped on $S^1 \times S^5$ in a twisted version of $\mathrm{AdS}_4 \times S^7$ background. This effectively confirms the suggestion for the "M5 brane index" made in arXiv:2007.05213 on the basis of indirect superconformal algebra considerations.

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https://doi.org/10.1016/j.nuclphysb.2024.116507
Received 10 December 2023; Accepted 28 February 2024

Available online 8 March 2024

1. Introduction

The superconformal index is the refined Witten index for radially quantized superconformal theories [1,2]. It has schematic form $\mathrm{I} = \mathrm{Tr}_{\mathrm{BPS}}[(-1)^{\mathrm{F}} q^H \prod_a u_a^{C_a}]$ where the trace is restricted to BPS states annihilated by some chosen supercharge Q. H and C_a are the Cartan generators of the symmetry superalgebra commuting with Q, and q and u_a are free parameters (fugacities). For a superconformal theory on $\mathbb{R}^d$, the index may be computed as a supersymmetric partition function on $S^1 \times S^{d-1}$ [3]. This partition function is defined by path integral with fermions being periodic in S^1 and including extra twists (rotations) of coordinates and/or fields that encode the presence of particular fugacities.

For the 6d superconformal $(2,0)$ theory describing the low-energy dynamics of a stack of N M5 branes, the corresponding index $\mathrm{I}_N^{(2,0)}(q)$ in the Schur limit [4] depends on a single fugacity q. In a recent paper [5] it was shown that the leading large N non-perturbative $\mathcal{O}(q^N)$ contribution to this index, suggested in [6,7] to be related to the contribution of M2 brane $S^1 \times S^2$ instanton in $\mathrm{AdS}_7 \times S^4$, is indeed reproduced by a semiclassical partition function of quantum M2 brane in a twisted version of $\mathrm{AdS}_7 \times S^4$ background.

Here we will demonstrate that the analogous fact is true for the leading large N non-perturbative contribution to the superconformal index of the $\mathcal{N}=8$ supersymmetric $k=1$ $U_k(N)\times U_{-k}(N)$ ABJM theory [8] describing low-energy dynamics of N coincident M2 branes, with the roles of M2 and M5 branes effectively interchanged. This is consistent with the expectation that supersymmetric partition function of the ABJM theory on $S^1 \times S^2$ should be described on the dual M-theory side by a quantum M2 brane path integral that may receive non-perturbative large N contributions from M5 brane instantons. Namely, the leading large N correction to the ABJM index will be found to be given by the semiclassical partition function of a quantum M5 brane wrapped on $S^1 \times S^5$ in a twisted version of $\mathrm{AdS}_4 \times S^7$ background. This will confirm an earlier suggestion of [6] based on superconformal algebra considerations.

The computation described below demonstrates that a semiclassical expansion of quantum M5 brane path integral (around a configuration with a non-degenerate induced metric) is well defined and leads to consistent results. It provides an M5 brane analog of similar computations of semiclassical M2 brane path integrals in several recent works [9,5,10].

Let us first recall the result for the index of the (2,0) theory. For $N=1$ the $(2,0)$ theory is that of a free 6d tensor multiplet and the index may be computed using representation theory [2]. For $N>1$, the absence of an intrinsic definition of the $(2,0)$ theory may be bypassed by identifying its index on $S^1_\beta \times S^5$ (β is the length of the circle) with the non-perturbative supersymmetric partition function of the maximally supersymmetric 5d SYM theory on S^5 (with the coupling proportional to β). This requires specific R-symmetry twists on $S^1_\beta \times S^5$ (corresponding to a particular choice of Cartan generators C_i in the definition of the index) in order to preserve 16 real supersymmetries. Using this strategy, the index $\mathrm{I}_N^{(2,0)}$ was computed by supersymmetric localization in [11,12].

At large N, the Schur index $\mathrm{I}_N^{(2,0)}(q)$ reduces to the generating function $\mathrm{I}_{\mathrm{KK}}^{(2,0)}(q)$ of Kaluza-Klein BPS states of supergravity on $\mathrm{AdS}_7 \times S^4$. Finite N corrections can be organized as an expansion in powers of q^N (with q-dependent coefficients)

$$\mathrm{I}_N^{(2,0)}(q) = \mathrm{I}_{\mathrm{KK}}^{(2,0)}(q)\left[1 - \frac{q}{(1-q)^2}\, q^N + \mathcal{O}(q^{2N})\right], \qquad q = e^{-\beta}\ . \tag{1.1}$$

Motivated by similar results for the index of $\mathcal{N}=4$ $U(N)$ 4d SYM theory [7] an alternative derivation of the large N correction in (1.1) attributing it to a contribution of an M2 brane wrapped on $S^1 \times S^2$ in $\mathrm{AdS}_7 \times S^4$ was suggested in [6]. In this approach, the $q^N = e^{-\beta N}$ factor originates from the classical value of the wrapped M2 brane action while the prefactor should be a superconformal index counting M2 brane BPS fluctuations.

This wrapped brane index was found in [6] using symmetry considerations, without analyzing in detail the world-volume theory.[2] Instead, ref. [5] directly computed the semiclassical partition function of the M2 brane $S^1\times S^2$ instanton in $\mathrm{AdS}_7\times S^4$ reproducing the $-\frac{q}{(1-q)^2}$ factor in (1.1) as the one-loop contribution to the M2 brane path integral.

Below we will consider the analog of the large N expansion (1.1) for the superconformal index of the $\mathcal{N}=8$ $k=1$ $U(N)_k\times U(N)_{-k}$ ABJM theory [8] dual to 11d M-theory on the $\mathrm{AdS}_4\times S^7$ background. Here the direct definition of the ABJM theory on $S^1_\beta\times S^2$ does not present a problem and the expression for the corresponding index can be found [14] using supersymmetric localization at finite N.

The $N\to\infty$ limit of the ABJM index again counts the supergravity BPS states on $\mathrm{AdS}_4\times S^7$. For finite N, the integral representation [14] for the index can be used to extract its small q expansion but it is not suitable to find its non-perturbative expansion analogous to (1.1). The latter was suggested in [6] to be related to the contributions of M5 branes wrapped on $S^1_\beta\times S^5$ inside $\mathrm{AdS}_4\times S^7$. To determine the corresponding M5 brane index ref. [6] suggested to use an analytic continuation relation to the index of a single (2,0) tensor multiplet in flat space that happens to have superconformal subalgebra compatible with the supercharge defining the index which is isomorphic to the unbroken supersymmetry algebra on the M5 brane.

Starting with the expression for the M5 brane index suggested in [6] and taking an unrefined limit (i.e. setting all fugacities apart from single q to be trivial) we found the following analog of (1.1)[3]

$$\mathrm{I}^{\rm ABJM}_N(q)=\mathrm{I}^{\rm ABJM}_{\rm KK}(q)\Big[1+\tfrac{1}{6}N^3\,q^N\big(G_0(q)+\mathcal{O}(N^{-1})\big)+\mathcal{O}(q^{2N})\Big], \tag{1.2}$$

$$G_0(q)=-q\prod_{n=1}^{\infty}(1-q^n)^{-7}\ , \qquad q=e^{-\frac12\beta}\ . \tag{1.3}$$

Note that the leading correction in (1.2) contains a peculiar N^3 enhancement factor that had no counterpart in the $(2,0)$ case in (1.1).[4]

The aim of the present paper is to show that the prefactor of the leading q^N term in (1.2) can be found also by the direct one-loop $S^1_\beta\times S^5$ M5 brane instanton computation, in full analogy with the M2 brane instanton computation in the case of the (2,0) index (1.1) in [5].

The unrefined ABJM index defined by a combination of the radial Hamiltonian and a rotation generator in one plane of $\mathbb{R}^3$ should be given by the supersymmetric partition function of the ABJM theory on a twisted product $S^1_\beta\times\widetilde{S}^2$ with an isometric angle of S^2 being "mixed" with the coordinate y of S^1_β. The dual M-theory description should be in terms of the $\mathrm{AdS}_4\times S^7$ background with the $S^1_\beta\times\widetilde{S}^2$ boundary (here $y\equiv y+\beta$)[5]

$$ds^2=\tfrac14 L^2\,(dx^2+\sinh^2x\,d\widetilde{S}_2+\cosh^2x\,dy^2)+L^2(dv^2+\cos^2v\,dS_5+\sin^2v\,dz^2)\ , \tag{1.4}$$

$$d\widetilde{S}_2=d\vartheta^2+\sin^2\vartheta\,(d\varphi+i\,dy)^2, \qquad C_3=-\tfrac{i}{8}L^3\,\sinh^3x\,dy\wedge\mathrm{vol}_{S^2}\ . \tag{1.5}$$

Below we will compute the semiclassical path integral for the M5 brane wrapped on $S^1_\beta\subset\mathrm{AdS}_4$ and $S^5\subset S^7$. As the effective M5 brane tension is given by $\mathrm{T}_5=L^6T_5=\frac{1}{\pi^3}N$, the q^N term in (1.2) will come from the classical action of the wrapped M5 brane. Its prefactor will be given by the one-loop M5 brane partition function, with the $N^3\sim(\sqrt{\mathrm{T}_5})^6$ factor being due to the zero-mode contribution and $G_0(q)$ originating from the combination of determinants of the M5 brane quadratic fluctuation operators.

The set of M5 brane fluctuations in the static gauge will generalize the one of the 6d $(2,0)$ tensor multiplet in flat target space, i.e. it will contain 3 scalars of AdS_4, 2 scalars of S^7, the rank 2 antisymmetric tensor with self-dual field strength and 4 Majorana-Weyl 6d fermions propagating on $S^1_\beta\times S^5$. We will separately compute the contributions of all of these fields that will thus involve all fluctuations, not just BPS one. Combined together the resulting supersymmetric partition function will effectively reproduce the expression for the supersymmetric M5 brane index conjectured in [6]. This result will crucially depend on the special structure of the supersymmetric M5 brane action in the background (1.4), (1.5) (e.g. the coupling to C_3 background via the "dual" C_6 potential and fermionic covariant derivatives, etc.).

We shall start in section 2 with a review of the structure of the large N expansion of the superconformal index of the $k=1$ ABJM theory. In section 3 we will consider the supersymmetric M5 brane path integral in semiclassical expansion near $S^1\times S^5$ solution in

[2] Explicitly, the single M2 brane superconformal algebra was mapped to that of the 3d $k=1$ ABJM theory or of the free theory of $\mathcal{N}=8$ 3d scalar multiplet (see, e.g., [13]). The superalgebra mapping implies an analytic continuation relation between the indices of the two 3d theories – the wrapped M2 brane and the free $\mathcal{N}=8$ scalar multiplet in flat space. This led the authors of [6] to conjecture a remarkable general formula for the finite N corrections to the index in terms of a sum over multiple M2 wrappings, reducing to (1.1) for a single wrapping.

[3] Note that the counterpart of G_0 in the large N expansion of (2,0) theory index in (1.1) had a much simpler rational form. In (1.1) this was the analytic continuation of the index of a free 3d $\mathcal{N}=8$ multiplet describing a single M2 brane in flat space, see a discussion in Appendix B and in particular (B.15).

[4] Such N-dependent factors are common in unrefined index expressions and are related to the "wall-crossing" effect [15–17]. In general, superconformal indices are protected against continuous deformations, but the coefficients of their small q expansions can jump across special surfaces in the space of global-symmetry fugacities.

[5] This may be compared to the twisted $\mathrm{AdS}_7\times S^4$ background in the $(2,0)$ theory case considered in [5] (see also [18]): $ds^2_{11}=L'^2(dx^2+\sinh^2x\,dS_5+\cosh^2x\,dy^2)+\frac14 L'^2\big[dv^2+\cos^2v\,dS_2+\sin^2v\,(dz+i\,dy)^2\big]$ supported by $C_3=-\frac18 L'^3\cos^2v\,\mathrm{vol}_{S^2}\wedge(dz+i\,dy)$.

the background (1.4), (1.5). We will find the quadratic fluctuation actions for all M5 brane fluctuations and present the results for the corresponding one-loop contributions to partition function.

The combined expression for the one-loop M5 brane supersymmetric partition function or free energy will be given in section 4. We will demonstrate that it matches the expression for the $N^3 G_0(q)$ prefactor of q^N in the unrefined ABJM index (1.2). In section 4.3 we will also consider a generalization to the case of non-trivial fugacities u_a implying the presence of extra twist angles in S^7 part of (1.4). We will find again a precise matching between the corresponding M5 brane partition and the expression for the M5 brane index suggested in [6].

In section 5 we will show how the general expressions for the fluctuation determinants of the fields of a single (2,0) multiplet on $S^1 \times S^5$ found in section 3 can be used to directly compute the corresponding supersymmetric partition function or the $N = 1$ (2,0) superconformal index.

In Appendix A we shall make some remarks about brane instanton (or "giant graviton") expansion of superconformal index on the example of $\mathcal{N} = 4$ SYM theory. In Appendix B we shall review the definition of the superconformal index of (2,0) theory. In Appendix C, starting with the general expression for the M5 brane index suggested in [6], we will show that taking the unrefined limit of the corresponding large N expansion of the ABJM index leads to the expression in (1.2). In Appendix D we will derive the free energy of a conformal scalar field on $S^1 \times S^5$ with an extra shift of the S^1 mode number that was used in section 4. Appendix E will contain details of the computation discussed in section 4.3.

2. Superconformal index of $k = 1$ ABJM theory at large N

The superconformal algebra of the 3d $\mathcal{N} = 8$ supersymmetric $U_k(N) \times U_{-k}(N)$ ABJM theory with Chern-Simons level $k = 1$ is $\mathfrak{osp}(8|4)$. Its bosonic subalgebra $\mathfrak{so}(2,3) \oplus \mathfrak{so}(8)$ is the sum of the 3d conformal algebra and the R-symmetry algebra with six Cartan generators

$$C_I = (H,\ J_{12},\ R_{12},\ R_{34},\ R_{56},\ R_{78}). \tag{2.1}$$

The superconformal index is defined in terms of the supercharge Q satisfying $[C_I, Q] = \frac{1}{2}\sigma_I\, Q$ with $\sigma_I = (1,-1,1,1,1,1)$ $(I = 1,\dots,6)$. The subalgebra commuting with Q is $\mathfrak{osp}(6|2) \oplus \mathfrak{u}(1)_\Delta$, where $\mathfrak{osp}(6|2)$ has bosonic algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(6)$ and the central factor $\mathfrak{u}(1)_\Delta$ is generated by

$$\Delta = \{Q, Q^\dagger\} = H - J_{12} - \tfrac{1}{2}(R_{12} + R_{34} + R_{56} + R_{78}). \tag{2.2}$$

The index is given by

$$\mathrm{I}_N^{\mathrm{ABJM}}(q, \boldsymbol{u}) = \operatorname*{Tr}_{\Delta=0} \left[(-1)^{\mathrm{F}}\, q^{2(H+J_{12})}\, u_1^{R_{12}}\, u_2^{R_{34}}\, u_3^{R_{56}}\, u_4^{R_{78}}\right], \qquad u_1 u_2 u_3 u_4 = 1, \tag{2.3}$$

where the trace is restricted to gauge singlet states with $\Delta = 0$ and u_a are R-symmetry fugacities. An explicit integral representation of (2.3) was obtained in [14] by using supersymmetric localization. It is valid also for the $\mathcal{N} = 6$ supersymmetric ABJM theory with generic level k, and for $N > 1$ includes the contribution of monopole operators.

