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Homogeneous Structures and Homogeneous Geodesics of the Hyperbolic Oscillator Group

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Abstract: In this paper, we study some homogeneity properties of a semi-direct extension of the Heisenberg group, known in literature as the hyperbolic oscillator (or Boidol) group, equipped with the left-invariant metrics corresponding to the ones of the oscillator group. We identify the naturally reductive case by the existence of the corresponding special homogeneous structures. For the cases where these special homogeneous structures do not exist, we exhibit a complete description of the homogeneous geodesics.

Keywords: hyperbolic oscillator group; semi-direct extensions; homogeneous structures; homogeneous geodesics

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1. Introduction

Homogeneous spaces are a central topic of geometry. Because of their uniform structure, investigations of the homogeneous spaces are very common. A pseudo-Riemannian manifold (M, g) is said to be *homogeneous* if for any points $p, q \in M$, there exists an isometry which maps p to q. In other words, the full group of isometries acts transitively on M. As the geometry of the manifold is the same around each point, analytic objects on M can be investigated algebraically.

A tensorial approach for the study of reductive homogeneous manifolds is given by homogeneous structures. This concept was first introduced on Riemannian manifolds by Ambrose and Singer in [1] and then developed in the pioneering book [2]. Gadea and Oubiña [3] extended homogeneous structures to pseudo-Riemannian settings. For recent developments in pseudo-Riemannian homogeneous structures, refer to [4].

Homogeneous structures allow several relevant geometric properties of homogeneous manifolds to be characterized. In particular, they can be used to characterize *naturally reductive homogeneous manifolds*, i.e., the ones for which the Levi-Civita connection of $(M = G/H, g)$ (where G and H denote a group of isometries acting transitively on M and the isotropy subgroup, respectively) and the canonical connection of the reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G have exactly the same geodesics.

All geodesics of a naturally reductive space are homogeneous. *Homogeneous geodesics* are the most natural geodesics of a homogeneous manifold. Given a homogeneous pseudo-Riemannian manifold $(M = G/H, g)$, a geodesic γ passing through a point $p \in M$ is called homogeneous if it is the orbit of some one-parameter subgroup. In general, the group G is not uniquely determined. However, if γ is homogeneous with respect to an isometry group, then it is so with respect to the connected maximal group of isometries. We may refer to [5] for a survey on homogeneous geodesics.

Semi-direct extensions of the Heisenberg group were introduced in arbitrary dimension in [6], as a natural generalization of the well-known *oscillator groups*. A detailed study of the geometry of four-dimensional examples, equipped with metrics that naturally extend to this class the Lorentzian metrics of the oscillator group, started in [7] and concerned several relevant geometric properties: Ricci solitons [7], critical metrics of quadratic curvature functionals [8], and conformal geometry [9]. The four-dimensional oscillator group, equipped with these left-invariant Lorentzian metrics, is a well-known example of a naturally reductive homogeneous Lorentzian manifold (see [10]). It is, then, natural to ask whether such results extend to other semi-direct extensions of the Heisenberg group.

Following ref. [7], there are three models of these semi-direct extensions having non-isomorphic Lie algebras. Besides the oscillator group, the remaining models are the hyperbolic oscillator group and a nilpotent group (see Proposition 5).

The *hyperbolic oscillator group* is also known in literature as the *Boidol group* or the *split oscillator group*. Recently, lattices of the hyperbolic oscillator group were classified in [11]. The purpose of this paper is to investigate the homogeneity properties of the hyperbolic oscillator group. Differently from the case of the oscillator group, not all the corresponding metrics are naturally reductive on the hyperbolic oscillator group. This leads in a natural way to consider the homogeneous geodesics of the remaining cases, which we completely determine. Finally, we also describe the corresponding results for the remaining model of semi-direct extensions of the Heisenberg group.

The paper is organized in the following way. In Section 2, we report some general definitions and results concerning homogeneous pseudo-Riemannian manifolds, pseudo-Riemannian homogeneous structures, and homogeneous geodesics. In Section 3, we introduce some needed information concerning semi-direct extensions of the Heisenberg group, with particular regard to the geometry of the hyperbolic oscillator group equipped with the one-parameter family of metrics corresponding to the ones of the oscillator group. In Section 4, we first investigate the homogeneous structures of these examples and establish which of them are naturally reductive. We then provide a complete classification of homogeneous geodesics for the cases which are not naturally reductive. Finally, in Section 5, we give the corresponding results for the nilpotent group, which provides the remaining model of semi-direct extensions of the Heisenberg group. Calculations were checked using the Maple 16© software.

2. Preliminaries

In this section, we briefly report the basic definitions and results that we use throughout the paper.

2.1. Homogeneous Pseudo-Riemannian Manifolds

As we already recalled in the Introduction, a pseudo-Riemannian manifold (M, g) is called homogeneous if for any pair (p, q) of points of M, there exists an isometry f of (M, g) such that $f(p) = q$. In other words, the full group of isometries acts transitively on M.

Given a homogeneous pseudo-Riemannian manifold (M, g), a *coset representation* of M is given by $M = G/H$, where G is a group of isometries acting transitively on M and H and the isotropy subgroup. The Lie algebra $\mathfrak{g}$ of G then splits as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. It is well known that the same homogeneous pseudo-Riemannian manifold can admit different coset representations.

