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Semiclassical quantization of M5 brane probes wrapped on $\mathrm{AdS}_3 \times S^3$ and defect anomalies

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Abstract: We consider two supersymmetric M5 brane probe solutions in $\mathrm{AdS}_7 \times S^4$ and one in $\mathrm{AdS}_4 \times S^7$ that all have the $\mathrm{AdS}_3 \times S^3$ world-volume geometry. The values of the classical action of the first two M5 probes (with S^3 in AdS_7 or in S^4) are related to the leading N^2 parts in the anomaly b-coefficient in the (2,0) theory corresponding to a spherical surface defect in symmetric or antisymmetric $SU(N)$ representations. We present a detailed computation of the corresponding one-loop M5 brane partition functions finding that they vanish (in a particular regularization). This implies the vanishing of the order N^0 part in the b-anomaly coefficients, in agreement with earlier predictions for their exact values. It remains, however, a puzzle of how to reproduce the non-vanishing order N terms in these coefficients within the semiclassical M5-brane probe setup.

Keywords: M-Theory, Anomalies in Field and String Theories, AdS-CFT Correspondence

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1 Introduction

The defect anomaly coefficients in 6d (2,0) theory (see, e.g., [1–5] and refs. there) can be studied via AdS/CFT correspondence by considering BPS M-brane probes in $AdS_7 \times S^4$ [6, 7] and semiclassically quantizing them [8–11].[1]

Ref. [8] considered an M2 brane probe wrapped on $AdS_3 \subset AdS_7$ in $AdS_7 \times S^4$ background that intersected the boundary over S^2. The effective dimensionless M2 tension in this case is $\mathsf{T}_2 = \frac{2}{\pi}N$ where N is the number of M5 branes forming the $AdS_7 \times S^4$ background (or

[1] Examples of one-loop computations for M-branes in AdS backgrounds were discussed also in [12–15].

rank of the (2,0) boundary CFT). A semiclassical expansion of the M2 brane free energy F then determines the large N expansion of the "central charge" or b-anomaly coefficient of the S^2 defect in the (2,0) theory. The resulting classical and one-loop M2 brane contributions were found to be [8] [2]

$$\mathrm{b} = 12N - 9 + \mathcal{O}(N^{-1})\,. \tag{1.1}$$

This turns out to be consistent with the expression for b-anomaly found from the entanglement entropy computation for the "bubbling" M5-M2 geometry [24]. The general expression for b-anomaly corresponding to a $\frac{1}{2}$-BPS surface defect operator in (2,0) theory in a $SU(N)$ representation with the Young tableau with a large number of boxes is [1–3]

$$\mathrm{b} = 24(\rho,\lambda) + 3(\lambda,\lambda)\,. \tag{1.2}$$

Here ρ is the Weyl vector of $SU(N)$ and λ is the highest weight of the $SU(N)$ representation.[3]

If we formally assume that (1.2) is valid not just for large representations but also for the ones with finite number of boxes then for a single M2 brane corresponding to the surface operator in the fundamental representation (with $(\rho,\lambda) = \frac{N-1}{2}$, $(\lambda,\lambda) = \frac{N-1}{N}$) one finds

$$\mathrm{b} = 12N\left(1 + \frac{1}{4}N^{-1}\right)(1 - N^{-1}) = 12N - 9 - 3N^{-1}\,. \tag{1.3}$$

The first two terms here match the classical and one-loop terms in (1.1) while the N^{-1} term should correspond to the 2-loop M2 brane correction.

In the case of k-symmetric and k-antisymmetric representations (1.2) gives (cf. (H.29), (H.30))

$$\mathrm{b}_{(k)} = 12kN\left(1 + \frac{1}{4}kN^{-1}\right)(1 - N^{-1}) = 12kN - 3k(4-k) - 3k^2N^{-1}\,, \tag{1.4}$$

$$\mathrm{b}_{[k]} = 12kN\left(1 + \frac{1}{4}N^{-1}\right)(1 - kN^{-1}) = 12kN - 3k(4k-1) - 3k^2N^{-1}\,, \tag{1.5}$$

which of course reduce to (1.3) for $k = 1$. The case $k > 1$ should correspond to a system of multiple M2 brane probes which it is not clear how to quantize directly. However, for

[2] To recall, in the presence of 2d defects in a CFT defined on a curved space its stress-tensor trace anomaly can be written as the sum of the ambient space contribution and the following additional term localized on the defect [3, 16–19]: $T^\mu_\mu\big|_{\text{defect}} = -\frac{1}{24\pi}\big(\mathrm{b}\,\widehat{R} + \mathrm{d}_1\Pi^\mu_{ij}\,\Pi^{ij}_\mu - \mathrm{d}_2\,W_{ij}{}^{ij}\big)$. Here $\widehat{R}$ is the Ricci scalar for the induced metric on the defect, Π^μ_{ij} is the traceless second fundamental form of the defect and $W_{ijk\ell}$ is the pull-back of the Weyl tensor. We follow [2] and denote the "central charge" coefficient as b. Following [20–22] one may compute holographically the entanglement entropy (EE) of a spherical region centred on the 2d defect or of a semi-circle centred on the 2d boundary. After subtracting the EE of the ambient CFT the coefficient of the logarithmic in the UV cutoff term may be denoted as $\frac{1}{3}b$. For a CFT$_d$ this "central charge" b obeys [2, 23] $b = \mathrm{b} - \frac{d-3}{d-1}\mathrm{d}_2$.

[3] To be precise, the status of (1.2) as an exact in N expression as found in [1] could still be viewed as conjecture. In [3] a similar expression for the d_2 anomaly coefficient (see (1.11), (H.25)) was derived as an exact result from a superconformal index computation (it also follows from the 5d Wilson Loop localization computation as in [25], see appendix H below). Given that b and d_2 appear on an equal footing in the spherical entanglement entropy [2, 26] one may expect that the expression for b should also be exact. Indeed, in [27] the same expression (1.2) was found using 't Hooft anomaly considerations.

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large $k \sim N \gg 1$ one may expect that such M2 brane configuration should "blow up" into a single M5 brane (wrapped on $S^3 \subset \mathrm{AdS}_7$ in the case of k-symmetric representation and on $S^3 \subset S^4$ in the case of k-antisymmetric one) with a non-zero world-volume 3-form flux representing the M2 brane charge k.

The two corresponding classical M5 brane probe solutions in $\mathrm{AdS}_7 \times S^4$ that have $\mathrm{AdS}_3 \times S^3$ world volume geometry were found in [6] (see also [25, 28]). We will refer to them as **Ia** and **Ib** probes below. These solutions should apply in the limit when

$$N, k \gg 1\,, \qquad \kappa^2 \equiv \frac{k}{2N} = \text{fixed}\,. \tag{1.6}$$

κ plays the role of a free parameter of an M5 brane solution related to its location in $\mathrm{AdS}_7 \times S^4$ and also to the value of the world-volume 3-form field H_3. Expressing (1.4), (1.5) in terms of N and κ we get

$$\mathrm{b}_{(k)} = 24N(N-1)\,\kappa^2\left(1+\frac{1}{2}\kappa^2\right) = 24N^2\kappa^2\left(1+\frac{1}{2}\kappa^2\right) - 24N\kappa^2\left(1+\frac{1}{2}\kappa^2\right)\,, \tag{1.7}$$

$$\mathrm{b}_{[k]} = 24N\left(N+\frac{1}{4}\right)\kappa^2(1-2\kappa^2) = 24N^2\kappa^2(1-2\kappa^2) + 6N\kappa^2\left(1-2\kappa^2\right)\,. \tag{1.8}$$

By analogy with the M2 brane probe case in (1.1) one may conjecture that (1.7), (1.8) may be reproduced by semiclassically quantizing the corresponding M5 brane probe.

The effective dimensionless M5 brane tension here is $\mathsf{T}_5 = \frac{2}{\pi^3}N^2 = \frac{1}{2\pi}(\mathsf{T}_2)^2$ (see (2.8)) and the leading N^2 terms in (1.7) and (1.8) are indeed reproduced by the values of the classical M5 brane action for the two corresponding solutions.

However, the subleading terms in (1.7), (1.8) do not appear to have a natural interpretation within the semiclassical M5 brane expansion, i.e. the expansion in powers of $(\mathrm{T}_5)^{-1} \sim N^{-2}$ for fixed κ. The order N terms in (1.7), (1.8) look as if they are coming, in fact, from a classical M2 brane action or "$\frac{1}{2}$-loop" order of M5 brane perturbation theory.[4]

Regardless the resolution of the puzzle of the order N terms, the expressions (1.7), (1.8) do not contain order N^0 terms implying that one-loop M5 brane corrections to the b-coefficient should be zero. Our aim here will be to demonstrate this by directly computing the one-loop corrections to the free energy of the corresponding two M5 brane probe solutions in $\mathrm{AdS}_7 \times S^4$ found in [6]. We will also consider a similar M5 brane solution in $\mathrm{AdS}_4 \times S^7$ having again the $\mathrm{AdS}_3 \times S^3$ world volume (this solution was found in [6] and also in [29] and the study of bosonic fluctuations around it was initiated in [30, 31]).

This will require a non-trivial extension of the earlier computations of one-loop partition functions of M5 branes wrapped on $S^1 \times S^5$ in a twisted version of $\mathrm{AdS}_4 \times S^7$ in [15] and on $\mathrm{AdS}_5 \times S^1$ in $\mathrm{AdS}_7 \times S^4$ in [11] to the cases with a non-zero H_3 world-volume field. The presence of the H_3 background introduces a complication due to the self-duality constraint on the world-volume 3-form field requiring to use the detailed structure of the M5 brane action [32–38] (see also [39–41]).

[4]Surprisingly, the κ-dependence of the leading (order N^2) and the subleading (order N) terms in (1.7), (1.8) happens to be the same. One could then conjecture that these expressions correspond to the classical M5 brane contribution but with "renormalized" M5 tension. It is not clear, however, how to justify this possibility given, in particular, that this "renormalization" happens to be different in the two cases in (1.7) and (1.8) (cf. also (1.12) and (1.13) below).

	Background	Probe	World-volume		One-loop correction
Ia	$\mathrm{AdS}_7 \times S^4$	$M5$	$\mathrm{AdS}_3 \times S^3$	$S^3 \subset \mathrm{AdS}_7$	here
Ib	$\mathrm{AdS}_7 \times S^4$	$M5$	$\mathrm{AdS}_3 \times S^3$	$S^3 \subset S^4$	here
I'	$\mathrm{AdS}_7 \times S^4$	$M5$	$\mathrm{AdS}_5 \times S^1$		[11]
I"	$\mathrm{AdS}_7 \times S^4$	$M2$	AdS_3		[8]
II	$\mathrm{AdS}_4 \times S^7$	$M5$	$\mathrm{AdS}_3 \times S^3$		here
II'	$\mathrm{AdS}_4 \times S^7$	$M2$	$\mathrm{AdS}_2 \times S^1$		[12]

Table 1. Brane probes in $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_4 \times S^7$ preserving 16 supersymmetries.

	Background	Supergroup
Ia	$\mathrm{AdS}_7 \times S^4$	$OSp(4^*\vert 2) \times OSp(4^*\vert 2) \subset OSp(8^*\vert 4)$
Ib	$\mathrm{AdS}_7 \times S^4$	$OSp(4^*\vert 2) \times OSp(4^*\vert 2) \subset OSp(8^*\vert 4)$
II	$\mathrm{AdS}_4 \times S^7$	$OSp(4\vert 2,\mathbb{R}) \times OSp(4\vert 2,\mathbb{R}) \subset OSp(8\vert 4,\mathbb{R})$

Table 2. Supersymmetry algebras preserved by the M5 brane probes with world-volume $\mathrm{AdS}_3 \times S^3$.

1.1 Review

To put the discussion in a broader context, let us review some facts about supersymmetric M-brane probes in $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_4 \times S^7$ and their relation to defect anomalies.

Supergravity in 11d admits two special maximally supersymmetric solutions [42] — $\mathrm{AdS}_7 \times S^4$ (near-horizon limit of a stack of M5 branes) and $\mathrm{AdS}_4 \times S^7$ (near-horizon limit of a stack of M2 branes). The dual 6d and 3d CFT's have total of 32 supersymmetries. For $\mathrm{AdS}_7 \times S^4$ the bosonic isometries are $SO(2,6) \times SO(5) \subset OSp(8^*|4)$, while for $\mathrm{AdS}_4 \times S^7$ they are $SO(2,3) \times SO(8) \subset OSp(8|4,\mathbb{R})$.

M-brane probe configurations in these backgrounds that preserve 16 supersymmetries are listed in table 1 below (see [6, 7]).[5] We introduced the labels (**Ia**, **Ib**, etc.) for the different probes that will be used below. We also included a column with references to the computations of the one-loop corrections to the corresponding M-brane partition functions.

Our focus will be on cases **Ia**, **Ib**, **II** that all have the $\mathrm{AdS}_3 \times S^3$ world-volume geometry. Their bosonic isometry is $SO(2,2) \times SO(4) \times SO(4)$ which is a part of the subalgebra of $OSp(8^*|4)$ in the cases **Ia, Ib**, and of $OSp(8|4,\mathbb{R})$ in the case **II**. The corresponding supergroups are given in table 2 (which is adapted from [7]).[6] In the cases **Ia** and **Ib** the

[5] The case **Ia** was discussed in [6] but was not mentioned explicitly in table 4 of [7].

[6] In the case **Ia** the brane is wrapped on $\mathrm{AdS}_3 \times S^3 \subset \mathrm{AdS}_7$ thus having $SO(2,2) \times SO(4)$ symmetry, and is also localized at a point in S^4 leading to the extra $SO(4)$ factor. In the case **Ib** the additional $SO(4)$ symmetry comes from the 3-sphere part of $ds^2_{\mathrm{AdS}_7} = L^2(du^2 + \cosh^2 u\, ds^2_{\mathrm{AdS}_3} + \sinh^2 u\, ds^2_{S^3})$, while in the case **II** it comes from the second S'^3 in $ds^2_{S^7} = L^2(d\theta^2 + \cos^2\theta\, ds^2_{S^3} + \sin^2\theta\, ds^2_{S'^3})$.

boundary of the AdS_3 part of M5 probe represents a 2-dimensional defect in the dual 6d $(2,0)$ CFT. In general, one may distinguish the two cases:

(i) the standard global AdS_7 with the boundary S^6 and thus with the boundary of AdS_3 being S^2;

(ii) "thermal" AdS_7 with the boundary $S^1_\beta \times S^5$ and thus with the boundary of the corresponding "thermal" AdS_3 being $S^1_\beta \times S^1$ (β is the length of the circle).

While here we will be mostly interested in the first case when S^2 corresponds to the spherical Wilson surface defect operator in 6d CFT (see [8] and refs. there), let us add some comments on the second (ii) case. There the dual (2,0) CFT will be defined on $S^1_\beta \times S^5$ and thus may be connected to the 5d SYM theory (with β related to the YM coupling constant). Then the $S^1_\beta \times S^1$ defect may be interpreted in terms of the S^1 Wilson loop (WL) in the 5d SYM.

In the case **I'** of table 1, i.e. of a single M2 brane probe in $AdS_7 \times S^4$, the corresponding WL should be in the fundamental representation [14]. When the defect is taken in a large-rank representation of the gauge group $SU(N)$, one may conjecture that its dual description may be in terms of a single M5 brane probe carrying M2 brane charge and wrapped on AdS_3 and also on $S^3 \subset AdS_7$ for the symmetric representation or $S^3 \subset S^4$ for the antisymmetric representation [25]. The connection to the 5d SYM then suggests to compare the localization prediction [25] for the corresponding S^1 WL expectation value following from the Chern-Simons matrix model (cf. [43]) to the M5 brane probe partition function.

The large β limit of the second (ii) case is related also to the d_2 anomaly coefficient of a surface defect in the (2,0) theory. In the present context of the M5 brane probes with the geometry $AdS_3 \times S^3$ one may expect that the anomaly coefficients b and d_2 can be extracted from the corresponding free energy $F = -\log Z$ [3] as follows.

If the boundary of AdS_3 is S^2, its regularized volume is $\mathrm{vol}(AdS_3) = -2\pi \log(r\Lambda_{IR})$, where r is the radius and Λ_{IR} an IR cutoff (the latter we shall not explicitly indicate in what follows). Then

$$\partial AdS_3 = S^2: \qquad F = -\frac{1}{3}\mathrm{b}\log r\,. \tag{1.9}$$

If the boundary of AdS_3 is $S^1_\beta \times S^1$, then $\mathrm{vol}(AdS_3) = -\frac{1}{2}\pi\beta$ and d_2 is proportional to the Casimir energy that corresponds to the large β asymptotics of F

$$\partial AdS_3 = S^1_\beta \times S^1: \qquad F_{\beta \gg 1} = -\frac{1}{12}\,\mathrm{d}_2\,\beta\,. \tag{1.10}$$

The exact expressions for d_2 for symmetric and antisymmetric representations are similar to the ones for the b-coefficient given in (1.4), (1.5) [1–3] (in general, the analog of (1.2) is $d_2 = 24(\rho,\lambda) + 6(\lambda,\lambda)$, see also (H.29), (H.30))

$$\mathrm{d}_{2\,(k)} = 12N\,k\left(1+\frac{1}{2}kN^{-1}\right)(1-N^{-1})\,, \qquad \mathrm{d}_{2\,[k]} = 12N\,k\left(1+\frac{1}{2}N^{-1}\right)(1-kN^{-1})\,. \tag{1.11}$$

The same expressions were found in [25] by a saddle point analysis of the localization matrix model that computes the corresponding 5d SYM WL expectation value ($F = -\log\langle W\rangle$).

As we will explain below in appendix H, the reason why the saddle point approximation is enough to get the exact value of d_2 is because the subleading at large N and β corrections to the $N, \beta \gg 1$ limit turn out to be exponentially suppressed.

The semiclassical M5 probe expansion corresponds to taking N and k large with κ defined in (1.6) being fixed. In this case we can express (1.11) in the same form as in (1.7), (1.8)

$$\mathrm{d}_{2\,(k)} = 24N(N-1)\,\kappa^2(1+\kappa^2) = 24N^2\kappa^2(1+\kappa^2)\; - 24N\kappa^2(1+\kappa^2)\,, \tag{1.12}$$

$$\mathrm{d}_{2\,[k]} = 24N\left(N+\frac{1}{2}\right)\,\kappa^2(1-2\kappa^2) = 24N^2\kappa^2(1-2\kappa^2)\; + 12N\kappa^2(1-2\kappa^2)\,. \tag{1.13}$$

The leading N^2 terms in these expressions are indeed reproduced by the classical actions of an M5 brane probe in the corresponding cases **Ia** and **Ib** with $\partial\mathrm{AdS}_3 = S^1_\beta \times S^1$ [25]. However, like in the b-anomaly case in (1.7), (1.8), the semiclassical M5 brane interpretation of the order N corrections in (1.12), (1.13) remains an open problem.

In general, to find the coefficient d_2 defined as in (1.10) from the semiclassical M-brane probe perspective beyond the leading large N order one may actually need to replace the $\mathrm{AdS}_7 \times S^4$ background by its "twisted" version (see [14] and refs. there). Indeed, to get the β-dependent WL expectation value on the M-theory side one should start with an M-brane in the product of the "thermal" $\mathrm{AdS}_{7,\beta}$ and "twisted" $\tilde{S}^4$ (with one angle $z \to z + i\tau$, where $\tau \in (0,\beta)$) is the "time" coordinate of $\mathrm{AdS}_{7,\beta}$). This deformation does not change the value of classical M-brane probe or the leading term in F in the semiclassical expansion (1.6), but may alter the subleading corrections.

This can be seen already in the $k = 1$ case of the fundamental representation, when the WL expectation value should be reproduced by a single M2 brane probe wrapped on $\mathrm{AdS}_{3,\beta}$ [14]. According to (1.11), in this case we should get $\mathrm{d}_2 = 12N - 6 - 6N^{-1}$. The leading term here is the same as in b in (1.3) but the subleading term is different. This implies that the subleading correction in d_2 cannot be reproduced by the one-loop M2 brane probe computation in $\mathrm{AdS}_7 \times S^4$ as in [8] with the only modification being that the AdS_3 wrapped by M2 has now the boundary $S^1_{\beta\gg 1} \times S^1$ instead of S^2.[7] In fact, it was shown in [14] that quantizing M2 brane in the "twisted" $\mathrm{AdS}_{7,\beta} \times \tilde{S}^4$ background one indeed reproduces the localization result $\langle W\rangle_{\beta\gg 1} = \exp[(N - \frac{1}{2} + \cdots)\beta]$. Hence, one finds (cf. (1.10)) $\mathrm{d}_2 = 12\beta^{-1}\log\langle W\rangle_{\beta\gg 1} = 12N - 6 + \cdots$, which is in agreement with (1.11).

1.2 Summary

As already stated above, our aim below will be to compute the one-loop corrections to the free energies of the $\mathrm{AdS}_3 \times S^3$ M5 brane probes in the three cases **Ia**, **Ib** and **II** with the boundary of AdS_3 assumed to be S^2.[8]

We will start with the M5 brane action and expand it to quadratic order in fluctuations near a given classical solution. The presence of a non-zero H_3 background will introduce

[7] Since for large β the space $S^1_{\beta\gg 1} \times S^1$ is the same as $\mathbb{R} \times S^1$ which is conformal to S^2 one would get the one-loop contribution to the free energy on AdS_3 (which is a homogeneous space) in a universal form $F^{(1)} = a\,\mathrm{vol}(\mathrm{AdS}_3)$. One would then find the same ratio of the leading (order N) and subleading (order N^0) terms in both b and d_2, but this is not the case.

[8] Note that the general expressions for the quadratic fluctuation actions and the resulting structure of the one-loop partition functions derived below will apply also to the case of AdS_3 with $S^1_\beta \times S^1$ boundary.

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a non-trivial complication due to a mixing between the fluctuations of the B_2 potential and one of the scalar coordinates.

Expanding all the 6d fluctuation fields in modes on S^3 (labelled by level ℓ) we will "diagonalize" that mixing and then organize the towers of fluctuations into a collection of massive short supermultiplets on AdS_3 thus maintaining the underlying supersymmetry. We will then evaluate the resulting one-loop AdS_3 determinants using the standard relations for the case when the AdS_3 boundary is S^2. Summing all the contributions together, we will find that there are many non-trivial cancellations with the final results for the one-loop corrections to the free energies given simply by

$$F^{(1)}_{\mathbf{Ia}} = F^{(1)}_{\mathbf{Ib}} = -\frac{3}{2\pi}\,\mathrm{vol}(\mathrm{AdS}_3)\sum_{\ell=1}^{\infty}\ell\,. \tag{1.14}$$

Note that despite the theory on M5 brane probe being a 6d one in a non-trivial background, there are no logarithmic divergences.[9] This is of course a necessary requirement for being able to compare M5 brane free energy with the one that determines the defect anomaly.

The sum over ℓ in **Ia** and **Ib** cases is quadratically divergent and thus the final result depends on a choice of regularization. Similar divergent sums requiring a specific regularization appeared in related contexts, see, e.g., [44–46] (cf. also a discussion in [47]). One regularization procedure used in the past that led to consistent results is to introduce a sharp cutoff $\ell \leq \Lambda$ and drop all power divergent terms (see a discussion below (4.12)). Adopting it here we conclude that the coefficient in (1.14) should be set to zero, so that

$$F^{(1)}_{\mathbf{Ia}} = F^{(1)}_{\mathbf{Ib}} = 0\,. \tag{1.15}$$

This conclusion is then consistent with the fact that the exact expressions for the "central charge" coefficients (1.7) and (1.8) corresponding to the cases **Ia** and **Ib** do not contain order N^0 term.

In the case **II** in table 1 where the M5 brane is embedded into $\mathrm{AdS}_4 \times S^7$ the classical action takes the form (cf. (1.6), (1.7), (1.9))

$$F^{(0)}_{\mathbf{II}} = \frac{1}{4\pi}\Big(N - \frac{1}{2}\mathrm{k}^2\Big)\,\mathrm{vol}(\mathrm{AdS}_3) = \frac{1}{4\pi}N(1-\varkappa^2)\,\mathrm{vol}(\mathrm{AdS}_3)\,, \qquad \varkappa^2 = \frac{\mathrm{k}^2}{2N}\,. \tag{1.16}$$

Here k is an integer parameter of H_3 that has an interpretation of the M2 brane charge carried by M5 brane [6] with $\varkappa^2$ being the fixed semiclassical parameter (here the 5-brane tension is proportional to N). In contrast to the cases **Ia** and **Ib**, the limit of $\mathrm{k}=1$ should not have a description in terms of a single M2 brane embedded into $\mathrm{AdS}_4 \times S^7$ as the intersection of M2 branes (of a probe one with N copies at the boundary) over a 2-surface is not a BPS one. The computation of the M5 brane one-loop correction to (1.16) is similar to the cases **Ia** and **Ib** and gives

$$F^{(1)}_{\mathbf{II}} = 0\,. \tag{1.17}$$

[9] We will verify this independently by showing that the corresponding Seeley coefficient vanishes. The same was found also in the case of the M5 brane probe with the $S^1 \times S^5$ [15] and the $S^1 \times \mathrm{AdS}_5$ [11] geometries (with $H_3 = 0$) which are conformally flat and have zero 6d Euler density. Note that logarithmic divergences are automatically absent at one-loop level in the case of the 3d theory on the M2 brane (as illustrated, e.g., by the examples considered in [12, 14]).

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Here the vanishing of the one-loop contribution happens before the summation over ℓ, i.e. at each S^3 level ℓ independently: the contributions of states in the AdS_3 supermultiplet with fixed ℓ cancel each other.

The plan of the rest of this paper is as follows. In section 2 we will present the three M5 brane classical solutions corresponding to the probes **Ia**, **Ib** and **II** in table 1 that have $\mathrm{AdS}_3 \times S^3$ world volume geometry with $\partial\mathrm{AdS}_3 = S^2$. We will compute the corresponding values of the M5 brane action reproducing the leading large N terms in the corresponding defect anomaly coefficients in (1.4) (**Ia**) and (1.5) (**Ib**) and also obtaining (1.16).

In section 3 we will study the quadratic fluctuations of the bosonic fields in the M5 brane action near the three probe solutions (the fermionic fluctuation operators will be found in appendix A). We will derive the general expressions for the corresponding fluctuation determinants that appear in the one-loop M5 brane partition function. We will then expand the 6d fluctuation fields in modes on S^3 and present the corresponding mass and scaling dimension spectra of the KK towers of fields on AdS_3.

