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Korovkin closures in cones of set-valued functions

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ABSTRACT

The aim of the paper is to establish a complete characterization of the Korovkin closures in cones of set-valued functions. Korovkin closures play a central role in Korovkin approximation theory and their complete characterization is an old problem which has been considered since the beginning of the Korovkin theory's development. However, until now this characterization is well-known only in spaces of continuous single real-valued functions. In order to give a more detailed description of the Korovkin closures, it is natural to introduce the notions of onesided external and internal Korovkin closures by separating two different roles of the approximating functions in a Korovkin system. Contrary to what one might expect and differently from the single-valued case, only the role of the external Korovkin closures will be decisive in the characterization of the Korovkin systems and this will allow us to give also a more precise description together with different examples of particular interest. Further characterizations are given in the cone of set-valued functions having real compact intervals as values. In this case the study of the Korovkin closure is deepened by introducing the upper and lower Korovkin closures.

1. Introduction and notation

In this paper we study in detail the Korovkin closure in cones of set-valued functions. This continues the study of the Korovkin approximation of set-valued functions which has become a theme of increasing interest in the last years.

Indeed, Korovkin approximation in cones of set-valued functions begins after the papers by Keimel and Roth Keimel-Roth [16,17].

Since then, many Korovkin-type results have been obtained both in cones of set-valued functions and in spaces of vector-valued functions, thanks to the close relationship between the two topics.

In Campiti [6,7], the notion of convexity monotone operator has been used in order to extend different characterizations of Korovkin subcones and subspaces. A more recent and detailed analysis has been performed in Campiti [8] even in the case where the limit operator is not assigned.

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Recently, the analysis of Korovkin approximation in cones of Hausdorff continuous set-valued functions has been continued through the papers Campiti [9,10,11,12].

In this paper we are interested in studying in detail the Korovkin closure in cones of Hausdorff continuous set-valued functions.

This is a classical problem in Korovkin approximation theory. Indeed, the study of Korovkin closures is essential in Korovkin approximation theory, since it allows us not only to characterize Korovkin systems, but also to obtain the largest subspace where we can ensure the convergence of suitable nets of operators.

For this reason, Korovkin closures have been considered since the beginning of the development of Korovkin approximation theory.

In spaces of continuous real-valued functions it was first studied in Baskakov [3], Bauer [4] and Berens-Lorent [5] (see also the Notes and References to Section 4.1 in Altomare-Campiti [2] and Chapter 6 in Donner [15]).

More recently, the Korovkin closure has been also studied in Banach algebras Altomare [1] and Lindenstrauss spaces Dieckmann [14].

However, again in spaces of continuous real-valued functions, in Campiti [13], it has been given a more detailed description of the Korovkin closure introducing the so-called upper and lower Korovkin closures and distinguishing their different roles.

On the contrary, in cones of Hausdorff continuous set-valued functions, we have not an explicit characterization of the Korovkin closure.

In this paper we are able to completely characterize the Korovkin closure with respect to monotone operators and we introduce the external and internal Korovkin closure. However, their role will be different from the single-valued case since only the external Korovkin closure will play a key role in the characterization of the Korovkin closure.

Moreover, in the case of set-valued functions with values in the cone of compact real intervals, we can give further detailed descriptions, by distinguishing the upper and lower Korovkin closures according to the single-valued case considered in Campiti [13].

First, we need some notation.

Let $\mathbf{B}$ denote the closed unit ball in $\mathbb{R}^d$ and let $\mathcal{K}(\mathbb{R}^d)$ be the cone of all non empty compact convex subsets of $\mathbb{R}^d$ endowed with the natural addition and multiplication by positive scalars. In $\mathcal{K}(\mathbb{R}^d)$ it is also defined the Hausdorff distance

$$d_H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}.$$

If X is a Hausdorff topological space, we shall deal with the cone $C(X,\mathcal{K}(\mathbb{R}^d))$ of all continuous set-valued functions; hence, $f \in C(X,\mathcal{K}(\mathbb{R}^d))$ if and only if for every $x_0 \in X$ and $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $f(x) \subset f(x_0) + \varepsilon\cdot\mathbf{B}$ and $f(x_0) \subset f(x) + \varepsilon\cdot\mathbf{B}$ whenever $x \in U$.

The cone $C(X,\mathcal{K}(\mathbb{R}^d))$ is naturally ordered by inclusion, that is

$$f \le g \;\Leftrightarrow\; \forall x \in X:\; f(x) \subset g(x),$$

whenever $f, g \in C(X,\mathcal{K}(\mathbb{R}^d))$.

Moreover, if $(f_i)^{\le}_{i\in I}$ is a net in $C(X,\mathcal{K}(\mathbb{R}^d))$ and if $f \in C(X,\mathcal{K}(\mathbb{R}^d))$, we shall write $\lim_{i\in I^{\le}} f_i = f$ if, for every $\varepsilon > 0$, there exists $\alpha \in I$ such that, for every $i \in I$, $i \ge \alpha$, and for every $x \in X$, we have

$$f(x) \subset f_i(x) + \varepsilon\cdot\mathbf{B}, \qquad f_i(x) \subset f(x) + \varepsilon\cdot\mathbf{B}.$$

An operator $L : C(X,\mathcal{K}(\mathbb{R}^d)) \to C(X,\mathcal{K}(\mathbb{R}^d))$ is called *linear* if preserves addition and multiplication by positive scalars and is called *monotone* if it preserves inclusions.

If $\varphi \in C(X, \mathbb{R}^d)$, it will be useful to denote by $\{\varphi\}$ the set-valued function (in $C(X, \mathcal{K}(\mathbb{R}^d))$) defined by setting $\{\varphi\}(x) = \{\varphi(x)\}$ for every $x \in X$.

Moreover, if $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, we shall denote by $\mathrm{Sel}(f)$ the convex subset of $C(X, \mathcal{K}(\mathbb{R}^d))$ consisting of all continuous selections of f.