A homogeneous pseudo-Riemannian manifold (M, g) is said to be *reductive* when it admits a coset representation $M = G/H$ such that, in the corresponding Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. If H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. While all homogeneous Riemannian manifolds are reductive, in dimension four and higher, there exist homogeneous pseudo-Riemannian manifolds that do not admit any reductive decomposition.

A homogeneous pseudo-Riemannian manifold (M, g) is *naturally reductive* if it admits a reductive coset representation $M = G/H$, such that the reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and the G-invariant metric g satisfy the equation

$$g([X, Y]_{\mathfrak{m}}, Z) + g([X, Z]_{\mathfrak{m}}, Y) = 0, \quad \text{for all } X, Y, Z \in \mathfrak{m}. \tag{1}$$

This equation is equivalent to requiring that the Levi-Civita connection of (M, g) and the canonical connection of the reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ have the same geodesics. In particular, this implies that a naturally reductive manifold is a *g.o. space* (i.e., a "geodesic orbit space"), that is, all of its geodesics are homogeneous.

2.2. Pseudo-Riemannian Homogeneous Structures

We start with the following definition and result.

Definition 1. *Let (M, g) denote a connected pseudo-Riemannian manifold, and ∇ and R denote the Levi-Civita connection and curvature tensor of (M, g) respectively. A (pseudo-Riemannian) homogeneous structure on (M, g) is a tensor field S of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - S$ satisfies*

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} S = 0. \tag{2}$$

Theorem 1 ([1,12])**.** *Let (M, g) be a connected, simply connected, and complete pseudo-Riemannian manifold. Then, (M, g) admits a pseudo-Riemannian homogeneous structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

In fact, such a homogeneous structure S defines a reductive decomposition of a suitable coset description of (M, g), and visa versa. As different notions of completeness are not equivalent in pseudo-Riemannian settings, we specify that in the above Theorem 1, "complete" refers to geodesic completeness (see for example [4]).

Now let S be a homogeneous structure on an n-dimensional pseudo-Riemannian manifold (M, g). We will denote by S both the $(1, 2)$-tensor field and its metric equivalent $(0, 3)$-tensor field, defined by $S_{XYZ} = g(S_X Y, Z)$.

By fixing a point $x \in M$ and an orthonormal basis of $T_x M$, we consider the vector space $V = \mathbb{R}^n$ endowed with the standard symmetric bilinear form $\langle\,,\,\rangle$ of signature (p, q) as a model of $(T_x M, g_x)$. We take the space of tensors $\mathcal{S}(V) \subset \otimes^3 V^*$ with the same symmetries as the homogeneous structure S, that is, defined by

$$\mathcal{S}(V) = \{S \in \otimes^3 V^* \,/\, S_{XYZ} + S_{XZY} = 0\}.$$

As a vector space, $\mathcal{S}(V)$ is isomorphic to $V^* \otimes \wedge^2 V^*$ and carries a non-degenerate symmetric bilinear form, defined by

$$\langle S, S' \rangle = \sum_{i,j,k=1}^{n} \varepsilon^i \varepsilon^j \varepsilon^k S_{e_i e_j e_k} S'_{e_i e_j e_k},$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of $(V, \langle\,,\,\rangle)$ and $\varepsilon^i = \langle e_i, e_i \rangle$.

In order to classify homogeneous structures, one decomposes the $O(p,q)$-module $\mathcal{S}(V)$ into irreducible submodules. If $n = \dim M \geq 3$, then the space $\mathcal{S}(V)$ decomposes into irreducible and mutually orthogonal $O(p,q)$-submodules as

$$\mathcal{S}(V) = \mathcal{S}_1(V) \oplus \mathcal{S}_2(V) \oplus \mathcal{S}_3(V),$$

where

$$\mathcal{S}_1 = \Big\{ S \in \mathcal{S} \;/\; S_{XYZ} = g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y),\ \varphi \in V^* \Big\},$$
$$\mathcal{S}_2 = \Big\{ S \in \mathcal{S} \;/\; S_{XYZ} + S_{YZX} + S_{ZXY} = 0,\ c_{12}(S) := \sum_{i=1}^{n} \varepsilon_i S_{e_i e_i \cdot} = 0 \Big\},$$
$$\mathcal{S}_3 = \Big\{ S \in \mathcal{S} \;/\; S_{XYZ} + S_{YXZ} = 0 \Big\}.$$

In particular, Equation (1) holds if and only if $S \in \mathcal{S}_3$, so that *a homogeneous (reductive) pseudo-Riemannian manifold (M,g) is naturally reductive if and only if it admits a homogeneous structure belonging to $\mathcal{S}_3$.*

2.3. Homogeneous Geodesics

We first recall the more general and accurate definition of homogeneous geodesics in pseudo-Riemannian settings.

Definition 2. *Let $(M = G/H, g)$ be a homogeneous pseudo-Riemannian manifold and p a point of M. A geodesic $\gamma(s)$ through p, defined in an open interval J (where s is an affine parameter) is said to be homogeneous if there exists the following:*

(1) A diffeomorphism $s = \varphi(t)$ between the real line $\mathbb{R}$ and the open interval J;
(2) A vector $V \in \mathfrak{g}$, such that $\gamma(\varphi(t))) = exp(tV)(p)$ for all $t \in \mathbb{R}$.