In section 4 we first organize these AdS_3 fields into supermultipets corresponding to the supergroups which represent the required symmetry in each of the three cases (details of these will be discussed in appendix C). We will then compute the corresponding free energies deriving the expressions in (1.14) and (1.17).

Some open questions will be mentioned in section 5. In appendix B we will work out the explicit form of the one-loop partition function of the gauge-invariant rank 2 antisymmetric tensor defined on $\mathrm{AdS}_3 \times S^3$ space with generic radii. In appendix D we will comment on a close analytic continuation relation between the fluctuation spectra in the **Ib** and **II** cases.

In appendix E we will discuss the structure of UV divergences of the one-loop free energies, explaining, in particular, why the logarithmic divergences are absent separately in the contributions of each of the 6d fluctuation fields. In appendix F we will recall some facts about spectra of p-form Laplacians on S^d and their decompositions into transverse and longitudinal parts. In appendix G we will discuss the values of the Casimir energies and the expressions for "thermal" single particle partition functions associated with the AdS_3 fluctuation field supermultiplets presented in table 3.

Finally, in appendix H we will discuss the large N expansion of the 5d SYM Wilson loop expectation value in the symmetric or antisymmetric representation of $SU(N)$ represented by the localization matrix model integral. We will demonstrate that in the large β limit the saddle-point result of [25] that matches the expressions for d_2 in (1.11) is, in fact, exact up to exponential corrections.

2 Classical solutions and actions for M5 brane probes

The bosonic part of the PST action for an M5 brane in 11d supergravity background is given by [32, 33, 38] (see also [34, 35])

$$S = T_5\Big[\int d^6\xi\,\Big(-\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|}+\frac{\sqrt{-|G|}}{4(\partial a)^2}\partial_i a \star\mathsf{H}^{ijk}\mathsf{H}_{jk\ell}\,\partial^\ell a\Big)+\int\Big(\frac{1}{2}H_3\wedge C_3+C_6\Big)\Big], \tag{2.1}$$

where $T_5 = \frac{1}{(2\pi)^5 \ell_P^6}$ and G_{ij}, C_3 and C_6 are pull-backs of the supergravity background fields to the world volume ($i, j, \ell, \ldots = 0, 1, \ldots, 5$)

$$G_{ij} = G_{MN}(X(\xi))\, \partial_i X^M \partial_j X^N , \qquad C_{ijk} = C_{MNL}(X(\xi))\, \partial_i X^M \partial_j X^N \partial_k X^L . \tag{2.2}$$

$X^M(\xi)$ and $H_3 = dB_2$ are the dynamical world-volume fields while the scalar $a(\xi)$ will be gauge-fixed as

$$a(\xi) = \xi^1 . \tag{2.3}$$

We use the definitions $(\partial a)^2 = G^{ij}\partial_i a \partial_j a$ and[10]

$$\mathsf{H}_{ijk} = H_{ijk} - C_{ijk} , \qquad \hat{\mathsf{H}}^{ij} = \frac{1}{6\sqrt{-|G|}} \frac{1}{\sqrt{-(\partial a)^2}}\, \varepsilon^{ijk\ell mn} \partial_k a\, \mathsf{H}_{\ell mn} . \tag{2.4}$$

The 11d field C_6 is defined in terms of the dual of F_4

$$dC_6 = \star F_4 - \frac{1}{2} C_3 \wedge F_4 , \qquad F_4 = dC_3 . \tag{2.5}$$

2.1 M5 branes wrapping $\mathrm{AdS}_3 \times S^3 \subset \mathrm{AdS}_7 \times S^4$

We will parametrize the $\mathrm{AdS}_7 \times S^4$ background as[11]

$$ds^2_{11} = L^2(du^2 + \cosh^2 u\, ds^2_{\mathrm{AdS}_3} + \sinh^2 u\, ds^2_{S^3}) + \frac{1}{4} L^2 ds^2_{S^4} , \tag{2.6}$$

$$F_4 = dC_3 = \frac{3}{8} L^3 \,\mathrm{vol}_{S^4} , \qquad L^3 = 8\pi N \ell_P^3 . \tag{2.7}$$

The corresponding dimensionless tensions of M2 and M5 brane probes in this background are then

$$\mathsf{T}_2 = L^3 T_2 = \frac{L^3}{(2\pi)^2 \ell_P^3} = \frac{2}{\pi} N , \qquad \mathsf{T}_5 = L^6 T_5 = \frac{L^6}{(2\pi)^5 \ell_P^6} = \frac{1}{2\pi} (\mathsf{T}_2)^2 = \frac{2}{\pi^3} N^2 . \tag{2.8}$$

We may assume the boundary of AdS_3 to be S^2 so that (in Euclidean coordinates)

$$ds^2_{\mathrm{AdS}_3} = dw^2 + \sinh^2 w\, ds^2_{S^2} . \tag{2.9}$$

We may also consider the "thermal" case of $\mathrm{AdS}_{3,\beta} \subset \mathrm{AdS}_{7,\beta}$ with the boundary $S^1_\beta \times S$ when

$$ds^2_{\mathrm{AdS}_3} = dw^2 + \cosh^2 w\, d\tau^2 + \sinh^2 w\, d\phi^2 , \qquad \tau \in (0, \beta), \quad \phi \in (0, 2\pi) . \tag{2.10}$$

In the limit $\beta \to \infty$ when the boundary of (2.10) becomes $\mathbb{R} \times S^1$ the metrics (2.10) and (2.9) are related by a coordinate transformation. The Minkowski signature version of (2.10) with the boundary $\mathbb{R} \times S^1$ is

$$ds^2_{\mathrm{AdS}_3} = -\cosh^2 w\, dt^2 + dw^2 + \sinh^2 w\, d\phi^2 . \tag{2.11}$$

[10] $\star\mathsf{H}$ in (2.1) is 6d dual 3-form; we will use $\star$ to denote also 11d dual forms. The 6d antisymmetric tensor with raised indices is assumed to be numerical with $\varepsilon^{012345} = +1$, while $\varepsilon_{i_1 \cdots i_6}$ is given by $G\, \varepsilon^{i_1 \cdots i_6}$, where $G \equiv |G| = \det G_{ij}$. In components, we thus have $6\sqrt{-|G|}\, (\star\mathsf{H})^{ijk} = \varepsilon^{ijkmnr} \mathsf{H}_{mnr}$.

[11] We will use the notation vol_M for the volume form of a space M and $\mathrm{vol}(M)$ for its integral.

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The metric of S^4 in (2.6) can be represented as

$$ds^2_{S^4} = d\theta^2 + \sin^2\theta\, ds^2_{S'^3}\,, \qquad\qquad \theta \in (0,\pi)\,. \tag{2.12}$$

We will consider the M5 brane probes that wrap $\mathrm{AdS}_3 \times S^3$ with $S^3 \subset \mathrm{AdS}_7$ ("probe **Ia**") or with $S^3 = S'^3 \subset S^4$ ("probe **Ib**").

In this subsection we shall label the coordinates in (2.6), (2.11), (2.12) as

$\mathrm{AdS}_3(t,w,\phi)$	S^3	u	θ	S'^3
$0,1,2$	$3,4,5$	6	7	$8,9,\natural$

(2.13)

2.1.1 Probe Ia

The solution for the M5 brane wrapped on $\mathrm{AdS}_3 \times S^3 \subset \mathrm{AdS}_7$ was found in [6]. As follows from (2.7), we have $F_4 \wedge C_3 = 0$ and then (2.5) gives[12]

$$dC_6 = \star F_4 = \frac{3}{8}L^3\left(\frac{2}{L}\right)^4 \star\left(e^7\wedge e^8\wedge e^9\wedge e^{\natural}\right) = 6L^6\cosh^3 u\,\sinh^3 u\, du \wedge \mathrm{vol}_{\mathrm{AdS}_3}\wedge \mathrm{vol}_{S^3}\,, \tag{2.14}$$

$$C_6 = L^6\left(\sinh^6 u + \frac{3}{2}\sinh^4 u\right)\,\mathrm{vol}_{\mathrm{AdS}_3}\wedge\mathrm{vol}_{S^3}\,, \tag{2.15}$$

where we fixed the integration constant so that $C_6(u=0)=0$.

The relevant BPS M5 brane solution is obtained by identifying the coordinates of $\mathrm{AdS}_3\times S^3$ (labelled by 0,1,2,3,4,5 above) with the world-volume coordinates ξ^i and also assuming that it is localized at one point in S^4, i.e. at $\theta = 0$. Then the remaining AdS_7 coordinate u should be fixed to a constant value u_0 and H_3 should be chosen to be proportional to the volume form of S^3

$$u = u_0\,, \qquad \sinh u_0 = \kappa\,, \qquad \theta = 0\,, \qquad H_3 = \kappa^2 L^3\,\mathrm{vol}_{S^3}\,, \qquad \kappa^2 = \frac{k}{2N}\,. \tag{2.16}$$

Here κ is a free parameter. It is related to $k \in \mathbb{Z}$ which is the M2 brane charge that is carried by the M5 brane due to the H_3 flux through S^3 being non-zero [6, 48]

$$T_2\int_{S^3} H_3 = 2\pi k\,, \qquad \text{i.e.} \qquad 2\pi\kappa^2\mathsf{T}_2 = k\,. \tag{2.17}$$

Here we used (2.8) and that $\mathrm{vol}(S^3) = \int \mathrm{vol}_{S^3} = 2\pi^2$. The resulting induced $\mathrm{AdS}_3\times S^3$ world-volume metric and the 6-form in (2.15) are given by

$$ds^2 = L^2\Big[(1+\kappa^2)ds^2_{\mathrm{AdS}_3} + \kappa^2 ds^2_{S^3}\Big]\,, \tag{2.18}$$

$$C_6 = \kappa^4\left(\kappa^2 + \frac{3}{2}\right)\,L^6\,\mathrm{vol}_{\mathrm{AdS}_3}\wedge\mathrm{vol}_{S^3}\,. \tag{2.19}$$

As the M5 brane is localised in S^4, the pull-back of C_3 in (2.7) is zero so that in (2.4)

$$\mathsf{H}_3 = H_3 = \kappa^2 L^3\,\mathrm{vol}_{S^3}\;. \tag{2.20}$$

As a result, the $\star\mathsf{HH}$ term in the action (2.1) vanishes.

[12] e^A is the canonically normalized basis of 1-forms for the metric (2.6).

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We shall use the notation g_A and g_S for the unit-radius metrics on AdS$_3$ and S^3, so that (2.18) implies

$$\sqrt{-|G|} = L^6\kappa^3(1+\kappa^2)^{3/2}\sqrt{-g_A}\sqrt{g_S}\,. \tag{2.21}$$

The components of the tensor $\hat{\mathsf{H}}^{ij}$ defined in (2.4) then are (using the gauge condition (2.3))

$$\hat{\mathsf{H}}^{ij} = \frac{1}{6\sqrt{-|G|}}\sqrt{-G_{11}}\varepsilon^{ij1\ell mn}\mathsf{H}_{\ell mn} = \frac{i}{L^2\kappa(1+\kappa^2)\sqrt{-g_A}}\varepsilon^{ij1}\,, \tag{2.22}$$

$$\hat{\mathsf{H}}_{02} = -\hat{\mathsf{H}}_{20} = G_{00}G_{22}\hat{\mathsf{H}}^{02} = iL^2\frac{1+\kappa^2}{\kappa}\sqrt{-g_A}\,. \tag{2.23}$$

Thus

$$G_{ij} + \hat{\mathsf{H}}_{ij} = \begin{pmatrix} -L^2(1+\kappa^2)\cosh^2 w & 0 & iL^2\frac{1+\kappa^2}{\kappa}\sinh w\cosh w & 0 \\ 0 & L^2(1+\kappa^2) & 0 & 0 \\ -iL^2\frac{1+\kappa^2}{\kappa}\sinh w\cosh w & 0 & L^2(1+\kappa^2)\sinh^2 w & 0 \\ 0 & 0 & 0 & L^2\kappa^2 g_S \end{pmatrix},$$

$$\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|} = L^6\kappa^2(1+\kappa^2)^2\sqrt{-g_A}\sqrt{g_S}\,, \qquad \sqrt{-g_A} = \sinh w\cosh w\,. \tag{2.24}$$

Using (2.24) and (2.19) we get from (2.1) the following value for the classical M5 brane action[13]

$$S = -T_5L^6\kappa^2\left(1+\frac{1}{2}\kappa^2\right)\int \mathrm{vol}_{\mathrm{AdS}_3}\wedge \mathrm{vol}_{S^3}\,. \tag{2.25}$$

Continuing to the Euclidean signature ($S \to -S_E$) and assuming that the AdS$_3$ has metric (2.9) with S^2 as its boundary we get for the tree-level contribution to the M5 brane free energy

$$F^{(0)}_{\mathbf{Ia}} = S_{\mathrm{E}} = \mathsf{T}_5\kappa^2\left(1+\frac{1}{2}\kappa^2\right)\mathrm{vol}(\mathrm{AdS}_3)\,\mathrm{vol}(S^3) = -8N^2\kappa^2\left(1+\frac{1}{2}\kappa^2\right)\log r\,, \tag{2.26}$$

where we used (2.8) and that $\mathrm{vol}(\mathrm{AdS}_3) = -2\pi\log r$. Comparing this with (1.9) we conclude that the corresponding leading large T_5 or large N, fixed κ contribution to the b-anomaly coefficient is

$$\mathrm{b}^{(0)}_{\mathbf{Ia}} = 24N^2\kappa^2\left(1+\frac{1}{2}\kappa^2\right) = 12kN\left(1+\frac{1}{4}kN^{-1}\right), \tag{2.27}$$

which reproduces the leading term in (1.7).

Naively, one could expect to get the same value of the action (2.25) in the case when AdS$_3$ has $S^1_\beta \times S^1$ boundary, now with $\mathrm{vol}(\mathrm{AdS}_3) = -\frac{1}{2}\pi\beta$ (see, e.g., (A.3) in [14]). It turns out, however, that there is a subtlety — one is to use a different gauge choice for C_6 [25, 26].

[13] Let us note that the value of the classical action for the M5 wrapped on AdS$_3$ with S^2 boundary and on S^3 (probe **Ia**) or on S'^3 (probe **Ib**) was not explicitly computed in [6]. The observation that it matches the large N value of the b defect anomaly coefficient [1, 2] for the k-symmetric or k-antisymmetric representations effectively follows from the entanglement entropy computation in [26].

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This then gives (2.25) with $1+\frac{1}{2}\kappa^2 \to 1+\kappa^2$ or[14]

$$S_{\rm E} = \mathsf{T}_5\kappa^2(1+\kappa^2)\,{\rm vol}({\rm AdS}_3)\,{\rm vol}(S^3) = -2N^2\kappa^2(1+\kappa^2)\,\beta = -Nk\Big(1+\frac{1}{2}kN^{-1}\Big)\,\beta. \tag{2.28}$$

Comparing to (1.10) this reproduces the leading semiclassical ($N,k\gg 1$, fixed k/N) contribution to the d_2 anomaly coefficient that matches the large N term in $\mathrm{d}_{2\,(k)}$ in (1.11), (1.12).[15]

2.1.2 Probe Ib

The M5 brane wrapped on $\mathrm{AdS}_3 \subset \mathrm{AdS}_7$ and $S^3 = S'^3 \subset S^4$ was discussed in [6, 25, 49]. From F_4 in (2.7) and (2.12) we get

$$F_4 = dC_3 = \frac{3}{8}L^3\sin^3\theta\;d\theta\wedge{\rm vol}_{S'^3}\,,\qquad\qquad C_3 = \frac{1}{8}L^3(2-3\cos\theta+\cos^3\theta)\;\;{\rm vol}_{S'^3}\,. \tag{2.29}$$

The M5 brane solution is represented by (cf. (2.16); we do not put prime on world-volume S^3)

$$u=u_0=0\,,\qquad \theta=\theta_0,\qquad \cos\theta_0 = 1-4\kappa^2,\qquad H_3=\kappa^2L^3\,{\rm vol}_{S^3}\,,\qquad \kappa^2=\frac{k}{2N}. \tag{2.30}$$

Here $k\le N$ is an integer fixed again from the quantization condition of the M2 brane charge carried by the M5 brane (cf. (2.16), (2.17)). The induced metric is

$$ds^2 = L^2\Big(ds^2_{{\rm AdS}_3}+\frac{1}{4}\sin^2\theta_0 ds^2_{S^3}\Big) = L^2\big[ds^2_{{\rm AdS}_3}+2\kappa^2(1-2\kappa^2)ds^2_{S^3}\big]\,. \tag{2.31}$$

Here

$$\mathsf{H}_3 = H_3 - C_3 = L^3 f(\kappa)\,{\rm vol}_{S^3}\,,\qquad\qquad f(\kappa)=\kappa^2(1-2\kappa^2)(1-4\kappa^2)\,. \tag{2.32}$$

On this solution $C_6=0$, $C_3\wedge H_3=0$ and the classical action (2.1) is given only by the first "volume" term (the $\star\mathsf{HH}$ term in (2.1) again does not contribute). As in (2.24), (2.25) we then find

$$\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|} = L^6\kappa^2(1-2\kappa^2)\sqrt{-g_A}\sqrt{g_S}\,, \tag{2.33}$$

$$S = -T_5L^6\kappa^2(1-2\kappa^2)\int {\rm vol}_{{\rm AdS}_3}\wedge{\rm vol}_{S^3}\;. \tag{2.34}$$

As a result (cf. (2.26))

$$F^{(0)}_{\bf Ib} = S_{\rm E} = \mathsf{T}_5\kappa^2(1-2\kappa^2)\,{\rm vol}({\rm AdS}_3)\,{\rm vol}(S^3) = \frac{4}{\pi}N^2\kappa^2(1-2\kappa^2)\,{\rm vol}({\rm AdS}_3)\,. \tag{2.35}$$

[14] Ref. [25] used different coordinates in which $ds^2_{{\rm AdS}_7} = L^2y^{-2}(dy^2+dr^2+r^2d\varphi^2+dr'^2+r'^2dS_3)$, $F_7=\star F_4 = dC_6 = 6L^6\,{\rm vol}_{{\rm AdS}_7} = 6L^6y^{-7}rr'^3\;dy\wedge dr\wedge d\varphi\wedge dr'\wedge{\rm vol}_{S^3}$, and have chosen $C_6 = -L^6y^{-6}rr'^3\;dr\wedge d\varphi\wedge dr'\wedge{\rm vol}_{S^3}$. This gauge choice does not respect the AdS_3 symmetry but is justified by the condition that the resulting M5 solution with AdS_3 having flat $\mathbb{R}^2$ boundary has zero action, as expected for a flat defect. Then transformed to the coordinates used here one gets (cf. (2.19)) $C_6 = L^6\kappa^4(1+\kappa^2)\;\,{\rm vol}_{{\rm AdS}_3}\wedge{\rm vol}_{S^3}+\delta C_6$ where δC_6 vanishes on the classical M5 brane solution. This leads to the value of the on-shell action in (2.28).

[15] The value of the M5 probe action in the case of AdS_3 with $S^1_\beta\times S^1$ boundary was matched to the leading-order matrix model result (the "Casimir energy" or linear in β part of the WL expectation value) in [25] but it was not explicitly acknowledged earlier that this coefficient is the same as the d_2 coefficient (as expected for the $S^1_\beta\times S^1$ defect [3]) for both symmetric and antisymmetric representations [1, 2].

In the case when $\partial\mathrm{AdS}_3 = S^2$ comparing to (1.9) we then get

$$\mathrm{b}^{(0)}_{\mathbf{Ib}} = 24\kappa^2(1-2\kappa^2) = 12Nk(1-kN^{-1})\,, \tag{2.36}$$

which is thus in agreement with the leading large N term in (1.5), (1.8).

In the case when $\partial\mathrm{AdS}_3 = S^1_\beta \times S^1$ the action has the same form (2.34) (here there is no subtlety with a different gauge choice of C_6). Using again that $\mathrm{vol}(\mathrm{AdS}_3) = -\frac{1}{2}\pi\beta$ and comparing with (1.10), (1.13) we reproduce the first leading large N term in $\mathrm{d}_{2\,[k]}$ in (1.13) which happens to be the same as in $\mathrm{b}_{[k]}$ in (1.13), i.e. is equal to $\mathrm{b}^{(0)}_{\mathbf{Ib}}$ in (2.36).

2.2 M5 brane wrapping $\mathbf{AdS_3 \times S^3 \subset AdS_4 \times S^7}$: probe II

The $\mathrm{AdS}_4 \times S^7$ background is described by (cf. (2.6), (2.7))

$$ds^2_{11} = \mathrm{L}^2(du^2 + \cosh^2 u\, ds^2_{\mathrm{AdS}_3}) + 4\mathrm{L}^2(d\theta^2 + \cos^2\theta\, ds^2_{S^3} + \sin^2\theta\, ds^2_{S'^3})\,, \tag{2.37}$$

$$F_4 = dC_3 = 3\mathrm{L}^3\, \cosh^3 u\, du \wedge \mathrm{vol}_{\mathrm{AdS}_3}\,, \qquad \mathrm{L}^6 = \frac{1}{2}\pi^2 N\, \ell_P^6\,, \tag{2.38}$$

$$C_3 = \mathrm{L}^3 \mathrm{f}(u)\, \mathrm{vol}_{\mathrm{AdS}_3}\,, \qquad \mathrm{f}(u) = 3\sinh u + \sinh^3 u\,, \tag{2.39}$$

where N is the number of M2 branes that form the $\mathrm{AdS}_4 \times S^7$ background. The effective 2-brane and 5-brane tensions here are (cf. (2.8))

$$\mathsf{T}_2 = \mathrm{L}^3 T_2 = \frac{1}{4\sqrt{2}\pi}\sqrt{N}\,, \qquad \mathsf{T}_5 = \mathrm{L}^6 T_5 = \frac{1}{2\pi}(\mathsf{T}_2)^2 = \frac{1}{64\pi^3}N\,. \tag{2.40}$$

In this case we shall label the coordinates in (2.37) as (cf. (2.13))

$$\begin{array}{ccccc} \hline \mathrm{AdS}_3(t,w,\phi) & u & \theta & S^3 & S'^3 \\ 0,1,2 & 3 & 4 & 5,6,7 & 8,9,\natural \\ \hline \end{array} \tag{2.41}$$

The corresponding solution for the $\frac{1}{2}$-BPS M5 brane probe wrapping $\mathrm{AdS}_3 \times S^3$ was found in [6]; an equivalent solution (in different coordinates) was constructed in [29]. Here one has (cf. (2.16), (2.30))[16]

$$u = u_0\,, \qquad \sinh u_0 = \varkappa\,, \qquad \theta = 0\,, \qquad H_3 = -8\mathrm{L}^3\varkappa\,\mathrm{vol}_{S^3}\,. \tag{2.42}$$

Like in (2.17) the free parameter $\varkappa$ can be expressed in terms of an integer M2 brane charge k carried by the M5 brane (cf. (2.40))[17]

$$\varkappa = \frac{\mathrm{k}}{\sqrt{2N}}\,, \qquad 8\varkappa\mathsf{T}_2\,\mathrm{vol}(S^3) = 2\pi\mathrm{k}\,. \tag{2.43}$$

Then the induced metric and H_3 are given by

$$ds^2 = \mathrm{L}^2\big[(1+\varkappa^2)\, ds^2_{\mathrm{AdS}_3} + 4ds^2_{S^3}\big]\,, \tag{2.44}$$

$$\mathsf{H}_3 = H_3 - C_3 = -\varkappa\mathrm{L}^3\,\big[(3+\varkappa^2)\,\mathrm{vol}_{\mathrm{AdS}_3} + 8\,\mathrm{vol}_{S^3}\big]\,. \tag{2.45}$$

[16] For simplicity, we set to zero the free parameter multiplying a possible "electric" ($\sim \mathrm{vol}_{\mathrm{AdS}_3}$) term in H_3 [6].

[17] The notation for the parameter determining u_0 used in [6] was $b = 4\varkappa$ and we corrected a typo there.

Here the projection of F_4 and thus of dC_6 to the brane are trivial so that we may take $C_6 = 0$. Then the M5 brane action (2.1) is given by (here $\mathrm{f}(u_0) = \varkappa(3 + \varkappa^2)$)

$$S = -\mathsf{T}_5 \int \Big([8(1+\varkappa^2)^2 - 4\varkappa\, \mathrm{f}(u_0)] - 4\varkappa\, \mathrm{f}(u_0) \Big) \ \mathrm{vol}_{\mathrm{AdS}_3} \wedge \mathrm{vol}_{S^3} \,, \tag{2.46}$$

where the first term is the "volume" part contribution, the second is that of the $\star\mathsf{HH}$ term and the last $4\varkappa\, f(u_0)$ one comes from the WZ part $C_3 \wedge H_3$. The resulting Euclidean action or the classical contribution to the M5 brane free energy is (cf. (2.25), (2.35))

$$F^{(0)}_{\mathbf{II}} = S_{\mathrm{E}} = \frac{1}{4\pi} N(1-\varkappa^2)\,\mathrm{vol}(\mathrm{AdS}_3) = \frac{1}{4\pi}\left(N - \frac{1}{2}\mathrm{k}^2\right)\mathrm{vol}(\mathrm{AdS}_3)\,. \tag{2.47}$$

This expression was already given in (1.16).

3 Quadratic fluctuations and one-loop partition function

In this section we will study the quadratic fluctuations near the three M5 probe solutions **Ia, Ib** and **II** and derive the general expressions for the corresponding one-loop fluctuation determinants. We will then expand the 6d fluctuation fields in modes on S^3 and present the corresponding mass spectra for the KK modes in AdS_3.

Here we will give details about the bosonic fluctuations while the fermionic fluctuation operator will be discussed in appendix A.

3.1 Probe Ia

We shall assume the static gauge in which the M5 coordinates along the $\mathrm{AdS}_3 \times S^3$ are identified with the world-volume ones ξ^i, i.e. they will not be fluctuating. The bosonic fluctuations will be those of the coordinate u in (2.6), the coordinates of S^4, and of the 2-form field defining H_3 (cf. (2.16)).

We shall denote the fluctuations of the five transverse coordinates as U and ζ_p where

$$u = u_0 + U\,, \qquad ds^2_{S^4} = \frac{d\zeta_p d\zeta_p}{(1+\frac{1}{4}\zeta^2)^2} = d\zeta_p d\zeta_p + \dots\,, \qquad p = 1,2,3,4\,. \tag{3.1}$$

Then the expression for the induced metric G_{ij} in (2.2) including the second-order terms in fluctuations may be written as

$$\begin{aligned} ds^2 = G_{ij} d\xi^i d\xi^j = L^2 \Big\{ & [(1+\kappa^2) ds^2_{\mathrm{AdS}_3} + \kappa^2 ds^2_{S^3}] \\ & + dU^2 + [2\kappa\sqrt{1+\kappa^2}\, U + (1+2\kappa^2)\, U^2]\, (ds^2_{\mathrm{AdS}_3} + ds^2_{S^3}) + \frac{1}{4} d\zeta_p d\zeta_p \Big\} + \dots\,, \end{aligned} \tag{3.2}$$

where dU^2 stands for $\partial_i U \partial_j U d\xi^i d\xi^j$ and similarly for $d\zeta_p d\zeta_p$.