It is well-known that (see e.g. Lemma 4.1. and Theorem 3.2 in Keimel-Roth [16])

$$f = \bigcup_{\varphi \in \mathrm{Sel}(f)} \{\varphi\} \tag{1}$$

for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, that is $f(x) = \bigcup_{\varphi \in \mathrm{Sel}(f)} \{\varphi(x)\}$ for every $x \in X$.

2. Korovkin closures in cones of continuous set-valued functions

We begin with the following definition.

Let $H \subset C(X, \mathcal{K}(\mathbb{R}^d))$; the *Korovkin closure* of H in $C(X, \mathcal{K}(\mathbb{R}^d))$ for the identity operator with respect to equicontinuous nets of monotone (respectively, monotone linear) operators is defined as the following subset of $C(X, \mathcal{K}(\mathbb{R}^d))$:

$$\begin{aligned} K(H) := \{f \in C(X, \mathcal{K}(\mathbb{R}^d)) \mid \; & \lim_{i \in I^{\le}} L_i(f) = f \textit{ for every equicontinuous net} \\ & (L_i)_{i \in I}^{\le} \textit{ of monotone (respectively, monotone} \\ & \textit{linear) operators from } C(X, \mathcal{K}(\mathbb{R}^d)) \textit{ into itself} \\ & \textit{satisfying } \lim_{i \in I^{\le}} L_i(h) = h \textit{ for every } h \in H\}\,. \end{aligned}$$

Moreover H is said to be a *Korovkin system* in $C(X, \mathcal{K}(\mathbb{R}^d))$ with respect to monotone (respectively, monotone linear) operators for the identity operator if $K(H) = C(X, \mathcal{K}(\mathbb{R}^d))$.

In the single-valued case a characterization of the Korovkin closure is stated in Theorem 4.1.4 in Altomare-Campiti [2]. In spaces of continuous real valued functions, a more detailed analysis has been performed in Campiti [13].

However, in cones of set-valued functions, we have only some sufficient conditions which ensure that $f \in K(H)$.

On the contrary, in the next result we state a complete characterization.

We recall that a subset H of $C(X, \mathcal{K}(\mathbb{R}^d))$ is said to be *cofinal* if, for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, there exists $k \in H$ such that $f \le k$ (i.e., for every $x \in X$, $f(x) \subset h(x)$).

Moreover, H is said to be *downward cofinal* if, for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, there exists $h \in H$ such that $h \le f$.

A downward cofinal subset necessarily contains the single-valued functions. This condition is not too restrictive since Korovkin systems in $C(X, \mathcal{K}(\mathbb{R}^d))$ are often obtained using Korovkin systems in $C(X, \mathbb{R}^d)$ which ensure the convergence on single-valued functions.

We have the following characterization of downward cofinal subsets of $C(X, \mathcal{K}(\mathbb{R}^d))$.

Proposition 1. *Let $H \subset C(X, \mathcal{K}(\mathbb{R}^d))$. Then, the following statements are equivalent:*

a) *H is downward cofinal.*
b) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, we have $f = \bigcup_{h \in H, h \le f} h$.*
c) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, $x_0 \in X$ and $\varepsilon > 0$, there exist $h_1, \dots, h_s \in H$ such that $h_r \le f$ for every $r = 1, \dots, s$ and*

$$f(x_0) \subset \bigcup_{r=1}^{s} h_r(x_0) + \varepsilon \cdot \boldsymbol{B}.$$

d) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $\varepsilon > 0$, there exist $h_1, \ldots, h_m \in H$ such that $h_j \leq f$ for every $j = 1, \ldots, m$ and*

$$f \subset \bigcup_{j=1}^{m} h_j + \varepsilon \cdot \boldsymbol{B}.$$

Proof. *a)* $\Rightarrow$ *b)* Let $f \in C(X, \mathcal{K}(\mathbb{R}^d))$. Since H is downward cofinal, every continuous selection of f is in H and hence condition b) follows from (1).

b) $\Rightarrow$ *c)* Let $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, $x_0 \in X$ and $\varepsilon > 0$. We observe that

$$f(x_0) = \bigcup_{h \in H, h \leq f} h(x_0) \subset \bigcup_{h \in H, h \leq f} h(x_0) + \varepsilon \cdot \mathring{\mathbf{B}},$$

where $\mathring{\mathbf{B}}$ denotes the open unit ball in $\mathbb{R}^d$. Since $f(x_0)$ is compact there exists a finite number of functions $h_1, \ldots, h_s \in H$ such that $h_r \leq f$ for every $r = 1, \ldots, s$ and satisfying

$$f(x_0) \subset \bigcup_{r=1}^{s} h_r(x_0) + \varepsilon \cdot \mathring{\mathbf{B}} \subset \bigcup_{r=1}^{s} h_r(x_0) + \varepsilon \cdot \mathbf{B}.$$

This completes the proof of c).

c) $\Rightarrow$ *d)* It follows from a straightforward argument on the compactness of X.

d) $\Rightarrow$ *a)* It is obvious. □

Now, we can state the main characterization of the Korovkin closure.

Theorem 1. *Let X be a compact Hausdorff topological space and let H be a subset of $C(X, \mathcal{K}(\mathbb{R}^d))$ which is both cofinal and downward cofinal.*

If $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, the following statements are equivalent:

a) $f \in K(H)$.

b) $f = \bigcap_{k \in H, f \leq k} k$.

c) *For every $x_0 \in X$ and $\varepsilon > 0$, there exist $k_1, \ldots, k_s \in H$ such that $f \leq k_r$ for every $r = 1, \ldots, s$ and*

$$\bigcap_{r=1}^{s} k_r(x_0) \subset f(x_0) + \varepsilon \cdot \boldsymbol{B}.$$

d) *For every $\varepsilon > 0$, there exist $k_1, \ldots, k_m \in H$ such that $f \leq k_j$ for every $j = 1, \ldots, m$ and*

$$\bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \boldsymbol{B}.$$

e) *For every $\varepsilon > 0$, there exist $h_1, \ldots, h_m, k_1, \ldots, k_m \in H$ such that $h_j \leq f \leq k_j$ for every $j = 1, \ldots, m$ and*

$$f \subset \bigcup_{j=1}^{m} h_j + \varepsilon \cdot \boldsymbol{B}, \qquad \bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \boldsymbol{B}.$$