We will denote by $\hat{\mathrm{I}}$ the single-particle index corresponding to the index I, i.e. related to it by the plethystic exponential (here $\boldsymbol{x}$ stands for all fugacities including q)

$$\mathrm{I}(\boldsymbol{x}) = \mathrm{PE}[\hat{\mathrm{I}}(\boldsymbol{x})] \equiv \exp \sum_{m=1}^{\infty} \frac{1}{m}\, \hat{\mathrm{I}}(\boldsymbol{x}^m)\,. \tag{2.4}$$

At leading order in large N, the superconformal index coincides with the one that counts the BPS states of supergravity in $\mathrm{AdS}_4 \times S^7$ (i.e. KK states of 11d supergravity compactified on S^7) [14]. It has the following explicit expression [2]

$$\begin{aligned} &\mathrm{I}_N^{\mathrm{ABJM}}(q, \boldsymbol{u}) \overset{N\to\infty}{\to} \mathrm{I}_{\mathrm{KK}}(q, \boldsymbol{u}) = \mathrm{PE}[\hat{\mathrm{I}}_{\mathrm{KK}}(q, \boldsymbol{u})], \\ &\hat{\mathrm{I}}_{\mathrm{KK}}(q, \boldsymbol{u}) = \frac{1}{(1-q^4)^2} \prod_{a=1}^{4} \frac{1 - q^3\, u_a^{-1}}{1 - q\, u_a} - \frac{1 - q^4 + q^8}{(1-q^4)^4}. \end{aligned} \tag{2.5}$$

To understand the structure of finite N corrections, it is useful to examine the small q expansion of the index at low values of N. From the integral representation in [14] one finds (here for simplicity we set all $u_a = 1$)

$$\begin{aligned} \mathrm{I}_{N=1}^{\mathrm{ABJM}}(q) &= \mathbf{1} + \mathbf{4}\, q + 10\, q^2 + 16\, q^3 + 19\, q^4 + 20\, q^5 + \cdots, \\ \mathrm{I}_{N=2}^{\mathrm{ABJM}}(q) &= \mathbf{1} + \mathbf{4}\, q + \mathbf{20}\, q^2 + 56\, q^3 + 139\, q^4 + 260\, q^5 + \cdots, \\ \mathrm{I}_{N=3}^{\mathrm{ABJM}}(q) &= \mathbf{1} + \mathbf{4}\, q + \mathbf{20}\, q^2 + \mathbf{76}\, q^3 + 239\, q^4 + 644\, q^5 + \cdots. \end{aligned} \tag{2.6}$$

Here $\mathrm{I}_{N=1}^{\mathrm{ABJM}}(q) = \mathrm{PE}[\frac{4q}{1+q^2}]$ (see, e.g., [14,13]). In (2.6) we highlighted the expansion coefficients that remain stable under further increase of N and whose contribution reproduces the KK index in (2.5)

$$\mathrm{I}_{\mathrm{KK}}(q, \boldsymbol{u} = 1) = 1 + 4q + 20q^2 + 76q^3 + 274q^4 + 900q^5 + 2826q^6 + 8400q^8 + \cdots. \tag{2.7}$$

The structure of large N expansion implied by (2.6) and its generalization to non-trivial $\boldsymbol{u}$ is the following

$$\mathrm{I}_N^{\rm ABJM}(q,\boldsymbol{u}) = \mathrm{I}_{\rm KK}(q,\boldsymbol{u})\Big[1+q^N\,\delta\mathrm{I}_N^{(1)}(q,\boldsymbol{u})+\mathcal{O}(q^{2N+6})\Big]\,. \tag{2.8}$$

Here $\delta\mathrm{I}_N^{(1)}(q,\boldsymbol{u})$ is the first term in an infinite series of non-perturbative corrections $\sum_n q^{nN}\,\delta\mathrm{I}^{(n)}(q,\boldsymbol{u})$ (*cf.* Appendix A).

It was suggested in [6] that in the dual M-theory description of the ABJM theory the large N corrections to the index may be interpreted as originating from contributions of M5 brane instantons in $\mathrm{AdS}_4\times S^7$. In particular, the q^{nN} term may come from M5 brane wrapped n times on $S^1\times S^5$ part of $\mathrm{AdS}_4\times S^7$.[6] Ref. [6] conjectured the following expression for $\delta\mathrm{I}_N^{(1)}(q,\boldsymbol{u})$

$$\delta\mathrm{I}_N^{(1)}(q,\boldsymbol{u}) = \sum_{a=1}^4 u_a^N\,\mathrm{I}_a^{\rm M5}(q,\boldsymbol{u})\,,\qquad \mathrm{I}_a^{\rm M5}(q,\boldsymbol{u})] = \mathrm{PE}[\widehat{\mathrm{I}}_a^{\rm M5}(q,\boldsymbol{u})], \tag{2.9}$$

where the suitably defined "M5 brane index" $\mathrm{I}_a^{\rm M5}$ should be counting BPS fluctuations of an M5 brane wrapped on $S^1\times S^5$ in $\mathrm{AdS}_4\times S^7$ with the embedding of $S^5\subset S^7$ corresponding to the choice of one plane $z_a=0$ in the definition $\sum_{c=1}^4|z_c|^2=1$ of $S^7\subset\mathbb{C}^4$.[7]

The explicit computation of the index $\mathrm{I}_a^{\rm M5}$ in (2.9) requires the analysis of the set of fluctuation modes on the wrapped M5 brane which was not carried out in full in [6]. Instead, ref. [6] suggested to exploit the isomorphism[8] between the superconformal algebras of the two 6d models – the quadratic fluctuation action of the wrapped M5 brane in $\mathrm{AdS}_4\times S^7$ space and free (2,0) tensor multiplet defined on $S^1\times S^5$. This allowed to find $\mathrm{I}_a^{\rm M5}$ from the known expression for the superconformal index of the $N=1$ (2,0) multiplet (see Appendix B)

$$\mathrm{I}_{N=1}^{(2,0)}(q',\boldsymbol{y},u') = \mathop{\mathrm{Tr}}_{\Delta=0}\Big[(-1)^{\rm F}\,q'^{\,H+\frac13(J_{12}+J_{34}+J_{56})}\,y_1^{J_{12}}\,y_2^{J_{34}}\,y_3^{J_{56}}\,u'^{\,R_{12}-R_{34}}\Big],\qquad y_1y_2y_3=1\,. \tag{2.10}$$

Here J_{ij} are the generators of the $SO(6)$ rotation group of the 6d space on which the (2,0) multiplet is defined and R_{ab} are the Cartan generators of the $SO(5)$ R-symmetry group. The reason why the two functions $\mathrm{I}_a^{\rm M5}$ and $\mathrm{I}_{N=1}^{(2,0)}$ should be related should be supersymmetry (as count of BPS states should require only group theory information). This argument appears to be rather heuristic, given that details of $\mathrm{I}_a^{\rm M5}$ should depend on a particular structure of the supersymmetric M5 brane action in curved $\mathrm{AdS}_4\times S^7$ background (with non-trivial coupling to the 4-form flux, etc.).

The isomorphism between the generators of the two superconformal algebras determines the identification of the two sets of fugacities needed to get $\mathrm{I}_a^{\rm M5}(q,u_1,u_2,u_3,u_4)$ from the $N=1$ (2,0) index in (2.10). For the $\mathrm{I}_1^{\rm M5}$ term in (2.9) that distinguishes the u_1 fugacity this relation reads

$$(q',\ y_1,\ y_2,\ y_3,\ u') = (q^{\frac34}u_1^{-\frac14},\ u_2u_1^{\frac13},\ u_3u_1^{\frac13},\ u_4u_1^{\frac13},\ q^{-\frac52}u_1^{-\frac12})\,. \tag{2.11}$$

As a result, one gets the following expression for the corresponding single-particle M5 brane index [6]

$$\widehat{\mathrm{I}}_1^{\rm M5}(q,\boldsymbol{u}) = \frac{q^{-1}u_1^{-1}-q^2u_1^{-1}(u_2^{-1}+u_3^{-1}+u_4^{-1})+q^3u_1^{-1}+q^4}{(1-qu_2)(1-qu_3)(1-qu_4)}. \tag{2.12}$$

From (2.3) we can identify q here with $e^{-\frac12\beta}$ where β is the length of the circle in the 3d space $S^1_\beta\times S^2$ on which ABJM theory is defined. $\widehat{\mathrm{I}}_a^{\rm M5}$ terms with $a=2,3,4$ in the sum in (2.9) are then obtained from (2.12) by the permutations of u_1,u_2,u_3,u_4.

As we show in Appendix C, in the simplest "unrefined" case of $\boldsymbol{u}\to1$ the leading term $\delta\mathrm{I}^{(1)}(q,\boldsymbol{u})$ in (2.9) has the following explicit form

$$\delta\mathrm{I}_N^{(1)}(q)\equiv\delta\mathrm{I}_N^{(1)}(q,\boldsymbol{u}=1) = \frac16N^3G_0(q)+N^2G_1(q)+NG_2(q)+G_3(q)\,, \tag{2.13}$$

where

$$G_0(q) = -q\prod_{n=1}^\infty\frac{1}{(1-q^n)^7} = -q\,\mathrm{PE}\Big[\frac{7q}{1-q}\Big]\,. \tag{2.14}$$

Below, in sections 3 and 4, we will reproduce (2.13), (2.14) by the direct computation of the one-loop supersymmetric partition function of the M5 brane instanton in the relevant twisted version of the $\mathrm{AdS}_4\times S^7$ background, confirming the conjecture of [6]. In section 4.3 we will also discuss a generalization to the case with non-trivial u_a.

6 The corresponding classical action then produces the factor of $q^{nN}u_a^N$ taking into account that the effective M5 brane tension is proportional to N, see below.

7 For the ABJM theory at level $k\ge2$, the dual geometry is the orbifold $AdS_4\times S^7/\mathbb{Z}_k$. For each of the four wrapping configurations $z_a=0$ $(a=1,\dots,4)$ the M5 brane world volume is $S^1\times(S^5/\mathbb{Z}_k)$. M5 branes have a topological wrapping number $b\in\mathbb{Z}_k$, corresponding to the non-trivial homology $H_5(S^7/\mathbb{Z}_k)=\mathbb{Z}_k$, and the index can be computed in each sector with given b. The $b=0$ case corresponds to the $U(1)_k\times U(1)_{-k}$ ABJM theory. Finite N corrections to the index start with multiple wound branes for $k=2,3$ and presumably for all $k\ge2$ [6]. Wrapped branes with $b\neq0$ correspond to baryonic operators in the gauge theory charged under a $\mathbb{Z}_k$ subgroup of the gauge symmetry [19].

8 Details of the construction are similar to the one in the M2 brane case discussed in footnote 2.

3. Semiclassical expansion of M5 brane path integral

Our aim will be to show that the leading $q^N\, \delta \mathrm{I}_N^{(1)}(q,\boldsymbol{u})$ term in the ABJM index in (2.8) is reproduced by the path integral contribution of an M5 brane $S^1_\beta \times S^5$ instanton in a particular twisted version of the $\mathrm{AdS}_4\times S^7$ M-theory background dual to the $k=1$ ABJM theory.

Here we shall focus on the simplest case of the index (2.3) with all $u_a=1$, i.e. $\mathrm{Tr}[(-1)^{\mathrm{F}} e^{-\beta(H+J_{12})}]$ that should be given by the supersymmetric partition function of the ABJM theory on $S^1_\beta\times \widetilde S^2$ (with periodic fermions and an extra shift of the $\widetilde S^2$ angle related to the shift of H by the J_{12} rotational generator, cf. [20]). The dual M-theory background $\widetilde{\mathrm{AdS}}_4\times S^7$ is then the following "twisted" analog of the Euclidean $\mathrm{AdS}_4\times S^7$

$$ds^2=\tfrac14 L^2\big(dx^2+\sinh^2 x\, d\widetilde S_2+\cosh^2 x\, dy^2\big)+L^2\big(dv^2+\cos^2 v\, dS_5+\sin^2 v\, dz^2\big)\ , \tag{3.1}$$

where the boundary of $\widetilde{\mathrm{AdS}}_4$ is $\widetilde S^2\times S^1_\beta$ with the S^1_β coordinate being $y\equiv y+\beta$ and

$$d\widetilde S_2=d\vartheta^2+\sin^2\vartheta\,(d\varphi+i\,dy)^2. \tag{3.2}$$

The topologically non-trivial shift of the 2π-periodic angle φ by iy is required by supersymmetry.[9] The metric (3.1) solves the 11d supergravity equations when supplemented by the "electric" 4-form flux

$$\begin{aligned} F_4 =&dC_3=-\tfrac{3i}{8}L^3\ \mathrm{vol}_{\widetilde{\mathrm{AdS}}_4}\,,\qquad \mathrm{vol}_{\widetilde{\mathrm{AdS}}_4}=\sinh^2 x\cosh x\, dx\wedge dy\wedge \mathrm{vol}_{\widetilde S^2}=\mathrm{vol}_{\mathrm{AdS}_4}\,,\\ C_3=&-\tfrac{i}{8}L^3\ \sinh^3 x\, dy\wedge \mathrm{vol}_{S^2}\,,\qquad\qquad L^6=32\pi^2 N\ell_{\mathrm P}^6, \end{aligned}\tag{3.3}$$

where the shift term in $\mathrm{vol}_{\widetilde S^2}=\sin\vartheta\, d\vartheta\wedge(d\varphi+i\,dy)$ does not contribute to the product with dy.

3.1. M5 brane action

The Euclidean action of a probe M5 brane in an 11d supergravity background may be written as [21–23] (see also [24,21,25])

$$S=T_5\int d^6\xi\Big[\sqrt{\det(G_{mn}+\widetilde{\mathrm H}_{mn})}-\tfrac{i}{4}\sqrt{G}\,\widetilde{\mathrm H}^{\star\, mn}\widetilde{\mathrm H}_{mn}\Big]+iT_5\int\Big(C_6+\tfrac12 \mathrm H\wedge C_3\Big)+S_{\mathrm F}(\theta), \tag{3.4}$$

$$\begin{aligned} &H_{mn\ell}=3\partial_{[m}A_{n\ell]},\qquad \mathrm H_{mn\ell}=H_{mn\ell}-C_{3,mn\ell},\qquad H^{\star\, mn\ell}=\frac{1}{6\sqrt{G}}\,\varepsilon^{mn\ell abc}H_{abc},\\ &\widetilde{\mathrm H}_{mn}=\mathrm H^\star_{mn\ell}U^\ell,\qquad \widetilde{\mathrm H}^\star_{mn}=\mathrm H_{mn\ell}U^\ell,\qquad U_\ell(\xi)\equiv\frac{1}{\sqrt{(\partial_r a)^2}}\,\partial_\ell a(\xi)\ . \end{aligned}\tag{3.5}$$

Here $G_{mn}=\partial_m X^M\partial_n X^N G_{MN}(X(\xi))$ is the induced metric (X^M are 11d coordinates), $H_{mn\ell}$ (which is self-dual on shell) is the field strength of the world-volume antisymmetric gauge field $A_{mn}(\xi)$ and θ denotes the 11d fermions ($S_{\mathrm F}(\theta)$ is given in (3.35) below). The 6-form C_6 is defined by[10]

$$dC_6=F_4^\star-\frac12 C_3\wedge F_4\ . \tag{3.6}$$

The auxiliary scalar $a(\xi)$ ensures the manifest 6d covariance of the action. It may be fixed by a gauge choice $a(\xi)=\xi^1$ [27].

Below we will consider a semiclassical expansion of the path integral for the M5 brane in the $\widetilde{\mathrm{AdS}}_4\times S^7$ background (3.1), (3.3)

$$\mathrm Z=\int[dX\,dA_2\,d\theta]\,e^{-S[X,A_2,\theta]}=\ Z_1\,e^{-S_{\mathrm{cl}}}\big[1+\mathcal O(T_5^{-1})\big]\ ,\qquad S_{\mathrm{cl}}=T_5\,\bar S_{\mathrm{cl}}\ , \tag{3.7}$$

$$Z_1=e^{-F}\ ,\qquad F=\tfrac12\sum_r(-1)^{\mathrm F_r}\log\det\Delta_r\ . \tag{3.8}$$

Here Δ_r are second-derivative operators of quadratic fluctuations.

The action (3.4) admits the classical solution corresponding to the M5 brane wrapped on $S^1\subset\widetilde{\mathrm{AdS}}_5$ and $S^5\subset S^7$ in (3.1) with other X^M coordinates, the A_{mn} gauge field, and the fermions being trivial

$$X^1=y=\xi^1,\qquad X^i(S^5)=\xi^i\ (i=2,...,6)\ ,\qquad x=0,\quad v=0,\quad A_{mn}=0,\quad \theta=0\ . \tag{3.9}$$

The induced metric (up to the overall L^2 factor) is then

$$G^{(0)}_{mn}d\xi^m d\xi^n=\tfrac14 d\xi_1^2+dS_5\,,\qquad dS_5=g_{ij}(\xi)d\xi^i d\xi^j,\qquad \xi^1\equiv\xi^1+\beta\ , \tag{3.10}$$

[9] As already mentioned in footnote 5, this may be compared to the 11d metric $\mathrm{AdS}_7\times\widetilde S^4$ in ref. [5] which discussed the M2 brane $S^1_\beta\times S^2$ instanton contribution to the large N expansion of the superconformal index of the (2,0) theory. The roles of S^2 and S^5 are thus effectively interchanged. Here the twist is in the AdS_4 part, while in [5] it was in the S^4 part of $\mathrm{AdS}_7\times S^4$ (as it was associated with the R-charge generator in the definition of the (2,0) index).

[10] $F_4^\star$ is the 11d dual of F_4. Note also that $d(dC_6)=0$ on the equations of motion for C_3 (assuming there is no 11d gravitino background, see, e.g., [26]).

where dS_5 is the metric of a unit-radius 5-sphere. Note that the presence of the $\frac{1}{4}$ factor in front of $d\xi_1^2$ means that the effective length of the "thermal" circle is $\frac{1}{2}\beta$ rather than β.