The C_3 field has support in S^4 (cf. (2.7)) while the classical solution is in $\mathrm{AdS}_3 \times S^3$ part of AdS_7 and thus the pull-back of C_3 will not contribute at the quadratic fluctuation level. As a result, we have from (2.4)

$$\mathsf{H}_{ijk} = H_{ijk} = H^{(0)}_{ijk} + \mathrm{h}_{ijk} = \kappa^2 L^3 [\mathrm{vol}_{S^3}]_{ijk} + \mathrm{h}_{ijk}\,, \tag{3.3}$$

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where $H^{(0)}_{ijk}$ is the classical value in (2.16) and $\mathrm{h}_3 = d\tilde{B}_2$ is the contribution of the 2-form fluctuation. In the gauge (2.3) we have

$$\hat{\mathsf{H}}^{ij} = \frac{1}{6\sqrt{-|G|}}\frac{1}{\sqrt{-G^{11}}}\,\varepsilon^{ij1\ell mn}\mathsf{H}_{ijk}\,, \tag{3.4}$$

which will thus depend on both h_3 and the coordinate fluctuations via G_{ij} in (3.2). Expanding $\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|}$ to quadratic fluctuation order we find (cf. (2.24))

$$\begin{aligned}\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|} &= L^6\kappa^2(1+\kappa^2)^2\,\sqrt{-g}\,\Big\{1+\frac{6\kappa}{\sqrt{1+\kappa^2}}U+\mathbf{h}\\ &+\frac{1}{8(1+\kappa^2)}g^{ij}\partial_i\zeta_p\partial_j\zeta_p+\frac{1}{2(1+\kappa^2)}\big[g^{ij}\partial_iU\partial_jU+18(1+2\kappa^2)U^2\big]+\mathbf{hh}\Big\}+\ldots\,.\end{aligned} \tag{3.5}$$

Here $\mathbf{h}$ and $\mathbf{hh}$ stand for terms linear and quadratic in h_{ijk} (see below). We observe that the fields U and ζ_p have kinetic terms corresponding to the equal-radius $\mathrm{AdS}_3\times S^3$ metric $g=(g_A,g_S)$ (with each factors normalized to have unit radius)[18]

$$g_{ij}(\xi)\,d\xi^i d\xi^j = ds^2_{\mathrm{AdS}_3}+ds^2_{S^3}\,,\qquad \sqrt{-g}=\sqrt{-g_A}\sqrt{g_S}\,,\qquad L_A=L_S=1\,. \tag{3.6}$$

While the standard induced metric in (2.18) had unequal radii, the effective metric that governs the quadratic fluctuation propagation receives contribution from the non-trivial background of the H_{ijk} field and as a result turns out to be an equal-radii one. The equality of the two radii is a special feature of the solution **Ia** where $\mathrm{AdS}_3\times S^3$ belongs to AdS_7. In the other two cases **Ib** and **II** the effective metric will have the ratio of the AdS_3 and S^3 radii again being parameter-independent and equal to 2 and $\frac{1}{2}$ respectively.[19]

The linear term $\mathbf{h}$ originates from a product of $H^{(0)}_3$ and h_3 in terms with higher-order powers of H_{ijk} and is given by

$$\mathbf{h} = \frac{1}{6L^3\kappa^2(1+\kappa^2)\sqrt{g_S}}\,\varepsilon^{012\ell mn}\mathrm{h}_{\ell mn} = \frac{1}{L^3\kappa^2(1+\kappa^2)\sqrt{g_S}}\;\mathrm{h}_{345}\,. \tag{3.7}$$

It vanishes after the integration over ξ in (3.5) being a total derivative ($\mathrm{h}_3 = dB_2$) as there are no factors depending on S^3 coordinates remaining in (3.5).

The expansion of the action (2.1) contains also another contribution linear in h_3 that comes from the second $\star\mathsf{HH}$ term that may be written as

$$\frac{\sqrt{-|G|}}{4(\partial a)^2}\partial_i a\,\star\mathsf{H}^{ijk}\,\mathsf{H}_{jk\ell}\partial^\ell a = \frac{1}{4}\sqrt{-|G|}(G^{11})^{-1}\,\star\mathsf{H}^{1jk}\,\mathsf{H}_{jk1}\,G^{11} = \frac{1}{4!}\varepsilon^{1ijk\ell m}\mathsf{H}_{1ij}\mathsf{H}_{k\ell m}\,. \tag{3.8}$$

The linear in h_3 term in its expansion is

$$\frac{1}{4!}\varepsilon^{1ijk\ell m}\mathsf{H}_{1ij}\mathsf{H}_{k\ell m}\quad\to\quad\frac{1}{4!}\varepsilon^{1ijk\ell m}\,\mathrm{h}_{1ij}H^{(0)}_{k\ell m} = \frac{1}{2}L^3\kappa^2\,\sqrt{g_S}\;\mathrm{h}_{012}\,. \tag{3.9}$$

[18] The κ-dependent prefactors in the quadratic terms in (3.5) can be rescaled away and will not contribute to the one-loop free energy.

[19] Let us note that here the effective metric is not simply related to $G_{ij}+\hat{\mathsf{H}}_{ij}$ in the M5 brane action. This is different from the case of the D3-brane probe with $\mathrm{AdS}_2\times S^2$ geometry discussed in [50, 51] where the equal-radii effective $\mathrm{AdS}_2\times S^2$ metric was related to the inverse of the symmetric part of $(G_{ij}+F_{ij})^{-1}$. Note also that a similar equal-radii effective $\mathrm{AdS}_3\times S^3$ M5 brane world-volume metric appeared in a different context in [52].

It again gives a total derivative in the action as its prefactor does not depend on AdS_3 coordinates.

In addition, we need to include terms coming from the expansion of the C_6 term in (2.1). Using (2.15) we get

$$C_6 = L^6\left[\kappa^4\left(\kappa^2+\frac{3}{2}\right)+6\kappa^3(1+\kappa^2)^{3/2}\,U+9\kappa^2(1+\kappa^2)(1+2\kappa^2)\,U^2+\cdots\right]\,\mathrm{vol}_{\mathrm{AdS}_3}\wedge\mathrm{vol}_{S^3}\,. \tag{3.10}$$

We observe that the linear in U contributions in (3.5) and in (3.10) cancel each other in the total action (1.9), which is a manifestation of the fact that the background (2.16) is indeed an extremum of the M5 brane action.

Moreover, the quadratic U^2 terms in (3.5) and in (3.10) also cancel each other so that U is also a massless fluctuation like ζ_p in (3.5). This is consistent with the fact that u_0 (or κ) is a free parameter of the solution (2.16) so that $U \to U$+const should be a symmetry of the fluctuation action.

The terms quadratic in h_3 come directly from (3.9), i.e. $\frac{1}{4!}\varepsilon^{1ijk\ell m}\mathsf{H}_{1ij}\mathsf{H}_{k\ell m} \to \frac{1}{4!}\varepsilon^{1ijk\ell m}\mathrm{h}_{1ij}\mathrm{h}_{k\ell m}$ and also from (3.5). The latter may be written as (the indices are contracted with the same effective metric g_{ij} as in (3.6))

$$\mathbf{hh} = \frac{1}{12L^6\kappa^2(1+\kappa^2)^2}\sum_{i,j,k\neq 1}\mathrm{h}_{ijk}\mathrm{h}^{ijk}\,. \tag{3.11}$$

In total, the expansion of the integrand in the PST action (1.9) then contains the following $\mathrm{h}_3\mathrm{h}_3$ terms

$$L_2^{(+)} = \frac{1}{24}\varepsilon^{1ijk\ell m}\mathrm{h}_{1ij}\mathrm{h}_{k\ell m} - \frac{1}{12}\sqrt{-g}\sum_{i,j,k\neq 1}\mathrm{h}_{ijk}\mathrm{h}^{ijk}\,. \tag{3.12}$$

Eq. (3.12) has the form of the non-covariant Lagrangian describing propagation of a free (anti) self-dual 3-form field on a curved 6d background g_{ij} [34, 53–55].[20] The equations of motion following from (3.12) imply (doing one integration under proper boundary conditions) the 6d (anti)self-duality condition $\mathrm{h}^{1ij} = -\frac{1}{3!\sqrt{-g}}\varepsilon^{1ijk\ell m}\mathrm{h}_{k\ell m}$ or $\star\mathrm{h}_3 = -\mathrm{h}_3$ (cf. [38]).

Using the (anti) self-duality condition to replace the factor $\mathrm{h}_{k\ell m}$ with h_{1ij} with in the first term in (3.12) one finds that (3.12) it takes the standard covariant form for the 2-form Lagrangian, i.e.

$$L_2 = -\frac{1}{12}\sqrt{-g}\,\mathrm{h}_{ijk}\mathrm{h}^{ijk}\,. \tag{3.13}$$

The corresponding partition function evaluated under the self-duality constraint on h_3 should then be given by the *square root* of the standard gauge 2-form partition function on a curved 6d background (discussed in [56–58] and refs. there). This can be shown, e.g., at a diagrammatic level [59].[21]

[20] With index 1 interpreted as the time-like one it has a "phase-space" form with the spatial B_{rs} components as coordinates and $\partial_1 B_{rs}$ as momenta (cf. [53]).

[21] This approach was applied in [58] to find the conformal anomaly coefficients of the $(2,0)$ tensor multiplet. It extends to the full partition function Z as a functional of curved metric. Z can be computed as a sum

The same conclusion should follow directly from the PST action or (3.12) provided the corresponding path integral over B_{ij} is defined with an appropriate measure factor (containing a determinant of a particular 1st-order differential operator not involving ∂_1 derivative).[22] In more detail, starting with the covariant Lagrangian (3.13) for B_{ij} and writing it in the phase-space form (considering flat space case, fixing $B_{0r}=0$ gauge, ignoring some trivial factors and using here $r,s,q=1,\dots,5$) we get $L=\frac{1}{2}p^{rs}\partial_0 B_{rs}-\frac{1}{4}p^{rs}p_{rs}-\frac{1}{12}\mathrm{h}_{rsq}\mathrm{h}^{rsq}$. We may then introduce a new ("dual") field $\tilde{B}_{rs}$ by setting $p^{rs}=\frac{1}{2}\varepsilon^{rsquv}\partial_q\tilde{B}_{uv}\equiv(\mathrm{D}\tilde{B})^{rs}$. Here D is a 1st-order operator containing only spatial derivatives. One can easily see that its square is $(\partial)^2_\perp$ defined on $B^\perp_{rs}$, i.e. is the same operator that appears in the spatial part $\mathrm{h}_{rsq}\mathrm{h}^{rsq}$ of the Lagrangian. The resulting path integral over B_{rs} and $\tilde{B}_{rs}$ will have the Jacobian factor $\det \mathrm{D}$. The action for B_{rs} and $\tilde{B}_{rs}$ can be re-written as a sum of the decoupled actions for $B^{(\pm)}_{rs}=B_{rs}\pm\tilde{B}_{rs}$ each similar to (3.12) (with the role of the indices 1 and 0 interchanged and gauge fixed as $B_{0r}=0$), i.e. having kinetic operators $\mathcal{O}_\pm\mathrm{D}$ with $\mathcal{O}_\pm=\partial_0\pm\mathrm{D}$ (which are direct analogs of $\partial_0\pm\partial_1$ in the 2d chiral scalar cases). The original path integral for the B_{ij} field then formally factorizes as $Z_2^{(+)}Z_2^{(-)}$ where each factor is defined by the integral over the "chiral" $B^{(\pm)}$ field with the measure containing the $[\det \mathrm{D}]^{1/2}$ factor. This factorization is then equivalent to $\det(\partial_0^2-\partial_\perp^2)=\mathcal{O}_+\mathcal{O}_-$.

This "square root" prescription for the one-loop partition function $Z_2^{(+)}$ of a self-dual 2-form in the M5 brane action was used in the $S^5\times S^1$ geometry case in [15] and in the $\mathrm{AdS}_5\times S^1$ case in [11]. In the present $\mathrm{AdS}_3\times S^3$ case the expression for Z_2 in terms of the determinants of the relevant 2nd-order operators is presented in appendix B.

In total, the scalar U and ζ_p terms in (3.5), (3.10) and (3.13) lead to the following action for the quadratic fluctuations of the bosonic fields

$$S_2=-\int d^6\xi\,\sqrt{-g}\,\Big(\frac{1}{2}g^{ij}\partial_i U\partial_j U+\frac{1}{2}g^{ij}\partial_i\zeta_p\partial_j\zeta_p+\frac{1}{12}\mathrm{h}_{ijk}\mathrm{h}^{ijk}\Big)\,. \tag{3.14}$$

To arrive at (3.14) we rescaled the fluctuation fields by constant κ-dependent factors. In contrast to the D3-brane case discussed in [51] here the Seeley coefficient b_6 of each of the

of diagrams with external graviton lines and implementing the projection to the (anti) self-dual 2-form component in each internal propagator [59]. Note that here we will be interested only in the real part of the partition function, i.e. will ignore the phase part related to the gravitational anomaly [59]. In general, both the self-dual 3-form and chiral 6d fermions of the M5 brane action will contribute to its gravitational anomaly (the cancellation of the M5 brane anomalies in the M-theory context was discussed in [60] and refs. there).

[22] Eq. (3.12) (with index "1" in (3.12) relabelled as "0") may be viewed as a 6d generalization of the Floreanini-Jackiw Lagrangian [61] for a chiral scalar $\varphi^{(+)}$ in 2 dimensions given (in flat 2d space) by $L=\partial_0\varphi^{(+)}\partial_1\varphi^{(+)}-\partial_1\varphi^{(+)}\partial_1\varphi^{(+)}=\partial_1\varphi^{(+)}\partial_-\varphi^{(+)}$. The corresponding equation of motion $\partial_1\partial_-\varphi^{(+)}=0$ implies (assuming $\partial_-\varphi^{(+)}=0$ holds at spatial infinity) that $\partial_-\varphi^{(+)}=0$, i.e. gives the 2d self-duality condition. Integrating over $\varphi^{(+)}$ with a measure containing the $[\det\partial_1]^{1/2}$ factor gives the chiral scalar partition function as $[\det\partial_-]^{-1/2}$. Generalized to curved space this is the same as the square root of the real scalar partition function $[\det(\partial_-\partial_+)]^{-1/2}$ (up to pure-phase gravitational anomaly factor in the Euclidean case). The reason for the $[\det\partial_1]^{1/2}$ measure factor can be understood as follows [62]. Starting with the real scalar Lagrangian $L_0=\partial_0\varphi\partial_0\varphi-\partial_1\varphi\partial_1\varphi$ and writing the corresponding path integral in phase-space form with $L=2p\partial_0\varphi-p^2-\partial_1\varphi\partial_1\varphi$ one can then set $p=\partial_1\tilde{\varphi}$ getting duality-symmetric Lagrangian $L=\partial_0\tilde{\varphi}\partial_1\varphi+\partial_1\tilde{\varphi}\partial_0\varphi-\partial_1\tilde{\varphi}\partial_1\tilde{\varphi}-\partial_1\varphi\partial_1\varphi$ (ignoring total derivative). The path integral over p and φ had canonical measure, so then one over φ and $\tilde{\varphi}$ should contain the Jacobian factor $\det\partial_1$. Introducing $\varphi^{(\pm)}=\varphi\pm\tilde{\varphi}$ one gets $L=\partial_1\varphi^{(+)}\partial_-\varphi^{(+)}+\partial_1\varphi^{(-)}\partial_+\varphi^{(-)}$ and thus the original path integral factorises into $Z^{(+)}Z^{(-)}$ with $Z^{(\pm)}=[\det\partial_\pm)]^{-1/2}$ each originating from path integral containing the measure factor $[\det\partial_1]^{1/2}$.

relevant differential operators, i.e. the coefficient of the logarithmic UV 6d divergence vanishes (see appendix E) and thus the rescaling of the fields (or local measure factors) do not contribute to the finite part of the resulting free energy. The same observation applies to the fermionic field contribution discussed in appendix A and also is true in the two other cases **Ib** and **II**.

Combining the $Z_2^{(+)}$ contribution for self-dual part of h_{ijk} with the contribution of the 5 massless scalars U and ζ_p and the fermions we end up with the partition function Z of the (2,0) tensor multiplet on the equal-radii $\mathrm{AdS}_3 \times S^3$ space. As this space is conformally flat and its 6d Euler density vanishes and as each field has Weyl-invariant Lagrangian, we conclude as in [58] that there is no conformal anomaly or no logarithmic UV divergence.

Explicitly, using the expression for $Z_2^{(+)}$ in (B.13) in appendix B the bosonic part of Z_{B} may be written as

$$\begin{aligned} Z_{\mathrm{B}} = Z_2^{(+)} Z_0^5 &= \big[\det \mathbf{\Delta}_{1\perp,1\perp}(0)\big]^{-1/4} \big[\det \mathbf{\Delta}_{0,0}(0)\big]^{-1/2} \big[\det \mathbf{\Delta}_{0,0}(0)\big]^{-5/2} \\ &= \big[\det \mathbf{\Delta}_{1\perp,1\perp}(0)\big]^{-1/4} \big[\det \mathbf{\Delta}_{0,0}(0)\big]^{-3} . \end{aligned} \tag{3.15}$$

We used (B.8) and that $L_A = L_S$. Here $\mathbf{\Delta}_{1\perp,1\perp} = -\nabla_A^2 - \nabla_S^2$ is defined on the B_{ar} field on $\mathrm{AdS}_3 \times S^3$ which is separately transverse in the AdS_3 index a and the S^3 index r. $\mathbf{\Delta}_{0,0}(0) = \nabla^2$ is a massless scalar Laplacian. The number of the bosonic degrees of freedom is thus the expected one: $\frac{1}{2}(2 \times 2) + 6 = 8$.

In addition, we have 8 fermionic degrees of freedom in the M5-action. A detailed analysis of the fermionic fluctuation contribution Z_{F} to the partition function is delegated to the appendix A. Z_{F} is given by the determinant of a massive Dirac operator on $\mathrm{AdS}_3 \times S^3$ defined using the same equal-radii effective metric as in (3.6)

$$Z_{\mathrm{F}} = \big[\det \mathcal{D}\big]^{1/2} , \qquad \mathcal{D} = i\nabla\!\!\!\!/ + \mathcal{M} . \tag{3.16}$$

Here we assume that $\mathcal{D}$ acts on 32 component spinor and thus (3.16) describes 8 real degrees of freedom. It turns out that in all the cases **Ia, Ib, II** the Dirac operator $\mathcal{D}$ in (3.16) has the form (see appendix A)

$$\mathcal{D} = i\nabla\!\!\!\!/ + \mathcal{M} = i\nabla\!\!\!\!/_A + i\nabla\!\!\!\!/_S + m_{\mathrm{F}} \widehat{\Gamma} , \qquad \mathcal{M} = m_{\mathrm{F}} \widehat{\Gamma} , \tag{3.17}$$

$$\widehat{\Gamma}^2 = 1 , \qquad [\nabla\!\!\!\!/_S, \widehat{\Gamma}] = \{\nabla\!\!\!\!/_A, \widehat{\Gamma}\} = 0 . \tag{3.18}$$

In the **Ia** case one finds that $m_{\mathrm{F}} = 0$ (see (A.17)).

To summarize, in the **Ia** case the one-loop partition function is the same as for the (2,0) multiplet (self-dual tensor, 5 massless scalars and 4 massless Weyl 6d fermions) defined on the equal-radius $\mathrm{AdS}_3 \times S^3$ space.

The spectra of the operators in (3.15), (3.16) can be found by first expanding in modes on S^3 and thus getting a tower of massive KK fields on AdS_3 labelled by level ℓ. As a result, their determinants can be expressed in terms of products of determinants of operators on AdS_3. For the scalar field in (3.1) we get (here for generality we assume that the radii of the two factors in $\mathrm{AdS}_3 \times S^3$ metric are L_A and L_S)

$$\mathbf{\Delta}_{0,0}(0) \;\; \rightarrow \;\; \mathbf{\Delta}_0 = -\nabla_A^2 + M_{0,\ell}^2 , \qquad M_{0,\ell}^2 = L_S^{-2} \ell(\ell+2) , \qquad d_\ell^{(0)} = (\ell+1)^2 , \tag{3.19}$$

where d_ℓ is the scalar degeneracy. Similarly, for the transverse vector operator in (3.15) (see (F.5))

$$\boldsymbol{\Delta}_{1\perp,1\perp}(0) \;\to\; \boldsymbol{\Delta}_{1\perp} = -\nabla_A^2 + M_{1,\ell}^2\,, \qquad M_{1,\ell}^2 = L_S^{-2}(\ell^2+4\ell+2)\,, \qquad d_\ell^{(1)} = 2(\ell+1)(\ell+3)\,. \tag{3.20}$$

Using a split basis for the 11d gamma matrices and expanding the fermions in the S^3 spinor spherical harmonics we get

$$\nabla\!\!\!\!/_S \;\to\; \pm i M_{\frac12,\ell}\,, \qquad M_{\frac12,\ell} = L_S^{-1}\left(\frac{2\ell+1}{2}+1\right), \qquad d_\ell^{(\frac12)} = (\ell+1)(\ell+2)\,, \tag{3.21}$$

where d_ℓ is the degeneracy of the single fermion mode on S^3 (see, e.g., eq. (3.44) in [63]). Squaring the resulting Dirac operator on AdS_3 gives

$$\boldsymbol{\Delta}_{\frac12} = -\nabla_A^2 + \frac14 R_A + \hat M_{\frac12,\ell}^2\,, \qquad \hat M_{\frac12,\ell} = M_{\frac12,\ell} \pm m_{\rm F}\,, \qquad R_A = -6L_A^{-2}\,. \tag{3.22}$$

The corresponding dual conformal dimensions of the AdS_3 scalar ($s=0$) and the transverse vector ($s=1$) with mass M_s can be found using the standard relations[23]

$$\Delta^{(0)}(\Delta^{(0)}-2) = L_A^2\, M_0^2\,, \qquad \Delta^{(1)}(\Delta^{(1)}-2) = 1 + L_A^2\, M_1^2\,. \tag{3.23}$$

As a result, from (3.19), (3.20) for $L_A = L_S$ we get for the dimensions of the AdS_3 fields in the scalar and vector KK towers

$$\Delta^{(0)} = \ell+2\,, \qquad \Delta^{(1)} = \ell+3\,. \tag{3.24}$$

For spin $s=\frac12$ fermions with the squared Dirac operator $\boldsymbol{\Delta}_{\frac12} = -\nabla_A^2 + \frac14 R_A + M_{\frac12}^2$ one has the corresponding AdS_3 dimension given by

$$\Delta^{(\frac12)} = 1 + \sqrt{1+s+L_A^2\, M_{\frac12}^2} = 1 + \sqrt{\frac32 + L_A^2\, M_{\frac12}^2}\,. \tag{3.25}$$

Then for the value of the mass in (3.22) we get

$$\Delta^{(\frac12)} = 1 + L_A L_S^{-1}\left(\ell+\frac32\right) \pm L_A m_{\rm F}\,. \tag{3.26}$$

In present case **Ia** with $m_{\rm F}=0$ and $L_A = L_S$ the dimensions of the AdS_3 fermionic fields are found to be

$$\Delta^{(\frac12)} = \ell + \frac52\,. \tag{3.27}$$

[23] To recall, the dual-field dimensions for a massive p-form field in AdS_{d+1} is $\Delta = \frac{d}{2} \pm \sqrt{(\frac{d}{2}-p)^2 + m^2 L_A^2}$ (see, e.g. [64]). Here m^2 is the term in addition to the standard Hodge-deRham structure of the Laplacian For $d=2$ and $p=1$ the dimension satisfies $\Delta(\Delta-2) = -1 + m^2 L_A^2$. For a vector $(\boldsymbol{\Delta}_{1\perp})_{ab} = -(\nabla_A^2)_{ab} + R_{ab} + m^2 g_{ab} = (-\nabla_A^2 + M^2) g_{ab}$, where $M^2 = -2L_A^{-2} + m^2$ and thus $\Delta(\Delta-2) = 1 + M^2 L_A^2$.