The vector V is then called a geodesic vector.

In the case of *reductive* homogeneous pseudo-Riemannian manifolds, the *Geodesic Lemma* [13] provides the following simple algebraic characterization: given a vector $V \in \mathfrak{g}$, the curve $\gamma(t) = exp(tV)(p)$ is geodesic if and only if

$$\langle [V,Z]_{\mathfrak{m}}, V_{\mathfrak{m}} \rangle = k \langle V_{\mathfrak{m}}, Z \rangle \quad \text{for all } Z \in \mathfrak{m}, \tag{3}$$

where k is a real constant. In particular, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic, and this case can occur only if $\gamma(t)$ is a light-like curve in a (properly) pseudo-Riemannian space.

On the other hand, the approach developed in [14] permits to investigate homogeneous geodesics in the more general framework of homogeneous affine manifolds.

Let ∇ denote an affine connection on a manifold M. A smooth vector field X on M is said to be *affine Killing* if its integral curves are geodesics with respect to ∇.

The pair (M, ∇) is said to be a *homogeneous affine manifold* if for any two points $p, q \in M$, there exists an affine transformation of M, mapping p into q. In a homogeneous affine manifold (M, ∇), a *homogeneous geodesic* is a geodesic that is an orbit of a one-parameter group of affine diffeomorphisms. (Here, the canonical parameter of the group need not be the affine parameter of the geodesic.)

The following results of [14] show how to determine the homogeneous geodesics in a homogeneous affine manifold. In particular, these results apply to any homogeneous pseudo-Riemannian manifold, whether reductive or not.

Proposition 1 ([14])**.** *A homogeneous affine manifold (M, ∇) admits $n = \dim M$ affine Killing vector fields $K_1, \ldots, K_n$, which are linearly independent at each point of some neighborhood U of p.*

It is well known that a geodesic through $p \in M$ is uniquely determined locally by its tangent vector X at p. Consider now $n = \dim M$ affine Killing vector fields $K_1, \ldots, K_n$, linearly independent in a neighborhood U of p, as in the above Proposition 1. Let $\mathcal{B} = \{K_1(p), \ldots, K_n(p)\}$ denote the corresponding basis of the tangent space T_pM. Each tangent vector $X \in T_pM$ is uniquely determined by its coordinates $(c_1, \ldots, c_n)$ with respect to the basis $\mathcal{B}$, and it determines the corresponding Killing vector field $\tilde{X} = c_1K_1 + \ldots + c_nK_n$ and the integral curve γ_X of $\tilde{X}$ through p.

The following results and definitions clarify the relationship between affine Killing vector fields and homogeneous geodesics of (M, ∇).

Proposition 2 ([14])**.** *Let $(M = G/H, \nabla)$ (where G acts transitively and effectively on M) be a homogeneous affine space. Then, each curve γ that is a regular ($\gamma'(t) \neq 0$ for all t) orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.*

Definition 3. *A nonvanishing smooth vector field V on M is said to be geodesic along its regular integral curve γ if $\gamma(t)$ is a geodesic, up to a possible reparametrization. If all regular integral curves of V are geodesics up to a reparametrization, then V is called a geodesic vector field.*

Proposition 3 ([14])**.** *Let (M, ∇) be a homogeneous affine manifold and V a nonvanishing Killing vector field on M.*

(1) V is geodesic along its integral curve γ if and only if

$$\nabla_{V_{\gamma(t)}} V = k_\gamma \cdot V_{\gamma(t)} \tag{4}$$

holds along γ, where k_γ is a real constant. If $k_\gamma = 0$, then t is the affine parameter of geodesic γ. If $k_\gamma \neq 0$, then the affine parameter is $s = e^{k_\gamma t}$;

(2) V is a geodesic Killing vector field if and only if

$$\nabla_V V = k \cdot V, \tag{5}$$

where k is a smooth function on M, constant along the integral curves of V.

Note that if (4) holds for $t = 0$, then it holds for the same k_γ for all sufficiently small values of t, and $\gamma(t)$ is a local geodesic. Moreover, such a local geodesic γ can be uniquely prolonged to a global homogeneous geodesic [15].

The Levi-Civita connection ∇ of a homogeneous pseudo-Riemannian manifold (M, g) is an invariant affine connection. Hence, the above results can be applied to the study of homogeneous geodesics and geodesic vector fields of (M, g), determining homogeneous geodesics through a chosen point $p \in M$ as the geodesic integral curves of Killing vector fields.

3. The Geometry of Semi-Direct Extensions of the Heisenberg Group

Denote by H the three-dimensional Heisenberg group and by $\mathfrak{h} = \mathrm{span}\{X, Y, U\}$ its Lie algebra, described by $[X, Y] = U$. Following the argument introduced in [6], each real matrix $\tilde{\mathcal{S}} \in \mathfrak{sp}(1, \mathbb{R})$ (the Lie algebra of the symplectic group $Sp(1, \mathbb{R})$ on $\mathbb{R}^2$), that is, of the form

$$\tilde{\mathcal{S}} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \tag{6}$$

determines a corresponding derivation

$$[\tilde{S},(z,u)] = (\tilde{\mathcal{S}}z, 0)$$

of $\mathfrak{h}$ and so, a one-dimensional semi-direct extension $\mathfrak{g} = \mathfrak{h} \rtimes (\mathbb{R}\tilde{S})$ of $\mathfrak{h}$. The corresponding connected, simply connected Lie group is then given by $G = G_{\tilde{\mathcal{S}}} = H \rtimes_{\tilde{\mathcal{S}}} \mathbb{R} = \mathbb{C} \times \mathbb{R} \times \mathbb{R}$.