Proof. *a)* $\Rightarrow$ *b)* Let $f \in K(H)$. By contradiction, assume that there exists $x_0 \in X$ such that

$$f(x_0) \subsetneq \bigcap_{k \in H, f \le k} k(x_0)$$

and consider the operator $L : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d))$ defined by setting, for every $g \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $x \in X$,

$$L(g)(x) := \bigcap_{k \in H, g \le k} k(x) \ .$$

Since H is cofinal, the operator is well-defined and obviously $L(g)(x) \in \mathcal{K}(\mathbb{R}^d)$ for every $x \in X$ and $L(g) \in C(X, \mathcal{K}(\mathbb{R}^d))$. Moreover, L is a monotone operator since the condition $g_1 \le g_2$ with $g_1, g_2 \in C(X, \mathcal{K}(\mathbb{R}^d))$ implies $\{k \in H \mid g_2 \le k\} \subset \{k \in H \mid g_1 \le k\}$.

For every $k \in H$, we have $L(k) = k$ but $L(f)(x_0) \ne f(x_0)$. Hence, we can consider a net $(L_i)_{i \in I}^{\le}$ of constant value $L_i = L$ for every $i \in I$ and we obtain that $(L_i(k))_{i \in I}^{\le}$ converges to k for every $k \in H$ but $(L_i(f))_{i \in I}^{\le}$ does not converge to f and this contradicts a).

b) $\Rightarrow$ *c)* Let $x_0 \in X$ and $\varepsilon > 0$. From b), we have

$$\bigcap_{k \in H, f \le k} k(x_0) \subset f(x_0) + \frac{\varepsilon}{3} \mathbf{B} \ .$$

The subset $C := (f(x_0) + \varepsilon \cdot \mathbf{B})) \setminus (f(x_0) + \frac{2}{3} \varepsilon \cdot \mathring{\mathbf{B}}))$, where $\mathring{\mathbf{B}}$ denotes the open unit ball in $\mathbb{R}^d$, is a compact subset of $\mathbb{R}^d$ and if put, for every $k \in H$ satisfying $f \le k$, $C_k := C \cap k(x_0)$, we have that every C_k is a closed subset of C and

$$\bigcap_{k \in H, f \le k} C_k = \emptyset \, .$$

Hence, it follows that there exists a finite number of functions $k_1, \ldots, k_r \in H$ such that $f \le k_r$ for every $r = 1, \ldots, s$ and

$$\bigcap_{r=1}^{s} C_{k_r} = \emptyset \, .$$

This yields

$$\bigcap_{r=1}^{s} k_r(x_0) \subset f(x_0) + \frac{2}{3} \varepsilon \cdot \mathbf{B} \subset f(x_0) + \varepsilon \cdot \mathbf{B}$$

and this completes the proof of c).

c) $\Rightarrow$ *d)* It follows from a straightforward argument on the compactness of X.

d) $\Rightarrow$ *e)* It is enough to take into account that H is downward cofinal and Proposition 1, d).

e) $\Rightarrow$ *a)* Let $(L_i)_{i \in I}^{\le}$ be an equicontinuous net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}^d))$ into itself satisfying $\lim_{i \in I^{\le}} L_i(h) = h$ for every $h \in H$. In order to show that $\lim_{i \in I^{\le}} L_i(f) = f$ we fix $\varepsilon > 0$ and consider $h_1, \ldots, h_m, k_1, \ldots, k_m \in H$ such that $h_j \le f \le k_j$ for every $j = 1, \ldots, m$ and

$$f \subset \bigcup_{j=1}^{m} h_j + \varepsilon \cdot \mathbf{B}\,, \qquad \bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \mathbf{B}\,.$$

Since $\lim_{i\in I^{\le}} L_i(h_j) = h_j$ and $\lim_{i\in I^{\le}} L_i(k_j) = k_j$ for every $j = 1, \dots, m$, there exists $\alpha \in I$ such that, for every $i \ge \alpha$, $x \in X$ and $j = 1, \dots, m$,

$$\begin{aligned} L_i(h_j)(x) &\subset h_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,, \qquad & h_j(x) &\subset L_i(h_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,. \\ L_i(k_j)(x) &\subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,, \qquad & k_j(x) &\subset L_i(k_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,. \end{aligned}$$

It follows, for every $i \ge \alpha$, $x \in X$ and $j = 1, \dots, m$,

$$L_i(f)(x) \subset L_i(k_j)(x) \subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}$$

and consequently

$$L_i(f)(x) \subset \bigcap_{j=1}^{m} k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset f(x) + \varepsilon \cdot \mathbf{B}\,.$$

On the other hand, for every $i \ge \alpha$, $x \in X$ and $j = 1, \dots, m$,

$$h_j(x) \subset L_i(f)(x) \subset L_i(h_j)(x) \subset h_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}$$

and consequently

$$f(x) \subset \bigcup_{j=1}^{m} h_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset \bigcup_{j=1}^{m} L_i(h_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset L_i(f)(x) + \varepsilon \cdot \mathbf{B}$$

and this completes the proof. □

The downward cofinality assumption on H may be avoided in some cases where we are interested only in sufficient conditions which ensure that H is a Korovkin system. Some examples in this direction have been obtained in Theorem 2.4 in Campiti [6] (see also Theorem 2.3 in Campiti [9] and Theorem 2.1 in Campiti [10]).

However, using the preceding Theorem 1, we are also able to give a characterization of Korovkin systems for the identity operator which is even more general than the results obtained in Campiti [6,9,10] and where it is not assumed that H is downward cofinal.