3.2 Probe Ib

Here (see (2.30)) we may use again the static gauge, setting to zero fluctuations of the coordinates along $\mathrm{AdS}_3 \subset \mathrm{AdS}_7$ and $S'^3 \subset S^4$. The remaining transverse fluctuations will then be of u and S^3 directions in the AdS_7 metric and of θ in (2.12). Since the expansion goes around $u = 0$ we may parametrize these 5 fluctuations like in (3.1) as ζ_p $(p = 1, 2, 3, 4)$ and Θ

$$ds^2 = du^2 + \sinh^2 u\, ds^2_{S^3} = \frac{d\zeta_p d\zeta_p}{(1 - \frac{1}{4}\zeta^2)^2} = d\zeta_p d\zeta_p + \dots\,, \qquad \theta = \theta_0 + \Theta\,. \tag{3.28}$$

The perturbed induced metric may then be written like in (3.2) as (cf. (2.31))

$$\begin{aligned} ds^2 = L^2 \Big\{ & ds^2_{\mathrm{AdS}_3} + 2\kappa^2(1-2\kappa^2) ds^2_{S^3} + d\zeta_p d\zeta_p + \zeta_p\zeta_p ds^2_{\mathrm{AdS}_3} + \frac{1}{4} d\Theta^2 \\ & + \Big[\kappa(1-4\kappa^2)\sqrt{2-4\kappa^2}\,\Theta + \Big(\frac{1}{4} - 4\kappa^2 + 8\kappa^4\Big)\,\Theta^2 \Big] ds^2_{S^3} \Big\} + \cdots\,. \end{aligned} \tag{3.29}$$

Here we again denote the world volume sphere S'^3 as S^3. Using the expression for C_3 in (2.29) the expansion of H_{ijk} in (2.4) may be written as (cf. (3.3))

$$\mathsf{H}_{ijk} = H^{(0)}_{ijk} - C_{ijk} + \mathrm{h}_{ijk} = L^3 \mathcal{U}(\Theta)\ [\mathrm{vol}_{S^3}]_{ijk} + \mathrm{h}_{ijk}, \tag{3.30}$$

$$\mathcal{U}(\Theta) = \kappa^2(1-2\kappa^2)\Big[(1-4\kappa^2) - 6\kappa\sqrt{2-4\kappa^2}\,\Theta - \frac{9}{2}\Big(1-4\kappa^2\Big)\,\Theta^2\Big] + \cdots\,. \tag{3.31}$$

The analog of the expansion of $\sqrt{-|G_{ij} + \hat{\mathsf{H}}_{ij}|}$ in (3.5) is found to be

$$\begin{aligned} \sqrt{-|G_{ij} + \hat{\mathsf{H}}_{ij}|} = L^6\, 8\kappa^2(1-2\kappa^2)\sqrt{-g}\,\Big[& 1 + \mathbf{h} \\ & + \frac{1}{8}(g^{ij}\partial_i\Theta\partial_j\Theta + 24\,\Theta^2) + \frac{1}{2}(g^{ij}\partial_i\zeta_p\partial_j\zeta_p + 3\zeta_p\zeta_p) + \mathbf{hh} + \mathbf{hT} \Big] + \dots\,. \end{aligned} \tag{3.32}$$

Here the effective $\mathrm{AdS}_3 \times S^3$ metric g_{ij} which the fluctuations are propagating in turns out to be independent of κ and when written in terms of the unit radius AdS_3 and S^3 metrics g_A, g_S is simply

$$g_{ij} d\xi^i d\xi^j = ds^2_{\mathrm{AdS}_3} + \frac{1}{4}\, ds^2_{S^3}\,, \qquad \sqrt{-g} = \frac{1}{8}\sqrt{-g_A}\sqrt{g_S}\,, \qquad L_A = 1\,,\ \ L_S = \frac{1}{2}\,. \tag{3.33}$$

The linear $\mathbf{h}$ and quadratic $\mathbf{hh}$ and $\mathbf{hT}$ terms are

$$\mathbf{h} = -\frac{1-4\kappa^2}{6L^3\kappa^2(1-2\kappa^2)\sqrt{g_S}}\ \varepsilon^{021\ell mn}\ \mathrm{h}_{\ell mn}, \qquad \mathbf{hh} = \frac{1}{96L^6\kappa^2(1-2\kappa^2)} \sum_{i,j,k\neq 1} \mathrm{h}_{ijk}\mathrm{h}^{ijk}\,, \tag{3.34}$$

$$\mathbf{hT} = \frac{1}{L^3\kappa\sqrt{\frac{1}{2} - \kappa^2}\,\sqrt{g_S}}\ \varepsilon^{021\ell mn}\ \mathrm{h}_{\ell mn}\,\Theta\,. \tag{3.35}$$

The indices in **hh** are contracted using g_{ij}. The linear term **h** is again a total derivative (as in (3.7)), i.e. it vanishes after the integration over 6d space. From (3.30) we find also that the second term in (2.1) here is (cf. (3.9))

$$\frac{1}{4!}\varepsilon^{1ijkmn}\mathsf{H}_{1ij}\mathsf{H}_{kmn} \quad\to\quad \frac{1}{2}L^3\sqrt{g_S}\,\mathrm{h}_{012}\,\mathcal{U}(\Theta) + \frac{1}{4!}\varepsilon^{1ijkmn}\mathrm{h}_{1ij}\mathrm{h}_{kmn}\,. \tag{3.36}$$

The C_6 term in the WZ part of the action (2.1) does not contribute at the quadratic fluctuation level while (cf. (3.30))

$$\frac{1}{2}\int \mathsf{H}_3\wedge C_3 = \frac{1}{2}L^3\int \mathsf{H}_3\wedge\left[\kappa^2-\mathcal{U}(\Theta)\right]\mathrm{vol}_{S^3} \;\to\; \frac{1}{2}L^3\int d^6\xi\,\sqrt{g_S}\,\mathrm{h}_{012}\left[\kappa^2-\mathcal{U}(\Theta)\right]. \tag{3.37}$$

Combining the integral of (3.36) with (3.36) we conclude that the $\mathcal{U}(\Theta)$ term cancels, i.e. there are no linear and quadratic in Θ terms remaining in the fluctuation action (and the integral of the remaining h_{012} term again vanishes).

The resulting bosonic fluctuation Lagrangian contains (i) 5 massive scalar fields Θ and ζ_p (cf. (3.32)); (ii) $\mathrm{h}_3\mathrm{h}_3$ term given by the combination of the **hh** term in (3.34) and the second term in (3.36); (iii) the **hT** mixing term. The $\mathrm{h}_3\mathrm{h}_3$ term is the same as in (3.12) in the **Ia** case describing the (anti) self-dual 2-form field. We may formally use the (anti) self-duality condition to rewrite it in the covariant form (3.13) assuming that the path integral is to be carried out over the (anti) self-dual fields only or, equivalently, using the "square root" prescription for the corresponding contribution to the partition function.

Rescaling the fields by constant factors the resulting action for the quadratic fluctuation fields on $\mathrm{AdS}_3\times S^3$ with the metric (3.33) may be written in the following κ-independent form (cf. (3.14))

$$S_2 = -\int d^6\xi\,\sqrt{-g}\left[\frac{1}{2}(g^{ij}\partial_i\Theta\partial_j\Theta+24\Theta^2)+\frac{1}{2}(g^{ij}\partial_i\zeta_p\partial_j\zeta_p+3\zeta_p\zeta_p)+\frac{1}{12}\mathrm{h}_{ijk}\mathrm{h}^{ijk}-\frac{48}{\sqrt{g_S}}\,\mathrm{h}_{345}\,\Theta\right]. \tag{3.38}$$

The remaining non-trivial problem is to diagonalize the last term representing the mixing (3.35) between the scalar Θ and the S^3 components of the field $\mathrm{h}_3 = dB_2$.

Using the explicit form of the gauge-fixed Lagrangian for the B_{ij} field on $\mathrm{AdS}_3\times S^3$ given in (B.6) and specifying it to the present case of the metric (3.33) with $L_A = 1$, $L_S = \frac{1}{2}$ we get

$$\frac{1}{12}\mathrm{h}_{ijk}\mathrm{h}^{ijk} \;\to\; \frac{1}{4}\left[B_{ab}(-\nabla^2-2)B^{ab} + B_{rs}(-\nabla^2+8)B^{rs} + 2B_{ar}(-\nabla^2+6)B^{ar}\right], \tag{3.39}$$

where the indices a, b correspond to AdS_3, the indices r, s to S^3 and $\nabla^2 = \nabla_A^2 + \nabla_S^2$. The "mixed" part of the Lagrangian in (3.38) that involves Θ and B_{rs} may be written as

$$L_{\rm mix} = -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\,[g^{ij}\partial_i\Theta\partial_j\Theta+24\Theta^2+\frac{1}{2}B_{rs}(-\nabla_A^2-\nabla_S^2+8)B^{rs}]+24\sqrt{-g_A}\,\Theta\,\varepsilon^{rst}\,\partial_r B_{st}\,. \tag{3.40}$$

We may replace B_{rs} by a 3-vector V^r given by

$$B_{rs} = \frac{8}{\sqrt{g_S}}\varepsilon_{rsu}V^u\,. \tag{3.41}$$

Here ε_{rst} is defined using the S^3 metric with radius $\frac{1}{2}$. Then (3.40) becomes

$$L_{\rm mix} = -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\left[g^{ij}\partial_i\Theta\partial_j\Theta + 24\Theta^2 + V_r(-\nabla_A^2 - \nabla_S^2 + 8)V^r + \ 12\,\Theta\,\nabla_r V^r\right], \quad (3.42)$$

where ∇_r is defined with respect to the S^3 part of g_{ij}, i.e. $\frac{1}{4}g_S$ where g_S is the unit radius metric. Next, it is useful to split V_r into the transverse and scalar part as

$$V_r = V_r^{\perp} + (\nabla_S)_r\, P\,, \quad (3.43)$$

so that (3.42) becomes ($\nabla^2 = \nabla_A^2 + \nabla_S^2$)

$$L_{\rm mix} = -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\; V_r^{\perp}(-\nabla^2 + 8)V^{\perp r} + L_{\Theta,P}\,, \quad (3.44)$$

$$L_{\Theta,P} = -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\,[\Theta(-\nabla^2)\Theta + 24\Theta^2 + P(-\nabla^2)(-\nabla_S^2)P + 12\,\Theta\,\nabla_S^2 P]\,. \quad (3.45)$$

Thus we are left only with the scalar (Θ, P) mixing described by $L_{\Theta,P}$. One may further redefine P to $P' = (-\nabla_S^2)^{1/2}P$ to get a 2-derivative mixed scalar action. The associated Jacobian cancels the one that is introduced by (3.43) and one finds that the partition function corresponding to (3.45) is

$$Z_{\Theta,P} = \left[\det(\nabla^4 - 24\nabla^2 + 36\nabla_S^2)\right]^{-1/2}. \quad (3.46)$$

It is convenient to separate the contribution of the scalar Θ and its mixing to B_{rs} from the total contribution of the B_{ij} field in (3.40). This amounts to normalizing (3.46) to the contribution of the $(-\nabla^2)$ factor in the P kinetic term in (3.45), i.e. to replacing (3.46) by

$$Z_{\Theta,\,\rm mix} = [\det(-\nabla^2)]^{1/2}\ [\det(\nabla^4 - 24\nabla^2 + 36\nabla_S^2)]^{-1/2}. \quad (3.47)$$

Then in the limit of no mixing, i.e. when the ∇_S^2 term in (3.45) and thus (3.47) is dropped, the partition function (3.47) reduces to the one of the decoupled massive scalar Θ only.

The total bosonic partition function will then be given by (3.47) combined with the contribution of the self-dual tensor and 4 massive scalars ζ_p in (3.38), which is analogous to the one in (3.15). Cancelling the common scalar det factor against the determinant of $\mathbf{\Delta}_{0,0}(0) = -\nabla^2 = -\nabla_A^2 - \nabla_S^2$ gives

$$Z_{\rm B} = Z_2^{(+)}\, Z_0^4\, Z_{\Theta,\,\rm mix} = \left[\det \mathbf{\Delta}_{1\perp,1\perp}(6)\right]^{-1/4}\left[\det \mathbf{\Delta}_{0,0}(3)\right]^{-2}\ \left[\det(\nabla^4 - 24\nabla^2 + 36\nabla_S^2)\right]^{-1/2}. \quad (3.48)$$

Here we specialised the general $\mathrm{AdS}_3 \times S^3$ self-dual field contribution $Z_2^{(+)}$ in (B.13) to the present case of $L_A = 1$, $L_S = \frac{1}{2}$ (so that $\mathbf{\Delta}_{1\perp,1\perp}(6)$ originates from the B_{ar} term in (3.39), etc.).

The expression is similar to (3.15) in the **Ia** case apart from the last factor (3.47) involving a more complicated 4-derivative operator factor. Once we expand in harmonics on S^3 the latter can be written in terms of a product of 2-derivative scalar operators on AdS_3. Indeed, doing the replacement $-\nabla_S^2 \to L_S^{-2}\ell(\ell+2)$ as in (3.19) we get

$$\nabla^4 - 24\nabla^2 + 36\nabla_S^2 = (\nabla_A^2 + \nabla_S^2)^2 + 24\nabla_A^2 + 60\nabla_S^2 \quad \to \quad (-\nabla_A^2 + M_{0+,\ell}^2)(-\nabla_A^2 + M_{0-,\ell}^2)\,,$$
$$M_{0+,\ell}^2 = 4(\ell+2)(\ell+3)\,, \qquad\qquad M_{0-,\ell}^2 = 4\ell(\ell-1)\,. \quad (3.49)$$

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Each of these factors enters with the scalar degeneracy $d_\ell^{(0)} = (\ell+1)^2$. Note that this simple factorization is a consequence of particular coefficients that appeared in the fluctuation Lagrangian in (3.42).

As a result, the last factor in (3.48) or (3.46) may be written as

$$Z_{\Theta,\,\mathrm{mix}} = \prod_\ell \Big[\det \mathbf{\Delta}_0(M^2_{0_+,\ell})\ \det \mathbf{\Delta}_0(M^2_{0_-,\ell}) \Big]^{-\frac{1}{2}d_\ell^{(0)}} , \tag{3.50}$$

where $\mathbf{\Delta}(M^2) = -\nabla_A^2 + M^2$, cf. (3.19). Similarly, we get (cf. (3.20))

$$\big[\det \mathbf{\Delta}_{1\perp,1\perp}(6)\big]^{-1/4}\ \big[\det \mathbf{\Delta}_{0,0}(3)\big]^{-2} = \prod_\ell \big[\det \mathbf{\Delta}_{1\perp}(M^2_{1,\ell})\big]^{-\frac{1}{4}d_\ell^{(1)}}\ \big[\det \mathbf{\Delta}_0(M^2_{0,\ell})\big]^{-2d_\ell^{(0)}} , \tag{3.51}$$

$$M^2_{1,\ell} = 6 + 4(\ell^2+4\ell+2)\,, \qquad\qquad M^2_{0,\ell} = 3 + 4\ell(\ell+2)\,. \tag{3.52}$$

The fermion contribution to the partition function has the same general form as in the **Ia** case, i.e. is given by (3.16), (3.17) (see appendix A). In the present case we get $m_{\rm F} = \frac{3}{2}$, see (A.21), (A.22). Thus after expanding in spinor harmonics on S^3 the effective mass of the fermions in AdS_3 appearing in the squared Dirac operator in (3.22) here is

$$\hat{M}_{\frac{1}{2},\ell} = M_{\frac{1}{2},\ell} \pm m_{\rm F} = (2\ell+3) \pm \frac{3}{2}\,. \tag{3.53}$$

The conformal dimensions corresponding to the above AdS_3 mass spectrum are found as in (3.23)–(3.27) using (3.33). For the four scalars ζ_p with mass $M^2_{\ell,0}$ in (3.73) and the two mixed scalars with masses in (3.49) we get

$$\Delta^{(0)}_{\zeta_p} = 2\ell+3\,, \qquad \Delta^{(0)}_{+} = 2\ell+6\,, \qquad \Delta^{(0)}_{-} = 2\ell\,. \tag{3.54}$$

For the transverse vector we get from (3.20), (3.23) and (3.52)

$$\Delta^{(1)} = 2\ell+5\,. \tag{3.55}$$

For the fermions from (3.26) and (3.53) we get $\Delta_\pm^{(\frac{1}{2})} = 1 + (2\ell+3) \pm \frac{3}{2}$, i.e.

$$\Delta_+^{(\frac{1}{2})} = 2\ell + \frac{11}{2}\,, \qquad\qquad \Delta_-^{(\frac{1}{2})} = 2\ell + \frac{5}{2}\,. \tag{3.56}$$

3.3 Probe II

The derivation of fluctuations near the solution (2.42) describing M5 brane wrapped on $\mathrm{AdS}_3 \subset \mathrm{AdS}_4$ and $S^3 \subset S^7$ is very similar to the one in the **Ib** case (the two cases are, in fact, closely related by an analytic continuation, see appendix D).[24]

In the static gauge where the fluctuations of AdS_3 and S^3 coordinates in (2.37) are set to zero we are left with the fluctuations of u and of θ and S'^3 (and also of the 2-form field B_{ij} and fermions). We parametrize them like in (3.1) or in (3.28) ($p = 1,2,3,4$)

$$ds^2 = d^2\theta + \sin^2\theta\, ds^2_{S'^3} = \frac{d\zeta_p d\zeta_p}{(1+\frac{1}{4}\zeta^2)^2} = d\zeta_p d\zeta_p + \dots\,, \qquad\qquad u = u_0 + U\,. \tag{3.57}$$

[24] Bosonic M5 brane fluctuations near this solution written in different coordinates [29] were previously studied in [30, 31].

The perturbed induced metric corresponding to (2.37) is then (cf. (3.29))

$$ds^2 = \mathrm{L}^2\Big\{(1+\varkappa^2)ds^2_{\mathrm{AdS}_3} + 4ds^2_{S^3} + dU^2 + \big[\varkappa^2 + 2\varkappa\sqrt{1+\varkappa^2}\,U + (1+2\varkappa^2)\,U^2\big]ds^2_{\mathrm{AdS}_3} + 4d\zeta_p d\zeta_p - 4\zeta_p\zeta_p ds^2_{S^3}\Big\} + \dots\,. \tag{3.58}$$

From (2.39) and (2.42) we get (cf. (3.30), (3.31))

$$\mathsf{H}_{ijk} = -\mathrm{L}^3\mathrm{f}(u)\,[\mathrm{vol}_{\mathrm{AdS}_3}]_{ijk} - 8\mathrm{L}^3\varkappa\,[\mathrm{vol}_{S^3}]_{ijk} + \mathrm{h}_{ijk}, \tag{3.59}$$

$$\mathrm{f}(u) = 3\sinh u + \sinh^3 u = \varkappa(3+\varkappa^2) + 3(1+\varkappa^2)^{3/2}\,U + \frac{9}{2}\varkappa\,(1+\varkappa^2)\,U^2 + \cdots\,. \tag{3.60}$$

Computing $\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|}$ we find for the quadratic fluctuation terms (linear terms will eventually drop out as in the two previous cases and so we omit them here)

$$\sqrt{-|G_{ij}+\hat{\mathsf{H}}_{ij}|} = 4\mathrm{L}^6(1+\varkappa^2)^2\sqrt{-g_A}\sqrt{g_S}\,\Big\{1 + \frac{1}{8(1+\varkappa^2)}\big[g^{ij}\partial_i U\partial_j U + 12(1+3\varkappa^2)\,U^2\big] + \frac{1}{2(1+\varkappa^2)}(g^{ij}\partial_i\zeta_p\partial_j\zeta_p - 3\zeta_p\zeta_p) + \mathbf{hh} + \mathbf{hT}\Big\} + \dots\,. \tag{3.61}$$

Here the effective $\mathrm{AdS}_3\times S^3$ metric g_{ij} is again independent of the parameter $\varkappa$, and is given by

$$g_{ij}d\xi^i d\xi^j = \frac{1}{4}ds^2_{\mathrm{AdS}_3} + ds^2_{S^3}, \qquad \sqrt{-g} = \frac{1}{8}\sqrt{-g_A}\sqrt{g_S}\,, \qquad L_A = \frac{1}{2}\,, \quad L_S = 1\,. \tag{3.62}$$

Also, we get (cf. (3.34), (3.35))

$$\mathbf{hh} = \frac{1}{768\mathrm{L}^6(1+\varkappa^2)^2}\sum_{i,j,k\neq 1}\mathrm{h}_{ijk}\mathrm{h}^{ijk}, \qquad \mathbf{hT} = \frac{\varkappa^2}{256\mathrm{L}^3(1+\varkappa^2)^{3/2}\sqrt{g_S}}\,\varepsilon^{021\ell mn}\mathrm{h}_{\ell mn}\,U\,. \tag{3.63}$$

The contribution of the WZ term in the M5 brane action (2.1) is (cf. (3.37))

$$\frac{1}{2}\int \mathsf{H}_3\wedge C_3 = \frac{1}{2}\int\big(-8\varkappa\mathrm{L}^3\,\mathrm{vol}_{S^3} + \mathrm{h}_3\big)\wedge \mathrm{L}^3\mathrm{f}(u)\,\mathrm{vol}_{\mathrm{AdS}_3}$$
$$\to\quad \mathrm{L}^3\int d^6\xi\,\sqrt{-g_A}\,\sqrt{g_S}\,\Big[18\varkappa^2(1+\varkappa^2)U^2 + \frac{3}{2\sqrt{g_S}}(1+\varkappa^2)^{3/2}\mathrm{h}_{345}U + \dots\Big]\,. \tag{3.64}$$

The U^2 term here cancels against the $\varkappa$-dependent part of the U^2 term in (3.61). As a result, putting $\mathrm{h}_3\mathrm{h}_3$ terms into the covariant form as in (3.38) and rescaling the fields we get for the bosonic part of the quadratic fluctuation action[25]

$$S_2 = -\int d^6\xi\sqrt{-g}\Big[\frac{1}{2}(g^{ij}\partial_i U\partial_j U + 12\,U^2) + \frac{1}{2}(g^{ij}\partial_i\zeta_p\partial_j\zeta_p - 3\zeta_p\zeta_p) + \frac{1}{12}\mathrm{h}_{ijk}\mathrm{h}^{ijk} + \frac{6}{\sqrt{g_S}}\,\mathrm{h}_{345}\,U\Big]. \tag{3.65}$$

This action has the same structure as in the **Ib** case in (3.38) so the derivation of the corresponding bosonic partition function follows the same steps as in (3.39)–(3.48) with the

[25]Closely related expression was found in [31].

difference being in the values of the coefficients. Namely, taking into account that here $L_A = \frac{1}{2}$, $L_S = 1$ we get (setting $B_{rs} = \frac{1}{\sqrt{g_S}}\varepsilon_{rsu}V^u$, $V_r = V_r^\perp + (\nabla_S)_r P$ and $\nabla^2 = \nabla_A^2 + \nabla_S^2$)

$$\frac{1}{12}\mathrm{h}_{ijk}\mathrm{h}^{ijk} \;\to\; \frac{1}{4}[B_{ab}(-\nabla^2-8)B^{ab} + B_{rs}(-\nabla^2+2)B^{rs} + 2B_{ar}(-\nabla^2-6)B^{ar}], \tag{3.66}$$

$$\begin{aligned} L_{\rm mix} &= -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\left[\partial^i U\partial_j U + 12U^2 + \frac{1}{2}B_{rs}(-\nabla_S^2 - \nabla_A^2 + 2)B^{rs}\right] + 3\sqrt{-g_A}\,U\,\varepsilon^{rst}\,\partial_r B_{st} \\ &= -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\left[\partial^i U\partial_j U + 12U^2 + V_r(-\nabla_A^2 - \nabla_S^2 + 2)V^r - 12\,U\,\nabla_r V^r\right] \\ &= -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\;V_r^\perp(-\nabla^2+2)V^{\perp r} + L_{U,P}\,, \end{aligned} \tag{3.67}$$

$$L_{U,P} = -\frac{1}{2}\sqrt{-g_A}\,\sqrt{g_S}\,[U(-\nabla^2)U + 12U^2 + P(-\nabla^2)(-\nabla_S^2)P - 12\,U\,\nabla_S^2 P]\,. \tag{3.68}$$

The resulting contribution to the partition function is also analogous to (3.46)–(3.48) (we use again the expression for $Z_2^{(+)}$ in (B.13) now with $L_A = \frac{1}{2}$, $L_S = 1$)

$$\begin{aligned} Z_{U,P} &= \left[\det(\nabla^4 - 12\nabla^2 + 36\nabla_S^2)\right]^{-1/2}, \\ Z_{U,\,\rm mix} &= [\det(-\nabla^2)]^{1/2}\,[\det(\nabla^4 - 12\nabla^2 + 36\nabla_S^2)]^{-1/2}, \end{aligned} \tag{3.69}$$

$$\begin{aligned} Z_{\rm B} &= Z_2^{(+)}\,Z_0^4\,Z_{U,\,\rm mix} \\ &= \left[\det \boldsymbol{\Delta}_{1\perp,1\perp}(-6)\right]^{-1/4}\left[\det \boldsymbol{\Delta}_{0,0}(-3)\right]^{-2}\,\left[\det(\nabla^4 - 12\nabla^2 + 36\nabla_S^2)\right]^{-1/2}. \end{aligned} \tag{3.70}$$

Expanding in harmonics on S^3 we get the following counterparts of the **Ib** relations (3.49)–(3.73):

$$\begin{aligned} &\nabla^4 - 12\nabla^2 + 36\nabla_S^2 \;\to\; (-\nabla_A^2 + M_{0+,\,\ell}^2)(-\nabla_A^2 + M_{0-,\,\ell}^2)\,, \\ &M_{0+,\,\ell}^2 = (\ell+2)(\ell+6)\,, \qquad M_{0-,\,\ell}^2 = \ell(\ell-4)\,, \end{aligned} \tag{3.71}$$

$$Z_{U,\,\rm mix} = \prod_\ell \left[\det \boldsymbol{\Delta}_0(M_{0+,\,\ell}^2)\;\det \boldsymbol{\Delta}_0(M_{0-,\,\ell}^2)\right]^{-\frac{1}{2}d_\ell^{(0)}}, \tag{3.72}$$

$$\begin{aligned} &\left[\det \boldsymbol{\Delta}_{1\perp,1\perp}(-6)\right]^{-1/4}\left[\det \boldsymbol{\Delta}_{0,0}(-3)\right]^{-2} = \prod_\ell \left[\det \boldsymbol{\Delta}_{1\perp}(M_{1,\,\ell}^2)\right]^{-\frac{1}{4}d_\ell^{(1)}}\left[\det \boldsymbol{\Delta}_0(M_{0,\,\ell}^2)\right]^{-2d_\ell^{(0)}}, \\ &M_{1,\ell}^2 = -6 + (\ell^2 + 4\ell + 2)\,, \qquad M_{0,\ell}^2 = -3 + \ell(\ell+2)\,. \end{aligned} \tag{3.73}$$

The fermion contribution to the partition function is given by (3.16), (3.17) with the same value $m_{\rm F} = \frac{3}{2}$ as in the **Ib** case (see (A.26), (A.27)). Hence the effective mass of the fermions in the squared Dirac operator in AdS$_3$ in (3.22) here is (cf. (3.21), (3.53))

$$\hat{M}_{\frac{1}{2},\ell} = M_{\frac{1}{2},\ell} \pm m_{\rm F} = \left(\ell + \frac{3}{2}\right) \pm \frac{3}{2}\,. \tag{3.74}$$

The dimensions corresponding to the above AdS$_3$ mass spectrum are found as in (3.23)–(3.27) (here $L_A = \frac{1}{2}$ according to (3.62)). For the scalars we get from (3.52), (3.71)

$$\Delta_{\zeta_p}^{(0)} = \frac{1}{2}\ell + \frac{3}{2}\,, \qquad \Delta_+^{(0)} = \frac{1}{2}\ell + 3\,, \qquad \Delta_-^{(0)} = \frac{1}{2}\ell\,, \tag{3.75}$$

while for the vector in (3.73) we have (cf. (3.23), (3.62))

$$\Delta^{(1)} = \frac{1}{2}\ell + 2\,. \tag{3.76}$$

For the fermions (3.26) and (3.74) give $\Delta_{\pm}^{(\frac{1}{2})} = 1 + (\ell + \frac{3}{2}) \pm \frac{3}{2}$, i.e.

$$\Delta_{+}^{(\frac{1}{2})} = \frac{1}{2}\ell + \frac{5}{2}\,, \qquad \Delta_{-}^{(\frac{1}{2})} = \frac{1}{2}\ell + 1\,. \tag{3.77}$$

4 One-loop free energies of Ia, Ib and II probes

To compute the free energy $F = -\log Z$ corresponding to the one-loop partition functions found above in the cases **Ia**, **Ib** and **II** it remains to sum up the contributions of each of the (scalar, vector, spinor) field on AdS_3 of the corresponding mass, or, equivalently, scaling dimension Δ (see (3.24), (3.27), (3.54)–(3.56) and (3.75)–(3.77)). The single-field contributions are given by the standard relations (see, e.g., [65, 66])[26]

$$\begin{aligned}
F_0(\Delta) &= -\frac{1}{12\pi}(\Delta-1)^3 \ \mathrm{vol}(AdS_3)\,,\\
F_1(\Delta) &= -\frac{1}{12\pi}(\Delta-1)[(\Delta-1)^2-3] \ \mathrm{vol}(AdS_3)\,,\\
F_{\frac{1}{2}}(\Delta) &= -\frac{1}{12\pi}(\Delta-1)\left[(\Delta-1)^2-\frac{3}{4}\right] \ \mathrm{vol}(AdS_3)\,.
\end{aligned} \tag{4.1}$$

Each AdS_3 field contribution should be taken with its S^3 degeneracy factor (given in (3.19), (3.20), (3.21)).