This construction includes as a special case of the well-known four-dimensional *oscillator algebra* (and the corresponding *oscillator group*), obtained by taking

$$\tilde{\mathcal{S}} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \quad \mu \neq 0.$$

The four-dimensional oscillator group has been investigated from several different points of view. We may refer to [12,16] and references within for several results on this relevant example. The study of semi-direct extensions $G_{\tilde{\mathcal{S}}} = H \rtimes_{\tilde{\mathcal{S}}} \mathbb{R}$ was undertaken in [7], generalizing to $G_{\tilde{\mathcal{S}}}$ the well-known family of left-invariant metrics of the oscilllator group. These metrics g_a, defined for any real value of a with $a^2 \neq 1$, have the form

$$g_a(e_1,e_1) = g_a(e_4,e_4) = a, \quad g_a(e_2,e_2) = g_a(e_3,e_3) = g_a(e_1,e_4) = g_a(e_4,e_1) = 1, \tag{7}$$

where $U = e_1$, $X = e_2$, $Y = e_3$, and $S = e_4$. The following explicit description was obtained in [7].

Proposition 4 ([7])**.** *Given $\tilde{\mathcal{S}} \in \mathfrak{sp}(1,\mathbb{R})$, described as in (6), the semi-direct extension $G = H \rtimes_{\tilde{\mathcal{S}}} \mathbb{R}$ can be realized as the four-dimensional subgroup of* $\mathrm{GL}(4,\mathbb{R})$*:*

$$G_{\tilde{\mathcal{S}}} = \{M_{\tilde{\mathcal{S}}}(x_1,x_2,x_3,x_4) \in \mathrm{GL}(4,\mathbb{R}) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\},$$

whose group elements have the form

$$M_{\tilde{\mathcal{S}}}(x_i) = \begin{pmatrix} 1 & x_2\, w(x_4) - x_3\, u(x_4) & x_2\, z(x_4) - x_3 v\,(x_4) & 2x_1 \\ 0 & u(x_4) & v(x_4) & x_2 \\ 0 & w(x_4) & z(x_4) & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where, depending on whether $\Delta = -\det(\tilde{\mathcal{S}}) = \alpha^2 + \beta\gamma$ is positive, null, or negative, we have

$$u(x_4) = \begin{cases} \cosh(\sqrt{\Delta}x_4) + \frac{\alpha}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ 1 + \alpha x_4 & \text{if } \Delta = 0, \\ \cos(\sqrt{-\Delta}x_4) + \frac{\alpha}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{8}$$

$$v(x_4) = \begin{cases} \frac{\beta}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ \beta x_4 & \text{if } \Delta = 0, \\ \frac{\beta}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{9}$$

$$w(x_4) = \begin{cases} \frac{\gamma}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ \gamma x_4 & \text{if } \Delta = 0, \\ \frac{\gamma}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{10}$$

$$z(x_4) = \begin{cases} \cosh(\sqrt{\Delta}x_4) - \frac{\alpha}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \textit{if } \Delta > 0, \\ 1 - \alpha x_4 & \textit{if } \Delta = 0, \\ \cos(\sqrt{-\Delta}x_4) - \frac{\alpha}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \textit{if } \Delta < 0, \end{cases} \quad (11)$$

As proved in [7], in coordinates (x_1, x_2, x_3, x_4), the metric g_a is explicitly given by

$$\begin{aligned} g_a \;=\; & a dx_1^2 + \left(\tfrac{a}{4}x_3^2 + w^2(x_4) + z^2(x_4)\right)dx_2^2 \\ & + \left(\tfrac{a}{4}x_2^2 + u^2(x_4) + v^2(x_4)\right)dx_3^2 + a dx_4^2 \\ & + a x_3 dx_1 dx_2 - a x_2 dx_1 dx_3 + 2dx_1 dx_4 \\ & - \tfrac{1}{2}\left(a x_2 x_3 + 4u(x_4)w(x_4) + 4v(x_4)z(x_4)\right)dx_2 dx_3 \\ & + x_3 dx_2 dx_4 - x_2 dx_3 dx_4. \end{aligned} \quad (12)$$

The description of the Lie algebra $\mathfrak{h} \rtimes (\mathbb{R}\tilde{S})$ up to isomorphisms is given in the following result.