The corresponding classical value of the M5 brane action may be written as

$$S_{\rm cl} = S_{\rm V} + S_{\rm WZ} = T_5 \int d^6\xi\, \sqrt{\det G} + i\, T_5 \int C_6\ . \tag{3.11}$$

Here the 5-brane tension enters together with the L^6 factor of the scale in (3.1), (3.3)[11]

$$T_5 = \frac{1}{(2\pi)^5\, \ell_{\rm P}^6}, \qquad \mathrm{T}_5 \equiv T_5 L^6 = \frac{1}{\pi^3} N\ . \tag{3.12}$$

To find C_6 in (3.6) we note that (3.3) implies that, written in terms of vielbein 1-forms, the expression of F_4 is[12]

$$F_4 = -6iL^3\, e_x \wedge dy \wedge e_\vartheta \wedge e_\varphi\ , \qquad F_4^\star = -6L^6\, e_v \wedge e_{S^5} \wedge e_z = -6L^6 \cos^5 v\, \sin v\, dv \wedge {\rm vol}_{S^5} \wedge dz, \tag{3.13}$$

where we used the explicit form of the S^7 metric in (3.1). As a result,

$$C_6 = L^6 \cos^6 v\ {\rm vol}_{S^5} \wedge dz\ . \tag{3.14}$$

The WZ term in (3.4), (3.11) then vanishes on the classical solution (3.9) (but it will contribute to the quadratic fluctuation action). Using that ${\rm vol}(S^5) = \pi^3$ we thus get

$$S_{\rm cl} = S_{\rm V} = \mathrm{T}_5\, \tfrac{1}{2}\beta\ {\rm vol}(S^5) = \tfrac{1}{2}\beta N\ . \tag{3.15}$$

Thus $e^{-S_{\rm cl}} = q^N$ where $q = e^{-\frac{1}{2}\beta}$. This reproduces the leading q^N term in (2.8) in agreement with the suggestion [6] that the $S^1_\beta \times S^5$ M5 instanton should be responsible for the leading large N correction to the ABJM index.

Below we will compute the one-loop correction (3.8) to the M5 brane partition function expanded near this classical solution. We will choose the static gauge identifying 6 of the X^M coordinates with the M5 world-volume coordinates ξ^m ($m = 1, \dots, 6$), i.e. setting to zero the fluctuations of X^1 and X^i in (3.9). We will also fix the κ-symmetry gauge (3.38) for the fermions. We will be left with 5 scalar fluctuation modes of the transverse coordinates, 3 physical modes of A_{mn} and 8 fermionic modes.[13]

The corresponding fluctuation operators $\Delta = -\nabla^2 + \dots$ in (3.8) will be defined on $S^1_\beta \times S^5$ with the metric (3.10) with periodic S^1_β boundary conditions for fermions.[14] Their contributions to the one-loop free energy (3.8) may be split into the "Casimir" part linear in the effective length $\frac{1}{2}\beta$ of the circle in (3.10) and a "thermal" part which is exponentially small at large β (see, e.g., [30])

$$F(\beta) = \tfrac{1}{2}\beta E_c + \bar F(\beta)\ . \tag{3.16}$$

The term $\bar F(\beta($ will have the form (here k is labelling eigenvalues of a fluctuation operator on S^5)

$$\bar F(\beta) = \sum_k c_k \log\big(1 - e^{-\frac{1}{2}\beta E_k}\big) = -\sum_{m=1}^\infty \frac{1}{m} \sum_k c_k\, e^{-\frac{1}{2}\beta m E_k}\ . \tag{3.17}$$

Thus the one-loop partition function in (3.7), (3.8) may be written as

$$Z_1 = e^{-\frac{1}{2}\beta E_c}\, \mathcal{Z}(q)\ , \qquad \mathcal{Z}(q) = e^{-\bar F} = \exp\Big[\sum_{m=1}^\infty \frac{1}{m} \widehat{\mathcal{Z}}(q^m)\Big] \equiv {\rm PE}[\widehat{\mathcal{Z}}(q)], \tag{3.18}$$

where the "single-particle" partition function $\widehat{\mathcal{Z}}(q)$ is (*cf.* (2.4))

$$\widehat{\mathcal{Z}}(q) = \sum_k c_k\, q^{E_k}, \qquad q \equiv e^{-\frac{1}{2}\beta}. \tag{3.19}$$

Each of the one-loop contribution of 6d fluctuation fields will have no logarithmic UV divergences with power divergences removed by zeta-function regularization. The aim will be to compare the total result for the partition function with the prefactor of the q^N term in the ABJM index in (2.8), (2.13), (2.14).

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[11] Note that the 5-brane tension is related to the 2-brane tension $T_2 = \frac{1}{(2\pi)^2 \ell_{\rm P}^3}$ as [28]: $T_5 = \frac{1}{2\pi} T_2^2$ and thus also $\mathrm{T}_5 = \frac{1}{2\pi} \mathrm{T}_2^2$ where $\mathrm{T}_2 \equiv T_2 L^3$.

[12] Note that $F_4^\star$ is defined originally in Minkowski space where F_4 is real. In the present case $F_4^\star$ is purely spatial and thus remains real after the rotation to the Euclidean signature.

[13] These physical degrees of freedom are the same as of a 6d $(2,0)$ tensor multiplet describing the leading 2-derivative term in the static-gauge M5 brane action in flat space, *i.e.* five scalars, a real antisymmetric 2-tensor with (anti)self-dual field strength, and four symplectic Majorana–Weyl 6d spinors. They transform, respectively, in the $\mathbf{5}, \mathbf{1}, \mathbf{4}$ representations of $USp(4)$ (see, e.g., [29]).

[14] The discussion of fluctuations for the bosons and fermions will be very similar to the one in the case of the $S^1_\beta \times S^2$ M2 brane instanton contribution to the index of the $(2,0)$ theory in [5].

3.2. Scalar fluctuations

Expanding the action (3.4) to quadratic order in fluctuations around the classical background (3.9) using the static gauge we observe that as the 3-form C_3 in (3.3) is of higher order in fluctuation fields its contribution to H_{mnk} as well as the second part of the WZ term may be ignored. The resulting scalar part of the M5 brane action then takes the form of (3.11).

To recall, in the static gauge $y = \xi_1 \in (0, \beta)$ and $\xi_2, \dots, \xi_6$ are coordinates on S^5. The 5 "transverse" angular coordinate fields $x(\xi), \theta(\xi), \varphi(\xi), v(\xi), z(\xi)$ can be expressed in terms of 5 "Cartesian" ones $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ of AdS_4 part and $\mathrm{Y}_1, \mathrm{Y}_2$ of S^7 part defined as

$$\mathrm{X}_1 = \sinh x \cos\vartheta, \qquad \mathrm{X}_2 = \sinh x \sin\vartheta \cos\varphi, \qquad \mathrm{X}_3 = \sinh x \sin\vartheta \sin\varphi,$$
$$\mathrm{Y}_1 = \sin v \cos z, \qquad \mathrm{Y}_2 = \sin v \sin z, \tag{3.20}$$

$$x = \mathrm{arcsinh}\sqrt{\mathrm{X}_1^2 + \mathrm{X}_2^2 + \mathrm{X}_3^2}, \qquad \vartheta = \arctan\frac{\sqrt{\mathrm{X}_2^2 + \mathrm{X}_3^2}}{\mathrm{X}_1}, \qquad \varphi = \arctan\frac{\mathrm{X}_3}{\mathrm{X}_2},$$
$$v = \arcsin\sqrt{\mathrm{Y}_1^2 + \mathrm{Y}_2^2}, \qquad z = \arctan\frac{\mathrm{Y}_2}{\mathrm{Y}_1}. \tag{3.21}$$

For generality, let us consider the case when the shift of φ by iy has a coefficient κ (we will set κ to its "supersymmetric" value $\kappa = 1$ at the end). To expand the volume S_{V} part of (3.4), (3.11), we note that according to (3.21)

$$\tfrac{1}{4}\Big[dx^2 + \sinh^2 x\left(d\vartheta^2 + \sin^2\vartheta\,(d\varphi + i\kappa\, d\xi_1)^2\right) + \cosh^2 x\, d\xi_1^2\Big]$$
$$= \tfrac{1}{4}d\xi_1^2\,[1 + \mathrm{X}_1^2 + (1-\kappa^2)(\mathrm{X}_2^2 + \mathrm{X}_3^2)] + \tfrac{1}{4}d\mathrm{X}_a^2 + \tfrac{i}{2}\kappa\, d\xi_1\,(\mathrm{X}_2 d\mathrm{X}_3 - \mathrm{X}_3 d\mathrm{X}_2) + \cdots, \tag{3.22}$$
$$dv^2 + \cos^2 v\, dS_5 + \sin^2 v\, dz^2 = dS_5\,(1 - \mathrm{Y}_a^2) + d\mathrm{Y}_a^2 + \cdots. \tag{3.23}$$

The perturbed induced metric is then

$$G_{mn}\, d\xi^m d\xi^n = (G_{mn}^{(0)} + G_{mn}^{(1)} + \cdots)\, d\xi^m\, d\xi^n, \qquad m, n = 1, \dots, 6\,, \tag{3.24}$$

where $G_{mn}^{(0)}$ is the metric of $S^1 \times S^5$ given in (3.10). We shall use $g_{ij} \equiv g_{ij}(S^5)$ $(i, j = 2, ..., 6)$ and denote the derivative over the S^1 direction ξ^1 as $\partial_1\Phi \equiv \dot\Phi$. After an integration by parts and rescaling X_p $(p = 1,2,3)$ and Y_r $(r = 1,2)$ fields by constant factors (including $(\mathrm{T}_5)^{-1/2}$) we get for the corresponding quadratic fluctuation contributions to S_{V}

$$\delta S_{\mathrm{V}} = \delta S_{\mathrm{V}}(\mathrm{X}) + \delta S_{\mathrm{V}}(\mathrm{Y}),$$
$$\delta S_{\mathrm{V}}(\mathrm{X}) = \int d^6\xi\,\sqrt{g}\,\Big(\tfrac{1}{2}g^{ij}\partial_i\mathrm{X}_p\partial_j\mathrm{X}_p + 2\mathrm{X}_1^2 + 2(1-\kappa^2)(\mathrm{X}_2^2 + \mathrm{X}_3^2) + 2\dot{\mathrm{X}}_a^2 + 8i\kappa\,\mathrm{X}_2\dot{\mathrm{X}}_3\Big), \tag{3.25}$$
$$\delta S_{\mathrm{V}}(\mathrm{Y}) = \int d^6\xi\,\sqrt{g}\,\Big(\tfrac{1}{2}g^{ij}\partial_i\mathrm{Y}_r\partial_j\mathrm{Y}_r + 2\,\dot{\mathrm{Y}}_r^2 - \tfrac{5}{2}\,\mathrm{Y}_r^2\Big)\,. \tag{3.26}$$

To find the contribution of the WZ term in (3.11) we use (3.14) (integrating by parts and again doing constant field rescaling)

$$iT_5\int C_6 \to \int \mathrm{vol}_{S^5} \wedge [3 \times 2 \times (\mathrm{Y}_2 d\mathrm{Y}_1 - \mathrm{Y}_1 d\mathrm{Y}_2)] = \int 12\;\mathrm{Y}_2\dot{\mathrm{Y}}_1\;\mathrm{vol}_{S^5} \wedge d\xi_1\,. \tag{3.27}$$

Combining this with (3.26) we get

$$\delta S_{\mathrm{V}}(\mathrm{Y}) + S_{\mathrm{WZ}}(\mathrm{Y}) = \int d^6\xi\,\sqrt{g}\,\Big(\tfrac{1}{2}g^{ij}\partial_i\mathrm{Y}_r\partial_j\mathrm{Y}_r + 2\,\dot{\mathrm{Y}}_r^2 - \tfrac{5}{2}\,\mathrm{Y}_r^2 + 12i\,\mathrm{Y}_2\dot{\mathrm{Y}}_1\Big). \tag{3.28}$$

Next, let us expand the fields X_p and Y_r in Fourier series in S^1, i.e. $\mathrm{X}_1 = \sum_{n=-\infty}^{\infty}\mathrm{X}_1^{(n)}\, e^{in_\beta\xi_1}$, etc., so that $\partial_1 \to in_\beta$, where

$$n_\beta \equiv \frac{2\pi n}{\beta}, \qquad n \in \mathbb{Z}. \tag{3.29}$$

As a result, we get towers of scalar fields on S^5 with the standard kinetic operators $-\nabla^2$ on S^5 and the effective mass terms depending on n_β. Explicitly, from (3.25) X_1 then leads to a scalar tower on S^5 with the mass squared

$$M_{\mathrm{X}_1}^2 = 4 + 4n_\beta^2\,. \tag{3.30}$$

Since the 6d Lagrangian for $\mathrm{X}_2, \mathrm{X}_3$ in (3.25) may be written as

$$\mathscr{L}_{\mathrm{X}} = g^{ij}\partial_i\bar{\mathrm{X}}\partial_j\mathrm{X} - 4\bar{\mathrm{X}}\left[\ddot{\mathrm{X}} + 2\kappa\dot{\mathrm{X}} + (\kappa^2 - 1)\mathrm{X}\right], \qquad \mathrm{X} = \frac{1}{\sqrt{2}}(\mathrm{X}_2 + i\mathrm{X}_3)\,, \tag{3.31}$$

the corresponding mass in the action on S^5 will be given by (we now specialize to the case of $\kappa = 1$)

$$M_{\mathrm{X}}^2 = 4 + 4\,(n_\beta + i\kappa)^2\Big|_{\kappa=1} = 4 + 4\,(n_\beta + i)^2\,. \tag{3.32}$$

Similarly, from (3.28) for Y_r we get

$$\mathscr{L}_{\mathrm{Y}} = g^{ij}\partial_i\bar{\mathrm{Y}}\partial_j\mathrm{Y} + \bar{\mathrm{Y}}(-5\mathrm{Y} - 4\ddot{\mathrm{Y}} - 12\dot{\mathrm{Y}}) , \qquad \mathrm{Y} = \frac{1}{\sqrt{2}}(\mathrm{Y}_1 + i\mathrm{Y}_2) , \tag{3.33}$$

corresponding to

$$M^2_{\mathrm{Y}} = -5 + 4n^2_\beta - 12i\, n_\beta = 4 + 4(n_\beta - \tfrac{3}{2}i)^2 . \tag{3.34}$$

In (3.30), (3.32), (3.34) we isolated the $M^2 = 4$ term that corresponds to a conformally coupled massless scalar in 6d space $S^1 \times S^5$ (cf. footnote 17 below). The factor 4 that multiplies the n^2_β terms in the masses in (3.30), (3.32), (3.34) originates from the $\frac{1}{4}$ in the induced metric (3.10)[15] and corresponds to the effective rescaling $\beta \to \frac{1}{2}\beta$ already mentioned above.

Note also that while the shift of n_β in (3.32) is a direct consequence of the twist of the S^2 angle in (3.2) implying ξ^1-dependent rotation in the $(\mathrm{X}_2, \mathrm{X}_3)$ plane, the non-trivial shift in (3.34) is due to the contribution of the WZ term (3.27). The two are effectively correlated as a result of the underlying supersymmetry of the problem.[16]

3.3. Fermionic action

The quadratic fermionic part of the κ-symmetric M5 brane action in a general background which is solution of 11d supergravity may be written as [21,22] (here $m, n = 1, ..., 6$ as in (3.4), X^M are 11d coordinates and θ is a 32 component 11d Majorana spinor)

$$S_F = iT_5 \int d^6\xi \Big[\sqrt{G}\, G^{mn}\partial_m X^M\, \bar\theta\Gamma_M \hat{D}_n\theta \\ - \tfrac{1}{5!}\varepsilon^{mnklpq}\partial_m X^M \partial_n X^N \partial_k X^K \partial_l X^L \partial_p X^R\, \bar\theta\Gamma_{MNKLR}\hat{D}_q\theta + \mathcal{O}(\theta^4)\Big] , \tag{3.35}$$

$$G_{mn} = \partial_m X^M \partial_n X^N G_{MN}(X), \qquad G_{MN} = E^A_M E^A_N, \qquad \Gamma_M = E^A_M(X)\Gamma_A , \tag{3.36}$$

$$\hat{D}_k = \partial_k X^M \hat{D}_M, \qquad \hat{D}_M = \mathrm{D}_M - \tfrac{1}{288}(\Gamma^{PNKL}{}_M + 8\Gamma^{PNK}\delta^L_M)F_{PNKL} . \tag{3.37}$$

$\hat{D}_M$ is the generalized 11d spinor covariant derivative [31] and $\mathrm{D}_M = \partial_M + \frac{1}{4}\Gamma_{AB}\omega^{AB}_M$. In particular cases of the maximally supersymmetric $\mathrm{AdS}_4 \times S^7$ or $\mathrm{AdS}_7 \times S^4$ backgrounds the fermionic part of the M5 brane action may be written in an explicit form including also higher orders in θ [32,25,33].

In the present case of the M5 brane wrapped on $S^1_\beta \times S^5$ in the "twisted" $\widetilde{\mathrm{AdS}}_4 \times S^7$ background (3.1), the computation of the fermionic contribution to the one-loop effective action is very similar to the one in the case of the M2 brane instanton wrapped on $S^1_\beta \times S^2$ in $\mathrm{AdS}_7 \times \widetilde{S}^4$ considered in [5]. As discussed above, we fix the static gauge $X^m = \xi^m$, $X^i = 0$ $(i = 2, ..., 6)$. A natural choice for a κ-symmetry gauge is like in the flat space case, i.e.

$$(1 - \Gamma)\theta = 0 , \qquad \Gamma \equiv \frac{1}{6!\sqrt{G}}\varepsilon^{mnklpq}\partial_m X^M \partial_n X^N \partial_k X^K \partial_l X^L \partial_p X^R \partial_q X^P \Gamma_{MNKLRP} . \tag{3.38}$$

Then the WZ term in (3.35) takes the same form as the Dirac term with the number of independent components of θ reduced to 16.