For consistency with the underlying supersymmetry (see table 2) the AdS_3 fields should be organized into short supermultiplets (labelled by ℓ) of the $OSp(4^*|2)\times OSp(4^*|2)$ superalgebra in the cases **Ia** and **Ib**, and of the $OSp(4|2,\mathbb{R}) \times OSp(4|2,\mathbb{R})$ in the case **II**. A detailed way how to do this is discussed in appendix C.

In all the three cases the resulting supermultiplets contain individual fields with masses or dimensions corresponding to particular shifted values of the S^3 level ℓ (or, equivalently, they include fields from different levels ℓ). Remarkably, these shifts turns out to be same as in the spin-1 short supermultiplet of the $SU(1,1|2) \times SU(1,1|2)$ supergroup that describes the standard massless $(2,0)$ tensor multiplet on the supersymmetric $AdS_3 \times S^3$ vacuum of $(2,0)$ 6d supergravity [67, 68]. The corresponding spectrum is formally the same as we found above in the **Ia** case, but the organization of states in the supermultiplet (i.e. the assignment of the Δ-values) is different as the corresponding superalgebras are different (see appendix C for details).

In table 3 we present the summary of the corresponding spectra in the three cases and also, for comparison, for the massless (2,0) tensor multiplet on $AdS_3 \times S^3$ corresponding to the $SU(1,1|2) \times SU(1,1|2)$ symmetry.

There $A_{\perp}^{(\ell)}$ stands for the transverse vector originating from the B_{ar} component of the 2-form field on $AdS_3 \times S^3$ or the operator $\boldsymbol{\Delta}_{1\perp,1\perp}$ upon reduction on S^3 (cf. (3.20)). Also,

[26] These are for a scalar, transverse AdS_3 vector and a single fermion counted as one real degree of freedom. We have total of 6 scalars in (3.15), (3.48) and (3.70). The vector corresponds to two degrees of freedom and the fermions carry a total of 8 degrees of freedom.

AdS$_3$ field	d_{S^3}	$(\mathbf{2},\mathbf{0})$ Δ	**Ia** Δ	**Ib** Δ	**II** Δ
$A_\perp^{(\ell)}$	$2(\ell+1)(\ell+3)$	$\ell+3$	$\ell+3$	$2\ell+5$	$\frac{1}{2}\ell+2$
$\varphi_-^{(\ell+2)}$	$(\ell+3)^2$	$\ell+2$	$\ell+4$	$2\ell+4$	$\frac{1}{2}\ell+1$
$\zeta_p^{(\ell+1)}$	$4\times(\ell+2)^2$	$\ell+3$	$\ell+3$	$2\ell+5$	$\frac{1}{2}\ell+2$
$\varphi_+^{(\ell)}$	$(\ell+1)^2$	$\ell+4$	$\ell+2$	$2\ell+6$	$\frac{1}{2}\ell+3$
$\psi_+^{(\ell+1)}$	$4\times(\ell+2)(\ell+3)$	$\ell+\frac{5}{2}$	$\ell+\frac{7}{2}$	$2\ell+\frac{9}{2}$	$\frac{1}{2}\ell+\frac{3}{2}$
$\psi_-^{(\ell)}$	$4\times(\ell+1)(\ell+2)$	$\ell+\frac{7}{2}$	$\ell+\frac{5}{2}$	$2\ell+\frac{11}{2}$	$\frac{1}{2}\ell+\frac{5}{2}$

Table 3. AdS$_3$ supermultiplet structure of the fluctuation spectra in **Ia**, **Ib**, **II** cases and also for the massless (2,0) tensor multiplet on the AdS$_3\times S^3$ space.

$\zeta_p^{(\ell)}$ represents the modes of the 4 scalar transverse fluctuations, $\varphi_\pm$ denote the two scalar modes corresponding to the scalar part of B_{rs} component of the 2-form field mixed with the transverse scalar U or Θ in (3.50) or (3.72), and $\psi_\pm$ are the two sets of the fermions having different masses in (3.53), (3.56) and (3.74), (3.77).

In **Ia** case there is no scalar mixing so that $\varphi_+^{(\ell)}$ and $\varphi_-^{(\ell)}$ are both massless and thus have equal dimension (given in (3.24)) for the equal values of ℓ; the same applies to the corresponding massless fermions ψ_+ and ψ_-. d_{S^3} denotes the total S^3 degeneracy including the factor of the relevant number n of the number of fields, i.e. $\mathrm{d}_{S^3}\to n\times d_\ell^{(s)}$.

In all these cases we can check the balance of the numbers of bosonic and fermionic states and also the validity of the two additional sum rules for each supermultiplet

$$\sum_{\text{multiplet}}(-1)^{\mathrm{F}}\mathrm{d}_{S^3}=0\,,\qquad \sum_{\text{multiplet}}(-1)^{\mathrm{F}}\mathrm{d}_{S^3}\,\Delta=0\,,\qquad \sum_{\text{multiplet}}(-1)^{\mathrm{F}}\mathrm{d}_{S^3}\,\Delta^2=2\,. \tag{4.2}$$

Here the sum is over the set of fields in the supermultiplet corresponding to a fixed value of ℓ.

To compute the total value of the free energy F we may first find the contribution F_ℓ of each supermultiplet with fixed ℓ and then sum over ℓ, i.e. $F=\sum_\ell F_\ell$. For example, in the case of the (2,0) multiplet in table 3 we get, using the expressions in (4.1),

$$\begin{aligned} F_\ell^{(\mathbf{2},\mathbf{0})} &= d_\ell^{(1)}F_{1\perp}(\ell+3)+d_{\ell+2}^{(0)}F_0(\ell+2)+4\,d_{\ell+1}^{(0)}F_0(\ell+3)+d_\ell^{(0)}F_0(\ell+4)\\ &\quad -4\,d_{\ell+1}^{(\frac{1}{2})}F_{\frac{1}{2}}\left(\ell+\frac{5}{2}\right)-4\,d_\ell^{(\frac{1}{2})}F_{\frac{1}{2}}\left(\ell+\frac{7}{2}\right)\,, \end{aligned} \tag{4.3}$$

where according to (3.19), (3.20), (3.21) the degeneracies are given by

$$d_\ell^{(0)}=(\ell+1)^2\,,\qquad d_\ell^{(1)}=2(\ell+1)(\ell+3)\,,\qquad d_\ell^{(\frac{1}{2})}=(\ell+1)(\ell+2)\,. \tag{4.4}$$

Using (4.1) we then find from the data in table 3 that

$$F_\ell^{(\mathbf{Ia})}=F_\ell^{(\mathbf{Ib})}=-\frac{3}{2\pi}(\ell+2)\ \mathrm{vol}(\mathrm{AdS}_3)\,,\qquad F_\ell^{(\mathbf{2},\mathbf{0})}=\frac{1}{3}F_\ell^{(\mathbf{Ia})}\,,\qquad F_\ell^{(\mathbf{II})}=0. \tag{4.5}$$

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These surprisingly simple expressions follow from cancellations that are due to the underlying supersymmetry.

Indeed, if one computes the free energy without taking into account the shifts of ℓ required by the supermultiplet structure, i.e. by just directly combining together the contributions of all modes with a given ℓ, one gets the following quartic polynomial expressions in ℓ (we use the notation F'_ℓ for these "no-shifts" values)

$$F'^{(\mathbf{2,0})}_\ell = \frac{1}{6\pi}(1+\ell)(33+34\ell+15\ell^2+5\ell^3)\ \mathrm{vol}(\mathrm{AdS}_3)\,, \tag{4.6}$$

$$F'^{(\mathbf{Ia})}_\ell = \frac{1}{6\pi}(1+\ell)(9+16\ell+15\ell^2+5\ell^3)\ \mathrm{vol}(\mathrm{AdS}_3)\,, \tag{4.7}$$

$$F'^{(\mathbf{Ib})}_\ell = \frac{1}{3\pi}(1+\ell)(63+91\ell+60\ell^2+20\ell^3)\ \mathrm{vol}(\mathrm{AdS}_3)\,, \tag{4.8}$$

$$F'^{(\mathbf{II})}_\ell = \frac{5}{48\pi}(1+\ell)(18+14\ell+3\ell^2+\ell^3)\ \mathrm{vol}(\mathrm{AdS}_3)\,. \tag{4.9}$$

Summing these expressions over ℓ with a UV cutoff at large ℓ one concludes that total free energy contains quintic (and lower-order) UV divergences. The logarithmic divergences cancel out as can be shown independently by starting with the general expression for the partition function on $\mathrm{AdS}_3 \times S^3$ (see appendix E). At the same time, the sums over ℓ in the first three cases in (4.5) are only quadratically divergent, which is obviously a consequence of maintaining supersymmetry-implied multiplet structure at each value of level ℓ.

Let us note that the multiplet structure presented in table 3 applies for $\ell \geq 0$ when all states are present (have positive d_{S^3}). For $\ell = -1$ there are additional states that also form a short supermultiplet (we include the factors of their degeneracies from table 3 taken at $\ell = -1$)

$$4 \times \varphi^{(1)}_+, \qquad 4 \times \zeta^{(0)}_+, \qquad 4 \times 2 \times \psi^{(0)}_+\,. \tag{4.10}$$

This is again a balanced multiplet since the total number of states is $4+4-8=0$. Note that the contribution of (4.10) to the total free energy is the same as of a general multiplet evaluated at $\ell = -1$ (at this value all extra states have zero degeneracy and thus do not contribute).

Taking this into account, the total free energy should be given by summing the expressions in (4.5) from $\ell = -1$ to ∞, or, equivalently, after shifting ℓ so that the sum starts from 1, we get

$$F_{\mathbf{II}} = \sum_\ell F^{(\mathbf{II})}_\ell = 0\,, \qquad F_{\mathbf{Ia}} = F_{\mathbf{Ib}} = 3F_{(\mathbf{2,0})} = \sum_{\ell=-1}^{\infty} F^{(\mathbf{Ia})}_\ell = -\frac{3}{2\pi}\mathcal{C}_1\,\mathrm{vol}(\mathrm{AdS}_3)\,, \tag{4.11}$$

$$\mathcal{C}_1 = \sum_{\ell=-1}^{\infty}(\ell+2) = \sum_{\ell'=1}^{\infty}\ell'\,. \tag{4.12}$$

We conclude that while the one-loop free energy is manifestly zero in the probe **II** case, it requires a particular regularization to define it in the other two cases.

The familiar Riemann ζ-function (with $\zeta(-1) = -\frac{1}{12}$) is not appropriate here as ℓ is the radial quantum number of S^3 (and not, e.g, an S^1 mode number). A more natural alternative is to use a prescription like $\sum_{\ell'=1}^{\infty} \ell' z^{\ell'}\big|_{z\to 1}$ or a sharp cutoff $\ell' < \Lambda$ as in [44–46, 69, 70]. Dropping singular terms one then finds that $\mathcal{C}_1 = 0$.

5 Open questions

The main open question that remains to be understood is if the subleading in N terms in the b-coefficients in (1.7), (1.8) can be reproduced in the framework of the semiclassical M5 brane **Ia** and **Ib** probes.[27] One speculative suggestion is that a classical probe action should actually be given by a combination of the M5 and M2 brane actions, with an on-shell value of the latter (wrapping AdS_3) reproducing the required order N terms in (1.7), (1.8) (recall that according to (2.8) the M2 brane tension is proportional to N). A problem with this idea is that it is not clear how to see why both the M5 and M2 contributions should have the same κ-dependent factors that are required to match the expressions in (1.7), (1.8).

Assuming there is some "classical action" explanation for the exact expressions in (1.7), (1.8) one would then to prove that there are no further quantum M5 brane corrections to the free energy and thus to the b-coefficient. At the one-loop level discussed above that would require the choice of a particular regularization that sets to zero the infinite-sum coefficient in (1.14), (4.12). Such a regularization was required in similar supersymmetric conformal anomaly contexts [44–46]. The choice of regularization should be motivated by some underlying symmetries of the quantum M5 brane theory on the $\mathrm{AdS}_3 \times S^3$ space that may not be manifest after the explicit expansion in modes on S^3. We already accounted for the AdS_3 supersymmetry by combining the contributions of all states in a supermultiplet at level ℓ before introducing a cutoff. Which are additional symmetries that select a specific choice of the subtraction procedure that leads to $\mathcal{C}_1 = 0$ remains to be understood.

As for the probe **II** case when the M5 brane is wrapped on $\mathrm{AdS}_3 \times S^3$ in $\mathrm{AdS}_4 \times S^7$, the one-loop correction was found to be manifestly zero (cf. (1.17), (4.5)). However, it remains to find the interpretation of the corresponding classical action (1.16) in the dual $k = 1$ ABJM theory, i.e. to see which is the corresponding $\frac{1}{2}$-BPS spherical defect and which is its b-anomaly coefficient. To try to shed light on this, it would be useful to generalize the discussion of the probe **II** case to the M5 brane probe wrapping $\mathrm{AdS}_3 \times S^3$ in the M-theory background $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ dual to the level k ABJM theory.

Similar issues appear in trying to go beyond the classical M5 brane probe action values [25] in order to match the exact expressions for d_2 in (1.12), (1.13). As was mentioned in the Introduction, that will involve considering quantum M5 branes in a "twisted" $\mathrm{AdS}_{7,\beta} \times \tilde{S}^4$ background with the $\mathrm{AdS}_{3,\beta}$ boundary being $S^1_\beta \times S^1$. It would be of interest to compute the corresponding one-loop M5 brane correction that should be related to order N^0 term in free energy and thus in d_2 and check that it again vanishes.

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[27] As already mentioned in the Introduction, it is not clear a priori if an effective description of a defect or multiple M2 brane in large $SU(N)$ representation in terms of an effective M5 brane may apply beyond the classical approximation.

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supersymmetry theories" held in Trani in July 2024 (funded by COST Action CA22113, INFN and Salento University). MB is supported by the INFN grant GAST. AAT is supported by the STFC Consolidated Grant ST/X000575/1.

A Fermionic fluctuation operator

In this appendix we will determine the structure of the analog of the Dirac operator appearing in the quadratic fermionic part of the M5 brane action. This fermionic part complements [33] the bosonic part of the PST action in (2.1). The determinant of this Dirac operator contributes (after a κ-symmetry gauge fixing) to the one-loop M5 brane partition function (cf. (3.16)).

This fermionic operator may be found from the covariant M5 brane equations of motion [35, 37].[28] The presence of a non-trivial 3-form H_3 background for the three solutions **Ia** (2.16), **Ib** (2.30) and **II** (2.42) makes the determination of the explicit form of this Dirac operator more complicated than in the previously discussed $S^1 \times S^5$ and $S^1 \times \mathrm{AdS}_5$ M5-brane cases [11, 15] where the background H_3 field was zero.

In this appendix we will denote the 11d target space indices by capital letters $M, N, P = 0, 1, 2, \ldots, 9, \natural$ and the 6d world-volume indices by lower-case Latin letters $i, j, m, n, \ldots = \hat{0}, \hat{1}, \ldots, \hat{5}$. We will use "hat" to indicate quantities induced on the world-volume, introducing, in particular, the notation $\hat{g}_{ij}$ for the induced metric which is equivalent to G_{ij} used in (2.2). We will underline the tangent space indices so that

$$ds^2_{11} = G_{MN} dx^M dx^N \,, \qquad G_{MN} = \eta_{\underline{MN}} e^{\underline{M}}_M e^{\underline{N}}_N \,, \qquad ds^2_{\mathrm{M5}} = \hat{g}_{ij} d\xi^i d\xi^j \,, \qquad \hat{g}_{ij} = \eta_{\underline{ij}} \hat{e}^{\underline{i}}_i \hat{e}^{\underline{j}}_j \,. \tag{A.1}$$

We will use the following definitions

$$\{\Gamma_{\underline{M}}, \Gamma_{\underline{N}}\} = 2\eta_{\underline{MN}} \,, \qquad \Gamma_{\underline{01\ldots9\natural}} = 1 \,, \qquad \nabla_M = \partial_M + \frac{1}{4}\omega_M^{\underline{RS}} \Gamma_{\underline{RS}} \,, \tag{A.2}$$

$$D_M = \nabla_M + \mathrm{F}_M \,, \qquad \mathrm{F}_M = -\frac{1}{288}\left(\Gamma_{PNKLM} - 8 G_{MP} \Gamma_{NKL}\right) F^{PNKL} \,. \tag{A.3}$$

For the two backgrounds of interest (I) $\mathrm{AdS}_7 \times S^4$ in (2.6), (2.7) and (II) $\mathrm{AdS}_4 \times S^7$ in (2.37), (2.38) we have

$$\mathrm{F}^{(I)}_M = e^{\underline{M}}_M \, \Gamma_{\underline{M}} \, \Gamma^{\underline{789\natural}} \cdot \begin{cases} -\frac{1}{2} L^{-1} & \mathrm{AdS}_7 \\ L^{-1} & S^4 \end{cases} , \qquad \mathrm{F}^{(\mathrm{II})}_M = e^{\underline{M}}_M \, \Gamma_{\underline{M}} \, \Gamma^{\underline{0123}} \cdot \begin{cases} \frac{1}{2} \mathrm{L}^{-1} & \mathrm{AdS}_4 \\ -\frac{1}{4} \mathrm{L}^{-1} & S^7 \end{cases} , \tag{A.4}$$

where we indicated the overall scaling factors on the two subspaces (i.e. for index M in AdS_d or S^d).[29]

Let us first recall the connection between the bosonic part of the PST action (2.1) and the covariant formulation of the M5 brane equations of motion in terms of the (anti) self-dual

[28] The fermionic equations are given in an implicit form in [35]. The explicit Dirac-like equation for the spinor variable ϑ was obtained (with manifest κ-symmetry and for generic fluxes) to linear order in ϑ in [37]. Let us note that the PST formalism produces a different κ-symmetry generator [33], but as was shown in [36], the two corresponding κ-symmetry transformations are equivalent up to a redefinition of the gauge parameter.

[29] In our notation (cf. (2.6), (2.37)) the coordinates and thus derivatives like ∇_M are dimensionless and these factors of L^{-1} or L^{-1} cancel against similar factors in the vielbein $e^{\underline{M}}_M$ in (A.1).

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3-form h_{ijk} [35].[30] One defines (cf. (2.4))

$$h^{ij} = h^{ijk}\, V_k = -\frac{1}{6\sqrt{-\hat{g}}}\varepsilon^{ijksmn}\, h_{smn}\, V_k\,, \qquad\qquad V_k \equiv \frac{1}{\sqrt{-(\partial a)^2}}\partial_k a\,, \tag{A.5}$$

where we used the anti-self-duality condition on h^{ijk}. As in (2.3), we will use the gauge $a = \xi^1$. The field h^{ij} is related to $\hat{\mathsf{H}}^{ij}$ in (2.4) by [33, 36]

$$h_{ij} = \frac{1}{4}\hat{\mathsf{H}}_{ij} + \frac{1}{4}\frac{\hat{\mathsf{H}}_{im}\,\hat{\mathsf{H}}^{mn}\,\hat{\mathsf{H}}_{nj} + \frac{1}{4}\hat{\mathsf{H}}_{mn}\hat{\mathsf{H}}^{mn}\hat{\mathsf{H}}_{ij}}{1+\sqrt{-|\hat{g}_{rs}+\hat{\mathsf{H}}_{rs}|}+\frac{1}{4}\hat{\mathsf{H}}_{rs}\hat{\mathsf{H}}^{rs}}\,. \tag{A.6}$$

The general expression for the linearized Dirac-like fermionic equation is given by [37][31]

$$e^i_{\underline{i}}\,\Gamma^{\underline{i}}\,\hat{D}_i\big[(1+\Gamma_*)\vartheta\big] = 0\,, \qquad\qquad \Gamma_* = \Gamma_{\underline{012345}} + \frac{1}{3}h^{\underline{ijk}}\Gamma_{\underline{ijk}}\,, \tag{A.7}$$

$$\Gamma_i = \partial_i x^M \Gamma_M\,, \qquad \hat{D}_i = \partial_i x^M D_M\,, \qquad (\Gamma_{\underline{012345}})^2 = 1 = (\Gamma_*)^2\,, \tag{A.8}$$

where D_M was defined in (A.3). Note that the second term in Γ_* in (A.7) squares to zero (h_{ijk} is assumed to be anti-self-dual). $e^i_{\underline{i}}$ in (A.7) is the *effective* sechsbein corresponding to the effective metric g_{ij} appearing in the Lagrangian for the bosonic fluctuations in (3.6), (3.33), (3.62). There is also the following relation between $e^i_{\underline{i}}$, the induced metric $\hat{g}_{ij}$ and h_{ac} [33]

$$\eta^{\underline{ij}}\, e^i_{\underline{i}}\, \hat{e}^j_{\underline{j}} = \hat{g}^{ij} + 2\big[\hat{g}^{ij}h^{mn}h_{mn} + 2V^iV^jh^{mn}h_{mn} + 4h^{ik}h_k{}^j - \frac{1}{\sqrt{-\hat{g}}}V^{(i}\varepsilon^{j)mnrsk}V_m h_{nr}h_{sk}\big]\,. \tag{A.9}$$

Below we will derive the explicit expression for the operator in (A.7) in the three cases **Ia**, **Ib** and **II**. We will show that this effective Dirac operator appearing in the quadratic fermionic action after a κ-symmetry gauge fixing has indeed the form given in (3.17), (3.18) with $m_{\rm F}^{(\mathbf{Ia})} = 0$ and $m^{(\mathbf{Ib})} = m_{\rm F}^{(\mathbf{II})} = \frac{3}{2}$.

Let us note that in the analysis of the bosonic fluctuations in section 3, when discussing the effective metric in (3.6), (3.33), (3.62) that enters the quadratic fluctuation Lagrangians in (3.14), (3.38), (3.65) we scaled out the overall factor of L or L (as well as numerical factors that can be absorbed into the fluctuation fields). Similarly, in (A.7) the overall scale of the effective $e^i_{\underline{i}}$ will be irrelevant (it can be absorbed into ϑ in the quadratic fermionic action) so we will also drop it in the expressions below.[32] We will use the labelling of the coordinates as given in (2.13) and (2.41).[33]

[30] It should not be confused with the fluctuation field h_{ijk} used in the text (as, e.g., in (3.3)).

[31] We follow here the notation for the M5-brane fermionic operator in [11, 15]. Note that the sign in front of the second term in Γ_* is convention-dependent and the final results will not depend on its choice.

[32] Note also that there is also no difference between the covariant derivatives for the effective and the induced $AdS_3 \times S^3$ metrics as they are different only by the values of the radii which the spin connection is independent of.

[33] Let us note for completeness that the global supersymmetry preserved by the M5 brane probes is determined by the solutions of the Killing spinor equations of the target space backgrounds (see, e.g., [6, 71]) $\nabla_M\varepsilon = -\mathrm{F}_M\varepsilon$ subject to the κ-symmetry gauge condition on the brane $\Gamma_*\varepsilon = \varepsilon$. For example, in the case **II** we get $\varepsilon = e^{\frac{1}{2}\theta\Gamma^{\underline{01234}}}\,\Sigma_{S^3}\,\Sigma_{S'^3}\,\Sigma_{\mathrm{AdS}_4}\varepsilon_0$ where Σ's are the matrices that define the Killing spinors on S^3, S'^3 and AdS_4 respectively. This solution for ε preserves $\mathcal{N}=8$ supersymmetry. Solving the κ-symmetry gauge condition imposes an additional projector and therefore halves the amount of supersymmetry preserved by the brane configuration.

Probe Ia.

From the form of the background in (2.6) and the evaluation of (A.9), we obtain the induced and the effective sechsbeins as

$$\hat{e}_i^{\underline{i}} = L\,(\sqrt{1+\kappa^2}\; e_{\mathrm{AdS}_3},\;\kappa\; e_{S^3})\,, \qquad\qquad e_i^{\underline{i}} = (e_{\mathrm{AdS}_3},\; e_{S^3})\,. \tag{A.10}$$

The corresponding metrics are the induced $\mathrm{AdS}_3 \times S^3$ one as in (3.2) and the effective one of the equal-radii $\mathrm{AdS}_3 \times S^3$ space as in (3.6). The pull-back of the covariant derivative on the world volume is

$$\hat{\nabla}_i = \partial_i x^M \nabla_M \equiv \nabla_i + \frac{1}{2}\omega_i^{\underline{i6}}\Gamma_{\underline{i6}}\,, \qquad\qquad \nabla_i \equiv \partial_i + \frac{1}{4}\omega_i^{\underline{jk}}\Gamma_{\underline{jk}}\,, \tag{A.11}$$

where there is no sum over the index $\underline{i}$ and ∇_i is the covariant spinor derivative corresponding to the effective $e_i^{\underline{i}}$ in (A.10).