Proposition 5 ([7])**.** *Consider an arbitrary $\tilde{S} \in \mathfrak{sp}(1, \mathbb{R})$ and the corresponding derivation $\tilde{S}$ of the Heisenberg Lie algebra $\mathfrak{h}$. Depending on whether $\Delta = -\det(\tilde{S}) = \alpha^2 + \beta\gamma$ is (A) positive, (B) null, or (C) negative, the one-dimensional extension $\mathfrak{h} \rtimes (\mathbb{R}\tilde{S})$ is isomorphic to the Lie algebra $\mathfrak{g}$ =span$\{e_1, e_2, e_3, e_4\}$, completely described by $[e_2, e_3] = e_1$ and*

$$\begin{aligned} (A): &\quad [e_4, e_2] = \mu e_2, \quad [e_4, e_3] = -\mu e_3, \quad \mu > 0; \\ (B): &\quad [e_4, e_2] = \mu e_3, \quad [e_4, e_3] = 0, \quad \mu \geq 0; \\ (C): &\quad [e_4, e_2] = \mu e_3, \quad [e_4, e_3] = -\mu e_2, \quad \mu > 0. \end{aligned}$$

Case (C) in Proposition 5 is the oscillator Lie algebra. The four-dimensional oscillator group, equipped with any left-invariant Lorentzian metric g_a, is a well-known example of naturally reductive homogeneous manifold (see [10]). In particular, all of its geodesics are homogeneous. Case (A) is known in the literature as the *hyperbolic oscillator group* or *Boidol group* [11]. Case (B) is a nilpotent group.

We now focus on the case of the hyperbolic oscillator group and describe the calculations that we use in the following sections.

According to the description given in the Proposition 5, we consider the case $\alpha = \mu > 0$ and $\beta = \gamma = 0$. Functions $u(x_4)$, $v(x_4)$, $w(x_4)$, and $z(x_4)$ of Proposition 4 then reduce to

$$u(x_4) = e^{\mu x_4}, \quad v(x_4) = 0, \quad w(x_4) = 0, \quad z(x_4) = e^{-\mu x_4}. \quad (13)$$

We now put $\partial_i = \frac{\partial}{\partial x_i}, i = 1, \cdots 4$ and determine the Levi-Civita connection ∇ of g_a with respect to $\{\partial_i\}$. Using the *Koszul formula*, we find that the possibly nonvanishing components of ∇ are given by

$$\begin{aligned} &\nabla_{\partial_1}\partial_2 = -\tfrac{a}{4}x_2 e^{-2\mu x_4}\partial_1 - \tfrac{a}{2}e^{-2\mu x_4}\partial_3, \\ &\nabla_{\partial_1}\partial_3 = -\tfrac{a}{4}x_3 e^{2\mu x_4}\partial_1 + \tfrac{a}{2}e^{2\mu x_4}\partial_2, \\ &\nabla_{\partial_2}\partial_2 = -\tfrac{1}{4(a^2-1)}e^{-2\mu x_4}(a^3 x_2 x_3 - a x_2 x_3 + 4\mu)\partial_1 - \tfrac{a}{2}x_3 e^{-2\mu x_4}\partial_3 + \tfrac{a}{a^2-1}\mu e^{-2\mu x_4}\partial_4, \\ &\nabla_{\partial_2}\partial_3 = -\tfrac{a}{8}(x_3^2 e^{2\mu x_4} - x_2^2 e^{-2\mu x_4})\partial_1 + \tfrac{a}{4}x_3 e^{2\mu x_4}\partial_2 + \tfrac{a}{4}x_2 e^{-2\mu x_4}\partial_3, \\ &\nabla_{\partial_2}\partial_4 = (-\tfrac{1}{4}x_2 e^{-2\mu x_4} + \tfrac{1}{2}x_3\mu)\partial_1 - \mu\partial_2 - \tfrac{1}{2}e^{-2\mu x_4}\partial_3, \\ &\nabla_{\partial_3}\partial_3 = \tfrac{1}{4(a^2-1)}e^{2\mu x_4}(a^3 x_2 x_3 - a x_2 x_3 + 4\mu)\partial_1 - \tfrac{a}{2}x_2 e^{2\mu x_4}\partial_3 - \tfrac{a}{a^2-1}\mu e^{2\mu x_4}\partial_4, \\ &\nabla_{\partial_3}\partial_4 = (\tfrac{1}{2}x_2\mu - \tfrac{1}{4}x_3 e^{2\mu x_4})\partial_1 + \tfrac{1}{2}e^{2\mu x_4}\partial_2 + \mu\partial_3. \end{aligned} \quad (14)$$

The curvature tensor is determined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ and the Riemann curvature tensor is the $(0,4)$-curvature tensor given by $R(X,Y,Z,W) = -g(R(X,Y)Z,W)$. Starting from (14) and taking into account its symmetries, we obtain that the Riemann curvature tensor is completely determined by the following possibly nonvanishing components $R_{ijkh} = R(\partial_i, \partial_j, \partial_k, \partial_h)$:

$$
\begin{array}{ll}
R_{1212} = \frac{a^2 e^{-2\mu x_4}}{4}, & R_{1223} = \frac{a^2 e^{-2\mu x_4} x_2}{8}, \\
R_{1224} = -\frac{a e^{-2\mu x_4}}{4}, & R_{1234} = \frac{a\mu}{2}, \\
R_{1313} = \frac{a^2 e^{2\mu x_4}}{4}, & R_{1323} = \frac{a^2 e^{2\mu x_4} x_3}{8}, \\
R_{1324} = \frac{a\mu}{2}, & R_{1334} = -\frac{a e^{2\mu x_4}}{4}, \\
\multicolumn{2}{l}{R_{2323} = \frac{a((a^3 x_2^2 - a x_2^2) e^{-2\mu x_4} + (a^3 x_3^2 - a x_3^2) e^{2\mu x_4} - 12a^2 + 16\mu^2 + 12)}{16(a^2-1)},} \\
R_{2324} = -\frac{a(x_2 e^{-2\mu x_4} - 2x_3\mu)}{8}, & R_{2334} = -\frac{a(x_3 e^{2\mu x_4} - 2x_2\mu)}{8}, \\
R_{2424} = -\frac{e^{-2\mu x_4}(4\mu^2-1)}{4}, & R_{2434} = -\mu, \\
R_{3434} = -\frac{e^{2\mu x_4}(4\mu^2-1)}{4}. &
\end{array}
$$