Like in the M2 brane case in [5] the resulting fermionic action may be represented as a collection of 4 MW 6d fermions (each with 4 real components) on $S^1 \times S^5$ with the kinetic operator $i\Gamma^k\nabla_k + \Pi$ coming from $\hat{D}_n$ in (3.35). The "mass" operator Π contains two operators P_1, P_2 built out of Γ-matrices that square to one and thus can be used to define projectors. One originates from the contribution to (3.37) of the F_4 flux along the AdS_4 directions in (3.3) (i.e. it is proportional to $\Gamma_{x\vartheta y\varphi}$) and the other is related to the contribution of the constant term in the Lorentz connection related to the rotation in (3.2) (i.e. it is proportional to $\Gamma_{y\varphi}$)

$$\Pi = \tfrac{3}{2}P_1 + P_2 , \qquad P_1^2 = P_2^2 = 1 , \qquad [P_1, P_2] = 0 . \tag{3.39}$$

Diagonalizing Π we get four 6d MW fermions with the effective masses $\pm\frac{3}{2} \pm 1$, i.e. $\pm\frac{1}{2}$ and $\pm\frac{5}{2}$.

Recalling that the vielbein component corresponding to the y direction in the induced 6d metric originating from (3.1) has an extra $\frac{1}{2}$ relative to the S^5 part (so that inverse vielbein comes with a factor of 2) we get $\Gamma^m\partial_m \to 2\Gamma^1\partial_1 + \Gamma^j\partial_j$ where j is the index of the S^5 directions. Expanding the 6d fermions in Fourier modes in ξ^1 implies that $\partial_1 \to in_\beta$ (cf. (3.29)) and we thus end up with four towers of the 5d fermionic fields on S^5 with the operators

$$\mathcal{D} = i\Gamma^j\nabla_j + M_\psi, \qquad M_\psi = 2(n_\beta + i\nu), \qquad \nu = \pm\tfrac{1}{4}, \ \pm\tfrac{5}{4} . \tag{3.40}$$

Since the S^1_β mode number n_β takes both positive and negative values we may assume that ν takes just positive values $\frac{1}{4}$ and $\frac{5}{4}$. The corresponding squared fermionic operator has the form

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15 Explicitly, we have $-\nabla^2_{S_1\times S_5} = -4\partial_1^2 - \nabla^2_{S^5}$.

16 Note that the signs of the shifts in (3.32) and (3.34) do not matter as n_β takes both positive and negative values.

$$\Delta_{\frac{1}{2}} = -\hat{\nabla}^2_{S^5} + \tfrac{1}{4}R + M_\psi^2 = -\hat{\nabla}^2_{S^5} + 5 + M_\psi^2 \ , \tag{3.41}$$

where $\hat{\nabla}$ contains spinor connection and we used that the scalar curvature is $R(S^1\times S^5) = R(S^5) = 20$.

3.4. One-loop free energies

Having found the scalar and fermion quadratic fluctuation operators on $S^1\times S^5$ we may now compute the corresponding contributions to the one-loop free energy in (3.8), (3.18), (3.19). We will then include also the contribution of the antisymmetric tensor field which is straightforward to find as it does not involve a twist or coupling to C_3.

3.4.1. Scalars

The 5 scalar fluctuations written in terms of the Fourier modes on S^1 as towers of fields on S^5 have effective masses (see (3.30), (3.32), (3.34))

$$M^2 = 4 + \bar{M}_n^2, \qquad \bar{M}_n^2 = 4\,(n_\beta + i\nu)^2, \qquad n_\beta = \frac{2\pi}{\beta}\, n, \ \ n\in\mathbb{Z}, \tag{3.42}$$

$$\nu(\mathrm{X}_1) = 0, \qquad \nu(\mathrm{X}_2) = \nu(\mathrm{X}_3) = 1, \qquad \nu(\mathrm{Y}_1) = \nu(\mathrm{Y}_2) = \tfrac{3}{2}\ . \tag{3.43}$$

The contribution to free energy from each scalar fluctuation field is then[17]

$$\begin{aligned} F_\phi^{S^1\times S^5}(\beta,\nu) &= \sum_{n\in\mathbb{Z}} F_\phi^{S^5}(\beta,\bar{M}_n) = \tfrac{1}{2}\sum_{n\in\mathbb{Z}} \log\det(-\nabla^2_{S^5} + 4 + \bar{M}_n^2) \\ &= \tfrac{1}{24}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}(k+1)(k+2)^2(k+3)\,\log\left[(k+2)^2 + 4(n_\beta+i\nu)^2\right]\ . \end{aligned} \tag{3.44}$$

The double sum in (3.44) is formally divergent and can be defined using the standard ζ-function regularization (see Appendix D). As a result, we get (cf. (3.16))

$$F_\phi^{S^1\times S^5}(\beta,\nu) = \tfrac{1}{2}\beta\, E_{c,\phi}(\nu) + \bar{F}_\phi^{S^1\times S^5}(\beta), \tag{3.45}$$

$$\bar{F}_\phi^{S^1\times S^5}(\beta,\nu) = \tfrac{1}{24}\sum_{\pm}\sum_{k=0}^{\infty}(k+1)(k+2)^2(k+3)\,\log\left[1 - e^{-(k+2\pm 2\nu)\frac{\beta}{2}}\right]\ , \tag{3.46}$$

$$E_{c,\phi}(\nu) = -\tfrac{31}{60480} + \tfrac{1}{18}\nu^4 - \tfrac{4}{45}\nu^6\ . \tag{3.47}$$

From $\bar{F}$, we read off the corresponding single-particle partition function (see (3.17), (3.19))

$$\hat{\mathcal{Z}}_\phi(q,\nu) = \tfrac{1}{24}\sum_{\pm}\sum_{k=0}^{\infty}(k+1)(k+2)^2(k+3)\,q^{k+2\pm 2\nu}, \qquad\qquad q = e^{-\frac{1}{2}\beta}. \tag{3.48}$$

Computing the sum, we obtain[18]

$$\hat{\mathcal{Z}}_\phi(q,\nu) = \frac{q^2(1+q)}{2(1-q)^5}\,(q^{2\nu} + q^{-2\nu}). \tag{3.49}$$

The total contribution of the five scalars corresponding to (3.43) is then

$$\hat{\mathcal{Z}}_{\rm scalar}(q) = \hat{\mathcal{Z}}_\phi(q,0) + 2\hat{\mathcal{Z}}_\phi(q,1) + 2\hat{\mathcal{Z}}_\phi(q,\tfrac{3}{2}) = \frac{(1+q)(1+q+q^3+q^5+q^6)}{q\,(1-q)^5}. \tag{3.50}$$

The sum of the scalar Casimir energies is given by

$$E_{c,\rm scalar} = E_{c,\phi}(0) + 2E_{c,\phi}(1) + 2E_{c,\phi}(\tfrac{3}{2}) = -\tfrac{92639}{60480}\ . \tag{3.51}$$

3.4.2. Fermions

The discussion of the fermionic contribution to free energy corresponding to (3.40), (3.41) is similar. We assume that fermions are periodic on S^1 as appropriate for a “supersymmetric” partition function.

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[17] We use label ϕ to indicate the scalar contribution. In general, the operator $-\nabla^2_{S^d}$ on a unit-radius d-sphere has eigenvalues λ_k with degeneracy d_k (see [34])

$$\lambda_k = k(k+d-1), \qquad \mathrm{d}_k = \frac{(2k+d-1)\,(k+d-2)!}{k!\,(d-1)!}, \qquad k = 0,1,2,\dots\ .$$

Thus for a conformally coupled (in $S^1\times S^d$) scalar operator $-\nabla^2_{S^d} + \frac{1}{4}(d-1)^2$ one gets $\lambda_k \to \lambda_k + \frac{1}{4}(d-1)^2 = (k+\frac{d-1}{2})^2$ or $(k+2)^2$ in the present case of $d=5$.

[18] Note that in the conformal coupling case, i.e. for $\nu=0$, the expression (3.48), (3.43), i.e. $\hat{\mathcal{Z}}_\phi(q,0) = \frac{q^2(1-q^2)}{(1-q)^6} = \frac{q^2(1+q)}{(1-q)^5}$ can be obtained by counting scalar operators on $\mathbb{R}^6$, see Appendix E.

Let us consider first a conformally coupled, i.e. massless, spinor field on $S^1 \times S^5$. Using that the eigenvalues and degeneracies of the operator $-\hat{\nabla}^2_{S^d}$ with spinor covariant derivative are given by [35]

$$\lambda_k = \Big(k+\frac{d}{2}\Big)^2 - \frac{d(d-1)}{4}, \qquad \mathrm{d}_k = \frac{(k+d-1)!}{k!(d-1)!}, \qquad k=0,1,2,\dots, \tag{3.52}$$

the eigenvalues of $-\hat{\nabla}^2_{S^d} + \frac{1}{4}R = -\hat{\nabla}^2_{S^d} + \frac{d(d-1)}{4}$ are simply $(k+\frac{d}{2})^2$ (with the same degeneracy). For $d=5$ this leads to the expression for the free energy contribution of a single MW spinor analogous to the one in the scalar case (3.45)[19]

$$\bar{F}_\psi^{S^1\times S^5}(\beta,\nu) = 4\times\tfrac{1}{24}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)(k+4)\,\log\big[(k+\tfrac{5}{2})^2+4(n_\beta+i\nu)^2\big]\,. \tag{3.53}$$

The sum over n here is computed as in the scalar case and one finds

$$\bar{F}_\psi^{S^1\times S^5}(\beta,\nu) = \tfrac{1}{12}\sum_{\pm}\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)(k+4)\log\big[1-e^{-(k+\frac{5}{2}\pm 2\nu)\frac{\beta}{2}}\big]\,. \tag{3.54}$$

The corresponding single-particle partition function is[20]

$$\hat{\mathcal{Z}}_\psi(q,\nu) = \frac{2q^{\frac{5}{2}}}{(1-q)^5}\left(q^{2\nu}+q^{-2\nu}\right). \tag{3.55}$$

The Casimir energy can be found as in (D.8) and $E_c^+(\nu)=E_c^-(\nu)$ so that

$$E_{c,\psi}(\nu) = \frac{1}{12}\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)(k+4)(k+\tfrac{5}{2}+2\nu)^s \overset{s\to 1}{\to} \tfrac{367}{96768} - \tfrac{3}{32}\nu^2 + \tfrac{5}{18}\nu^4 - \tfrac{8}{45}\nu^6\,. \tag{3.56}$$

According to (3.40) the total contribution of two MW spinors with $\nu=\frac{1}{4}$ and two with $\nu=\frac{5}{4}$ is then given by (we reverse the sign to account for fermion statistics)

$$-2\left[\hat{\mathcal{Z}}_\psi(q,\tfrac{1}{4})+\hat{\mathcal{Z}}_\psi(q,\tfrac{5}{4})\right] = -\frac{4(1+q^2+q^3+q^5)}{(1-q)^5}\,, \qquad -2\left[E_{c,\psi}(\tfrac{1}{4})+E_{c,\psi}(\tfrac{5}{4})\right] = \tfrac{2173}{7560}\,. \tag{3.57}$$

3.4.3. Antisymmetric tensor

The contribution of the A_{mn} gauge field in (3.4) to the one-loop partition function on $S^1\times S^5$ is straightforward to find by first relaxing the self-duality condition on its strength H_{mnk} and taking half of the resulting value of free energy at the end (*cf.* [37] and references there). Rather than computing the relevant determinants directly one may use the underlying conformal symmetry (and the absence of conformal anomaly in the $S^1\times S^5$ case) to map the problem to scaling dimension counting on $\mathbb{R}^6$ (*cf.* footnote 18) and then use the expression for the corresponding partition function in [38,36].

As a result, the single-particle partition function for the self-dual antisymmetric tensor on $S^1_\beta\times S^5$ is found to be[21]

$$\hat{\mathcal{Z}}_A(q) = \tfrac{1}{4}\sum_{k=0}^{\infty}(k+1)(k+2)(k+4)(k+5)q^{k+3} = \frac{10q^3-15q^4+6q^5-q^6}{(1-q)^6} = \frac{q^3(10-5q+q^2)}{(1-q)^5}\,. \tag{3.58}$$

The corresponding contribution to the free energy is[22]

$$\bar{F}_A^{S^1\times S^5}(\beta) = -\sum_{m=1}^{\infty}\frac{1}{m}\hat{\mathcal{Z}}_A(q^m) = \tfrac{1}{4}\sum_{k=0}^{\infty}(k+1)(k+2)(k+4)(k+5)\log(1-q^{k+3})\,. \tag{3.59}$$

The Casimir energy is [36]

$$E_{c,A} = -\tfrac{191}{4032}. \tag{3.60}$$

[19] Here the factor 4 corresponds to 4 real components of a MW spinor. Note that for large k the summand here is the same as 4 times the one in real scalar case in (3.44).

[20] For $\nu=0$ the single-particle partition function of a massless 6d fermion on $S^1\times S^5$ can be found by conformal mapping to $\mathbb{R}^6$ and operator counting giving $\hat{\mathcal{Z}}_\psi(q,0) = \frac{4q^{5/2}(1-q)}{(1-q)^6} = \frac{4q^{5/2}}{(1-q)^5}$, as in Eq. (A.4) in [36].

[21] Here again $q=e^{-\frac{1}{2}\beta}$ as in (3.48), taking into account that the radii of S^1 and S^5 differ by 1/2.

[22] Note that the large k limit of the summand here is 3 times the one for a real scalar in (3.46), in agreement with the fact that a 2-form field with self-dual field strength has 3 physical degrees of freedom.

Table 1
Single-particle partition function, shift ν and Casimir energy for the fluctuation fields.

	$\hat{\mathcal{Z}}(q,\nu)$	ν	E_c
scalar	$\frac{q^2(1+q)}{2(1-q)^5}\,(q^{2\nu}+q^{-2\nu})$	$0,\ 2\times 1,\ 2\times\frac{3}{2}$	$-\frac{31}{60480}+\frac{1}{18}\nu^4-\frac{4}{45}\nu^6$
self dual tensor	$\frac{q^3(10-5q+q^2)}{(1-q)^5}$	0	$-\frac{191}{4032}$
fermion	$\frac{2q^{\frac{5}{2}}}{(1-q)^5}\,(q^{2\nu}+q^{-2\nu})$	$2\times\frac{1}{4},\ 2\times\frac{5}{4}$	$\frac{367}{96768}-\frac{3\nu^2}{32}+\frac{5}{18}\nu^4-\frac{8}{45}\nu^6$

Table 2
Scalar and fermion zero modes and their multiplicity.

field	ν	k	no. $\times\, \mathrm{d}_k$
$2\times\phi$	1	0	2×1
$2\times\phi$	$\frac{3}{2}$	1	2×6
$2\times\psi$	$\frac{5}{4}$	0	2×4

3.4.4. Summary

To summarize, we collect in Table 1 the expressions for the single-particle partition functions and Casimir energies for all of the above fields.[23]

These are to be combined taking into account that fermions contribute with opposite sign, i.e.

$$\hat{\mathcal{Z}}_{\rm tot}=\sum(-1)^{\rm F}\,\hat{\mathcal{Z}}\,,\qquad\qquad E_{c,\rm tot}=\sum(-1)^{\rm F}\,E_c\,. \tag{3.61}$$

4. Partition function of M5 brane instanton and ABJM index

As we already saw in (3.15), the classical action contribution $e^{-S_{\rm cl}}$ of the $S^1\times S^5$ M5 brane reproduces the factor q^N in the ABJM index (2.8). Let us now present the total result for the corresponding one-loop partition function by combining the contributions of the individual fluctuations found in the previous section. We will then compare it with the prefactor of the q^N term in the ABJM index (1.2), (2.13) which is the $\boldsymbol{u}=\mathbf{1}$ limit of the $q^N\,\delta\mathrm{I}^{(1)}_N$ part of (2.8).

4.1. One-loop partition function including the zero-mode contribution

Using the data in Table 1 or combining together (3.50), (3.54), (3.56) and (3.51), (3.57), (3.60) we get for the one-loop free energy (3.8) (see (3.16)–(3.19))

$$F=\tfrac{1}{2}\beta E_c+\bar{F},\qquad\qquad \bar{F}=-\sum_{m=1}^{\infty}\frac{1}{m}\hat{\mathcal{Z}}(q^m), \tag{4.1}$$

$$\hat{\mathcal{Z}}(q)=\hat{\mathcal{Z}}_\phi(q,0)+2\hat{\mathcal{Z}}_\phi(q,1)+2\hat{\mathcal{Z}}_\phi(q,\tfrac{3}{2})+\hat{\mathcal{Z}}_A(q)-2\left[\hat{\mathcal{Z}}_\psi(q,\tfrac{1}{4})+\hat{\mathcal{Z}}_\psi(q,\tfrac{5}{4})\right]=\frac{7q}{1-q}+3+q^{-1}-q, \tag{4.2}$$

$$E_c=E_{c,\phi}(0)+2E_{c,\phi}(1)+2E_{c,\phi}(\tfrac{3}{2})+E_{c,A}-2\Big[E_{c,\psi}(\tfrac{1}{4})+E_{c,\psi}(\tfrac{5}{4})\Big]=-\tfrac{31}{24}. \tag{4.3}$$

The presence of the constant term 3 in (4.2) implies that the sum in the expression for $\bar{F}$ in (4.1) is formally divergent. This divergence can be attributed to the contribution of the zero modes of the fluctuation operators that were not separated in our formal treatment of the "thermal" part $\bar{F}$ of the free energy in the previous section. The degeneracy implying the presence of these zero modes can be "resolved" by switching on extra twists in the background (corresponding to non-trivial values of fugacities in the definition of the index) leading to a finite expression for $\bar{F}$ (see section 4.3 below).