Theorem 2. *Let X be a compact Hausdorff topological space and let H be a cofinal subset of $C(X, \mathcal{K}(\mathbb{R}^d))$. Then, the following statements are equivalent:*

a) *H is a Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$ for the identity operator with respect to equicontinuous nets of monotone operators.*
b) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, we have $f = \bigcap_{k\in H, f\le k} k$.*
c) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, $x_0 \in X$ and $\varepsilon > 0$, there exist $k_1, \dots, k_s \in H$ such that $f \le k_r$ for every $r = 1, \dots, s$ and*

$$\bigcap_{r=1}^{s} k_r(x_0) \subset f(x_0) + \varepsilon \cdot \boldsymbol{B}\,.$$

d) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $\varepsilon > 0$, there exist $k_1, \ldots, k_m \in H$ such that $f \le k_j$ for every $j = 1, \ldots, m$ and*

$$\bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \boldsymbol{B}.$$

Proof. The implications $a) \Rightarrow b)$, $b) \Rightarrow c)$ and $c) \Rightarrow d)$ can be shown using exactly the same arguments of the corresponding implications of Theorem 1 applied to every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$.

Hence we have only to show that $d) \Rightarrow a)$. To this end, let $(L_i)_{i \in I}^{\le}$ be an equicontinuous net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}^d))$ into itself satisfying $\lim_{i \in I^{\le}} L_i(h) = h$ for every $h \in H$.

Let $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $\varepsilon > 0$. First consider the case where $f = \{\varphi\}$ with $\varphi \in C(X, \mathbb{R}^d)$. From d) we can consider $k_1, \ldots, k_m \in H$ such that $f \le k_j$ for every $j = 1, \ldots, m$ and

$$\bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \mathbf{B}.$$

Since $\lim_{i \in I^{\le}} L_i(k_j) = k_j$ for every $j = 1, \ldots, m$, there exists $\alpha \in I$ such that, for every $i \ge \alpha$, $x \in X$ and $j = 1, \ldots, m$,

$$L_i(k_j)(x) \subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,, \qquad k_j(x) \subset L_i(k_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,.$$

In particular, for every $i \ge \alpha$, $x \in X$ and $j = 1, \ldots, m$,

$$L_i(f)(x) \subset L_i(k_j)(x) \subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}$$

and consequently

$$L_i(f)(x) \subset \bigcap_{j=1}^{m} k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset f(x) + \varepsilon \cdot \mathbf{B}\,.$$

Since f is single-valued, the proof is complete in this case.

Now, consider the general case. For every $x_0 \in X$ and $y_0 \in f(x_0)$, there exists a continuous selection $\varphi_{x_0,y_0} \in C(X, \mathbb{R}^d)$ of f such that $\varphi_{x_0,y_0}(x_0) = y_0$. Hence

$$f(x) \subset \bigcup_{x_0 \in X, y_0 \in f(x_0)} \{\varphi_{x_0,y_0}(x)\} + \varepsilon \cdot \mathbf{B}$$

and since the graph of f is compact in $X \times \mathbb{R}^d$, there exist a finite number of functions $\varphi_1, \ldots, \varphi_m \in C(X, \mathbb{R}^d)$ such that

$$f(x) \subset \bigcup_{j=1}^{m} \{\varphi_j(x)\} + \varepsilon \cdot \mathbf{B}\,.$$

From the first part we have $\lim_{i \in I^{\le}} L_i(\varphi_j) = \varphi_j$ for every $j = 1, \ldots, m$ and therefore, condition d) of Theorem 1 is satisfied and this yields the proof. $\square$

The preceding result furnishes a sufficient condition on H in order to be a Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$ and suggests also a further characterization of functions in $K(H)$ using only conditions b), c) or d) in the preceding Theorem 2 and assuming that the cofinal subset H contains the single-valued functions.

More generally, we may assume that H contains the single-valued functions in a Korovkin system in $C(X, \mathbb{R}^d)$ for the identity operator with respect to equicontinuous nets of operators; in this case, we may require any property to the equicontinuous net since it is important only that the Korovkin system ensures the converges for every single-valued continuous function. In Campiti [6,7,8,9,10] we have considered for example convexity monotone operators.

As a consequence, we may state different examples of Korovkin systems in $C(X, \mathcal{K}(\mathbb{R}^d))$.

For example, if $X \subset \mathbb{R}^p$, from Corollary 1.13 in Campiti [6] we have the following result.

Corollary 1. *Let X be a compact subset of $\mathbb{R}^p$ and let H be a cofinal subset of $C(X, \mathcal{K}(\mathbb{R}^d))$. Let $\Gamma \subset C(X, \mathbb{R})$ be defined as follows*

$$\Gamma := \left\{ \boldsymbol{1}, \mathrm{pr}_1, \dots, \mathrm{pr}_p, \sum_{q=1}^{p} \mathrm{pr}_q^2 \right\}$$

where pr_q denotes the canonical q-projection on $\mathbb{R}^p$ and assume that the subset

$$\Lambda := \big\{ \{\varphi\} \mid \varphi \in C(X, \mathbb{R}^d),\ \exists\, \psi \in \Gamma\,,\ \exists\, q = 1, \dots, p\ : \\ \mathrm{pr}_q \circ \varphi = \psi\,, \mathrm{pr}_r \circ \varphi = 0\,, r = 1, \dots, p, r \neq q \big\}\,,$$

of $C(X, \mathcal{K}(\mathbb{R}^d))$ is contained in H.

If $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, then the following statements are equivalent:

a) *$f \in K(H)$.*
b) *$f = \bigcap_{k \in H, f \le k} k$.*
c) *For every $x_0 \in X$ and $\varepsilon > 0$, there exist $k_1, \dots, k_s \in H$ such that $f \le k_r$ for every $r = 1, \dots, s$ and*

$$\bigcap_{r=1}^{s} k_r(x_0) \subset f(x_0) + \varepsilon \cdot \boldsymbol{B}\,.$$

d) *For every $\varepsilon > 0$, there exist $k_1, \dots, k_m \in H$ such that $f \le k_j$ for every $j = 1, \dots, m$ and*

$$\bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \boldsymbol{B}\,.$$

In order to analyze in more details the characterization of the Korovkin closure, it may be interesting to consider separately the role of the functions h_j from that of the functions k_j in conditions e) of Theorem 1.