In particular, the Ricci tensor is then described by

$$
\begin{aligned}
\varrho = {} & \tfrac{a^2}{2} dx_1^2 + \tfrac{a}{8}(a x_3^2 - 4e^{-2\mu x_4}) dx_2^2 + \tfrac{a}{8}(a x_2^2 - 4e^{2\mu x_4}) dx_3^2 \\
& -(2\mu^2 - \tfrac{1}{2}) dx_4^2 + \tfrac{a^2}{2} x_3 dx_1 dx_2 - \tfrac{a^2}{2} x_2 dx_1 dx_3 + a dx_1 dx_4 \\
& -\tfrac{a^2}{4} x_2 x_3 dx_2 dx_3 + \tfrac{a}{2} x_3 dx_2 dx_4 - \tfrac{a}{2} x_2 dx_3 dx_4.
\end{aligned} \tag{15}
$$

4. Homogeneous Structures and Geodesics of the Hyperbolic Oscillator Group

We consider the hyperbolic oscillator group, equipped with the family of left-invariant metrics g_a described by Equations (12) and (13) of Case (A).

Set $S = T_{ijk} dx^i \otimes dx^j \otimes dx^k$ as an arbitrary homogeneous structure on $G_{\mathfrak{F}}$, where the coefficients T_{ijk} are arbitrary smooth functions on $G_{\mathfrak{F}}$. Note that $\tilde{\nabla} g = 0$ necessarily yields $T_{ijk} = -T_{ikj}$, so that we only need to determine the components T_{ijk} with $j < k$. Observe that if $S \in \mathcal{S}_3$, then we have the additional symmetry $T_{ijk} = -T_{jik}$. We separately consider two cases, depending on whether $a = 0$.

The first case: $a \neq 0$.

In this case, applying Equations (2), we obtain a complete description of all homogeneous structures. In fact, one has

$$
(\tilde{\nabla}_{\partial_i} R)(\partial_1, \partial_2)\partial_4 = \tfrac{a\mu}{a^2-1} e^{-2\mu x_4} T_{i14} \partial_3, \quad i = 1, \cdots, 4, \tag{16}
$$

where $T_{i14} = 0$ for all indices $i = 1, \ldots, 4$. The Ricci tensor is determined by (15), and using a direct calculation, we find

$$
\begin{array}{ll}
(\tilde{\nabla}_{\partial_1} \varrho)(\partial_1, \partial_2) = -a T_{112}, & (\tilde{\nabla}_{\partial_1} \varrho)(\partial_1, \partial_3) = -a T_{113}, \\
(\tilde{\nabla}_{\partial_2} \varrho)(\partial_1, \partial_2) = -a T_{212}, & (\tilde{\nabla}_{\partial_2} \varrho)(\partial_1, \partial_3) = -\tfrac{a}{2}(2T_{213} + a), \\
(\tilde{\nabla}_{\partial_3} \varrho)(\partial_1, \partial_2) = \tfrac{a}{2}(a - 2T_{312}), & (\tilde{\nabla}_{\partial_3} \varrho)(\partial_1, \partial_3) = -a T_{313}, \\
(\tilde{\nabla}_{\partial_4} \varrho)(\partial_1, \partial_2) = -a T_{412}, & (\tilde{\nabla}_{\partial_4} \varrho)(\partial_1, \partial_3) = -a T_{413}.
\end{array}
$$

Since we necessarily have $\tilde{\nabla}\varrho = 0$, the above equations immediately yield $T_{112} = T_{113} = T_{212} = T_{313} = T_{412} = T_{413} = 0$ and $T_{213} = -T_{312} = -\frac{a}{2}$. Using these conditions, from (16) we obtain