As follows from the expressions for the fluctuation determinants in (3.44), (3.53), (3.59), the zero modes appear only for the scalars with $\nu=1,\ \frac{3}{2}$ and fermions with $\nu=\frac{5}{4}$ and only in the sector with $n_\beta=0$ (i.e. on the "ground" level of the S^5 KK modes). The corresponding values of the S^5 mode number k and their multiplicities are summarized in Table 2. We thus get 14 bosonic and 8 fermionic zero modes that should have been separated in the one-loop determinants.[24]

Assuming that the resulting $0^8/0^8$ ambiguity can be resolved, the effective number of the remaining (bosonic) zero modes is 6. This number matches the value 3 of the constant term in (4.2). Indeed, regularizing the $k=0$ zero-mode term in the scalar case in (3.46) by setting $\nu=1-\varepsilon,\ \varepsilon\to 0$ we get $\bar{F}_0=\frac{1}{2}\log(1-e^{-\varepsilon\beta})=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}e^{-\varepsilon\beta m}$. Thus each zero mode contributes $\frac{1}{2}$ to a constant term in $\hat{\mathcal{Z}}$.

[23] To recall, here the fermions are assumed to be periodic on S^1_β so these are contributions to a supersymmetric partition function and supersymmetric Casimir energy. Note that ref. [36] considered instead the standard thermal partition function of a (2,0) multiplet with antiperiodic fermions.

[24] As usual, their presence reflects (super)symmetries preserved by the corresponding $S^1\times S^5$ solution embedded into AdS4 $\times\, S^7$ in (3.1).

Let us note that, as follows from (3.44), for $\nu = \frac{3}{2}$ we get also one negative mode for each of the two Y-scalars in (3.33), (3.50), (4.2) representing the transverse S^7 fluctuations of M5 brane wrapped on $S^5 \subset S^7$. This reflects the expected instability of this M5 brane configuration. Analytically continued, each of these negative modes produces an imaginary contribution to the resulting partition function. Their contribution corresponds to the q^{-1} term in the single-particle partition function (3.49) (or (3.50) and thus (4.2)). It translates into $\sum_{m=1} \frac{1}{m} q^{-m} = -\log(1-q^{-1})$ term in the free energy which is then indeed imaginary for $q < 1$. This issue may be formally ignored by assuming an analytic continuation in q as is usually done when computing the index.

With the standard normalization of the M5 brane path integral (3.7), each zero mode is expected to contribute a factor of $\sqrt{\mathrm{T}_5} \sim \sqrt{N}$ (cf. (3.4), (3.12)). This suggests that the resulting expression for the one-loop partition function in (3.7) may be written as

$$\mathrm{Z}_1 = Z_1 e^{-S_{\rm cl}} = c_0\, N^3\, q^N\, q^{-\frac{31}{24}}\, e^{-\bar{F}'} \,, \tag{4.4}$$

$$\bar{F}' = -\sum_{n=1}^{\infty} \frac{1}{m} \big[\widehat{\mathcal{Z}}(q^m) - 3\big] = -\log(-q) - 7\sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m} \,. \tag{4.5}$$

In (4.4) we explicitly included the zero-mode, classical, and the Casimir energy q^{E_c} (4.3) contributions and in (4.5) we subtracted the constant term in (4.2).[25] As a result,

$$\mathrm{Z}_1 = c_0\, N^3\, q^{N-\frac{31}{24}} \Big[-q \exp\Big(7\sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m}\Big)\Big] \,. \tag{4.6}$$

4.2. Matching to large N correction in unrefined ABJM index

Comparing the partition function (4.6) to $q^N \delta \mathrm{I}_N^{(1)}$ part of (2.8) which in the $\boldsymbol{u} = \mathbf{1}$ case takes the form of the leading N^3 term in (1.2), (2.13) we observe a close similarity. Removing the Casimir energy contribution $q^{-\frac{31}{24}}$ (as we are after an index that counts, according to the conjecture of [6], the BPS fluctuations of the wrapped M5 brane)[26] and assuming that $c_0 = \frac{1}{6}$ we match the non-trivial q-dependent prefactor of the "classical" q^N contribution. Indeed, we have

$$\exp\Big[\sum_{m=1}^{\infty} \frac{1}{m} \frac{7q^m}{1-q^m}\Big] = \exp\Big[\sum_{m=1}^{\infty} \frac{1}{m} \sum_{m'=1}^{\infty} 7q^{m'}\Big] = \exp\Big[-7\sum_{m=1}^{\infty} \log(1-q^m)\Big] = \prod_{n=1}^{\infty} (1-q^n)^{-7}. \tag{4.7}$$

This demonstrates that the result of the direct computation of the $S^1 \times S^5$ M5 partition function in $\mathrm{AdS}_4 \times S^7$ reproduces the q^N prefactor in the ABJM superconformal index in (2.13).

A related observation is that the single-particle partition function $\widehat{\mathcal{Z}}(q)$ in (4.2) matches the $\boldsymbol{u} \to \mathbf{1}$ limit of the single-particle "M5 brane superconformal index" (2.12), obtained in [6] by an analytic continuation of the expression for the superconformal index of the free (2,0) 6d multiplet in flat space

$$\widehat{\mathcal{Z}}(q) = \frac{q^{-1} - 3q^2 + q^3 + q^4}{(1-q)^3} = \widehat{\mathrm{I}}_1^{\rm M5}(q, \mathbf{1}) \,. \tag{4.8}$$

4.3. Generalization to the case with non-trivial R-symmetry fugacities u_a

In the above discussion we considered the supersymmetric partition function of wrapped $S^1 \times S^5$ M5 brane in $\widetilde{\mathrm{AdS}}_4 \times S^7$ corresponding to the contribution to the ABJM index (2.3), (2.8), (2.9) with $u_1 = u_2 = u_3 = u_4 = 1$. To include the dependence on the $SO(8)$ R-symmetry fugacities we would need to add extra u_a dependent twists in the S^7 angles in (3.1).

Let us represent S^7 as a subspace $\sum_{a=1}^4 |z_a|^2 = 1$ in $\mathbb{C}^4$ with the metric $dS_7 = \sum_{a=1}^4 |dz_a|^2 = \sum_{a=1}^4 (dn_a^2 + n_a^2 d\phi_a^2)$, $\sum_{a=1}^4 n_a^2 = 1$. Adding the y-shifts to the angles ϕ_a should reflect the presence of the R-charge fugacities u_a in the ABJM index in (2.3) (cf. (3.2))

$$dS_7 = \sum_{a=1}^4 \big[dn_a^2 + n_a^2 (d\phi_a + i\alpha_a dy)^2\big] \,, \qquad \sum_{a=1}^4 n_a^2 = 1 \,. \tag{4.9}$$

Here $y \equiv y + \beta$ is the $S^1_\beta \subset \mathrm{AdS}_4$ coordinate in (3.1). Taking into account the relative normalizations of the factors in (2.3) we should set (note the constraint $u_1 u_2 u_3 u_4 = 1$)

$$u_a = e^{-\beta\alpha_a} = q^{2\alpha_a} \,, \qquad q = e^{-\frac{1}{2}\beta} \,, \qquad \sum_{a=1}^4 \alpha_a = 0 \,. \tag{4.10}$$

[25] We also used the analytic continuation to define $\sum_{m=1}^{\infty} \frac{q^{-m} - q^m}{m} = \log(-q)$.

[26] Note that as there is no Casimir energy term in 3d, similar factor did not appear in the case of the M2 brane supersymmetric partition function in [5].

Let us now assume that M5 brane is wrapping $S^5 \subset S^7$ defined by $z_1 = 0$. In the parametrization used in (3.1) we may identify the angle z with ϕ_1 and thus have it shifted as $dz \to dz + i\alpha_1 dy$, with the wrapped M5 brane solution still described by (3.9). The corresponding classical M5 action will be given by (3.11) where the volume part will not depend on α_a (as $dy = d\xi^1$ shifts of differentials of the S^5 will not contribute the product with $d\xi^1$ corresponding to S^1_β). However, now we will get a non-zero contribution from the WZ term as C_6 in (3.14) will contain $dz \to dz + i\alpha_1 dy$ with $v = 0$ for the solution (3.9) (cf. (3.12), (3.15))[27]

$$S_{\rm WZ} = iT_5 \int C_6 = iT_5 \int {\rm vol}_{S^5} \wedge (i\alpha_1 dy) = N\alpha_1 \beta\ . \tag{4.11}$$

As a result, we will get $e^{-S_{\rm WZ}} = e^{-N\alpha_1\beta} = u_1^N$, reproducing the overall factor u_1^N in the first term in the sum in $\delta {\rm I}^{(1)}_N(q, \boldsymbol{u})$ in (2.9).

To get the factor ${\rm I}_1^{\rm M5}(q, \boldsymbol{u})$ in (2.9) we need to compute the one-loop correction for non-zero shifts (4.10). In view of the condition $u_1u_2u_3u_4 = 1$ in (2.3), the simplest non-trivial case to consider is[28]

$$u_1 = u_4 = 1, \qquad u_2 = u_3^{-1} = u \equiv q^{2\alpha}\ , \quad \text{i.e.} \quad \alpha_1 = \alpha_4 = 0, \quad \alpha_2 = -\alpha_3 = \alpha\ . \tag{4.12}$$

The resulting generalization of the S^5 part of the S^7 metric in (3.1) is then (we rename the coordinates compared to (4.9))

$$d\widetilde{S}_5 = dn_1^2 + dn_2^2 + dn_3^2 + n_1^2(d\phi_1 + i\alpha dy)^2 + n_2^2(d\phi_2 - i\alpha dy)^2 + n_3^2 d\phi_3^2\ , \qquad n_1^2 + n_2^2 + n_3^2 = 1\ . \tag{4.13}$$

Considering the analog of the $S^1_\beta \times S^5$ M5 brane solution in the resulting twisted-product space[29] $\widetilde{\rm AdS}_7 \times \widetilde{S}^7$ we should expect to find that the corresponding one-loop partition function should have the single-particle counterpart that generalizes (4.8) in such a way that it matches the relevant special case of (2.12). Namely, we should find the following generalization of (4.8)

$$\widehat{\mathcal{Z}}(q, u) = \frac{q^{-1} - q^2(1 + u + u^{-1}) + q^3 + q^4}{(1-q)(1-u\,q)(1-u^{-1}q)} = \widehat{\rm I}_1^{\rm M5}(q, 1, u, u^{-1}, 1)\ . \tag{4.14}$$

To show this one may "disentangle" the effect of the J_{12}-related (cf. (2.3)) fixed shift of the $S^2 \subset {\rm AdS}_4$ angle φ in (3.2) from the dependence on the α-parameter in (4.13). This is possible in the static gauge: when evaluating the ν-shift effects on the one-loop fluctuation operators due to the φ twist in (3.2) as in section 3 the dependence on α can be ignored as it can be absorbed into the non-fluctuating S^5 coordinates.

Then the problem of finding the dependence on α becomes formally the same as computing the supersymmetric partition function of a single (2,0) tensor multiplet on the 6d space which is the twisted product of S^1_β and $\widetilde{S}^5$ with the metric (4.13). We describe this computation in Appendix E. From (E.13), (E.19), (E.18) we get the following set of the $u = q^{2\alpha}$ dependent single-particle partition functions generalizing the $\nu = 0$ expressions (3.49), (3.55), (3.58) for the scalar, fermion, and the antisymmetric tensor fields

$$\widehat{\mathcal{Z}}_\phi(q, 0; u) = \frac{q^2(1-q^2)}{(1-q)^2(1-u\,q)^2(1-u^{-1}q)^2} = \frac{q^2(1+q)}{(1-q)(1-u\,q)^2(1-u^{-1}q)^2}\ , \tag{4.15}$$

$$\widehat{\mathcal{Z}}_\psi(q, 0; u) = \frac{2c_\psi(u)\, q^{5/2}}{(1-q)(1-u\,q)^2(1-u^{-1}q)^2}\ , \qquad c_\psi(u) = \tfrac{1}{4}(u + 2 + u^{-1})\ , \tag{4.16}$$

$$\widehat{\mathcal{Z}}_A(q; u) = \frac{q^3[c_1(u) - c_2(u)\, q + q^2]}{(1-q)(1-u\,q)^2(1-u^{-1}q)^2}\ , \tag{4.17}$$
$$c_1(u) = u^2 + 2u + 4 + 2u^{-1} + u^{-2}\ , \qquad c_2(u) = 2u + 1 + 2u^{-1}\ .$$

Using the u-dependent partition functions in (4.15), (4.16), (4.17) and the including the effect of the ν shifts in Table 1 (which amounts to multiplication of the scalar and fermion contributions by $\frac{1}{2}(q^{2\nu} + q^{-2\nu})$) we can check that total single-particle partition function indeed reproduces the expression in (4.14) generalizing (4.8) to $u \neq 1$

$$\begin{aligned}\widehat{\mathcal{Z}}(q, u) =& \widehat{\mathcal{Z}}_\phi(q, 0; u) + 2\widehat{\mathcal{Z}}_\phi(q, 1; u) + 2\widehat{\mathcal{Z}}_\phi(q, \tfrac{3}{2}; u) + \widehat{\mathcal{Z}}_A(q; u) - 2\Big[\widehat{\mathcal{Z}}_\psi(q, \tfrac{1}{4}; u) + \widehat{\mathcal{Z}}_\psi(q, \tfrac{5}{4}; u)\Big] \\ =& \frac{1}{(1-q)(1-uq)(1-u^{-1}q)}\Big[q^{-1} - q^2(1 + u + u^{-1}) + q^3 + q^4\Big]\ .\end{aligned} \tag{4.18}$$

Note that expanding (4.18) in q we get

$$\widehat{\mathcal{Z}}(q, u) = q^{-1} + u + 1 + u^{-1} + \mathcal{O}(q). \tag{4.19}$$

The q-independent part $u + 1 + u^{-1}$ here is the $u \neq 1$ generalization of the term 3 in (4.2) which, as we discussed above, leads to a formal divergence in $\bar{F}$ in (4.1) that should be attributed to the contribution of the zero modes. Since $\sum_{m=1}^\infty \frac{1}{m}(u^m + u^{-m}) =$

[27] We assume a particular orientation on $S^1 \times S^5$ so that $\int dy \wedge {\rm vol}_{S^5} = \pi^3\beta$.

[28] For this choice the WZ term contribution to the classical action is trivial, $u_1^N = 1$.

[29] This space is thus an analog of the twisted-product space ${\rm AdS}_7 \times \widetilde{S}^4$ considered in connection with the M2 brane instanton contribution to the large N superconformal index of (2,0) theory in [5], cf. footnote 5.

$-\log[(1-u)(1-u^{-1})]$, here the contributions of $u+u^{-1}$ can be formally regularized using the analytic continuation in u. The divergence due to the remaining term 1 in (4.19) should be associated to the contribution of two bosonic zero modes that should produce a prefactor $(\sqrt{\mathrm{T}_5})^2 \sim N$ (cf. (4.4)).

Indeed, analyzing the expression for the ABJM index (2.9) in the limit of $\boldsymbol{u} \to (1, u, u^{-1}, 1)$ and large N (generalizing the discussion in Appendix C in the $\boldsymbol{u} \to 1$ case leading to (1.2), (2.13)) we get[30]

$$\mathrm{I}_N^{\mathrm{ABJM}}(q,u) = \mathrm{I}_{\mathrm{KK}}^{\mathrm{ABJM}}(q,u)\left[1 - \frac{N}{(1-u)(1-u^{-1})}\,\mathrm{G}(q,u)\,q^N + \mathcal{O}(N^0 q^N)\right]. \tag{4.20}$$

Thus the leading large N contribution has the expected factor $\frac{N}{(1-u)(1-u^{-1})}$ associated with the presence of the q-independent term in (4.19). The function $\mathrm{G}(q,u)$ in (4.20) is given by (here $n_{123} \equiv n_1+n_2+n_3$)

$$\begin{aligned}\mathrm{G}(q,u) = &-\frac{q}{1-q}\prod_{n_1,n_2,n_3\geq 0,\, n_{123}\geq 2}^{\infty}\frac{1}{1-q^{n_{123}-1}u^{n_2-n_3}}\\ &\times\prod_{n_1,n_2,n_3=0}^{\infty}\frac{(1-q^{n_{123}+2}u^{n_2-n_3})(1-q^{n_{123}+2}u^{n_2-n_3+1})(1-q^{n_{123}+2}u^{n_2-n_3-1})}{(1-q^{n_{123}+3}u^{n_2-n_3})(1-q^{n_{123}+4}u^{n_2-n_3})}.\end{aligned} \tag{4.21}$$

Switching on all 4 non-trivial fugacities u_a we should find that the zero mode contribution is completely regularized[31] and, correspondingly, there is no N-dependent prefactor in the analog of the q^N term in (4.20).