This conducts to the following concepts of Korovkin closures which are different from the onesided Korovkin closure considered in the single-valued case in Campiti [13].

If $(L_i)_{i \in I}^{\le}$ is a net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}^d))$ into itself and if $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, then we can consider, for every $x \in X$,

$$\liminf_{i \in I^{\le}} L_i(f)(x) := \bigcup_{\alpha \in I} \bigcap_{i \in I, i \ge \alpha} L_i(f)(x)\,,$$

$$\limsup_{i \in I^{\le}} L_i(f)(x) := \bigcap_{\alpha \in I} \bigcup_{i \in I, i \ge \alpha} L_i(f)(x)\,.$$

Of course, we can also consider the set-valued functions $\liminf_{i \in I^{\le}} L_i(f)$ and $\limsup_{i \in I^{\le}} L_i(f)$ defined on $C(X, \mathcal{K}(\mathbb{R}^d))$ as above.

We can now give the following definition.

Definition 1. Let X be a compact Hausdorff topological space and let $H \subset C(X, \mathcal{K}(\mathbb{R}^d))$. Then, the *external Korovkin closure* (respectively, the *internal Korovkin closure*) of H in $C(X, \mathcal{K}(\mathbb{R}^d))$ with respect to monotone operators for the identity operator is the subset $K(H)_e$ (respectively, $K(H)_i$) defined as follows

$$\begin{aligned} K(H)_e := \{f \in C(X, \mathcal{K}(\mathbb{R}^d)) \mid\ & \limsup_{i \in I^{\leq}} L_i(f) \leq f \textit{ for every equicontinuous net} \\ & (L_i)_{i \in I}^{\leq} \textit{ of monotone operators from } C(X, \mathcal{K}(\mathbb{R}^d)) \\ & \textit{into itself satisfying } \lim_{i \in I^{\leq}} L_i(h) = h \textit{ for every} \\ & h \in H\}\,. \end{aligned}$$

(respectively,

$$\begin{aligned} K(H)_i := \{f \in C(X, \mathcal{K}(\mathbb{R}^d)) \mid\ & f \leq \liminf_{i \in I^{\leq}} L_i(f) \textit{ for every equicontinuous net} \\ & (L_i)_{i \in I}^{\leq} \textit{ of monotone operators from } C(X, \mathcal{K}(\mathbb{R}^d)) \\ & \textit{into itself satisfying } \lim_{i \in I^{\leq}} L_i(h) = h \textit{ for every} \\ & h \in H\}\,)\,. \end{aligned}$$

Moreover, H is said to be an *external Korovkin system* (respectively, an *internal Korovkin system*) in $C(X, \mathcal{K}(\mathbb{R}^d))$ with respect to monotone operators for the identity operator if $K(H)_e = C(X, \mathcal{K}(\mathbb{R}^d))$ (respectively, $K(H)_i = C(X, \mathcal{K}(\mathbb{R}^d))$.

In the following proposition, we justify the above definition of external and internal Korovkin closure pointing out their relation with the Korovkin closure.

Proposition 2. *Let X be a compact Hausdorff topological space and let $H \subset C(X, \mathcal{K}(\mathbb{R}^d))$.*
Then

$$K(H) = K(H)_e \cap K(H)_i\,.$$

Proof. The inclusion $K(H) \subset K(H)_e \cap K(H)_i$ is obvious.

Conversely, we take $f \in K(H)_e \cap K(H)_i$ and show that $f \in K(H)$.

Let $(L_i)_{i \in I}^{\leq}$ be an equicontinuous net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}^d))$ into itself satisfying $\lim_{i \in I^{\leq}} L_i(h) = h$ for every $h \in H$. We have, for every $x \in X$,

$$\bigcap_{\alpha \in I} \bigcup_{i \in I, i \geq \alpha} L_i(f)(x) \subset f(x) \subset \bigcup_{\alpha \in I} \bigcap_{i \in I, i \geq \alpha} L_i(f)(x)\,.$$

Let $\varepsilon > 0$ and $x_0 \in X$. From the first inclusion, there exists $\alpha_1 \in I$ such that

$$\bigcup_{i \in I, i \geq \alpha} L_i(f)(x_0) \subset f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B}$$

and consequently, for every $i \in I$, $i \geq \alpha_1$, $L_i(f)(x_0) \subset f(x_0) + \frac{\varepsilon}{2} \cdot \mathbf{B}$.

Similarly, from the second inclusion, there exists α_2 *inI* such that

$$f(x_0) \subset \bigcap_{i\in I, i\geq\alpha} L_i(f)(x_0) + \frac{\varepsilon}{2}\cdot \mathbf{B}$$

and consequently, for every $i \in I$, $i \geq \alpha_2$, $f(x_0) \subset L_i(f)(x_0) + \frac{\varepsilon}{2}\cdot \mathbf{B}$.

Now, if $\alpha \in I$ satisfies $\alpha \geq \alpha_1$ and $\alpha \geq \alpha_2$, we have, for every $i \in I$, $i \geq \alpha$,

$$L_i(f)(x_0) \subset f(x_0) + \frac{\varepsilon}{2}\cdot \mathbf{B}\,, \quad f(x_0) \subset L_i(f)(x_0) + \frac{\varepsilon}{2}\cdot \mathbf{B}\,.$$

At this point, a straightforward argument based on the compactness of X and the equicontinuity of the net $(L_i)_{i\in I}^{\leq}$ and the arbitrarily of $\varepsilon > 0$ yields the proof. □

As regards to the internal Korovkin closure, we can easily characterize the case where it is an internal Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$ with respect to monotone operators for the identity operator.

Indeed, using Proposition 1, we can show that this happens exactly when H is downward cofinal.