$$
\begin{aligned}
&(\tilde{\nabla}_{\partial_1} R)(\partial_1,\partial_2)\partial_1 = -\tfrac{a^2}{4(a^2-1)}T_{124}\partial_1, && (\tilde{\nabla}_{\partial_1} R)(\partial_1,\partial_2)\partial_4 = \tfrac{a\mu}{2}(a+2T_{123})\partial_2\\
&(\tilde{\nabla}_{\partial_1} R)(\partial_1,\partial_3)\partial_1 = -\tfrac{a^2}{4(a^2-1)}T_{134}\partial_1, && (\tilde{\nabla}_{\partial_2} R)(\partial_1,\partial_2)\partial_1 = \tfrac{a^2}{4(a^2-1)}(\mu e^{-2\mu x_4} - T_{224})\partial_1,\\
&(\tilde{\nabla}_{\partial_2} R)(\partial_1,\partial_2)\partial_4 = \tfrac{a\mu}{2}(ax_3+2T_{223})\partial_2, && (\tilde{\nabla}_{\partial_2} R)(\partial_1,\partial_3)\partial_1 = -\tfrac{a^2}{8(a^2-1)}(2T_{234}-1)\partial_1,\\
&(\tilde{\nabla}_{\partial_3} R)(\partial_1,\partial_2)\partial_1 = -\tfrac{a^2}{8(a^2-1)}(2T_{324}+1)\partial_1, && (\tilde{\nabla}_{\partial_3} R)(\partial_1,\partial_2)\partial_4 = -\tfrac{a\mu}{2}(ax_2-2T_{323})\partial_2,\\
&(\tilde{\nabla}_{\partial_3} R)(\partial_1,\partial_3)\partial_1 = -\tfrac{a^2}{4(a^2-1)}(\mu e^{2\mu x_4} + T_{334})\partial_1, && (\tilde{\nabla}_{\partial_4} R)(\partial_1,\partial_2)\partial_1 = -\tfrac{a^2}{4(a^2-1)}T_{424}\partial_1,\\
&(\tilde{\nabla}_{\partial_4} R)(\partial_1,\partial_2)\partial_4 = \tfrac{a\mu}{2}(2T_{423}+1)\partial_2, && (\tilde{\nabla}_{\partial_4} R)(\partial_1,\partial_3)\partial_1 = -\tfrac{a^2}{4(a^2-1)}T_{434}\partial_1,
\end{aligned}
$$

where as $\tilde{\nabla}R = 0$, we conclude that the non-zero components $T_{ijk}, j < k$ of an arbitrary homogeneous structure S are given by

$$
\begin{aligned}
&T_{123} = -\tfrac{1}{2}a, && T_{213} = -\tfrac{1}{2}a, && T_{223} = -\tfrac{1}{2}ax_3, && T_{224} = \mu e^{-2\mu x_4},\\
&T_{234} = \tfrac{1}{2}, && T_{312} = \tfrac{1}{2}a, && T_{323} = \tfrac{1}{2}ax_2, && T_{324} = -\tfrac{1}{2},\\
&T_{334} = -\mu e^{2\mu x_4}, && T_{423} = -\tfrac{1}{2}.
\end{aligned}
\tag{17}
$$

It is easy to check that $\tilde{\nabla}S = 0$ is now satisfied. Therefore, Equation (17) describes all homogeneous structures of the hyperbolic oscillator group equipped with a left-invariant metric $g_a, a \neq 0$. Indeed, just one homogeneous structure exists, for any initial data corresponding to $a \neq 0$ and $\mu > 0$. This is coherent with the fact that, as we shall see in Equation (19), the Lie algebra of Killing vector fields is four-dimensional. Correspondingly, for $a \neq 0$ and $\mu > 0$, there is only one group acting transitively on the hyperbolic oscillator group manifold, that is, the group itself.

Clearly, in this case, homogeneous structures of class $\mathcal{S}_3$ do not exist, since the corresponding symmetry $T_{ijk} = -T_{jik}$ does not hold (for example, $T_{224} = \mu e^{-2\mu x_4} \neq 0$).

The second case: $a = 0$**.**

In this case, a full classification of homogeneous structures is a much more difficult computational problem, corresponding to the fact that the full isometry group is larger. However, as long as the additional symmetry condition $T_{ijk} = -T_{jik}$ is satisfied, it is easy to check that tensor $S = T_{ijk}dx^i \otimes dx^j \otimes dx^k$, completely determined by the non-zero components $T_{ijk}, j < k$,

$$T_{234} = -T_{324} = T_{423} = -\frac{1}{2}, \tag{18}$$

satisfies Equation (2) and is a homogeneous structure belonging to class $\mathcal{S}_3$. Therefore, we proved the following.

Theorem 2. *The hyperbolic oscillator group, corresponding to the semi-direct extension of the three-dimensional Heisenberg group with Lie algebra (A) described in Proposition 5, equipped with the family of metrics g_a given by (12), is naturally reductive if and only if $a = 0$. For $a = 0$, Equation (18) describes a pseudo-Riemannian homogeneous structure $S \in \mathcal{S}_3$.*

For $a \neq 0$, Equation (17) describes the unique homogeneous structure of g_a (which is not of type $\mathcal{S}_3$).

We now focus on the hyperbolic oscillator group equipped with left-invariant metrics g_a, with $a \neq 0, \pm 1$, as g_0 is naturally reductive. We shall calculate the homogeneous geodesics passing through the base point $p = (0,0,0,0)$.