5. Supersymmetric partition function of free (2,0) multiplet on $S^1 \times S^5$

The expression for the superconformal index of a free $(2,0)$ tensor multiplet in flat 6d space [11] was used in [6] to find the "M5 brane index" (2.12) by an analytic continuation and re-identification of fugacities. For completeness, let us show how obtain the index of the $N=1$ (2,0) theory by directly computing the supersymmetric partition function of this 6d tensor multiplet on $S^1\times S^5$.

The simplest (unrefined) version of the (2,0) superconformal index is defined as $\mathrm{Tr}\left[(-1)^{\mathrm{F}}\, q^{H-R_{12}}\right]$ where R_{12} is a generator of the R-symmetry $SO(5)$ group. Hence to preserve supersymmetry one is to include a particular flat connection in the scalar and fermion kinetic operators on $S^1\times S^5$ corresponding to a rotation in the "target-space" 12-plane [11]. Like in the discussion in section 3 this will lead to particular shifts of the S^1 mode numbers in the "mass" terms in the corresponding differential operators on S^5 (cf. (3.34)).

As a result, we get three scalars with no shift, two scalars shift $v=\frac{1}{2}$ and four MW 6d fermions with shift $v=\frac{1}{4}$.[32] Using the data in Table 1 we then find (cf. (4.2), (4.3))

$$\widehat{\mathcal{Z}}(q) = 3\widehat{\mathcal{Z}}_\phi(q,0) + 2\widehat{\mathcal{Z}}_\phi(q,\tfrac{1}{2}) + \widehat{\mathcal{Z}}_A(q) - 4\,\widehat{\mathcal{Z}}_\psi(q,\tfrac{1}{4}) = \frac{q}{1-q}, \tag{5.1}$$

$$E_{c,\mathrm{tot}} = 3E_{c,\phi}(0) + 2E_{c,\phi}(\tfrac{1}{2}) + E_{c,A} - 4E_{c,\psi}(\tfrac{1}{4}) = -\tfrac{1}{24}\,. \tag{5.2}$$

These expressions agree with the ones for the Schur index (and supersymmetric Casimir energy) for a single 6d $(2,0)$ tensor multiplet (see also (B.11)) obtained in [11] by counting BPS states with R-charge dependent weights.

We may also consider a generalization to the case of two $SO(5)$ R-symmetry generators

$$Z_\eta(q) = \mathrm{Tr}\left[(-1)^{\mathrm{F}}\, q^{H-R_\eta}\right], \qquad R_\eta = \frac{\eta}{2}R_{12} + \frac{1-\eta}{2}R_{34}\,, \tag{5.3}$$

where η is a free parameter (denoted by Δ in [11]). In this case there are two orthogonal components in the flat connection (and two Γ-matrix projectors in the spinor covariant derivative). As a result we get one scalar with $v=0$, two with $v=\frac{1}{2}\eta$ and two with $v=\frac{1}{2}-\frac{\eta}{2}$ as well as two fermions with $v=\frac{1}{4}$ and two with $v=\frac{1}{4}-\frac{\eta}{2}$. The resulting single-particle partition function is

$$\begin{aligned}\widehat{\mathcal{Z}}_\eta(q) &= \widehat{\mathcal{Z}}_\phi(q,0) + 2\widehat{\mathcal{Z}}_\phi(q,\tfrac{\eta}{2}) + 2\widehat{\mathcal{Z}}_\phi(q,\tfrac{1}{2}-\tfrac{\eta}{2}) + \widehat{\mathcal{Z}}_A(q) - 2\left[\widehat{\mathcal{Z}}_\psi(q,\tfrac{1}{4}) + \widehat{\mathcal{Z}}_\psi(q,\tfrac{1}{4}-\tfrac{\eta}{2})\right]\\ &= \frac{q^{1+\eta}+q^{2-\eta}-3q^2+q^3}{(1-q)^3}\,.\end{aligned} \tag{5.4}$$

Note that for $\eta=-2$ and $\eta=3$ this coincides with (4.2)(4.8) (see also below).

Comparing (5.4) to the general expression for the single-particle superconformal index of one (2,0) multiplet $\widehat{\mathrm{I}}^{(2,0)}_{N=1}(q,\boldsymbol{y},u)$ in (2.10), (B.3), (B.9) we conclude that

$$\widehat{\mathcal{Z}}_\eta(q) = \widehat{\mathrm{I}}^{(2,0)}_{N=1}(q^{\frac{3}{4}},1,1,1,q^{\frac{1}{2}-\eta})\,, \tag{5.5}$$

[30] There are also terms proportional to u^N and u^{-N} but they have subleading coefficients $\sim N^0$.

[31] Note that the small q expansion of (2.12) is $\widehat{\mathrm{I}}_1^{\mathrm{M5}}(q,\boldsymbol{u}) = q^{-1}u_1^{-1} + (u_2+u_3+u_4)u_1^{-1} + \mathcal{O}(q)$ (cf. (4.19)).

[32] Here the length of S^1 is β, not $\frac{1}{2}\beta$, and the shift is still relative to $n_\beta = \frac{2\pi}{\beta}$.

which corresponds to $\mathrm{I}^{(2,0)}_{N=1}(q^{\frac{3}{4}},1,1,1,q^{\frac{1}{2}-\eta}) = \mathrm{Tr}\left[(-1)^{\mathrm{F}}\, q^{\frac{3}{4}[H+\frac{1}{3}(J_{12}+J_{34}+J_{56})]+(\frac{1}{2}-\eta)(R_{12}-R_{34})}\right]$. Using that Δ in (B.2) should be set to 0 in (B.3), this simplifies indeed to (5.3). Combining (5.5) with the analytic continuation rule (2.11) that gives $\widehat{\mathrm{I}}^{\mathrm{M5}}_1(q,\boldsymbol{u})$ in (2.12) the relation in (5.5) may be written also as

$$\hat{\mathcal{Z}}_\eta(q) = \widehat{\mathrm{I}}^{\mathrm{M5}}_1(q^{\frac{1+\eta}{4}}, q^{\frac{3\eta-9}{4}}, q^{\frac{3-\eta}{4}}, q^{\frac{3-\eta}{4}}, q^{\frac{3-\eta}{4}})\ . \tag{5.6}$$

Thus for $\eta = 3$ it should reproduce the "unrefined" $\boldsymbol{u} \to \mathbf{1}$ limit of the M5 brane index. Indeed, for $\eta = 3$ the values of the individual shifts and the total expression in (5.4) become the same as (4.2) that we found above by the direct analysis of fluctuations of the wrapped M5 brane in $\widetilde{\mathrm{AdS}}_4 \times S^7$.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Matteo Beccaria reports financial support was provided by National Institute of Nuclear Physics Section of Lecce. Arkady Tseytlin reports financial support was provided by Science and Technology Facilities Council. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We are grateful to S. Giombi for an initial collaboration and many useful discussions. MB was supported by the INFN grants GSS and GAST. AAT was supported in part by the STFC Consolidated Grants ST/T000791/1 and ST/X000575/1.

Appendix A. Brane instanton expansion of superconformal index

In a superconformal theory with an AdS dual the large gauge group rank N corrections to superconformal index may be interpreted in terms of contributions of brane instantons. Considering for simplicity the dependence on a single fugacity q, the structure of N dependence is encoded in the expansions like (1.1), (1.2), i.e.[33]

$$\mathrm{I}_N(q) = I_\infty(q)\left[1 + \sum_k q^{kN}\,\delta\mathrm{I}_k(q)\right], \tag{A.1}$$

where $\delta\mathrm{I}_k(q)$ is regular at small q and the only explicit dependence on N is in the factor q^{kN}. Such structure is observed in many special cases [39,40]. For historical reasons it is usually called "giant graviton expansion" (see, e.g., [15]) although a more appropriate name would be a "brane instanton expansion".

The paradigmatic case is that of the 4d $\mathcal{N} = 4$ $U(N)$ SYM theory corresponding to a system of N coincident D3 branes in type IIB superstring model. The large N limit of its index $\mathrm{I}_\infty(q)$ counts the S^5 Kaluza-Klein BPS states of $\mathrm{AdS}_5 \times S^5$ supergravity. The structure of finite N corrections from the state counting perspective is non-trivial [41]. They are important, e.g., for the corresponding BPS black hole entropy count since the asymptotic growth of the index (*i.e.* the number of states at increasing charge) is faster than that of a gas of the KK modes [42]. Finite N corrections to the index take the form (A.1) where each term in the square brackets is the effect of k wrapped D3 branes (or "giant gravitons" [43]).

The functions $\delta\mathrm{I}_k(q)$ can be computed by considering branes multiply wrapped around topologically trivial 3-cycles in the internal S^5 space [7]. If we represent S^5 as $|\mathrm{z}_1|^2 + |\mathrm{z}_2|^2 + |\mathrm{z}_3|^2 = 1$, the three 3-cycles are defined by $\mathrm{z}_i = 0$. Denoting the wrapping numbers by (n_1, n_2, n_3), the theory on the wrapped D3 branes is $U(n_1) \times U(n_2) \times U(n_3)$ gauge theory with bi-fundamental multiplets in a ring quiver diagram. Here q^{kN} in (A.1) is given by $q^{n_1 N} q^{n_2 N} q^{n_3 N}$ and represents the total classical prefactor coming from the classical charges and energy of the wrapped brane system. Also, we get[34]

$$\delta\mathrm{I}_k(q) = \sum_{\substack{n_1,n_2,n_3=0\\ n_1+n_2+n_3=k}}^{\infty} \delta\mathrm{I}_{n_1,n_2,n_3}(q)\ . \tag{A.2}$$

[33] More generally one has $q^{kN} \to q^{a_k N}$ where a_k is some increasing sequence of positive integers.

[34] In general, at mathematical level, the expansions like (A.1) are not unique. For instance, an exact expansion for the $\mathcal{N} = 4$ SYM index was given in [44], but it differs from the ones arising from wrapped D3 branes [45].

The actual evaluation of the so-called “brane indices” $\delta \mathrm{I}_{n_1,n_2,n_3}(q)$ goes in two steps: (a) first, one finds the single-letter index $\widehat{\delta \mathrm{I}}_{n_1,n_2,n_3}(q)$ of the brane world-volume superconformal theory[35]; (b) second, to get the brane index, one has to integrate the plethystic exponential $\mathrm{PE}[\widehat{\delta \mathrm{I}}_{n_1,n_2,n_3}]$ over the $U(n_1)\times U(n_2)\times U(n_3)$ gauge holonomies.

The second step is far from trivial because it is unclear which contour should be used to integrate the holonomies for the analytically continued single-particle index. For instance, representing the holonomies as $U(1)$ phases w_i, it is known that the standard cycles $|w_i|=1$ give a wrong result. In some cases, it is possible to match the finite N index by some special *ad hoc* choices, at least up to some total wrapping level k. A prescription working up to $k=3$ was proposed in [7] and other examples treated by the same approach can be found in [39,6,40]. For recent discussions of the analytical continuation for generic wrapping see [15,46,16,17,47].

Appendix B. Superconformal index of (2, 0) theory and its Schur limit

The 6d $(2,0)$ superconformal algebra is $\mathfrak{osp}(8^*|4)$ with the bosonic subalgebra $\mathfrak{so}(2,6)\oplus\mathfrak{so}(5)$ having six Cartan generators

$$C=(H,\ J_{12},\ J_{34},\ J_{56},\ R_{12},\ R_{34})\ . \tag{B.1}$$

The superconformal index of the $(2,0)$ theory is defined in terms of a supercharge Q satisfying $[C_I,Q]=\frac{1}{2}\sigma_I\,Q,\ I=1,\dots,6$ with $\sigma_I=(1,-1,-1,-1,1,1)$. The subalgebra commuting with Q is $\mathfrak{osp}(6|2)\oplus\mathfrak{u}(1)_\Delta$ with the bosonic algebra $\mathfrak{su}(1,3)\oplus\mathfrak{su}(2)$ and the central factor $\mathfrak{u}(1)_\Delta$ having the generator

$$\Delta=\{Q,Q^\dagger\}=H-(J_{12}+J_{34}+J_{56})-2(R_{12}+R_{34})\ . \tag{B.2}$$

The superconformal index is then

$$\mathrm{I}^{(2,0)}_N(q,\boldsymbol{y},u)=\operatorname*{Tr}_{\Delta=0}\Big[(-1)^{\mathrm{F}}\,q^{H+\frac{1}{3}(J_{12}+J_{34}+J_{56})}\,y_1^{J_{12}}\,y_2^{J_{34}}\,y_3^{J_{56}}\,u^{R_{12}-R_{34}}\Big],\qquad y_1y_2y_3=1. \tag{B.3}$$

The trace is restricted to the states with $\Delta=0$ (contributions of states with $\Delta>0$ cancel in pairs).

The Schur limit of the index (B.3) is defined by imposing the condition on the fugacities

$$q^{\frac{2}{3}}\,u\,y_1^{-1}=1\ , \tag{B.4}$$

that implies invariance under an additional supercharge Q' satisfying $[C_I,Q']=\frac{1}{2}\sigma_I\,Q',\ I=1,\dots,6$ with $\sigma_I=(1,-1,1,1,1,-1)$. It is associated with the second $\mathfrak{u}(1)$ with the generator

$$\Delta'=\{Q',Q'^\dagger\}=H-(J_{12}-J_{34}-J_{56})-2(R_{12}-R_{34}). \tag{B.5}$$

A convenient parametrization for the independent fugacities (obeying (B.4) and $y_1y_2y_3=1$) is

$$q=q'\,x',\quad y_1=q'^{\,\frac{2}{3}}\,x'^{\,-\frac{4}{3}},\qquad y_2=q'^{\,-\frac{1}{3}}\,x'^{\,\frac{2}{3}}\,y,\qquad y_3=q'^{\,-\frac{1}{3}}\,x'^{\,\frac{2}{3}}\,y^{-1}\ ,\qquad u=x'^{\,-2}\ . \tag{B.6}$$

In this limit the index (B.3) reads

$$\mathrm{I}^{(2,0)}_N(q',y)=\operatorname*{Tr}_{\Delta=\Delta'=0}\Big[(-1)^{\mathrm{F}}\,q'^{\,H+J_{12}}\,y^{J_{34}-J_{56}}\Big]. \tag{B.7}$$

From the condition $\Delta=\Delta'=0$ we get $H+J_{12}=2\,(H-R_{12})$. As a result, the unrefined ($y=1$) Schur index is given by

$$\mathrm{I}^{(2,0)}_N(q')\equiv\mathrm{I}^{(2,0)}_N(q',1)=\operatorname*{Tr}_{\Delta=\Delta'=0}\Big[(-1)^{\mathrm{F}}\,q'^{\,2(H-R_{12})}\Big]\ ,\qquad q'^{\,2}=e^{-\beta}\ . \tag{B.8}$$

The leading large N correction (A.1) to this index scales as $\sim q'^{\,2N}=\exp(-\beta N)$ [6].

The Schur index $\mathrm{Tr}[(-1)^{\mathrm{F}}e^{-\beta(H-R_{12})}]$ may be computed as a supersymmetric partition function on $S^1_\beta\times\widetilde{S}^5$ with periodic fermions and R-charge related twist (cf. [20]). As was demonstrated in [5], the leading non-perturbative contribution to the large N expansion of this index may represented in the dual M-theory description as the semiclassical partition function of M2 brane in $\mathrm{AdS}_7\times\widetilde{S}^4$ where $\widetilde{S}^4$ has one angle mixed ($z\to z+iy$) with the coordinate of $S^1_\beta\subset\mathrm{AdS}_7$.

Let us note also that for $N=1$ (2,0) tensor multiplet the single-particle index corresponding to (B.3) reads (see, e.g., [2,6])[36]

$$\widehat{\mathrm{I}}^{(2,0)}_{N=1}(q,\boldsymbol{y},u)=\frac{q^2(u+u^{-1})-q^{\frac{8}{3}}(y_1^{-1}+y_2^{-1}+y_3^{-1})+q^4}{(1-q^{\frac{4}{3}}y_1)(1-q^{\frac{4}{3}}y_2)(1-q^{\frac{4}{3}}y_3)}. \tag{B.9}$$

In the “unrefined” case $\boldsymbol{y},u\to 1$ it is given by

[35] In favourable cases, this may be accomplished by an analytic continuation of the index of the superconformal theory in flat space with the same superconformal algebra.