Proposition 3. *Let H be a closed subset of $C(X, \mathcal{K}(\mathbb{R}^d))$. Then, the following statements are equivalent:*

a) *H is an internal Korovkin system in $C(X, \mathcal{K}(\mathbb{R}^d))$ with respect to monotone operators for the identity operator.*
b) *H is downward cofinal.*
c) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, we have $f = \bigcup_{h\in H, h\leq f} h$.*
d) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, $x_0 \in X$ and $\varepsilon > 0$, there exist $h_1, \ldots, h_s \in H$ such that $h_r \leq f$ for every $r = 1, \ldots, s$ and*

$$f(x_0) \subset \bigcup_{r=1}^{s} h_r(x_0) + \varepsilon \cdot \boldsymbol{B}\,.$$

e) *For every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $\varepsilon > 0$, there exist $h_1, \ldots, h_m \in H$ such that $h_j \leq f$ for every $j = 1, \ldots, m$ and*

$$f \subset \bigcup_{j=1}^{m} h_j + \varepsilon \cdot \boldsymbol{B}\,.$$

Proof. The equivalence among conditions b)-e) follows from Proposition 1. Hence, we can only show that *a*) ⇒ *b*) and *e*) ⇒ *a*).

a) ⇒ *b*) Let $f \in C(X, \mathcal{K}(\mathbb{R}^d))$. If there exists $h \in H$ such that $f \cap h \neq \emptyset$ (i.e., $f(x) \cap h(x) \neq \emptyset$ for every $x \in X$), we set

$$\rho(f) := \inf_{h\in H, f\cap h\neq\emptyset} d_H(f\cap h, h)\,, \qquad \lambda(f) := \frac{1}{2}\,\frac{\rho(f)}{1+\rho(f)}\,,$$

otherwise we set $\lambda(f) := \frac{1}{2}$. Consider the operator $L : C(X, \mathcal{K}(\mathbb{R}^d)) \to C(X, \mathcal{K}(\mathbb{R}^d))$ defined by setting, for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $x \in X$,

$$L(f)(x) := \frac{1}{2} f(x) + \frac{1}{2}\left\{y \in f(x) \mid y + \lambda(f)\cdot \mathbf{B} \subset f(x)\right\}\,.$$

It can be readily shown that L is well-defined and is continuous and monotone. Moreover, for every $h \in H$, we have $\rho(h) = \lambda(h) = 0$ and consequently $L(h) = h$. Applying a) to a net of constant value $L_i := L$, we obtain that for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, we must have $f \leq \liminf_{i \in I^{\leq}} L_i(f) = L(f)$. From the definition of L, this yields $\rho(f) = \lambda(f) = 0$ and since this happens if and only if there exists $h \in H$ such that $h \leq f$, condition b) must be satisfied.

e) $\Rightarrow$ *a)* Let $(L_i)_{i\in I}^{\leq}$ be an equicontinuous net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}^d))$ into itself satisfying $\lim_{i \in I^{\leq}} L_i(h) = h$ for every $h \in H$. Moreover, let $f \in C(X, \mathcal{K}(\mathbb{R}^d))$ and $\varepsilon > 0$. From d), there exist $h_1, \ldots, h_m \in H$ such that $h_j \leq f$ for every $j = 1, \ldots, m$ and

$$f \subset \bigcup_{j=1}^{m} h_j + \frac{\varepsilon}{2} \cdot \mathbf{B}\,.$$

Since $\lim_{i \in I^{\leq}} L_i(h_j) = h_j$ for every $j = 1, \ldots, m$, we can find $\alpha \in I$ such that, for every $i \geq \alpha$ and $j = 1, \ldots, m$, $h_j \leq L_i(h_j) + \frac{\varepsilon}{2}\mathbf{B}$, and using the monotonicity of L_i,

$$f \subset \bigcup_{j=1}^{m} L_i(h_j) + \varepsilon \cdot \mathbf{B} \subset L_i(f) + \varepsilon \cdot \mathbf{B}\,.$$

Hence $f \subset \bigcap_{i \geq \alpha} L_i(f) + \varepsilon \cdot \mathbf{B} \subset \bigcup_{\alpha \in I} \bigcap_{i \geq \alpha} L_i(f) + \varepsilon \cdot \mathbf{B}$. Since $\varepsilon > 0$ is arbitrary, we obtain $f \leq \liminf_{i \in I^{\leq}} L_i(f)$. $\square$

From the preceding proposition, the assumption that H is downward cofinal automatically implies that $f \in K(H)_i$ for every $f \in C(X, \mathcal{K}(\mathbb{R}^d))$.

As regards to the external Korovkin closure, we have a different situation which is completely described in the next result.

Theorem 3. *Let X be a compact Hausdorff topological space and let H be a cofinal subset of $C(X, \mathcal{K}(\mathbb{R}^d))$. If $f \in C(X, \mathcal{K}(\mathbb{R}^d))$, the following statements are equivalent:*

a) $f \in K(H)_e$.
b) $f = \bigcap_{k \in H, f \leq k} k$.
c) *For every $x_0 \in X$ and $\varepsilon > 0$, there exist $k_1, \ldots, k_s \in H$ such that $f \leq k_r$ for every $r = 1, \ldots, s$ and*

$$\bigcap_{r=1}^{s} k_r(x_0) \subset f(x_0) + \varepsilon \cdot \boldsymbol{B}$$

d) *For every $\varepsilon > 0$, there exist $k_1, \ldots, k_m \in H$ such that $f \leq k_j$ for every $j = 1, \ldots, m$ and*

$$\bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \boldsymbol{B}\,.$$

Proof. The proofs of the implications *a)* $\Rightarrow$ *b)*, *b)* $\Rightarrow$ *c)* and *c)* $\Rightarrow$ *d)* are exactly the same as in the proof of Theorem 2.