Using (A.4) and (A.11) the two terms ∇_M and F_M in D_M in (A.3) lead to the following two contributions to the operator $e_{\underline{i}}^i\, \Gamma^{\underline{i}}\, \hat{D}_i$ in (A.7):

$$\nabla\!\!\!/ + \frac{3}{2}(\sqrt{1+\kappa^2}+\kappa)\Gamma_{\underline{6}}\,, \qquad e_{\underline{i}}^i\, \Gamma^{\underline{i}}\, \mathrm{F}_i = -\frac{3}{2}(\sqrt{1+\kappa^2}+\kappa)\Gamma^{\underline{789\natural}}\,, \qquad \nabla\!\!\!/ \equiv e_{\underline{i}}^i\, \Gamma^{\underline{i}}\nabla_i\,. \tag{A.12}$$

As a result, the fermionic equation of motion in (A.7) may be written as

$$\mathcal{D}[(1+\Gamma_*)\vartheta] = 0\,, \qquad\qquad \mathcal{D} = i\nabla\!\!\!/ - \frac{3}{2}i(\sqrt{1+\kappa^2}+\kappa)\,(\Gamma_{\underline{012345}}-1)\Gamma_{\underline{6}}\,, \tag{A.13}$$

where $i\nabla\!\!\!/$ is the Dirac operator corresponding to the effective $\mathrm{AdS}_3 \times S^3$ sechsbein in (A.10). Here we used that $\Gamma_{\underline{6789\natural}} = \Gamma_{\underline{012345}}$, which follows from $\Gamma_{\underline{01\ldots9\natural}} = 1$ in (A.2).

From (A.6) and the explicit form of H_3 in (2.16) we find that the only nonzero components of h_{ijk} correspond to $h_{\underline{\hat{0}\hat{2}}}$ (we scale out L^2 factor and $\underline{\hat{0}}, \underline{\hat{1}}, \ldots$ are the values of the world-volume indices)

$$h_{\underline{\hat{0}\hat{2}}} = \frac{1}{2}\,(\sqrt{1+\kappa^2}-\kappa)\,, \qquad\qquad h_{\underline{\hat{0}\hat{2}}} = h^{\underline{\hat{0}\hat{1}\hat{2}}} = -h^{\underline{\hat{3}\hat{4}\hat{5}}}\,. \tag{A.14}$$

Then Γ_* in (A.7) takes the form

$$\Gamma_* = \Gamma_{\underline{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}}} + h_{\underline{\hat{0}\hat{2}}}(\Gamma_{\underline{\hat{0}\hat{1}\hat{2}}} - \Gamma_{\underline{\hat{3}\hat{4}\hat{5}}}) = \Gamma_{\underline{012345}} + h_{\underline{\hat{0}\hat{2}}}\Gamma_{\underline{012}}(1-\Gamma_{\underline{012345}})\,, \tag{A.15}$$

where we used that $(\Gamma_{\underline{012}})^2 = 1$. Here all terms in Γ_* anticommute with each other and the term proportional to $h_{\underline{\hat{0}\hat{2}}}$ squares to zero, consistently with $(\Gamma_*)^2 = 1$ in (A.9).

We fix the κ-symmetry gauge by the condition $\Gamma_*\vartheta = \vartheta$. Using the explicit form of Γ_* in (A.15) this condition can be rewritten as

$$(1 - h_{\underline{\hat{0}\hat{2}}}\Gamma_{\underline{012}})\,(\Gamma_{\underline{012345}}-1)\vartheta = 0\,, \tag{A.16}$$

and is then solved by $(\Gamma_{\underline{012345}}-1)\vartheta = 0$. As a consequence, the mass term in (A.13) vanishes and we are left simply with massless Dirac operator which is independent of the parameter κ of the M5 brane solution

$$\mathcal{D} = i\nabla\!\!\!/\,. \tag{A.17}$$

This is the operator that appeared in (3.16), (3.17) corresponding to $m_{\mathrm{F}} = 0$ in (3.26).

Probe Ib.

In this case from (3.29) and (3.33) we find instead of (A.10)

$$\hat{e}^{\hat{i}}_{i} = L\,(\,e_{\mathrm{AdS}_3},\ \kappa\sqrt{2-4\kappa^2}\ e_{S^3}\,)\,, \qquad e^{\underline{i}}_{i} = (L_A\, e_{\mathrm{AdS}_3},\ L_S\, e_{S^3})\,, \qquad L_A = 1,\ \ L_S = \frac{1}{2}\,. \tag{A.18}$$

Since the resulting fermionic operator should depend only on the effective $\mathrm{AdS}_3 \times S^3$ geometry and thus should not depend on the value of κ (after a rescaling of the fermion field)[34] to simplify the discussion we may just set $\kappa = \frac{1}{2}$ from the start (corresponding to the choice of $\theta_0 = \frac{\pi}{2}$ in (2.30)). Then there will be no extra term from the spin connection compared to (A.11), i.e. $\partial_i x^M \nabla_M = \nabla_i$, the latter being the effective spinor covariant derivative on the world-volume. Furthermore, in this case the expression in (2.32) implies that $\mathsf{H}_{ijk} = 0$ and then from (A.6) it follows also that $h_{ij} = 0$.

Again, we may scale out factors of L but it is useful to keep the values of L_A and L_S in (A.18) generic till the very end. Then we find for the F_M term in (A.3) and the matrix Γ_* in (A.7)

$$e^{i}_{\underline{i}}\, \Gamma^{\underline{i}}\, \mathrm{F}_i = \frac{3}{2}(L_S^{-1} - L_A^{-1})\, \Gamma^{\underline{789\natural}}\,, \qquad \Gamma_* = \Gamma_{\underline{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}}} = \Gamma_{\underline{01289\natural}}\,. \tag{A.19}$$

As a consequence, the fermionic equation in (A.7) takes the form

$$\mathcal{D}[(1+\Gamma_*)\vartheta] = 0\,, \qquad \mathcal{D} = i\nabla\!\!\!\!/ + \frac{3}{2}i(L_S^{-1} - L_A^{-1})\, \Gamma^{\underline{789\natural}}\,, \tag{A.20}$$

where $i\nabla\!\!\!\!/$ corresponds to the effective $e^{i}_{\underline{i}}$ in (A.18).

Choosing the κ-symmetry gauge $\Gamma_*\vartheta = \vartheta$, we get $\Gamma_{\underline{89\natural}}\vartheta = \Gamma_{\underline{012}}\vartheta$ and thus (A.20) reduces to[35]

$$\mathcal{D}\vartheta = 0\,, \qquad \mathcal{D} = i\nabla\!\!\!\!/ + \mathcal{M}\,, \qquad \mathcal{M} = -\frac{3}{2}\,i(L_S^{-1} - L_A^{-1})\, \Gamma^{\underline{0127}}\,, \qquad (\Gamma^{\underline{0127}})^2 = -1\,. \tag{A.21}$$

In the present case of (A.18) with $L_A = 1$, $L_S = \frac{1}{2}$ we thus find that the mass matrix is given by

$$\mathcal{M} = -\frac{3}{2}\,i\Gamma^{\underline{0127}}\,, \qquad \mathcal{M}^2 = \frac{9}{4}\,. \tag{A.22}$$

Thus $\mathcal{M}$ has the eigenvalues $\pm\frac{3}{2}$ corresponding to $m_{\mathrm{F}} = \frac{3}{2}$ used in (3.53).

Probe II.

In this case from (3.58) and (3.62) we get (cf. (A.18))

$$\hat{e}^{\hat{i}}_{i} = \mathrm{L}\,(\sqrt{1+\varkappa^2}\, e_{\mathrm{AdS}_3},\ 2\ e_{S^3})\,, \qquad e^{\underline{i}}_{i} = (L_A\, e_{\mathrm{AdS}_3},\ L_S\, e_{S^3})\,, \qquad L_A = \frac{1}{2},\ \ L_S = 1\,. \tag{A.23}$$

---

[34] Note that a similar conclusion that like the bosonic fluctuation operators the fermionic operator does not have a non-trivial dependence on the κ-parameter of the classical solution was reached in analogous examples discussed in [11, 50].

[35] Note that here $\underline{0}, \underline{1}, \underline{2}$ are also indices on the world volume, while $\underline{7}$ is the index from the transverse space.

Again, the fermionic fluctuation spectrum should not depend on the parameter $\varkappa$ of the solution so we may simplify the discussion by setting $\varkappa = 0$ (corresponding to the choice of $u_0 = 0$ in (2.42)) (and also scale out factors of L). Then according to (2.45) we get $\mathsf{H}_{ijk} = 0$ and thus also $h_{ij} = 0$ from (A.6).

Instead of (A.19) and (A.21) here we find

$$e^i_{\underline{i}}\,\Gamma^{\underline{i}}\,\mathrm{F}_i = \frac{3}{2}(L_A^{-1} - L_S^{-1})\,\Gamma^{\underline{0123}}, \qquad \Gamma_* = \Gamma_{\underline{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}}} = \Gamma_{\underline{012567}}, \tag{A.24}$$

$$\mathcal{D}[(1+\Gamma_*)\vartheta] = 0, \qquad \mathcal{D} = i\bar{\nabla\!\!\!/} + \frac{3}{2}i(L_A^{-1} - L_S^{-1})\Gamma^{\underline{0123}}, \tag{A.25}$$

where $i\bar{\nabla\!\!\!/}$ corresponds to the effective $e^{\underline{i}}_{i}$ in (A.23). Using the κ-symmetry gauge $\Gamma_*\vartheta = \vartheta$ we get

$$\mathcal{D}\vartheta = 0\,, \qquad \mathcal{D} = i\bar{\nabla\!\!\!/} + \mathcal{M}\,, \qquad \mathcal{M} = \frac{3}{2}i(L_A^{-1} - L_S^{-1})\Gamma^{\underline{0123}}, \qquad (\Gamma^{\underline{0123}})^2 = -1\,. \tag{A.26}$$

In the present case of (A.23), i.e. for $L_A = 1,\ L_S = \frac{1}{2}$, we thus conclude that (cf. (A.17), (A.22))

$$\mathcal{M} = \frac{3}{2}\,i\Gamma^{\underline{0123}}, \qquad \mathcal{M}^2 = \frac{9}{4}\,. \tag{A.27}$$

This matrix $\mathcal{M}$ has the eigenvalues $\pm\frac{3}{2}$ corresponding to the value of $m_{\rm F} = \frac{3}{2}$ used in (3.74).

To conclude, in all the three cases discussed above $\mathcal{M}$ happens at the end to be proportional to $(L_A^{-1} - L_S^{-1})$ (cf. (A.21), (A.26)) thus leading to the values $m_{\rm F} = 0$ for the **Ia** probe and $m_{\rm F} = \frac{3}{2}$ for the **Ib** and **II** probes.

B Partition function of 2-form field on AdS$_3 \times S^3$

Here we consider the partition function of a 2-form field in 6d with the standard gauge-invariant Lagrangian (3.13) where $\mathrm{h}_{ijk} = 3\partial_{[i}B_{jk]}$ defined on the AdS$_3 \times S^3$ space with generic radii (cf. (3.6))

$$ds^2 = g_{ij}d\xi^d\xi^j = L_A^2 ds^2_{\rm AdS_3} + L_S^2 ds^2_{S^3}\,. \tag{B.1}$$

The corresponding 6d curvature factorizes into the AdS$_3$ and S^3 parts in the obvious way

$$R_{ijk\ell} = \mp L_{A,S}^{-2}(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})\,, \qquad R_{ij} = \mp 2L_{A,S}^{-2}g_{ij}\,, \qquad R_{A,S} = \mp 6L_{A,S}^{-2}\,. \tag{B.2}$$

Starting with $L_2 = \frac{1}{3!}\mathrm{h}^{ijk}\mathrm{h}_{ijk}$ where $\mathrm{h}_3 = dB_2$ and following the standard procedure (see, e.g., [58] and references there), we add to it the covariant gauge fixing term $(\nabla^i B_{ij})^2$ to get[36]

$$L_2 = \frac{1}{3!}\mathrm{h}^{ijk}\mathrm{h}_{ijk} \quad \to \quad L_2' = \frac{1}{2}B_{ij}(\Delta_2)^{ij}_{kl}B^{k\ell}\,, \tag{B.3}$$

where Δ_2 is the standard Hodge-deRham operator defined on 2-forms. Including the ghost factors, the corresponding partition function is

$$Z_2 = \frac{\det \Delta_1}{(\det \Delta_2)^{1/2}\,(\det \Delta_0)^{3/2}}\,, \tag{B.4}$$

$$(\Delta_2)^{ij}_{kl} = -\nabla^2\delta^{ij}_{k\ell} + 2R^{[i}_{[k}\delta^{j]}_{\ell]} - R_{k\ell}{}^{ij}\,, \qquad (\Delta_1)^i_j = -\nabla^2\delta^i_j + R^i_j\,, \qquad \Delta_0 = -\nabla^2. \tag{B.5}$$

[36] We include $\frac{1}{n!}$ factor in the definition of antisymmetrization, i.e. $T_{[ab]} = \frac{1}{2!}(T_{ab} - T_{ba})$, etc.

Let a, b be the AdS$_3$ and r, s the S^3 indices. Then L'_2 in (B.3) may be written as

$$L'_2 = \frac{1}{2}\Big[B_{ab}(-\nabla^2-2L_A^{-2})B^{ab}+2B_{ar}(-\nabla^2+2L_S^{-2}-2L_A^{-2})B^{ar}+B_{rs}(-\nabla^2+2L_S^{-2})B^{rs}\Big]\,. \tag{B.6}$$

One can do a similar split for the ghost Δ_1 operator. Then Z_2 in (B.4) may be written as

$$Z_2 = \frac{\det \mathbf{\Delta}_{1,0}(-2L_A^{-2})\;\det \mathbf{\Delta}_{0,1}(2L_S^{-2})}{\big[\det \mathbf{\Delta}_{2,0}(-2L_A^{-2})\;\det \mathbf{\Delta}_{1,1}(2L_S^{-2}-2L_A^{-2})\;\det \mathbf{\Delta}_{0,2}(2L_S^{-2})\big]^{1/2}\;\big[\det \mathbf{\Delta}_{0,0}(0)\big]^{3/2}}\,, \tag{B.7}$$

where

$$\mathbf{\Delta}_{p,q}(M^2) \equiv -\nabla^2 + M^2 = -\nabla_A^2 - \nabla_S^2 + M^2\,, \tag{B.8}$$

denotes an operator defined on a field in AdS$_3 \times S^3$ which has p-form indices in AdS$_3$ and q-form indices in S^3. Since in 3d a rank 2 antisymmetric tensor is algebraically equivalent to a vector we have

$$\det \mathbf{\Delta}_{2,0} = \det \mathbf{\Delta}_{1,0}\,, \qquad\qquad \det \mathbf{\Delta}_{0,2} = \det \mathbf{\Delta}_{0,1}\,, \tag{B.9}$$

and thus (B.7) may be rewritten as

$$Z_2 = \Big[\frac{\det \mathbf{\Delta}_{1,0}(-2L_A^{-2})\;\det \mathbf{\Delta}_{0,1}(2L_S^{-2})}{\det \mathbf{\Delta}_{1,1}(2L_S^{-2}-2L_A^{-2})\;\big(\det \mathbf{\Delta}_{0,0}(0)\big)^3}\Big]^{1/2}\,. \tag{B.10}$$

We may further split the fields with a vector index in AdS$_3$ or S^3 into the transverse and the longitudinal (scalar) parts which gives

$$\begin{aligned}
\det \mathbf{\Delta}_{1,0}(-2L_A^{-2}) &= \det \mathbf{\Delta}_{1\perp,0}(-2L_A^{-2})\;\det \mathbf{\Delta}_{0,0}(0)\,,\\
\det \mathbf{\Delta}_{0,1}(2L_S^{-2}) &= \det \mathbf{\Delta}_{0,1\perp}(2L_S^{-2})\;\det \mathbf{\Delta}_{0,0}(0)\,,\\
\det \mathbf{\Delta}_{1,1}(2L_S^{-2}-2L_A^{-2}) &= \det \mathbf{\Delta}_{1\perp,1\perp}(2L_S^{-2}-2L_A^{-2})\det \mathbf{\Delta}_{0,1\perp}(2L_S^{-2})\\
&\qquad \det \mathbf{\Delta}_{1\perp,0}(-2L_A^{-2})\det \mathbf{\Delta}_{0,0}(0)\,.
\end{aligned} \tag{B.11}$$

Then (B.10) becomes simply

$$Z_2 = \big[\det \mathbf{\Delta}_{1\perp,1\perp}(2L_S^{-2}-2L_A^{-2})\big]^{-1/2}\;\big[\det \mathbf{\Delta}_{0,0}(0)\big]^{-1}\,. \tag{B.12}$$

The (modulus of) partition function of a self-dual 2-form field is given by the square root of (B.4) or of (B.12), i.e.

$$Z_2^{(+)} = \big[\det \mathbf{\Delta}_{1\perp,1\perp}(2L_S^{-2}-2L_A^{-2})\big]^{-1/4}\;\big[\det \mathbf{\Delta}_{0,0}(0)\big]^{-1/2}\,. \tag{B.13}$$

C Short supermultiplets of fluctuation fields in AdS$_3$

Here we will discuss the supermultiplet structure of the fluctuation modes as fields on AdS$_3$ presented in table 3.

In table 3 some fields have shifted values of level ℓ. These shifts are the same as in table 4 of [67] for spin-1 short supermultiplet of $SU(1,1|2) \times SU(1,1|2)$ corresponding to the tensor product $[\ell+3]_L \otimes [\ell+3]_R$. Here $[k+1]$ (with $k = \ell+2$) denotes a spin-1 short multiplet of $SU(1,1|2)$ (see eq. (4.14) in [68]). This supermultiplet describes states of a (2,0) tensor multiplet in the supersymmetric $\mathrm{AdS}_3 \times S^3$ vacuum of 6d supergravity.

The bosonic subgroup of $SU(1,1|2)$ is $SL(2,\mathbb{R}) \times SU(2)$. Supercharges are in a doublet of R-symmetry automorphism $SU(2)_{\mathcal{R}}$[37] and mix different representations of $SL(2,\mathbb{R}) \times SU(2)$. Using labels $|\Delta; j, j'\rangle$ of $SO(2,2) \times SU(2) \times SU(2)_{\mathcal{R}}$ the spin-1 short multiplet of $SU(1,1|2)$ contains four $SL(2,\mathbb{R}) \times SU(2)$ representations

$$[k+1]^{SU(1,1|2)} \quad = \quad \begin{array}{cccc} \hline \text{STATES} & j & j' & \Delta \\ \hline |0\rangle & \frac{k}{2} & 0 & \frac{k}{2} \\ Q^{\pm}|0\rangle & \frac{k-1}{2} & \frac{1}{2} & \frac{k+1}{2} \\ Q^{\pm}Q^{\pm}|0\rangle & \frac{k-2}{2} & 0 & \frac{k+2}{2} \\ \hline \end{array} \tag{C.1}$$

One can check that taking the tensor product $[\ell+3]_L \otimes [\ell+3]_R$ (see, e.g., table I in [72]) one reproduces the data for the (2,0) tensor multiplet entry in table 3 (namely, the spin $|\Delta_L - \Delta_R|$, the S^3 degeneracy $(2j_L+1)(2j_R+1)$, field multiplicity $(2j'_L+1)(2j'_R+1)$, and $\Delta = \Delta_L + \Delta_R$). For example, the four ζ_p scalar states correspond to

$$|\frac{k+1}{2}; \frac{k-1}{2}, \frac{1}{2}\rangle \otimes |\frac{k+1}{2}; \frac{k-1}{2}, \frac{1}{2}\rangle \overset{k=\ell+2}{=} |\frac{\ell+3}{2}; \frac{\ell+1}{2}, \frac{1}{2}\rangle \otimes |\frac{\ell+3}{2}; \frac{\ell+1}{2}, \frac{1}{2}\rangle\,, \tag{C.2}$$

that have spin $\Delta_L - \Delta_R = 0$, the S^3 degeneracy $(\ell+2)^2$, multiplicity $(2 \times \frac{1}{2} + 1)^2 = 4$, and $\Delta = \ell + 3$. Another example is the "mixed" scalar φ_- that corresponds to

$$|\frac{k}{2}; \frac{k}{2}, 0\rangle \otimes |\frac{k}{2}; \frac{k}{2}, 0\rangle \overset{k=\ell+2}{=} |\frac{\ell+2}{2}; \frac{\ell+2}{2}, 0\rangle \otimes |\frac{\ell+2}{2}; \frac{\ell+2}{2}, 0\rangle\,, \tag{C.3}$$

which has, indeed, the spin $\Delta_L - \Delta_R = 0$, the S^3 degeneracy $(\ell+3)^2$, the multiplicity $(2 \times 0 + 1)^2 = 1$, and $\Delta = \ell + 2$.

However, in the cases **Ia**, **Ib** and **II** the dimensions of states with values of ℓ given in table 3 (as found from the explicit mass spectrum of fluctuations) do not match those of the (2,0) tensor multiplet. This is due to the fact that in these cases the corresponding supergroups are not $SU(1,1|2) \times SU(1,1|2)$ but the ones given in table 2. Let us discuss the three cases **Ia**, **II** and **Ib** in turn.

Case Ia: here the relevant short-multiplet factor is not of $SU(1,1|2)$ but of $OSp(4^*|2)$ discussed in [73]. The bosonic subgroup is $SL(2,\mathbb{R}) \times SU(2) \times SU(2)_{\mathcal{R}}$ and fermions are doublets of $SU(2)_{\mathcal{R}}$ (we use that $SU(1,1) = SL(2,\mathbb{R})$). The corresponding multiplet is[38]

$$[\,n\,]^{OSp(4^*|2)} \quad = \quad \begin{array}{ccc} \hline j & j' & \Delta \\ \hline \frac{n}{2} & \frac{1}{2} & \frac{n}{2}+1 \\ \frac{n-1}{2} & 0 & \frac{n+1}{2} \\ \frac{n+1}{2} & 0 & \frac{n+3}{2} \\ \hline \end{array} \tag{C.4}$$

[37] Here we use sub-index $\mathcal{R}$ for R-symmetry to distinguish it from the R-index (for "right") below.

[38] See eq. (5.12) in [73] where we corrected a typo.

Taking the product of such two multiplets with $n = \ell + 1$, i.e. $[\ell + 1] \times [\ell + 1]$, one finds the same states as in the above (2,0) case but with some interchanged values of Δ, in agreement with the **Ia** entries in table 3. For example, to get φ_- we are to consider

$$|\frac{n+3}{2};\frac{n+1}{2},0\rangle \otimes |\frac{n+3}{2};\frac{n+1}{2},0\rangle \overset{n=\ell+1}{=} |\frac{\ell+4}{2};\frac{\ell+2}{2},0\rangle \otimes |\frac{\ell+4}{2};\frac{\ell+2}{2},0\rangle\,. \tag{C.5}$$

This state has the spin $\Delta_L - \Delta_R = 0$, the S^3 degeneracy $(\ell+3)^2$, the multiplicity $(2\times 0+1)^2 = 1$, and $\Delta \equiv \Delta_L + \Delta_R = \ell + 4$ as for the $\varphi_-^{(\ell+2)}$ entry in the case of **Ia** in table 3.

Case II: here the building block is the short multiplet of the real form $OSp(4|2, \mathbb{R})$ discussed in [74, 75]. The bosonic subgroup is $USp(2) \times SO(4^*)$. We have $USp(2) = SU(2)_{\mathcal{R}}$ and $SO(4^*) = SU(1, 1) \times SU(2) = SL(2, \mathbb{R}) \times SU(2)$, so we find again the symmetry $SL(2, \mathbb{R}) \times SU(2) \times SU(2)_{\mathcal{R}}$ with the supercharges being doublets of $SU(2)_{\mathcal{R}}$. The general unitary multiplet is presented in eq. (3.2) in [75].[39] The shortening condition is $2\Delta - j - j' = 0$. If we solve it by setting $\Delta = \ell+2$ and $j = 2\Delta$, $j' = 0$ we get the states of the following short multiplet

$$[\,\ell\,]^{OSp(4|2,\mathbb{R})} \quad = \quad \begin{array}{ccc} \hline j & j' & \Delta \\ \hline \frac{\ell+1}{2} & \frac{1}{2} & \frac{\ell}{4}+1 \\ \frac{1}{2}\ell+1 & 0 & \frac{\ell}{4}+\frac{1}{2} \\ \frac{1}{2}\ell & 0 & \frac{\ell}{4}+\frac{3}{2} \\ \hline \end{array} \tag{C.6}$$

One can check that the states in the corresponding product $[\ell] \times [\ell]$ all have the same quantum numbers as before, with the exception of the total Δ values which are in agreement with the case **II** entries in table 3. For example, $\varphi_-^{(\ell+2)}$ originates from

$$|\frac{\ell}{4}+\frac{1}{2};\frac{\ell}{2}+1,0\rangle \otimes |\frac{\ell}{4}+\frac{1}{2};\frac{\ell}{2}+1,0\rangle\,, \tag{C.7}$$

that has the spin $\Delta_L - \Delta_R = 0$, the S^3 degeneracy $[2(\frac{\ell}{2}+1)+1]^2 = (\ell+3)^2$, the multiplicity $(2 \times 0 + 1)^2 = 1$, and the total $\Delta = \frac{1}{2}\ell + 1$.

Case Ib: this case should be related by an analytic continuation to case **II** and the corresponding states should fit the tensor product of the two short multiplets of $OSp(4^*|2)$. To show this we may exploit the low-rank isomorphism $OSp(4^*|2) \simeq D(2, 1; c)$ with $c = -2$ or $-\frac{1}{2}$ (the two cases are equivalent). Shortened representations of $D(2, 1; c)$ play a role in determining the BPS spectrum of states in string theory in $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ (where c is a ratio of two S^3 radii) and were discussed, e.g., in [76, 77].[40] The multiplet in eq. (A.13) of [77] obeys the shortening condition

$$h_0 = \frac{1}{1+c}\,j^- + \frac{c}{1+c}\,j^+ \qquad \overset{c=-2}{\longrightarrow} \qquad \Delta = 2j - j'\,, \tag{C.8}$$

[39]The notation there is related to the present one as $(K, R, S) \equiv (\Delta, j, j')$.
[40]The relation of their notation to ours is $(h_0, j^+, j^-) \equiv (\Delta, j, j')$.

and has the following content

$$[j,j']^{D(2,1;-2)} \quad = \quad \begin{array}{ccc} \hline j & j' & \Delta \\ \hline j^+ & j^- & 2j^+ - j^- \\ j^+ - \frac{1}{2} & j^- - \frac{1}{2} & 2j^+ - j^- + \frac{1}{2} \\ j^+ - \frac{1}{2} & j^- + \frac{1}{2} & 2j^+ - j^- + \frac{1}{2} \\ j^+ + \frac{1}{2} & j^- - \frac{1}{2} & 2j^+ - j^- + \frac{1}{2} \\ j^+ - 1 & j^- & 2j^+ - j^- + 1 \\ j^+ & j^- - 1 & 2j^+ - j^- + 1 \\ j^+ & j^- & 2j^+ - j^- + 1 \\ j^+ - \frac{1}{2} & j^- - \frac{1}{2} & 2j^+ - j^- + \frac{3}{2} \\ \hline \end{array} . \tag{C.9}$$

The following tensor product of such multiplets $[\frac{\ell}{2}+1,0]\times[\frac{\ell}{2}+1,0]$ contains states with all the quantum numbers being the same as in the above cases, with the exception of the total values of Δ which indeed match the **Ib** entries in table 3. To illustrate this let us consider again the example of φ_- that originates from the tensor product of two copies of the first state in (C.9)

$$|2j^+ - j^-; j^+, j^-\rangle \otimes |2j^+ - j^-; j^+, j^-\rangle \overset{j^+=\frac{\ell}{2}+1,\, j^-=0}{=} |\ell+2; \frac{\ell}{2}+1, 0\rangle \otimes |\ell+2; \frac{\ell}{2}+1, 0\rangle \,. \tag{C.10}$$

It has the spin $\Delta_L - \Delta_R = 0$, the S^3 degeneracy $(\ell+3)^2$, the multiplicity $(2\times 0+1)^2 = 1$, and the total $\Delta = 2(\ell+2) = 2\ell+4$. The same agreement can be checked for all other states.