[36] This expression agrees with Eq.(3.35) in [2], with a suitable identification of the fugacities, i.e. $x=q^{1/3},u=z^{1/2}$, etc.

$$\widehat{\mathrm{I}}^{(2,0)}_{N=1}(q) = \widehat{\mathrm{I}}^{(2,0)}_{N=1}(q,\mathbf{1},1) = \frac{2q^2 - 3q^{\frac{8}{3}} + q^4}{(1-q^{\frac{4}{3}})^3}. \tag{B.10}$$

To find its Schur limit, we change the variables as in (B.6) and set $x' = y = 1$ getting

$$\widehat{\mathrm{I}}^{(2,0)\,\mathrm{Schur}}_{N=1}(q') = \frac{q'^2}{1-q'^2}. \tag{B.11}$$

Let us add a comment on a comparison of the prefactors of the q^N terms in the expansion of the indices of the (2,0) (1.1) and the ABJM (1.2) theories. As was reviewed in the Introduction, the leading large N correction to the superconformal index of the (2,0) theory in (1.1) has a prefactor $-\frac{q}{(1-q)^2}$ with a simple dependence on q. At the same time, the counterpart of this prefactor in the ABJM index is proportional to the $G_0(q)$ function in (1.2) that has a complicated dependence on q. This may be puzzling as both were shown to originate from the one-loop partition functions – of the $S^1\times S^2$ M2 brane in the twisted $\mathrm{AdS}_7\times S^4$ background in the case of (1.1) [5] and of the $S^1\times S^5$ M5 brane in the twisted $\mathrm{AdS}_4\times S^7$ in the case of (1.2) (see section 4).

One may understand the reason for the simplicity of the prefactor in (1.1) from its alternative derivation in [6]. According to the proposal in [6], an indirect way to compute this prefactor is to exploit the isomorphism between the superconformal algebras of the two 3d models: the quadratic fluctuation action for the wrapped M2 brane in curved $\mathrm{AdS}_7\times S^4$ target space and of a single $\mathcal{N}=8$ 3d scalar multiplet in flat space (or, equivalently, of the $N=1$ ABJM model) defined on $S^1\times S^2$. For the latter theory, the expression for the index with non-trivial chemical potentials reads[37]

$$\mathrm{I}^{\mathrm{ABJM}}_{N=1}(q,\boldsymbol{u}) = \mathrm{PE}\Big[\frac{q(u_1+u_2+u_3+u_4) - q^3(u_1^{-1}+u_2^{-1}+u_3^{-1}+u_4^{-1})}{1-q^4}\Big]. \tag{B.12}$$

To relate fugacities of this model to those of the (2,0) theory index we may still use (2.11).[38] Explicitly, in the case of the index of the ABJM model on $S^1\times S^3$ *and* the index of M5 brane wrapped on $S^1\times S^5$ this is $su(2|4)\times u(1)$. In the case of the index of the (2,0) theory on $S^1\times S^5$ *and* M2 brane wrapped on $S^1\times S^3$ this is $su(4|2)\times u(1)$. The observation of [6] is that $su(2|4)\times u(1)$ and $su(4|2)\times u(1)$ are isomorphic. Adopting the Schur parametrization (B.6), we then get

$$(q,u_1,u_2,u_3,u_4) = (q'^{\frac{1}{2}}x', q'^{-\frac{5}{2}}x'^{-1}, q'^{\frac{3}{2}}x'^{-1}, q'^{\frac{1}{2}}x'y, q'^{\frac{1}{2}}x'y^{-1}). \tag{B.13}$$

In the unrefined Schur limit $x' = y = 1$ this reads

$$(q,u_1,u_2,u_3,u_4) = (q'^{\frac{1}{2}}, q'^{-\frac{5}{2}}, q'^{\frac{3}{2}}, q'^{\frac{1}{2}}, q'^{\frac{1}{2}}). \tag{B.14}$$

Using (B.14) and assuming as usual an analytic continuation in computing the plethystic exponential, we get

$$\mathrm{I}^{\mathrm{ABJM}}_{N=1}(q') = \mathrm{PE}[q'^2 + q'^{-2}] = -\frac{q'^2}{(1-q'^2)^2}\ . \tag{B.15}$$

This is the same as the prefactor in (1.1), taking into account the definition of the (2,0) index in (B.8).

Appendix C. Leading large N correction to ABJM index in unrefined limit

Here we will discuss the leading term $\delta\mathrm{I}^{(1)}_N(q,\boldsymbol{u})$ in (2.8) and large N expansion in the unrefined limit $\boldsymbol{u}\to 1$ deriving (2.13), (2.14).

To find $\delta\mathrm{I}^{(1)}_N(q,\boldsymbol{u})$ we need to compute $\mathrm{PE}[\widehat{\mathrm{I}}^{\mathrm{M5}}_a(q,\boldsymbol{u})]$ in (2.9) (see, e.g., [7]). Let us focus on the coefficient $\mathrm{PE}[\widehat{\mathrm{I}}^{\mathrm{M5}}_1]$ of the u_1^N term in (2.9). Expanding it at small q gives

$$\begin{aligned}\widehat{\mathrm{I}}^{\mathrm{M5}}_1(q,\boldsymbol{u}) =& \frac{1}{u_1 q} + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \Big(\frac{u_2^2}{u_1} + \frac{u_2u_3}{u_1} + \frac{u_3^2}{u_1} + \frac{u_2u_4}{u_1} + \frac{u_3u_4}{u_1} + \frac{u_4^2}{u_1}\Big)q \\ &+ \Big(-\frac{1}{u_1u_2} + \frac{u_2^3}{u_1} - \frac{1}{u_1u_3} + \frac{u_2^2u_3}{u_1} + \frac{u_2u_3^2}{u_1} + \frac{u_3^3}{u_1} - \frac{1}{u_1u_4} + \frac{u_2^2u_4}{u_1} + \frac{u_2u_3u_4}{u_1} \\ &+ \frac{u_3^2u_4}{u_1} + \frac{u_2u_4^2}{u_1} + \frac{u_3u_4^2}{u_1} + \frac{u_4^3}{u_1}\Big)q^2 + \mathcal{O}(q^3).\end{aligned} \tag{C.1}$$

The terms with positive/negative coefficients correspond to the contribution of bosonic/fermionic BPS states. The full plethystic exponential (2.4) may be written as a product of contributions of monomials

$$\mathrm{PE}[\widehat{\mathrm{I}}^{\mathrm{M5}}_1(q,\boldsymbol{u})] = \mathrm{PE}\Big[\frac{1}{u_1 q}\Big]\ \mathrm{PE}\Big[\frac{u_2}{u_1}\Big]\ \cdots. \tag{C.2}$$

[37] See Eqs. (40) and (17) in [6]. This expression reduces to the $N=1$ case in (2.6) for $\boldsymbol{u}=1$.

[38] The isomorphism used in [6] is between the (i) unbroken part of the superconformal algebra on the wrapped branes (M2 or M5 in the two cases), and (ii) the subalgebra respecting the supercharge used to define the corresponding index (*cf.* also footnote 2).

For a single monomial with coefficient ± 1, one has

$$\mathrm{PE}[\pm u^a q^b] = (1-u^a q^b)^{\mp 1}, \tag{C.3}$$

which is to be understood in terms of an analytic continuation in the fugacities.[39] As a result, we get

$$\begin{aligned}\mathrm{PE}[\widehat{\mathrm{I}}_1^{\mathrm{M5}}(q,\boldsymbol{u})] = &-\frac{u_1^4}{(u_1-u_2)(u_1-u_3)(u_1-u_4)}\,q\\ &-\frac{u_1^3\left[u_1^2+u_2^2+u_3^2+u_3u_4+u_4^2+u_2(u_3+u_4)\right]}{(u_1-u_2)(u_1-u_3)(u_1-u_4)}\,q^2+\mathcal{O}(q^3).\end{aligned} \tag{C.4}$$

The unrefined limit $\boldsymbol{u}\to 1$ appears to be singular, but, in fact, the poles cancel after summing over $a=1,\dots,4$ in (2.9) and one may check the agreement with the explicit expansions in (2.6) for low values of N.

It is important to note that the expansion coefficients in the unrefined expression for $\delta\mathrm{I}_N^{(1)}(q)$ in (2.6) depend on N. This is due to the factors u_a^N that produce an N-dependent leftover after the pole cancellation.[40] Indeed, we find that for generic N

$$\begin{aligned}\delta\mathrm{I}_N^{(1)}(q) = &-\tfrac{1}{6}(N+2)(N+3)(N+4)\,q-\tfrac{1}{6}(N+1)(N+3)(7N+8)\,q^2\\ &-\tfrac{5}{6}(N+1)(N+2)(7N-12)\,q^3-\tfrac{10}{3}(N+1)(7N^2-25N-9)\,q^4+\cdots.\end{aligned} \tag{C.5}$$

Thus for large N the leading enhancement in degeneracy is N^3 one

$$\delta\mathrm{I}_N^{(1)}(q) = \tfrac{1}{6}\,N^3\,G_0(q)+\mathcal{O}(N^2), \tag{C.6}$$

$$G_0(q) = -q-7q^2-35q^3-140q^4-490q^5-1547q^6-4522q^7-12405q^8-32305q^9+\cdots \tag{C.7}$$

To determine the closed form of $G_0(q)$, we go back to the general expression for $\widehat{\mathrm{I}}_1^{\mathrm{M5}}(q,\boldsymbol{u})$ in (2.12)

$$\widehat{\mathrm{I}}_1^{\mathrm{M5}}(q,\boldsymbol{u}) = \sum_{n_2,n_3,n_4=0}^{\infty} q^{n_2+n_3+n_4}u_2^{n_2}u_3^{n_3}u_4^{n_4}\left[q^{-1}u_1^{-1}-q^2u_1^{-1}(u_2+u_3+u_4)+q^3u_1^{-1}+q^4\right]. \tag{C.8}$$

Then (here $n_{234}\equiv n_2+n_3+n_4$)

$$\mathrm{PE}[\widehat{\mathrm{I}}_1^{\mathrm{M5}}] = \prod_{n_2,n_3,n_4=0}^{\infty}\frac{(1-q^{n_{234}+2}u_1^{-1}u_2^{n_2+1}u_3^{n_3}u_4^{n_4})(1-q^{n_{234}+2}u_1^{-1}u_2^{n_2}u_3^{n_3+1}u_4^{n_4})(1-q^{n_{234}+2}u_1^{-1}u_2^{n_2}u_3^{n_3}u_4^{n_4+1})}{(1-q^{n_{234}-1}u_1^{-1}u_2^{n_2}u_3^{n_3}u_4^{n_4})(1-q^{n_{234}+3}u_1^{-1}u_2^{n_2}u_3^{n_3}u_4^{n_4})(1-q^{n_{234}+4}u_2^{n_2}u_3^{n_3}u_4^{n_4})}. \tag{C.9}$$

The pole at $\boldsymbol{u}\to\mathbf{1}$ comes from the first factor in the denominator when $n_{234}=1$. Examining the residue we obtain

$$\begin{aligned}G_0(q) &= \prod_{\substack{n,m,\ell=0\\ n+m+\ell\neq 1}}^{\infty}\frac{1}{1-q^{n+m+\ell-1}}\prod_{n,m,\ell=0}^{\infty}\frac{(1-q^{n+m+\ell+2})^3}{(1-q^{n+m+\ell+3})(1-q^{n+m+\ell+4})}\\ &=\frac{1}{1-q^{-1}}\prod_{n=2}^{\infty}\left[\frac{1}{1-q^{n-1}}\right]^{\frac{(n+1)(n+2)}{2}}\prod_{m=0}^{\infty}\left[\frac{(1-q^{m+2})^3}{(1-q^{m+3})(1-q^{m+4})}\right]^{\frac{(m+1)(m+2)}{2}}.\end{aligned} \tag{C.10}$$

After some simplification this gives the expression in (2.14)

$$G_0(q) = -q\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^7} = -q^{\frac{31}{24}}\left[\eta(q)\right]^{-7}, \qquad \eta(q)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{C.11}$$

Note that the $q^{\frac{31}{24}}$ factor here happens to be the inverse of the supersymmetric Casimir energy factor in the one-loop M5 brane partition function in (4.4).

Appendix D. Free energy of a conformal scalar field on $S^1\times S^5$

Here we present the computation of the free energy (3.44) of the conformally coupled massless 6d scalar on $S^1_\beta\times S^5$ with an extra shift ν of the S^1 mode number $n_\beta=\frac{2\pi n}{\beta}$. That shift may be due to the presence of a flat connection or $y=\xi^1$ dependent rotation of a complex scalar. Since $S^1\times S^5$ is conformally flat, there are no logarithmic UV divergences in the corresponding one-loop effective action or free energy. One should still subtract power divergences using ζ-function regularization.

---

[39] If the coefficient of a monomial is not ± 1, we use that $\mathrm{PE}[2X]=(\mathrm{PE}[X])^2$, etc.

[40] Such a degeneracy enhancement at special points in the fugacity space is a common phenomenon (see, for instance, Eq. (4.4) in [16] and also [17]).

For the case of a conformally coupled massless scalar on $S^1\times S^5$ one has (see also footnote 17)[41]

$$F=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\mathrm{d}_k\,\log\lambda_{n,k}\ ,\qquad\qquad \mathrm{d}_k=\tfrac{1}{12}(k+1)(k+2)^2(k+3)\ , \tag{D.1}$$

$$\lambda_{n,k}=(\frac{2\pi n}{\bar\beta})^2+\omega_k^2\ ,\qquad \omega_k=k+2\ ,\qquad\qquad \bar\beta\equiv\tfrac{1}{2}\beta\ . \tag{D.2}$$

One may define the spectral zeta function $\zeta_\Delta(z)=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\mathrm{d}_n\lambda_{n,k}^{-z}$ in terms of which one finds that $\zeta_\Delta(0)=0$ (S^5 is odd-dimensional) and (see, e.g., [48])

$$F=-\tfrac{1}{2}\zeta'_\Delta(0)=F_c+\bar F\ ,\qquad F_c=\bar\beta E_c=\tfrac{1}{2}\bar\beta\sum_{k=0}^{\infty}\mathrm{d}_n\,\omega_k\ ,\qquad \bar F=\sum_{k=0}^{\infty}\mathrm{d}_k\log(1-e^{-\bar\beta\omega_k})\ . \tag{D.3}$$

In the present case of (3.44) we have instead of $\lambda_{n,k}$ (D.2) their "shifted" analog that can be written as

$$\lambda_{n,k}(\nu)=\omega_k^2+(\frac{2\pi n}{\bar\beta}+2i\nu)^2=(\omega_k^+ +i\frac{2\pi n}{\bar\beta})(\omega_k^- -i\frac{2\pi n}{\bar\beta})\ ,\qquad \omega_k^\pm\equiv\omega_k\pm 2\nu=k+2\pm 2\nu\ . \tag{D.4}$$

We observe that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\log\lambda_{n,k}(\nu)&=\sum_{n=-\infty}^{\infty}\log\big(\omega_k^+ +i\frac{2\pi n}{\bar\beta}\big)+\sum_{n=-\infty}^{\infty}\log\big(\omega_k^- -i\frac{2\pi n}{\bar\beta}\big)\\ &=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\log\big[(\omega_k^+)^2+(\frac{2\pi n}{\bar\beta})^2\big]+\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\log\big[(\omega_k^-)^2+(\frac{2\pi n}{\bar\beta})^2\big]\ .\end{aligned} \tag{D.5}$$

We can thus apply (D.3) to get the following representation for the free energy in (3.44)

$$F(\nu)=F_c(\nu)+\bar F(\nu)=\frac{1}{2}\big[F^+(\nu)+F^-(\nu)\big]\ ,\qquad\qquad F^\pm(\nu)=F_c^\pm+\bar F^\pm\ , \tag{D.6}$$

$$F_c^\pm=\bar\beta E_c^\pm=\frac{1}{2}\bar\beta\sum_{k=0}^{\infty}\mathrm{d}_k\,\omega_k^\pm\ ,\qquad\qquad \bar F^\pm=\sum_{k=0}^{\infty}\mathrm{d}_k\log\big(1-e^{-\bar\beta\omega_k^\pm}\big)\ . \tag{D.7}$$

We thus find that $\bar F=\frac{1}{2}(\bar F^+ +\bar F^-)$ is given in (3.46). To obtain a finite expression for the Casimir energy we may use the standard "energy" zeta-function regularization prescription[42]

$$E_c^\pm(\nu)=\frac{1}{2}\sum_{k=0}^{\infty}\mathrm{d}_k\,(\omega_k^\pm)^s\Big|_{s\to 1}\ \to\ -\tfrac{31}{60480}+\tfrac{1}{18}\nu^4-\tfrac{4}{45}\nu^6\ . \tag{D.8}$$

This is even in ν so we get the expression for $E_c=\frac{1}{2}(E_c^+ +E_c^-)$ as in (3.45), (3.47).

Appendix E. Partition functions of fields of (2,0) multiplet on twisted $S^1\times\widetilde S^5$

Here we shall find the partition functions (4.15), (4.16), (4.17) for the fields of the 6d tensor multiplet defined on the $S^1\times\widetilde S^5$ space with the metric (cf. (3.10), (4.13) and [49]; see also [50])

$$ds^2_{S^1\times\widetilde S^5}=\tfrac{1}{4}dy^2+dn_1^2+dn_2^2+dn_3^2+n_1^2(d\varphi_1+i\alpha dy)^2+n_2^2(d\varphi_2-i\alpha dy)+n_3^2d\varphi_3^2\ , \tag{E.1}$$

$$y\in(0,\beta),\qquad\qquad n_1^2+n_2^2+n_3^2=1\ . \tag{E.2}$$

We shall start with computing the conformally coupled scalar free energy directly from the determinant of its kinetic operator and also show how to obtain the same result by the operator counting method in "twisted" $\mathbb{R}^6$. We shall then apply the latter approach to the fermion and the antisymmetric tensor cases.