d) $\Rightarrow$ *a)* Let $(L_i)_{i\in I}^{\leq}$ be an equicontinuous net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}^d))$ into itself satisfying $\lim_{i \in I^{\leq}} L_i(h) = h$ for every $h \in H$. In order to show that $\limsup_{i \in I^{\leq}} L_i(f) \leq f$ we fix $\varepsilon > 0$ and consider $k_1, \ldots, k_m \in H$ such that $f \leq k_j$ for every $j = 1, \ldots, m$ and

$$\bigcap_{j=1}^{m} k_j \subset f + \varepsilon \cdot \mathbf{B}\,.$$

Since $\lim_{i\in I^{\le}} L_i(k_j) = k_j$ for every $j = 1, \dots, m$, there exists $\alpha \in I$ such that, for every $i \ge \alpha$, $x \in X$ and $j = 1, \dots, m$,

$$L_i(k_j)(x) \subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,, \qquad k_j(x) \subset L_i(k_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,.$$

It follows, for every $i \ge \alpha$, $x \in X$ and $j = 1, \dots, m$,

$$L_i(f)(x) \subset L_i(k_j)(x) \subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}$$

and consequently

$$L_i(f)(x) \subset \bigcap_{j=1}^{m} k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B} \subset f(x) + \varepsilon \cdot \mathbf{B}\,.$$

Hence we have also

$$\bigcup_{i\ge\alpha} L_i(f)(x) \subset f(x) + \varepsilon \cdot \mathbf{B}$$

and finally

$$\bigcap_{i\in I} \bigcup_{i\ge\alpha} L_i(f)(x) \subset f(x) + \varepsilon \cdot \mathbf{B}$$

which completes the proof. □

From Theorem 2, it follows that if H is a closed subset of $C(X, \mathcal{K}(\mathbb{R}^d))$ which is both cofinal and downward cofinal, then

$$K(H) = C(X, \mathcal{K}(\mathbb{R}^d)) \iff K(H)_e = C(X, \mathcal{K}(\mathbb{R}^d))\,.$$

Hence, the property of being a Korovkin system depends only on the external Korovkin closure.

3. Upper and lower Korovkin closures in cones of set-valued continuous functions

In spaces of continuous real functions, in Campiti [13] we have introduced the upper and lower Korovkin closures in order to give a more detailed description of the Korovkin closure and Korovkin systems.

In cones of set-valued continuous functions, the analogous of these concepts can be considered in the case $d = 1$ so in this section we shall restrict ourselves to this case.

If $f \in C(X, \mathcal{K}(\mathbb{R}))$, we can consider the two functions $\varphi_f, \psi_f \in C(X, \mathbb{R})$ defined by setting, for every $x \in X$,

$$\varphi_f(x) := \max f(x)\,, \qquad \psi_f(x) := \min f(x)\,.$$

Of course, if $(L_i)_{i\in I}^{\le}$ is a net of monotone operators from $C(X, \mathcal{K}(\mathbb{R}))$ into itself and if $f \in C(X, \mathcal{K}(\mathbb{R}))$ we have $\lim_{i\in I^{\le}} L_i(f) = f$ if and only if

$$\lim_{i\in I^{\leq}} \varphi_{L_i(f)} = \varphi_f \,, \qquad \lim_{i\in I^{\leq}} \psi_{L_i(f)} = \psi_f \,.$$

Hence, we can give the following definition.

Definition 2. Let X be a compact Hausdorff topological space and let $H \subset C(X, \mathcal{K}(\mathbb{R}))$. Then, the *upper Korovkin closure* (respectively, the *lower Korovkin closure*) of H with respect to monotone operators for the identity operator is the subset $K(H)^+$ (respectively, $K(H)^-$) defined as follows

$$\begin{aligned} K(H)^+ := \{f \in C(X, \mathcal{K}(\mathbb{R})) \mid\ & \lim_{i\in I^{\leq}} \psi_{L_i(f)} = \psi_f \textit{ for every equicontinuous net} \\ & (L_i)_{i\in I}^{\leq} \textit{ of monotone operators from } C(X, \mathcal{K}(\mathbb{R})) \\ & \textit{into itself satisfying } \lim_{i\in I^{\leq}} L_i(h) = h \textit{ for every } h \in H\} \,. \end{aligned}$$

(respectively,

$$\begin{aligned} K(H)^- := \{f \in C(X, \mathcal{K}(\mathbb{R})) \mid\ & \lim_{i\in I^{\leq}} \varphi_{L_i(f)} = \varphi_f \textit{ for every equicontinuous net} \\ & (L_i)_{i\in I}^{\leq} \textit{ of monotone operators from } C(X, \mathcal{K}(\mathbb{R})) \\ & \textit{into itself satisfying } \lim_{i\in I^{\leq}} L_i(h) = h \textit{ for every } h \in H\}\,) \,. \end{aligned}$$

Moreover, H is said to be an *upper Korovkin system* (respectively, a *lower Korovkin system*) in $C(X, \mathcal{K}(\mathbb{R}))$ with respect to monotone operators for the identity operator if $K(H)^+ = C(X, \mathcal{K}(\mathbb{R}))$ (respectively, $K(H)^- = C(X, \mathcal{K}(\mathbb{R}))$).

Of course, we have $K(H) = K(H)^+ \cap K(H)^-$.

Remark 1. We could further decompose the upper and lower Korovkin closures by distinguishing their external and internal parts.

Their description can be easily obtained from the following results and those in the preceding section and therefore, for the sake of brevity, we shall not give further details. □

In the following main result, we state a characterization of the upper and lower Korovkin closures.