D Analytic continuation between Ib and II cases

The two 11d backgrounds $AdS_7 \times S^4$ in (2.6), (2.7) and $AdS_4 \times S^7$ in (2.37), (2.38) are related by an analytic continuation (see, e.g., [78]). This is a consequence of the relation between the metrics of S^d and AdS_d, up to an overall sign change and an interchange of the values of the radii.

This implies also a formal relation between the M5 brane solutions wrapped on $AdS_3 \subset AdS_7$ and $S^3 \subset S^4$ in $AdS_7 \times S^4$ case (probe **Ib**) and on $AdS_3 \subset AdS_4$ and $S^3 \subset S^7$ in the $AdS_4 \times S^7$ (probe **II**).

Scaling out the overall factors of L in (2.6) and L in (2.37) these two M5 brane $AdS_3 \times S^3$ cases have the effective metrics in (3.33) and (3.62) also related by interchanging the two factors and the radii L_A and L_S. This translates into a relation between the M5 brane fluctuation spectra discussed in sections 2.2 and 3.3.

The scalar operator $-\nabla_A^2 - \nabla_S^2 + M^2$ on $AdS_3 \times S^3$ is mapped by this analytic continuation into itself with $\nabla_A^2 \leftrightarrow -\nabla_S^2$ provided also $M^2 \to -M^2$. Equivalently, since its eigenvalues are given by

$$\mathcal{O}_2 = -\nabla_A^2 - \nabla_S^2 + M^2 \quad \to \quad -\Delta(\Delta-2)L_A^{-2} + \ell(\ell+2)L_S^{-2} + M^2 \,, \tag{D.1}$$

the transformation

$$L_A \leftrightarrow L_S \,, \qquad\qquad \Delta \leftrightarrow -\ell \,, \tag{D.2}$$

is a formal symmetry provided one also reverses the sign of the mass term.

In the case of the 4 scalar fluctuations ζ_p in (3.28), (3.32) in **Ib** case and in (3.57), (3.61) their masses are indeed related by $M^2 \to -M^2$ (this has to do with the reversed sign of the curvature of the "transverse" subspaces $\mathrm{AdS}_4 \subset \mathrm{AdS}_7$ in the **Ib** case and of $S^4 \subset S^7$ in the **II** case).

For the mixed scalars, the quartic operators in (3.46) (for the scalar Θ representing the transverse fluctuation in S^4 in the **Ib** case mixed with the scalar part P of B_{rs}) and (3.69) (for the scalar U representing the transverse fluctuation in AdS_4 in the **II** case mixed with the scalar P) are

$$\mathcal{O}_4^{(\mathbf{Ib})} = \nabla^4 - 24\nabla^2 + 36\nabla_S^2\,, \qquad \mathcal{O}_4^{(\mathbf{II})} = \nabla^4 - 12\nabla^2 + 36\nabla_S^2\,. \tag{D.3}$$

Here $\nabla^2 = \nabla_A^2 + \nabla_S^2$ so that like in (D.1) their $\mathrm{AdS}_3 \times S^3$ spectra may be represented as

$$\mathcal{O}_4^{(\mathbf{Ib})} \to \quad [\Delta(\Delta-2)L_A^{-2} - \ell(\ell+2)L_S^{-2}]^2 - 24\Delta(\Delta-2)L_A^{-2} - 12\ell(\ell+2)L_S^{-2}\,, \tag{D.4}$$

$$\mathcal{O}_4^{(\mathbf{II})} \to \quad [\Delta(\Delta-2)L_A^{-2} - \ell(\ell+2)L_S^{-2}]^2 - 12\Delta(\Delta-2)L_A^{-2} - 24\ell(\ell+2)L_S^{-2}\,. \tag{D.5}$$

Thus they are also related by the transformation (D.2).

Similar formal analytic continuation applies also to the fermionic Dirac operator on $\mathrm{AdS}_3 \times S^3$ which turns out to have the structure (see (A.21), (A.26))

$$i\nabla\!\!\!\!/_A + i\nabla\!\!\!\!/_S \;+ \frac{3}{2}\,(L_A^{-1} - L_S^{-1})\hat{\Gamma}\,, \qquad \hat{\Gamma}^2 = 1\,, \tag{D.6}$$

and thus is covariant under $\nabla\!\!\!\!/_A \leftrightarrow \nabla\!\!\!\!/_S$ and $L_A \leftrightarrow L_S$.

This formal correspondence between the spectra of fluctuations in the **Ib** and **II** cases does not, of course, imply that the values of the corresponding one-loop free energies given by the sums over the spectrum should be the same (cf. section 4).

E Comments on one-loop divergences

Using heat kernel regularization the UV divergent term of one-loop free energy in 6d has the following form

$$F_\infty^{(1)} = -\frac{1}{(4\pi)^3}\int d^6\xi\,\sqrt{g}\left(\frac{1}{6}b_0\Lambda^6 + \frac{1}{4}b_2\Lambda^4 + \frac{1}{2}b_4\Lambda^2 + b_6\log\Lambda\right), \qquad \Lambda\to\infty\,, \tag{E.1}$$

where the Seeley's coefficients b_{2n} are local expressions in terms of the curvature of the 6d metric, gauge connection and mass matrix (see, e.g., [58] and references there). In all the cases discussed in section 4 there were no logarithmic divergences. In this appendix we will discuss the structure of divergences and their cancellation in more detail.

Logarithmic divergences.

For a massive scalar with an operator $-\nabla^2 + M^2$ defined on $\mathrm{AdS}_3 \times S^3$ with radii L_A and L_S we find that

$$b_6^{(0)}(M^2) = -\frac{1}{6}\big(L_A^{-2} - L_S^{-2} + M^2\big)^3. \tag{E.2}$$

For the self-dual 2-form, taking half of the expression in eq. (2.26) in [58] we get

$$b_6^{(2+)} = -\frac{1}{6}\big(L_A^{-2} - L_S^{-2}\big)^3 \,. \tag{E.3}$$

In the cases **Ib** and **II** where there are mixed scalars we have also a "compensating" massless scalar contribution in the numerator of (3.47) and (3.69) that, according to (E.2), is given by

$$-b_6^{(0)}(0) = \frac{1}{6}\big(L_A^{-2} - L_S^{-2}\big)^3 \,. \tag{E.4}$$

It thus cancels against the one in (E.3).

From the above expressions one finds that in the **Ia** case where the bosonic part of the one-loop partition function is given by (3.15) with $L_A = L_S = 1$ the corresponding coefficient of the log divergence is $b_6^{(\mathbf{Ia})} = 0$. This can be easily understood on general grounds: here we have a massless (conformally-invariant) (2,0) multiplet defined on the equal-radii $\mathrm{AdS}_3 \times S^3$ space. This space is conformally-flat (has zero Weyl tensor) and also its 6d Euler density vanishes. This implies that its conformal anomaly vanishes.

In the cases **Ib** and **II** we need to account also for the contribution to b_6 of the 4th order operators in (3.47) and (3.69). These can be found using the expression for this Seeley coefficient for a general covariant 4-th order operator in 6d space given in [79, 80]. Indeed, given the operator on $\mathrm{AdS}_3 \times S^3$ of the same form as in (3.47), (3.69), i.e.

$$\mathcal{O}_4 = \nabla^4 + c_1 \nabla_A^2 + c_2 \nabla_S^2, \tag{E.5}$$

one may represent it in the 6d covariant form using that

$$c_1\nabla_A^2 + c_2\nabla_S^2 = \big(p_1 R^{ij} + p_2 R\, g^{ij}\big)\nabla_i\nabla_j\,, \qquad \begin{aligned} c_1 &= -2(p_1+3p_2)L_A^{-2} + 6p_2 L_S^{-2}\,,\\ c_2 &= 2(p_1+3p_2)L_S^{-2} - 6p_2 L_A^{-2}\,. \end{aligned}$$

One then finds that for the operator (E.5)

$$\begin{aligned} b_6^{(\mathcal{O}_4)} = &-\frac{1}{192}\big(4L_A^{-2} - 4L_S^{-2} - c_1 - c_2\big)\Big[16\big(L_A^{-2} - L_{-S}^{2}\big)^2 + 7(c_1^2 + c_2^2) + 2c_1c_2 \\ &+ 16c_1\big(L_A^{-2} + 2L_S^{-2}\big) - 16c_2\big(2L_A^{-2} + L_S^{-2}\big)\Big]\,. \end{aligned} \tag{E.6}$$

In the **Ib** case where $L_A = 1$, $L_S = \frac{1}{2}$ the 4 decoupled scalars and the 2-form plus massless scalar contributions to b_6 separately vanish. From (3.48) here the operator $\mathcal{O}_4$ has $(c_1, c_2) = (-24, 12)$ and as a result (E.6) vanishes too. Thus the total $b_6^{(\mathbf{Ib})} = 0$. The same conclusion is reached in the "analytically-continued" **II** case where $L_A = \frac{1}{2}$, $L_S = 1$ and the operator $\mathcal{O}_4$ has $(c_1, c_2) = (-12, 24)$, so that $b_6^{(\mathbf{II})} = 0$.

One can show that the fermionic contribution to the coefficient of the logarithmic divergences also vanishes separately in all the three cases. Since it is not straightforward to square the fermionic Dirac operator in (3.17) one may first expand in modes on S^3 and then see if the resulting sum of individual mode contributions (cf. (3.22)) to the free energy over will not have logarithmic divergence.

The vanishing of the coefficient of the logarithmic divergence can be understood as being a consequence of the associated conformal dimensions Δ being simply linear in the S^3 mode number ℓ. For example, for a scalar operator $-\nabla^2+M^2$ on $\mathrm{AdS}_3\times S^3$ with radii L_A, L_S we get for the dimension of the AdS_3 field representing a mode with fixed ℓ (cf. (3.23))

$$\Delta_\ell(M)=1+\sqrt{1+L_A^2\big[M^2+\ell(\ell+2)L_S^{-2}\big]}\,. \tag{E.7}$$

The corresponding contribution to the free energy is given by $F_0(\Delta_\ell)$ in (4.1) (including also the degeneracy factor $d_\ell^{(0)}=(\ell+1)^2$ in (3.19)), i.e. it is proportional to $(\Delta_\ell-1)^3=[1+L_A^2(M^2+\ell(\ell+2)L_S^{-2})]^{3/2}$. Its large ℓ expansion contains the term $\sim\big(L_A^{-2}-L_S^{-2}+M^2\big)^3\,\ell^{-1}$ that leads to the logarithmic divergence in the sum $\sum_\ell F_0(\Delta_\ell)$. This log divergence is absent if

$$M^2=L_S^{-2}-L_A^{-2}\,,\qquad\qquad \Delta_\ell=1+L_AL_S^{-1}(\ell+1)\,. \tag{E.8}$$

This relation is indeed what we have found in all the three cases in (3.24), (3.54), (3.75) where the resulting sum over ℓ contained only a quadratic divergence.

In general, one can check that the coefficient of the log divergence in the sum over ℓ is given by (ignoring here power divergencies)

$$\sum_\ell^\Lambda F_0(\Delta_\ell)=-B_6\log\Lambda+\ldots\,,\qquad B_6=\frac{1}{(4\pi)^3}\,\mathrm{vol}(\mathrm{AdS}_3\times S^3)\,b_6\,,\quad b_6=-\frac{1}{6}\big(L_A^{-2}-L_S^{-2}+M^2\big)^3\,. \tag{E.9}$$

This is in agreement with the direct 6d computation of b_6 in (E.1), (E.2) which was not using the expansion in modes on S^3.[41]

Similar conclusion is reached also in the case of the mixed scalar operator in (E.5), (E.6) where the special values of the coefficients c_1,c_2 for which the conformal dimensions are linear in ℓ (cf. (3.52), (3.54) and (3.73), (3.75)) imply also the vanishing of b_6 in (E.6).

This discussion applies also to a 6d fermionic field on $\mathrm{AdS}_3\times S^3$ with the squared Dirac operator taken in the form $-\nabla^2+M^2$ where ∇^2 acts on a 6d spinor. In this case the direct computation of the b_6 Seeley coefficient (as, e.g., in [58]) gives

$$b_6=-\frac{1}{24}\big(3L_A^{-2}-3L_S^{-2}+2M^2\big)\big(3L_A^{-2}L_S^{-2}-3M^2L_A^{-2}+3M^2L_S^{-2}-2M^4\big)\,. \tag{E.10}$$

Here we ignored the factor of the number of fermionic components. Considering first the expansion in spinor harmonics on S^3 we get $-\nabla^2=-\nabla_A^2-\nabla_S\to-\nabla_A^2+L_S^{-2}[(\ell+\frac{3}{2})^2-\frac{3}{2}]$ (with degeneracy $d_\ell^{(\frac{1}{2})}=(\ell+1)(\ell+2)$). Then the corresponding conformal dimension of the AdS_3 ℓ-mode is (cf. (3.25))

$$\Delta_\ell(M)=1+\sqrt{\frac{3}{2}+L_A^2\left(M^2+L_S^{-2}\left[\left(\ell+\frac{3}{2}\right)^2-\frac{3}{2}\right]\right)}\,. \tag{E.11}$$

Computing $\sum_\ell^\Lambda F_{\frac{1}{2}}(\Delta_\ell)$ using (4.1) (and accounting for the fermionic minus sign factor that was not included in (4.1)) we get again the same first two relations in (E.9) where b_6 now matches the expression in (E.10). We conclude that the log divergence is absent if

$$M^2=M_{\frac{1}{2}}^2\equiv\frac{3}{2}\big(L_S^{-2}-L_A^{-2}\big)\,. \tag{E.12}$$

[41] Note that such a direct agreement between the two procedures applies only to the coefficient of the universal log divergence; coefficients of power divergent terms are sensitive to a particular choice of a UV cutoff.

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This is also the condition when the scaling dimension (E.11) becomes linear in ℓ[42]

$$\Delta_\ell(M_{\frac{1}{2}}) = 1 + L_A L_S^{-1}\left(\ell + \frac{3}{2}\right). \tag{E.13}$$

This is indeed the expression we had (up to the m_{F} part) in (3.26), (3.53), (3.74).

Power divergences.

Let us now comment on the coefficients of power divergences in (E.1). In general, in a supersymmetric theory one finds $b_0 = 0$ due to balance of the numbers of the bosonic and fermionic degrees of freedom. The coefficient b_2 for an operator like $-\nabla^2 + M^2$ is given by $\mathrm{tr}(\frac{1}{6}R - M^2)$. In the **Ia** case or for a massless (2,0) tensor multiplet on $\mathrm{AdS}_3 \times S^3$ it also vanishes due to an effective supersymmetric mass sum rule. Then the only potentially non-zero divergences are the quadratic ones controlled by the b_4 coefficient and they do not vanish in general. In the case **Ia** we may directly compute b_4 as for the (2,0) tensor multiplet on $\mathrm{AdS}_3 \times S^3$ from its known expression for the 2nd order 6d Laplacian (see eq. (3.64) in [11])

$$b_4^{(2,0)} = \frac{1}{4}R_{mnk\ell}R^{mnk\ell} - \frac{1}{2}R_{mn}R^{mn} + \frac{1}{10}R^2\,. \tag{E.14}$$

Specifying to the equal-radii $\mathrm{AdS}_3 \times S^3$ that gives $b_4^{(2,0)} = -6$, i.e. there should be a non-zero quadratic divergence.[43] This is consistent with the presence of a quadratic divergence in the sum over ℓ in (4.11), (4.12).

To analyse the quartic and quadratic divergences in the cases **Ib** and **II** we need to find the mixed-scalar quartic operator (E.5) contributions to them. An important point to emphasize is that the standard heat kernel cutoff does not regularize power divergences for the operators of different orders (e.g. 2nd and 4th) in a homogeneous manner. As a result, the corresponding Seeley coefficients cannot be simply combined together to find the total values of the coefficients of the power divergences.

A way to circumvent this problem is by evaluating the contributions of the 2nd order operators by the same algorithm as for the quartic ones by first squaring them. Then all the operators will have the same order and their contributions to divergences can be combined together directly. Thus, for the standard scalar operator $-\nabla^2 + M^2$ we may define the divergence coefficients in terms of those of its square as

$$b_n^{(0)}(-\nabla^2 + M^2) \equiv \frac{1}{2}b_n(\nabla^4 - 2M^2\nabla^2 + M^4)\,, \tag{E.15}$$

and similarly for the vector and 2-form Laplacians. Here b_n for the 4th-order operator is to be computed according to the prescription in [81]. In general, this will give the same value of b_6 coefficient due to the universal nature of the logarithmic divergence. From (E.15) we

[42] To be precise, the observation is that whenever Δ_ℓ is polynomial in ℓ we have no log divergence in the sum over ℓ, i.e. vanishing b_6. The converse need not be true. Indeed, from (E.10) one can see that there are other values of M^2 such that $b_6 = 0$. For these values Δ is not polynomial in ℓ, but the large ℓ expansion of F_ℓ still happens to have a vanishing coefficient of the ℓ^{-1} term.

[43] The same conclusion was reached in the case of the (2,0) multiplet defined on $\mathrm{AdS}_5 \times S^1$ [11].

find using the expressions for b_2 and b_4 for the quartic operator in [81][44]

$$b_2^{(0)} = -\frac{1}{4}(M^2 + L_A^{-2} - L_S^{-2})\,, \qquad b_4^{(0)} = \frac{\sqrt{\pi}}{8}(M^2 + L_A^{-2} - L_S^{-2})^2\,. \tag{E.16}$$

Similarly, for the self-dual 2-form contribution we get the same b_6 as in (E.3) and also

$$b_2^{(2+)} = \frac{3}{4}\left(L_A^{-2} - L_S^{-2}\right), \qquad b_4^{(2+)} = \frac{\sqrt{\pi}}{8}\left(-18L_A^{-2}L_S^{-2} + L_A^{-4} + L_S^{-4}\right)\,. \tag{E.17}$$

For the quartic operator in (E.5) we find in addition to b_6 in (E.6)

$$b_2^{(\mathcal{O}_4)} = \frac{1}{8}\left(c_1 + c_2 - 4L_A^{-2} + 4L_S^{-2}\right)\,, \tag{E.18}$$

$$b_4^{(\mathcal{O}_4)} = \frac{\sqrt{\pi}}{512}\left[15c_1^2 + 18c_1c_2 + 15c_2^2 + 32(3c_1 + c_2)L_S^{-2} - 32(c_1 + 3c_2)L_A^{-2} + 128\left(L_A^{-2} - L_S^{-2}\right)^2\right]. \tag{E.19}$$

Using these values we conclude that the bosonic contributions to the coefficients of the quartic and quadratic divergences in cases **Ia**, **Ib** and **II** are given by

$$b_2^{(\mathbf{Ia})} = 0, \qquad b_2^{(\mathbf{Ib})} = -3, \qquad b_2^{(\mathbf{II})} = 3, \tag{E.20}$$

$$b_4^{(\mathbf{Ia})} = -2\sqrt{\pi}\,, \qquad b_4^{(\mathbf{Ib})} = -\frac{337}{32}\sqrt{\pi}, \qquad b_4^{(\mathbf{II})} = -\frac{337}{32}\sqrt{\pi}. \tag{E.21}$$

The non-zero bosonic contributions to the quartic divergence coefficient b_2 in the **Ib** and **II** cases should be cancelled against the fermionic contributions, while the total values of b_4 should be non-zero for consistency with the quadratic divergence in the sum in (4.12).

F Spectra and decompositions of Laplacians on p-forms

Let us recall the spectrum of the Hodge-de Rham (HdR) operator Δ_p on co-exact p-forms on a unit sphere S^d, i.e. of the corresponding Laplacian on transverse antisymmetric tensors. For $p > 0$ the eigenvalues and degeneracies are given by (see, e.g., [82])

$$\lambda_\ell^{(p)} = (\ell + p + 1)(\ell + d - p)\,, \qquad d_\ell^{(p)} = \frac{(\ell + d)!(2\ell + d + 1)}{p!\ell!(d - p - 1)!(\ell + d - p)(\ell + p + 1)}\,, \qquad \ell = 0, 1, 2, \ldots\,. \tag{F.1}$$

In particular, in the scalar case (including the contribution of the zero mode)

$$\lambda_\ell^{(0)} = \ell(\ell + d - 1)\,, \qquad d_\ell^{(0)} = \frac{(\ell + d - 2)!\,(2\ell + d - 1)}{(d - 1)!\,\ell!}\,. \tag{F.2}$$

From here we may also find the spectrum of the part $(-\nabla^2)_{p\perp}$ of the HdR operator that does not include extra curvature terms. For a vector we have

$$V_i(\mathbf{\Delta}_1)^i_j V^i = V_i\big[-\nabla^2 + (d - 1)\big]V^i\,, \tag{F.3}$$

[44] In this case we observe a proportionality between the standard b_2 and b_4 for a 2nd order operator and their values defined via (E.15). This is due to the simplicity of the scalar operator, and is not true in general, cf. [81]. Note that unfamiliar $\sqrt{\pi}$ coefficient in b_4 is due to its definition for a 4th-order operator that was used in [81].

so that the corresponding eigenvalues of $(-\nabla^2)_1$ are $(\ell+2)(\ell+d-1)-(d-1)$. For a 2-form

$$V_{ij}(\boldsymbol{\Delta}_2)^{ij}_{rs}V^{rs} = V^{ij}\big(-\nabla^2\delta^{ij}_{rs} + 2R^{[r}_{[i}\delta^{s]}_{j]} - R_{ij}{}^{rs}\big)V_{rs} = V_{ij}\big[-\nabla^2 + 2(d-2)\big]V^{ij}\,, \qquad \text{(F.4)}$$

and thus the eigenvalues of $(-\nabla^2)_2$ are $(\ell+3)(\ell+d-2)-2(d-2)$. In case of S^3 we then have

Spectrum of $(-\nabla^2)_{p\perp}$ on S^3		
p	$\lambda^{(p)}_\ell$	$d^{(p)}_\ell$
0	$\ell(\ell+2)$	$(\ell+1)^2$
1	$(\ell+2)^2-2$	$2(\ell+1)(\ell+3)$
2	$(\ell+1)(\ell+3)-2$	$(\ell+2)^2$

(F.5)

Let us also recall some standard relations for the determinants $\det\Delta_p$ and $\det\Delta_{p\perp}$ defined on transverse tensors (see, e.g., [83] and references there). Let us consider AdS$_d$ ($\varepsilon=-1$) or S^d ($\varepsilon=1$) with radius L, i.e. with

$$R_{ijrs} = \frac{\varepsilon}{L^2}(g_{ir}g_{js} - g_{is}g_{jr})\,, \qquad R_{ij} = \frac{\varepsilon}{L^2}(d-1)g_{ij}\,, \qquad R = \frac{\varepsilon}{L^2}d(d-1)\,, \qquad \text{(F.6)}$$

and the operator

$$\boldsymbol{\Delta}_p = -\nabla^2_p + M^2, \qquad p = 0,1,2\,, \qquad \text{(F.7)}$$

acting on scalars, vectors or antisymmetric 2-tensors. For a vector we set $V_i = V_{i\perp} + \nabla_i V$ and then

$$V_i(-\nabla^2+M^2)V^i = V_{i\perp}(-\nabla^2+M^2)V^i_\perp + V(-\nabla^2)\Big[-\nabla^2 + M^2 - \frac{\varepsilon(d-1)}{L^2}\Big]V\,, \qquad \text{(F.8)}$$

$$\det\boldsymbol{\Delta}_1(M^2) = \det\boldsymbol{\Delta}_{1\perp}(M^2)\ \det\boldsymbol{\Delta}_0\Big(M^2 - \varepsilon\frac{d-1}{L^2}\Big)\,, \qquad \text{(F.9)}$$

where $\det(-\nabla^2)^{1/2}$ is cancelled against the Jacobian of the transformation $V_i \to (V_{i\perp}, V)$.[45]

Similarly, in the 2-form case setting $V_{ij} = V_{ij\perp} + \nabla_i V_{j\perp} - \nabla_j V_{i\perp}$ leads to

$$V_{ij}V^{ij} = V_{ij\perp}V^{ij\perp} + 2V_{i\perp}\left[-\nabla^2 + \frac{\varepsilon(d-1)}{L^2}\right]V^{i\perp} \qquad \text{(F.10)}$$

and thus the associated Jacobian is $J_2 = \big[\det\boldsymbol{\Delta}_{1\perp}(\varepsilon\frac{d-1}{L^2})\big]^{1/2}$. Also, we have

$$\begin{aligned} V_{ij}(-\nabla^2+M^2)V^{ij} = {}& V_{ij\perp}(-\nabla^2+M^2)V^{ij}_\perp + 2\nabla_i V_{j\perp}(-\nabla^2+M^2)\nabla^i V^j_\perp \\ & - 2\nabla_j V_{i\perp}(-\nabla^2+M^2)\nabla^i V^j_\perp. \end{aligned} \qquad \text{(F.11)}$$

Integrating by parts and commuting the covariant derivatives the last two terms may be written as

$$\begin{aligned} & -2V_{j\perp}\nabla_i(-\nabla^2+M^2)\nabla^i V^j_\perp + 2V_{i\perp}\nabla_j(-\nabla^2+M^2)\nabla^i V^j_\perp \\ &= 2V_{i\perp}\bigg[\Big(-\nabla^2 + \frac{\varepsilon(d-1)}{L^2}\Big)\Big(-\nabla^2 + M^2 - \frac{\varepsilon(d-3)}{L^2}\Big)\bigg]V^i_\perp \end{aligned}$$