E.1. Scalar free energy on $S^1\times\widetilde S^5$

Let us first consider the simplest case of a scalar on $S^1_\beta\times\widetilde S^1$ with the metric $\frac{1}{4}dy^2+(d\varphi+i\alpha dy)^2$, where $\varphi\in(0,2\pi)$. Introducing $\varphi'=\varphi+i\alpha y$ one finds that $-\nabla^2=-4\partial_y^2-\partial_{\varphi'}^2$ has the eigenfunctions $f_k(p)\exp(ipy+ik\varphi')=\exp[i(p+i\alpha k)y+ik\varphi]$ which are

[41] To recall, compared to the standard normalization we are considering a scalar field on $S^1\times S^5$ with an extra $\frac{1}{4}$ in the metric (3.10), i.e. with the effective length of the "temperature circle" S^1 being $\bar\beta=\frac{1}{2}\beta$.

[42] Equivalently, instead of $(\omega_k^\pm)^s$ one may use $\omega_k^\pm e^{-\epsilon\omega_k^\pm}$, do the sum and drop terms that are singular in the limit $\epsilon\to 0$. The expression in (D.8) generalizes to $\nu\neq 0$ the value found, e.g., in [30,36].

periodic under $y \to y + \beta$ and $\varphi \to \varphi + 2\pi$ if $p + i\alpha k = n_\beta = \frac{2\pi n}{\beta}$ where n and k are integers. The corresponding eigenvalue is $4p^2 + k^2 = 4(n_\beta - i\alpha k)^2 + k^2$, i.e.

$$F = \tfrac{1}{2} \log \det(-\nabla^2) = \tfrac{1}{2} \sum_{n,k\in\mathbb{Z}} \log \left[4(n_\beta - i\alpha k)^2 + k^2\right] . \tag{E.3}$$

The sum over n here can be done as in section 3.4 or Appendix D with the shift $v = \alpha k$. We find for the thermal part of the free energy

$$\bar{F} = \tfrac{1}{4} \sum_{\pm} \sum_{k=1}^{\infty} \log(1 - e^{-\frac{1}{2}\beta k(1\pm 2\alpha)}). \tag{E.4}$$

The corresponding single-particle partition function is then (cf. (4.12))

$$\hat{\mathcal{Z}}_\phi(q,u) = \sum_{\pm} \sum_{k=1}^{\infty} q^{k(1\pm 2\alpha)} = \sum_{\pm} \frac{q^{1\pm 2\alpha}}{1 - q^{1\pm 2\alpha}} = \sum_{\pm} \frac{u^{\pm 1} q}{1 - u^{\pm 1} q} , \qquad q = e^{-\frac{1}{2}\beta}, \quad u = q^{2\alpha} . \tag{E.5}$$

Next, let us consider a more complicated case of $S^1_\beta \times \widetilde{S}^3$ subset of (E.1) with the metric $\frac{1}{4}dy^2 + d\chi^2 + \cos^2\chi\,(d\varphi_1 + i\alpha dy)^2 + \sin^2\chi\,(d\varphi_2 - i\alpha dy)^2$. A convenient basis for the eigenfunctions of the Laplacian on the standard S^3 is (see, e.g., [51])

$$\Phi_{k,r_1,r_2} = F_{k,r_1,r_2}(\chi)\, e^{i(r_1+r_2)\varphi_1}\, e^{i(r_2-r_1)\varphi_2}, \qquad r_1, r_2 = -\frac{1}{2}k, \dots, \frac{1}{2}k , \quad k = 0, 1, \dots . \tag{E.6}$$

Including the effect of the two α-shifts is then achieved as in the above example and we find the following analog of (E.3) for a conformally coupled massless scalar[43]

$$\begin{aligned} F = \tfrac{1}{2} \log \det(-\nabla^2 + 1) &= \tfrac{1}{2} \sum_{k=0}^{\infty} \sum_{r_1,r_2=-k/2}^{k/2} \log \left[4\big(n_\beta + i\alpha(r_1 + r_2) - i\alpha(r_1 - r_2)\big)^2 + k(k+2) + 1\right] \\ &= \tfrac{1}{2} \sum_{k=0}^{\infty} \sum_{m=-k/2}^{k/2} (k+1) \log \left[4(n_\beta + 2i\alpha m)^2 + (k+1)^2\right] . \end{aligned} \tag{E.7}$$

The corresponding thermal partition function and the single-particle partition function are then

$$\bar{F} = \tfrac{1}{4} \sum_{\pm} \sum_{k=0}^{\infty} \sum_{m=-k/2}^{k/2} (k+1) \log[1 - e^{-\frac{1}{2}\beta(k+1\pm 4\alpha m)}] , \tag{E.8}$$

$$\begin{aligned} S^1 \times \widetilde{S}^3 : \qquad \hat{\mathcal{Z}}_\phi(q,u) &= \tfrac{1}{2} \sum_{\pm} \sum_{k=0}^{\infty} \sum_{m=-k/2}^{k/2} (k+1) q^{k+1\pm 4\alpha m} = \frac{q - q^3}{(1 - q^{1+2\alpha})^2 (1 - q^{1-2\alpha})^2} \\ &= \frac{q(1-q^2)}{(1 - u\,q)^2 (1 - u^{-1}\,q)^2} , \qquad q = e^{-\frac{1}{2}\beta}, \quad u = q^{2\alpha} . \end{aligned} \tag{E.9}$$

The computation in the case of $S^1_\beta \times \widetilde{S}^5$ we are interested in is similar. One is first to find a convenient labelling of the eigenfunctions of the Laplacian on S^5 represented by k-symmetric traceless tensors built out of Cartesian coordinates of $\mathbb{R}^6$. More precisely, we need the weights of the k-symmetric traceless representation S_k of $SO(6)$.

The solution to this problem is based on the embedding $SU(2) \times SU(2) \times U(1) \subset SO(6)$. The 15 generators of $SO(6)$ are J_{mn} with $m, n = 1, \dots, 6$ with the Cartan subalgebra generators J_{12}, J_{34}, J_{56}. The two $SU(2)$'s are the factors in $SO(4) = SU(2) \times SU(2)$ where the generators of $SO(4)$ are J_{ab} with $a, b = 1, \dots, 4$. The generator of $U(1)$ is J_{56}. Thus the states in a generic representation of $SO(6)$ are $|j, j'; m, m'; s\rangle$, $m = -j, \dots, j$, $m' = -j', \dots, j'$, where j, j' are integer or half-integer, m, m' have unit spacing, and the eigenvalue s of J_{56} is integer. For the S_k representation of $SO(6)$ one has $j' = j$.

All states are degenerate with respect to the ladder operators of the two $SU(2)$'s, *i.e.* $J_\pm$ and $J'_\pm$. Thus we can restrict our attention to the states $|j, j; j, j; s\rangle$ that contribute to the spectrum with multiplicity $(2j+1)^2$. As shown in [52], such states appearing in S_k have quantum numbers

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots, \tfrac{k}{2}, \qquad -(k - 2j) \le s \le k - 2j, \qquad s \text{ in steps of 2.} \tag{E.10}$$

The number of s values for a given j is thus $k - 2j + 1$, reproducing the multiplicity d_k of the eigenvalues of the Laplacian on S^5

$$\sum_{j=0,1/2,\dots,k/2} (k - 2j + 1)(2j+1)^2 = \sum_{n=0}^{k} (k - n + 1)(n+1)^2 = \tfrac{1}{12}(k+1)(k+2)^2(k+3) . \tag{E.11}$$

[43] For the standard Laplacian on S^3 we have $\lambda_k = k(k+2)$ and $\mathrm{d}_k = (k+1)^2$ and for a conformally coupled scalar we need to add 1 to λ_k (see footnote 17).

The free energy for a conformally coupled massless scalar on $S^1 \times \widetilde{S}^5$ is then found to be[44]

$$
\begin{aligned}
F &= \tfrac{1}{2}\log\det(-\nabla^2+4) \\
&= \tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{j=0,1/2,\dots,k/2}(k-2j+1)\sum_{r_1,r_2=-j}^{j}\log\Big[4\big(n_\beta+i\alpha(r_1+r_2)-i\alpha(r_1-r_2)\big)^2+(k+2)^2\Big] \\
&= \tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{j=0,1/2,\dots,k/2}(k-2j+1)(2j+1)\sum_{r_2=-j}^{j}\log\big[2(n_\beta+2i\alpha r_2)^2+(k+2)^2\big]. \qquad \text{(E.12)}
\end{aligned}
$$

The corresponding single particle partition function reads

$$
\begin{aligned}
S^1\times\widetilde{S}^5: \qquad \widehat{\mathcal{Z}}_\phi(q,u) &= \tfrac{1}{2}\sum_{\pm}\sum_{k=0}^{\infty}\sum_{j=0,1/2,\dots,k/2}(k-2j+1)(2j+1)\sum_{m=-j}^{j}q^{k+2\pm4\alpha m} \\
&= \frac{q^2(1-q^2)}{(1-q)^2(1-q^{1+2\alpha})^2(1-q^{1-2\alpha})^2} = \frac{q^2(1-q^2)}{(1-q)^2(1-u\,q)^2(1-u^{-1}q)^2}\ . \qquad \text{(E.13)}
\end{aligned}
$$

As in (E.9), here the only dependence on u is in the factors in the denominator. Eq. (E.13) thus reproduces the expression in (4.15).

E.2. Single-particle partition functions from operator counting on twisted $\mathbb{R}^6$

Let us note that the same results can be found by first mapping the conformal scalar on $S^1 \times \widetilde{S}^d$ to flat $\mathbb{R}^{d+1}$ space and then applying the operator counting method (see, e.g., [53,38,54]).[45] In the untwisted case $\alpha = 0$ one starts with a free massless scalar in $\mathbb{R}^6$ that has dimension 2. This corresponds to the factor q^2 in (E.13). The factor $1/(1-q)^6$ counts the number of descendant fields obtained by applying, in all possible ways, the six ∂_m derivatives to the scalar ϕ. The subtraction of q^4 term in (E.13) accounts for the fact that the trivial operators proportional to the equations of motion $\partial^2\phi$ should not be included. Similar interpretation is true for (E.9).

To include the twists one may separate coordinates into 1+2 rotation planes and multiply the relevant derivatives by the corresponding rotation phases. The numerator in (E.13) is then unchanged as the equation of motion is rotationally invariant. The same applies to (E.9).[46]

This operator counting method in twisted $\mathbb{R}^6$ can be applied also to find the single-particle partition functions the fermion (4.16) and the antisymmetric tensor (4.17). For the latter with the field strength H_{mnk} of dimension 3 in the untwisted case one has [38,54]

$$
\widehat{\mathcal{Z}}_A(q) = \frac{10q^3-15q^4+6q^5-q^6}{(1-q)^6} = \frac{q^3(10-5q+q^2)}{(1-q)^5}\ . \qquad \text{(E.14)}
$$

Here the coefficient 10 of the q^3 term in the numerator corresponds to the number $\frac{1}{2}\times\frac{6\cdot5\cdot4}{3!}=10$ of the H_{mnk} components subject to the self-duality condition. The two α-twists are taken into account by assigning to the six derivatives ∂_m of $\mathbb{R}^6$ the R-charge weights

$$
r_k = (u,u,u^{-1},u^{-1},1,1)\ , \qquad \text{(E.15)}
$$

corresponding to the opposite rotations in the 12 and 34 planes (cf. (E.1)). Then, before imposing the self-duality, the 20 components of H_{mnk} have the following rotation weights

3-form components	weight
H_{125}, H_{126}	$2\times u^2$
$H_{123}, H_{124}, H_{156}, H_{256}$	$4\times u$
$H_{135}, H_{136}, H_{145}, H_{146}, H_{235}, H_{236}, H_{245}, H_{246}$	$8\times u^0$
$H_{134}, H_{234}, H_{356}, H_{456}$	$4\times u^{-1}$
H_{345}, H_{346}	$2\times u^{-2}$

As a result, the $10q^3$ term in (E.14) gets generalized to

$$
10q^3 \ \to\ c_1(u)\,q^3, \qquad c_1(u) = u^2+2u+4+2u^{-1}+u^{-2}. \qquad \text{(E.16)}
$$

[44] We use the explicit relation between the two $SU(2)$ generators and the $SO(4)$ generators $J_3 = \frac{1}{2}(J_{12}+J_{34})$, $J_3' = \frac{1}{2}(J_{12}-J_{34})$.

[45] To recall, one has $(d\log r)^2 + dS_5 = \frac{1}{r^2}ds^2_{\mathbb{R}^6}$ so that the energies in $S^1\times S^5$ may be identified with the scaling dimensions in $\mathbb{R}^6$.

[46] Note that in the case of $S^1\times\widetilde{S}^1$ with only one twist the expression in (E.5) may be also interpreted in terms of the operator counting on $\mathbb{R}^2$ as the contribution of one tower of fields starting with dimension 1 primary $\partial_z\phi$ and another corresponding to its conjugate $\partial_{\bar z}\phi$, with the twist introducing factors of u and u^{-1} respectively.

The term $-15q^4$ in (E.14) corresponds to the subtraction of the contribution of the equation of motion operator[47] $\partial^m H_{mnk} \equiv \tilde{H}_{nk}$ for which similar assignments of the twist factors read

$\tilde{H}_{mn}$ components	weight
$\tilde{H}_{12}$	u^2
$\tilde{H}_{15}, \tilde{H}_{16}, \tilde{H}_{25}, \tilde{H}_{26}$	$4 \times u$
$\tilde{H}_{13}, \tilde{H}_{14}, \tilde{H}_{23}, \tilde{H}_{24}, \tilde{H}_{56}$	$5 \times u^0$
$\tilde{H}_{35}, \tilde{H}_{36}, \tilde{H}_{45}, \tilde{H}_{46}$	$4 \times u^{-1}$
$\tilde{H}_{34}$	u^{-2}

As a result, we get

$$-15q^4 \ \to\ -(u^2+4u+5+4u^{-1}+u^{-2})\,q^4 = -[c_1(u)+c_2(u)]\,q^4, \qquad c_2(u) = 2u+1+2u^{-1}. \tag{E.17}$$

One concludes that (E.14) is to be replaced by

$$\hat{\mathcal{Z}}_A(q,u) = \frac{q^3[c_1(u) - c_2(u)\,q + q^2]}{(1-q)(1-u\,q)^2(1-u^{-1}q)^2}\ , \tag{E.18}$$

which reproduces (4.17).

Similarly, in the case of the MW 6d fermion of dimension $\frac{5}{2}$ one finds the expression in (4.16), i.e.

$$\hat{\mathcal{Z}}_\psi(q) = \frac{2q^{5/2}(1-q)}{(1-q)^6} \to \hat{\mathcal{Z}}_\psi(q,u) = \frac{2c_\psi(u)\,q^{5/2}}{(1-q)(1-u\,q)^2(1-u^{-1}q)^2}, \qquad c_\psi(u) = \tfrac{1}{4}(u+2+u^{-1}). \tag{E.19}$$

Here in addition to the twist factors for the derivatives leading to the u dependence in the denominator we get also the factor $c_\psi(u) = \frac{1}{4}(u+1+1+u^{-1}) = \frac{1}{4}(u^{1/2}+u^{-1/2})^2$ that accounts for the rotation weights of different components of ψ. The effect of rotation in the two planes is represented by the matrix $\exp(\frac{i}{2}\delta\varphi_1\Gamma_{12} + \frac{i}{2}\delta\varphi_2\Gamma_{34})$. It can be diagonalized to give the factor $\exp\left[\frac{1}{2}(\pm\varphi_1 \pm \varphi_2)\right]$ with four independent sign combinations. For two opposite twists one has $\delta\varphi_1 = -\delta\varphi_2 = \beta\alpha$ and this leads to the sum of the terms $u+2+u^{-1}$ in $c_\psi(u)$.

Following the same logic, it is straightforward also to write down the partition functions for the case of the two independent twists α and α' in the two isometric angles in (E.1). Generalizing (E.12) and (E.13), for the scalar field we get (here $u = q^{2\alpha}$ and $u' = q^{2\alpha'}$)

$$\begin{aligned}\hat{\mathcal{Z}}_\phi(q,u,u') &= \tfrac{1}{2}\sum_{\pm}\sum_{k=0}^{\infty}\sum_{j=0,1/2,\ldots,k/2}(k-2j+1)\sum_{r_1,r_2=-j}^{j} q^{k+2\pm(2\alpha(r_1+r_2)+2\alpha'(r_1-r_2))} \\ &= \frac{q^2(1-q^2)}{(1-q)^2(1-q^{1+2\alpha})(1-q^{1-2\alpha})(1-q^{1+2\alpha'})(1-q^{1-2\alpha'})}.\end{aligned} \tag{E.20}$$

For the tensor field one has to generalize (E.15) to $r_k = (u, u', u^{-1}, u'^{-1}, 1, 1)$. For the fermion, considering two independent twists gives (E.19) with $c_\psi(u,u') = \frac{1}{4}(u^{1/2}+u^{-1/2})(u'^{1/2}+u'^{-1/2})$.

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