Theorem 4. *Let X be a compact Hausdorff topological space and let H be a subset of $C(X, \mathcal{K}(\mathbb{R}))$ which is both cofinal and downward cofinal.*

If $f \in C(X, \mathcal{K}(\mathbb{R}))$, the following statements are equivalent:

a) $f \in K(H)^+$ *(respectively, $f \in K(H)^-$).*

b) $\psi_f = \sup\left(\bigcup_{h\in H, h\leq f} h\right) = \sup\left(\bigcap_{k\in H, f\leq k} k\right)$ *(respectively, $\varphi_f = \inf\left(\bigcup_{h\in H, h\leq f} h\right) = \inf\left(\bigcap_{k\in H, f\leq k} k\right)$).*

c) *For every $\varepsilon > 0$, there exist $h_1, \dots, h_m, k_1, \dots, k_m \in H$ such that $h_j \leq f \leq k_j$ for every $j = 1, \dots, m$ and*

$$\psi_f \leq \sup\left(\bigcup_{j=1}^{m} h_j\right) + \varepsilon \,, \qquad \sup\left(\bigcap_{j=1}^{m} k_j\right) \leq \psi_f + \varepsilon$$

(respectively,

$$\inf\left(\bigcup_{r=1}^{s} h_r\right) \le \varphi_f + \varepsilon\,, \qquad \varphi_f \le \inf\left(\bigcap_{r=1}^{s} k_r\right) + \varepsilon\,).$$

Proof. *a)* $\Rightarrow$ *b)* Let $f \in K(H)^+$. By contradiction, assume that there exists $x_0 \in X$ such that

$$\psi_f(x_0) < \sup\left(\bigcap_{k\in H, f\le k} k(x_0)\right)$$

and consider the operator $L : C(X,\mathcal{K}(\mathbb{R})) \to C(X,\mathcal{K}(\mathbb{R}))$ defined by setting, for every $g \in C(X,\mathcal{K}(\mathbb{R}))$ and $x \in X$,

$$L(g)(x) := \bigcap_{k\in H, g\le k} k(x)\,.$$

Since H is cofinal, the operator is a well-defined monotone operator. For every $k \in H$, we have $L(k) = k$ but $L(f)(x_0) \ne f(x_0)$. Hence we can consider a net $(L_i)_{i\in I}^{\le}$ of constant value $L_i = L$ for every $i \in I$ and we obtain that $(L_i(k))_{i\in I}^{\le}$ converges to k for every $k \in H$ but $(L_i(\{\psi_f\}))_{i\in I}^{\le}$ cannot converge to $\{\psi_f\}$ and this contradicts a).

The other equality can be shown in a similar way using the fact that H is downward cofinal and considering the operator $L : C(X,\mathcal{K}(\mathbb{R})) \to C(X,\mathcal{K}(\mathbb{R}))$ defined by setting, for every $g \in C(X,\mathcal{K}(\mathbb{R}))$ and $x \in X$,

$$L(g)(x) := \bigcup_{h\in H, h\le g} h(x)\,.$$

Observe that L is well-defined since H is downward cofinal.

b) $\Rightarrow$ *c)* Let $\varepsilon > 0$ and $x_0 \in X$.

First, we observe that

$$\psi_f(x_0) = \sup\left(\bigcup_{h\in H, h\le f} h(x_0)\right) < \sup\left(\bigcup_{h\in H, h\le f} h(x_0)\right) + \varepsilon$$

and since X is compact there exists a finite number of functions $h_1,\dots,h_m \in H$ such that $h_j \le f$ for every $j = 1,\dots,m$ and satisfying

$$\psi_f(x_0) \le \sup\left(\bigcup_{j=1}^{m} h_j(x_0)\right) + \varepsilon\,.$$

In a similar way we can obtain the second inequality.

c) $\Rightarrow$ *a)* Let $(L_i)_{i\in I}^{\le}$ be an equicontinuous net of monotone operators from $C(X,\mathcal{K}(\mathbb{R}))$ into itself satisfying $\lim_{i\in I^{\le}} L_i(h) = h$ for every $h \in H$. In order to show that $\lim_{i\in I^{\le}} L_i(\{\psi_f\}) = \{\psi_f\}$ we fix $\varepsilon > 0$ and from c) we consider $h_1,\dots,h_m,k_1,\dots,k_m \in H$ such that $h_j \le f \le k_j$ for every $j = 1,\dots,m$ and

$$\psi_f \le \sup\left(\bigcup_{j=1}^{m} h_j\right) + \varepsilon\,, \qquad \sup\left(\bigcap_{j=1}^{m} k_j\right) \le \psi_f + \varepsilon$$

Since $\lim_{i\in I^{\le}} L_i(h_j) = h_j$ and $\lim_{i\in I^{\le}} L_i(k_j) = k_j$ for every $j = 1,\dots,m$, there exists $\alpha \in I$ such that, for every $i \ge \alpha$, $x \in X$ and $j = 1,\dots,m$,

$$L_i(h_j)(x) \subset h_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,, \qquad h_j(x) \subset L_i(h_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,.$$
$$L_i(k_j)(x) \subset k_j(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,, \qquad k_j(x) \subset L_i(k_j)(x) + \frac{\varepsilon}{2} \cdot \mathbf{B}\,.$$

It follows, for every $i \geq \alpha$, $x \in X$ and $j = 1, \ldots, m$,

$$\sup(L_i(f)(x)) \leq \sup(L_i(k_j)(x)) \leq \sup(k_j(x)) + \frac{\varepsilon}{2}$$

and consequently

$$\sup(L_i(f)(x)) \leq \sup\left(\bigcap_{j=1}^{m} k_j(x)\right) + \frac{\varepsilon}{2} \leq \psi_f(x) + \varepsilon\,.$$

On the other hand, for every $i \geq \alpha$, $x \in X$ and $j = 1, \ldots, m$,

$$\sup(h_j(x)) \leq \sup(L_i(f)(x)) \leq \sup(L_i(h_j)(x)) \leq \sup(h_j(x)) + \frac{\varepsilon}{2}$$

and consequently

$$\psi_f(x) \leq \sup\left(\bigcup_{j=1}^{m} h_j(x)\right) + \frac{\varepsilon}{2} \leq \sup\left(\bigcup_{j=1}^{m} L_i(h_j)(x)\right) + \frac{\varepsilon}{2} \leq \sup(L_i(f)(x)) + \varepsilon$$

and this completes the proof. $\square$

Also in this case it is possible to show that if H is a subset of $C(X, \mathcal{K}(\mathbb{R}))$ we have

$$K(H) = C(X, \mathcal{K}(\mathbb{R})) \iff K(H)_e^+ = K(H)_e^- = C(X, \mathcal{K}(\mathbb{R}))$$

and the preceding characterizations can be rewritten using only the external upper and lower Korovkin closures.

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