[45]To recall, $1 = \int DV_i e^{-\int V_iV^i} = \int DV_{i\perp}DV\ J_1\ e^{-\int[V_{i\perp}V^{i\perp} + V(-\nabla^2)V]}$ gives $J_1 = \det(-\nabla^2)^{1/2}$.

so that we get

$$\det \mathbf{\Delta}_2(M^2) = \det \mathbf{\Delta}_{2\perp}(M^2)\ \det \mathbf{\Delta}_{1\perp}\left(M^2 - \varepsilon\frac{d-3}{L^2}\right), \tag{F.12}$$

where again one factor was canceled against J_2. In the special cases of AdS$_3$ and S^3 this gives

$$\det \mathbf{\Delta}_1(M^2) = \det \mathbf{\Delta}_{1\perp}(M^2) \det \mathbf{\Delta}_0 \left(M^2 + 2L_A^{-2}\right), \quad \det \mathbf{\Delta}_2(M^2) = \det \mathbf{\Delta}_{2\perp}(M^2), \tag{F.13}$$

$$\det \mathbf{\Delta}_1(M^2) = \det \mathbf{\Delta}_{1\perp}(M^2) \det \mathbf{\Delta}_0 \left(M^2 - 2L_S^{-2}\right), \quad \det \mathbf{\Delta}_2(M^2) = \det \mathbf{\Delta}_{2\perp}(M^2). \tag{F.14}$$

G Casimir energy and "thermal" partition function of AdS$_3$ multiplets

In section 4 we focused on the free energy for the multiplets of fluctuations corresponding to the cases **Ia, Ib, II** defined on AdS$_3$ with S^2 boundary. Given a collection of AdS$_3$ fields corresponding to (Δ, s) representation of $SO(2,2)$ we can similarly compute their Casimir energy (see, e.g., eq. (F.2) in [84])[46]

$$E_c(\Delta, s) = \frac{1}{24}(-1)^{2s}(\Delta-1)^2\big[2(\Delta-1)^2 - 1\big]\,. \tag{G.1}$$

Using the data in table 3 we then get

$$E_c^{(\mathbf{Ia})} = E_c^{(\mathbf{Ib})} = \frac{3}{4}(\ell+2)\,, \qquad\qquad E_c^{(\mathbf{II})} = 0. \tag{G.2}$$

One may also consider the 2d central charge associated to a single "chiral" component of a spin s field in 3d which is given by (see, e.g., eq. (F.3) in [84])

$$\mathrm{c}(\Delta, s) = -\frac{1}{2}(\Delta-1)\big[(\Delta-1)^2 - 3s^2\big]\,. \tag{G.3}$$

This is to be taken with minus sign in the fermionic case. c is directly related to the free energy in (4.1), i.e.

$$F_s(\Delta) = \frac{1}{6\pi}\mathrm{c}(\Delta, s)\,\mathrm{vol}(\mathrm{AdS}_3)\,. \tag{G.4}$$

Note that in this case of AdS$_3$ the central charge c in (G.4) is formally the same as the b-anomaly coefficient in (1.9). One also finds that after summing over the contributions of states in a AdS$_3$ supermultiplet E_c and c are related by

$$E_c = -\frac{1}{12}\mathrm{c}\,. \tag{G.5}$$

One may also compute the value of the "thermal" single particle partition function or, equivalently, of the character of a $\Delta > |s|$ massive $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ representation associated to a field in AdS$_3$ having dual-field conformal dimension Δ and spin s. It is

[46] Here $s = 1$ corresponds to a self-dual tensor. This expression contains a factor of $-\frac{1}{2}$ compared to eq. (F.2) in [84] to get the Casimir energy of the AdS$_3$ fields with Dirichlet boundary conditions. This genuinely AdS$_3$ quantity was denoted as E_c^+ in [84] (the same applies to the related c-anomaly mentioned below).

given by $\mathrm{Tr}\, q^{L_0+\overline{L}_0} = \frac{q^\Delta}{(1-q)^2}$ (here $q = e^{-\beta} < 1$). Summing over the over fields in a fixed-ℓ supermultiplets in table 3 including the S^3 degeneracy we may consider

$$\mathcal{Z}_\ell(q) = \sum_{\text{multiplet}} (-1)^{\mathrm{F}} \mathrm{d}_{S^3} \frac{q^\Delta}{(1-q)^2} . \tag{G.6}$$

Then computing (G.6) for the 4 multiplets in table 3 we find

$$\mathcal{Z}_\ell^{(\mathbf{2,0})}(q) = \frac{q^{2+\ell}}{(1-q)^2}\Big[3 + \ell(1-\sqrt{q})^2 - 4\sqrt{q} + q\Big]^2, \tag{G.7}$$

$$\mathcal{Z}_\ell^{(\mathbf{Ia})}(q) = \frac{q^{2+\ell}}{(1-q)^2}\Big[1 + \ell(1-\sqrt{q})^2 - 4\sqrt{q} + 3q\Big]^2, \tag{G.8}$$

$$\mathcal{Z}_\ell^{(\mathbf{Ib})}(q) = \frac{q^{4+2\ell}}{(1-q)^2}\Big[3 + \ell(1-\sqrt{q})^2 - 4\sqrt{q} + q\Big]^2, \tag{G.9}$$

$$\mathcal{Z}_\ell^{(\mathbf{II})}(q) = \frac{q^{1+\frac{1}{2}\ell}}{(1-q)^2}\Big[3 + \ell(1-\sqrt{q})^2 - 4\sqrt{q} + q\Big]^2. \tag{G.10}$$

One observes the following curious relations (they follow also directly from the relations between Δ's in the supermultiplets)

$$\mathcal{Z}_\ell^{(\mathbf{2,0})}(q) = q^{1+\frac{1}{2}\ell}\, \mathcal{Z}_\ell^{(\mathbf{II})}(q) = q^{-2-\ell}\, \mathcal{Z}_\ell^{(\mathbf{Ib})}(q) , \qquad \mathcal{Z}_\ell^{(\mathbf{2,0})}(q) = q^{4+2\ell}\, \mathcal{Z}_\ell^{(\mathbf{Ia})}(q^{-1}) , \tag{G.11}$$

$$[\mathcal{Z}_\ell^{(\mathbf{II})}(q)]^2 = \mathcal{Z}_\ell^{(\mathbf{Ia})}(q^{-1})\ \mathcal{Z}_\ell^{(\mathbf{Ib})}(q) . \tag{G.12}$$

We note that the sums over the whole S^3 tower of modes

$$\mathcal{Z}(q) = \sum_{\ell=0}^{\infty} \mathcal{Z}_\ell(q) , \qquad\qquad q = e^{-\beta} , \tag{G.13}$$

are well-defined, i.e. are finite for $q < 1$. The total Casimir energy can be extracted from the small β expansion of (G.13) (see, e.g., appendix B in [85])

$$\mathcal{Z}(e^{-\beta}) = C_1\beta^{-1} + C_2 - 2E_c\,\beta + \mathcal{O}(\beta^2). \tag{G.14}$$

We find

$$\mathcal{Z}^{(\mathbf{2,0})}(e^{-\beta}) = \frac{13}{8}\beta^{-1} - \frac{3}{2} + \frac{101}{192}\beta + \cdots , \qquad \mathcal{Z}^{(\mathbf{Ia})}(e^{-\beta}) = \frac{5}{8}\beta^{-1} - \frac{3}{2} + \frac{313}{192}\beta + \cdots , \tag{G.15}$$

$$\mathcal{Z}^{(\mathbf{Ib})}(e^{-\beta}) = \frac{41}{64}\beta^{-1} - \frac{3}{2} + \frac{2491}{1536}\beta + \cdots , \qquad \mathcal{Z}^{(\mathbf{II})}(e^{-\beta}) = 5\beta^{-1} - \frac{3}{2} - \frac{1}{12}\beta + \cdots , \tag{G.16}$$

that gives

$$E_c^{(\mathbf{2,0})} = -\frac{101}{384} , \qquad E_c^{(\mathbf{Ia})} = -\frac{313}{384} , \qquad E_c^{(\mathbf{Ib})} = -\frac{2491}{3072} , \qquad E_c^{(\mathbf{II})} = \frac{1}{24} . \tag{G.17}$$

The value for $E_c^{(\mathbf{2,0})}$ is the same as in eq. (5.16) in [84]. However, the values in the other 3 cases do not appear to be compatible with (G.2) summed over ℓ. That may be related to the fact that the procedure based on using (G.13) may not be manifestly consistent with underlying supersymmetry.[47]

[47] Also, first introducing a finite β, then performing sum over ℓ (which is well defined for finite β) and then taking the limit $\beta \to 0$ may not be equivalent to first expanding in β and then summing over ℓ.

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Let us note also that in all four cases the values of E_c are reproduced by the procedure described in section 5.2 of [84]. Namely, writing $\frac{q^\Delta}{(1-q)^2} = \sum_{n=0}^{\infty}(n+1)q^{\Delta+n}$ and summing over zero-point energies $\frac{1}{2}(n+1)(\Delta+n)$ with a factor of d_{S^3} implies that one may compute E_c from the finite part of the small ε expansion of the following sum

$$E_c = \frac{1}{2}\sum_{n,\,\ell=0}^{\infty}\sum_{\text{multiplet}} (-1)^{\mathrm{F}}\,\mathrm{d}_{S^3}\;(n+1)\big[\Delta(\ell)+n\big]\,e^{-\varepsilon[\Delta(\ell)+n]}\Big|_{\text{finite part}}\,. \tag{G.18}$$

The simplicity of (G.2) has a counterpart here: if we fix n, ℓ and set $\varepsilon=0$ we find

$$\sum_{\text{multiplet}} (-1)^{\mathrm{F}}\mathrm{d}_{S^3}\;(n+1)\big[\Delta(\ell)+n\big] = 0\,, \tag{G.19}$$

in all four cases. This relation follows also from the sum rules in (4.2).

H Wilson loop in 5d SYM and d_2 defect anomaly

As discussed in the Introduction, the free energy F for the M5 brane probes **Ia** and **Ib** with AdS_3 boundary being $S^1_\beta \times S^1$ which may be related as in (1.10) to the d_2 defect anomaly coefficient of the dual (2,0) theory may be interpreted as $F(\beta) = -\log \mathrm{W}$ where $\mathrm{W} = \langle W\rangle$ is the expectation value of supersymmetric circular Wilson loop in the 5d SYM theory related to (2,0) theory on $S^1_\beta \times \mathbb{R}^5$. W itself may be computed exactly by localization methods [86] using Chern-Simons matrix model [87].

Here we will review and elaborate on the large N expansion of W in the symmetric or antisymmetric representation of $SU(N)$. In particular, we will show that in the large $N, \beta \gg 1$ limit the expression for the Wilson loop agrees, up to exponentially small corrections, with its saddle-point evaluation in [25], i.e. the latter does not receive $1/N^p$ corrections. This explains an apparent puzzle of why that saddle-point calculation reproduced the exact values [1–3] of the defect anomaly d_2 coefficients in (1.11).

Wilson loop in general representation.

Following [88], we start with the exact expression for the partition function of level k $U(N)$ CS theory on S^3 (here a_n are real eigenvalues of a matrix $\mathbf{a}$)

$$Z_N = \frac{1}{N!}\int \prod_{n=1}^{N} da_n\, e^{-ik\pi a_n^2} \prod_{n\neq m}^{N} 2\sinh\big[\pi(a_n-a_m)\big] \tag{H.1}$$

$$= (-1)^{\frac{1}{2}N(N-1)}\; e^{-\frac{1}{4}i\pi N^2}\; e^{-\frac{i\pi}{6k}N(N^2-1)}\; k^{-\frac{1}{2}N} \prod_{n=1}^{N-1}\Big(2\sin\frac{\pi n}{k}\Big)^{N-n}\,. \tag{H.2}$$

The expectation value of the circular Wilson loop in fundamental representation is obtained by inserting $\mathrm{Tr}(e^{2\pi\mathbf{a}}) = \sum_{n=1}^{N} e^{2\pi a_n}$ under the integral in (H.1) which leads to [88]

$$\mathrm{W}_N = e^{-\frac{i\pi N}{k}}\,\frac{\sin\frac{\pi N}{k}}{\sin\frac{\pi}{k}}\,. \tag{H.3}$$

Upon the analytic continuation

$$\frac{2\pi}{k} \;\;\to\;\; i\beta \tag{H.4}$$

this gives, for large N,

$$\mathrm{W}_N(\beta) = e^{\frac{N\beta}{2}} \frac{\sinh\frac{N\beta}{2}}{\sinh\frac{\beta}{2}} = \frac{1}{2\sinh\frac{\beta}{2}}\, e^{N\beta} + \mathcal{O}(1). \tag{H.5}$$

More generally, for the Wilson loop defined by a matrix **a** in a representation R of $U(N)$ the result can be found in [89] (ignoring the finite renormalization $k \to k+N$ which is absent in supersymmetric case)

$$\mathrm{W}_N^{\mathsf{R}} = \exp\left(\frac{\beta}{2} C_{\mathsf{R}}\right) s_R(x_1,\dots,x_N), \qquad x_n = \exp\left[-\frac{\beta}{2}(N-2n+1)\right]. \tag{H.6}$$

Here s_{R} is the Schur polynomial associated with the Young tableau of R represented by the partition $(\ell_1, \ell_2, \dots)$ with $\ell_i \geq \ell_{i+1}$, $i \geq 1$.[48] The factor $\exp(\frac{\beta}{2}C_{\mathrm{R}})$ is (the analytic continuation of) the framing phase [89] where

$$C_{\mathsf{R}} = (N+1)|\mathsf{R}| + \sum_r (\ell_r^2 - 2r\ell_r), \qquad |\mathsf{R}| = \sum_r \ell_r. \tag{H.7}$$

In particular, for the fundamental representation $s_{(1)}(x_1,\dots,x_N) = \sum_{n=1}^N x_n$, $C_{(1)} = N$, and thus

$$\mathrm{W}_N^{(1)} = e^{\frac{\beta N}{2}} \sum_{n=1}^N \exp\left[-\frac{\beta}{2}(N-2n+1)\right] = e^{\frac{N\beta}{2}} \frac{\sinh\frac{N\beta}{2}}{\sinh\frac{\beta}{2}}, \tag{H.8}$$

in agreement with (H.3)–(H.5).

Case of symmetric and antisymmetric representations of $U(N)$.

The generating functions and dimensions of the rank-p symmetric/antisymmetric representations $[p]_\pm$ are

$$\sum_{p=0}^\infty s_{[p]_\pm}(x_1,\dots,x_N) s^p = \prod_{n=1}^N (1 \mp s x_n)^{\mp 1}, \tag{H.9}$$

$$\dim[p]_\pm = s_{[p]_\pm}(1,\dots,1) = (1\mp s)^{\mp N}\big|_{s^p} = (\mp 1)^p \binom{\mp N}{p}, \qquad \dim[p]_- = \binom{N}{p},$$

$$\dim[p]_+ = \binom{N+p-1}{p}.$$

As in the Introduction we will use also the notation

$$[p]_+ = (p), \qquad [p]_- = [p]. \tag{H.10}$$

As is well known,

$$s_{(p)}(x_1,\dots,x_N) = \sum_{i_1 \leq i_2 \leq \dots \leq i_p} x_{i_1} \cdots x_{i_p}, \qquad s_{[p]}(x_1,\dots,x_N) = \sum_{i_1 < i_2 < \dots < i_p} x_{i_1} \cdots x_{i_p}. \tag{H.11}$$

[48] The Schur polynomial $s_{\mathsf{R}}(x_1,\dots,x_N)$ is simply the character of R, i.e. $\mathrm{ch}_{\mathsf{R}}[\mathbf{a}] = s_{\mathsf{R}}(x_1,\dots,x_N)\big|_{x_n = e^{a_n}}$.

Following [90] one can also represent $s_{(p)}$ as

$$s_{(p)}(x_1,\dots,x_N)=\sum_{n=1}^{N}x_n^p\prod_{m\neq n}\left(1-\frac{x_m}{x_n}\right)^{-1}, \tag{H.12}$$

and then (H.6) gives

$$\mathrm{W}_N^{(p)}(\beta)=e^{\frac{\beta}{2}C_{(p)}}\sum_{n=1}^{N}\exp\left[-\frac{\beta p}{2}(N-2n+1)\right]\prod_{m\neq n}\frac{1}{1-e^{\beta(m-n)}}. \tag{H.13}$$

From (H.7) we get

$$\begin{aligned} C_{(p)}&=(N+1)p+p^2-2p=Np+p(p-1)\,,\\ C_{[p]}&=(N+1)p+\sum_{r=1}^{p}(1-2r)=(N+1)p-p^2\,. \end{aligned} \tag{H.14}$$

We can obtain an exact representation of $\mathrm{W}_N^{(p)}$ and $\mathrm{W}_N^{[p]}$ starting from the identities[49]

$$\sum_{1\le i_1\le i_2\le\cdots\le i_p\le N}q^{i_1+\cdots+i_p}=\prod_{n=1}^{p}\frac{q-q^{N+n}}{1-q^n}\,,\qquad \sum_{1\le i_1<i_2<\cdots<i_p\le N}q^{i_1+\cdots+i_p}=\prod_{n=1}^{p}\frac{q^n-q^{N+1}}{1-q^n}\,. \tag{H.15}$$

We then get

$$\mathrm{W}_N^{(p)}=q^{-Np-\frac{1}{2}p(p-2)}\prod_{n=1}^{p}\frac{1-q^{N+n-1}}{1-q^n}\,,\qquad \mathrm{W}_N^{[p]}=q^{-Np-\frac{1}{2}p(1-2p)}\prod_{n=1}^{p}\frac{1-q^{N-n+1}}{1-q^n}\,,\qquad q\equiv e^{-\beta}\,. \tag{H.16}$$

Expanded at large N these give

$$\mathrm{W}_N^{(p)}=q^{-Np-\frac{1}{2}p(p-2)}\prod_{n=1}^{p}\frac{1}{1-q^n}\left[1-\frac{1-q^p}{1-q}q^N+\mathcal{O}(q^{2N})\right], \tag{H.17}$$

$$\mathrm{W}_N^{[p]}=q^{-Np-\frac{1}{2}p(1-2p)}\prod_{n=1}^{p}\frac{1}{1-q^n}\left[1-q^{1-p}\frac{1-q^p}{1-q}q^N+\mathcal{O}(q^{2N})\right], \tag{H.18}$$

where $q^N=e^{-\beta N}$, etc., terms in the square brackets represent exponentially suppressed corrections. Taking also the large β ($q\to 0$) limit we find the following simple expressions

$$\mathrm{W}_N^{(p)}\overset{N,\beta\gg 1}{=}q^{-Np-\frac{1}{2}p(p-2)}+\cdots\,,\qquad \mathrm{W}_N^{[p]}\overset{N,\beta\gg 1}{=}q^{-Np-\frac{1}{2}p(1-2p)}+\cdots\,, \tag{H.19}$$

where dots stand for corrections that are exponentially suppressed in both N and β.

[49]They can be proved using the generating function in (H.9). For instance, in symmetric representation case, we need to prove $\prod_{n=1}^{N}\frac{1}{1-sq^n}\Big|_{s^p}=\prod_{n=1}^{p}\frac{q-q^{N+n}}{1-q^n}$. In terms of the q-Pochhammer symbol $(a,q)_n=\prod_{k=0}^{n-1}(1-aq^k)$ and using the q-binomial theorem we have $\prod_{n=1}^{N}\frac{1}{1-sq^n}=\frac{1}{(sq,q)_N}=\frac{(sq^{N+1},q)_\infty}{(sq,q)_\infty}=\sum_{p=0}^{\infty}\frac{(q^N,q)_p}{(q,q)_p}(sq)^p$. Hence $\prod_{n=1}^{N}\frac{1}{1-sq^n}\Big|_{s^p}=q^p\frac{(q^N,q)_p}{(q,q)_p}=q^p\prod_{n=0}^{p-1}\frac{1-q^Nq^n}{1-q^{n+1}}=q^p\prod_{n=1}^{p}\frac{1-q^Nq^{n-1}}{1-q^n}=\prod_{n=1}^{p}\frac{q-q^{N+n}}{1-q^n}$ as in the first relation in (H.15). A similar procedure proves the second relation for the antisymmetric representation case.

$SU(N)$ case and relation to the d_2 anomaly coefficient.

It is easy to derive a simple relation between the Wilson loop in the $U(N)$ gauge theory and the one in the $SU(N)$ theory. We can split the matrix integration into the trace and traceless parts as follows (a_n are eigenvalues of $\mathbf{a}$)

$$\underbrace{\int \prod_n da_n}_{U(N)} \mathcal{F}(\mathbf{a}) = \int dt \prod_n da_n\, \delta(\sum_n a_n - t)\ \mathcal{F}(\mathbf{a}) = \underbrace{\int dt}_{U(1)} \underbrace{\int \prod_n da_n\, \delta(\sum_n a_n)}_{SU(N)} \mathcal{F}(\mathbf{a} + tN^{-1})\,, \tag{H.20}$$

where we redefined $a_n \to a_n + tN^{-1}$. This translation does not change the functions of $a_n - a_m$ in the integrand like in (H.1). Thus we have only to account for a change in the Gaussian factors in (H.1) and in $\mathrm{Tr}(e^{2\pi \mathbf{a}}) = \sum_{n=1}^N e^{2\pi a_n}$ in W[50]

$$e^{-\frac{2\pi^2}{\beta}\sum_n (a_n + \frac{t}{N})^2} \chi_R(e^{2\pi(a_n+\frac{t}{N})}) = e^{-\frac{2\pi^2}{\beta}\frac{t^2}{N}} e^{2\pi \frac{t}{N}|R|}\, e^{-\frac{2\pi^2}{\beta}\sum_n a_n^2} \chi_R(e^{2\pi a_n})\,, \tag{H.21}$$

where we have used the traceless condition $\sum_n a_n = 0$ in the $SU(N)$ case. This implies

$$\frac{\mathrm{W}^{\mathsf{R}}_{U(N)}}{\mathrm{W}^{\mathsf{R}}_{SU(N)}} = \frac{\int dt \exp\big(-\frac{2\pi^2}{\beta}\frac{t^2}{N} + 2\pi \frac{t}{N}|R|\big)}{\int dt \exp\big(-\frac{2\pi^2}{\beta}\frac{t^2}{N}\big)} = e^{\frac{\beta}{2N}|R|^2} = q^{-\frac{1}{2N}|R|^2}\,. \tag{H.22}$$

For the symmetric representation $\mathsf{R} = (p)$ one has $|\mathsf{R}| = p$ and thus, up to the exponential corrections,

$$\mathrm{W}^{(p)}_{SU(N)} \overset{N,\beta\gg 1}{=} q^{-Np-\frac{1}{2}p(p-2)+\frac{p^2}{2N}} = q^{-\frac{1}{12}\mathrm{d}_{2(p)}}\,, \qquad \mathrm{d}_{2\,(p)} = Np\left(1+\frac{p}{2N}\right)\left(1-\frac{1}{N}\right), \tag{H.23}$$

where $\mathrm{d}_{2(p)}$ is thus the same as in (1.11). Similarly, for $\mathsf{R} = [p] = (1,\dots,1)$ we have again $|\mathsf{R}| = p$ and

$$\mathrm{W}^{[p]}_{SU(N)} \overset{N,\beta\gg 1}{=} q^{-Np-\frac{1}{2}(1-2p)+\frac{p^2}{2N}} = q^{-\frac{1}{12}\mathrm{d}_{2[p]}}\,, \qquad \mathrm{d}_{2\,[p]} = Np\left(1+\frac{1}{2N}\right)\left(1-\frac{p}{N}\right). \tag{H.24}$$

These are also the same expressions as found in [25] using a saddle-point evaluation of the matrix model integral. This proves that in the $N, \beta \gg 1$ limit these expressions are actually exact if one ignores the exponentially small corrections.

Defect anomaly coefficients b and d_2 for the (n, m) representation of $SU(N)$.

For completeness, let us record also the derivation of the explicit form of b and d_2 for the more general (n, m) representation. For an $SU(N)$ representation with the Young tableaux corresponding to a partition $P = (\ell_1, \ell_2, \dots)$ one has [1]

$$\mathrm{b} = 24(\rho,\lambda) + 3(\lambda,\lambda)\,, \qquad \mathrm{d}_2 = 24(\rho,\lambda) + 6(\lambda,\lambda)\,, \tag{H.25}$$

[50] From the bi-alternant formula of Jacobi for Schur polynomials one has $\chi_R(sx_1,\dots,sx_n) = s^{|R|}\chi_R(x_1,\dots,x_n)$.

where $\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_n = \ell_n - \ell_{n+1}$ are the Dynkin labels and ρ is the Weyl vector so that

$$(\rho, \lambda) = \frac{1}{2} \sum_{q \geq 1} (N-q) q \lambda_q \, , \qquad (\lambda, \lambda) = \frac{1}{N} \sum_{q \geq 1} (N-q) \lambda_q \Big(-q\lambda_q + 2 \sum_{r=1}^{q} r \lambda_r \Big) . \tag{H.26}$$

For a representation with n rows and m columns we have $P = (m, m, \dots, m, 0, 0, \dots)$ with n entries equal to m. Hence, $\lambda = (0, 0, \dots, 0, m, 0, 0, \dots)$, i.e. we get a single non-zero component $\lambda_n = m$. Then (H.25) implies that

$$\mathrm{b} = 12(N-n)nm + \frac{3}{N}(N-n)m(-mn+2mn) = 12N\, nm \left(1 - \frac{n}{N}\right)\left(1 + \frac{m}{4N}\right), \tag{H.27}$$

$$\mathrm{d}_2 = 12(N-n)nm + \frac{6}{N}(N-n)m(-mn+2mn) = 12N\, nm \left(1 - \frac{n}{N}\right)\left(1 + \frac{m}{2N}\right). \tag{H.28}$$

In particular, for the special cases of (n, m) representations which are the symmetric $(1, k)$ and the antisymmetric $(k, 1)$ ones we reproduce the expressions used in (1.4), (1.5) and (1.11), i.e.

$$\mathrm{b}_{(k)} = 12N\, k \left(1 - \frac{1}{N}\right)\left(1 + \frac{k}{4N}\right), \qquad \mathrm{d}_{2\,(k)} = 12N\, k \left(1 - \frac{1}{N}\right)\left(1 + \frac{k}{2N}\right), \tag{H.29}$$

$$\mathrm{b}_{[k]} = 12N\, k \left(1 - \frac{k}{N}\right)\left(1 + \frac{1}{4N}\right), \qquad \mathrm{d}_{2\,[k]} = 12N\, k \left(1 - \frac{k}{N}\right)\left(1 + \frac{1}{2N}\right). \tag{H.30}$$

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