Consider an arbitrary smooth vector field $V = \sum_{i=1}^{4} X_i(x_1, \cdots, x_4)\partial_i$ on $(G_{\tilde{S}}, g_a)$. With a direct calculation using (14), we find that V is a Killing vector field if and only if the following system of equations is satisfied:

$$\begin{cases} (\mathrm{E}_1)\ 2\partial_1 X_4 - ax_2\partial_1 X_3 + ax_3\partial_1 X_2 + 2a\partial_1 X_1 = 0, \\ (\mathrm{E}_2)\ \partial_2 X_2(ax_3^2 + 4e^{-2\mu x_4}) - 4\mu X_4 e^{-2\mu x_4} + x_3(2a\partial_2 X_1 - ax_2\partial_2 X_3 + aX_3 + 2\partial_2 X_4) = 0, \\ (\mathrm{E}_3)\ \partial_3 X_3(ax_2^2 + 4e^{2\mu x_4}) + 4\mu X_4 e^{2\mu x_4} - x_2(2a\partial_3 X_1 + aX_3\partial_3 X_2 - aX_2 + 2\partial_3 X_4) = 0, \\ (\mathrm{E}_4)\ 2a\partial_4 X_4 - x_2\partial_4 X_3 + x_3\partial_4 X_2 + 2\partial_4 X_1 = 0, \\ (\mathrm{E}_5)\ ax_3\partial_4 X_2 - ax_2\partial_4 X_3 + 2a(\partial_4 X_1 + \partial_1 X_4) - x_2\partial_1 X_3 + x_3\partial_1 X_2 + 2\partial_4 X_4 + 2\partial_1 X_1 = 0, \\ (\mathrm{E}_6)\ \partial_1 X_2(ax_3^2 + 4e^{-2\mu x_4}) - ax_2x_3\partial_1 X_3 + 2ax_3(\partial_1 X_1 + \partial_2 X_2) - 2ax_2\partial_2 X_3 \\ \qquad +4a\partial_2 X_1 + 2aX_3 + 2x_3\partial_1 X_4 + 4\partial_2 X_4 = 0, \\ (\mathrm{E}_7)\ \partial_3 X_2(ax_3^2 + 4e^{-2\mu x_4}) + \partial_2 X_3(ax_2^2 + 4e^{2\mu x_4}) - 2ax_2\partial_2 X_1 + 2ax_3\partial_3 X_1 \\ \qquad -ax_2x_3(\partial_2 X_2 + \partial_3 X_3) - 2x_2\partial_2 X_4 + 2x_3\partial_3 X_4 - a(x_2X_3 + x_3X_2) = 0, \\ (\mathrm{E}_8)\ \partial_4 X_2(ax_3^2 + 4e^{-2\mu x_4}) - ax_2x_3\partial_4 X_3 + 2ax_3\partial_4 X_1 + 4a\partial_2 X_4 + 2x_3(\partial_2 X_2 + \partial_4 X_4) \\ \qquad -2x_2\partial_2 X_3 + 4\partial_2 X_1 + 2X_3 = 0, \\ (\mathrm{E}_9)\ \partial_1 X_3(ax_2^2 + 4e^{2\mu x_4}) - ax_2x_3\partial_1 X_2 - 2ax_2(\partial_1 X_1 + \partial_3 X_3) + 2ax_3\partial_3 X_2 + 4a\partial_3 X_1 \\ \qquad -2aX_2 - 2x_2\partial_1 X_4 + 4\partial_3 X_4 = 0, \\ (\mathrm{E}_{10})\ \partial_4 X_3(ax_2^2 + 4e^{2\mu x_4}) - ax_2x_3\partial_4 X_2 - 2x_2(\partial_3 X_3 + \partial_4 X_4) - 2ax_2\partial_4 X_1 + 4a\partial_3 X_4 \\ \qquad +2x_3\partial_3 X_2 + 4\partial_3 X_1 - 2X_2 = 0. \end{cases}$$

In order to solve the above system of equations, we first obtain some simpler conditions by differentiating some of them and taking the appropriate linear combinations. So, we obtain:

$$\begin{aligned} &\partial_4 \mathrm{E}_1 - a\partial_1 \mathrm{E}_4 : && 2(1-a^2)\partial_{14}^2 X_4 = 0, \\ &a\partial_4 \mathrm{E}_1 - \partial_1 \mathrm{E}_4 : && (a^2-1)\partial_{14}^2(X_2x_3 - X_3x_2 + 2X_1) = 0, \\ &\partial_4 \mathrm{E}_5 - \tfrac{1}{2(a^2-1)}\mathrm{E}_{12} - \tfrac{a}{2}\partial_4 \mathrm{E}_4 - \tfrac{a}{2(1-a^2)}\mathrm{E}_{11} : && (1-a^2)\partial_{44}^2 X_4 = 0. \end{aligned}$$

Integration of the three above equations yields

$$\begin{aligned} X_4 &= f_1(x_2, x_3)x_4 + f_2(x_1, x_2, x_3), \\ X_3 &= f_3(x_1, x_2, x_3) + f_4(x_2, x_3, x_4) + \tfrac{1}{x_2}(x_3X_2 + 2X_1), \end{aligned}$$

where $f_1(x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)$, and $f_4(x_2, x_3, x_4)$ are arbitrary functions. We substitute the above expressions of X_3 and X_4 into (E_1). Integrating, we have

$$f_3(x_1, x_2, x_3) = \frac{2}{ax_2} f_2(x_1, x_2, x_3) + f_5(x_2, x_3),$$

where $f_5(x_2, x_3)$ is arbitrary. (E_4) then reads $2af_1(x_2, x_3) - x_2\partial_4 f_4(x_2, x_3, x_4) = 0$, where

$$f_4(x_2, x_3, x_4) = \frac{2ax_4}{x_2} f_1(x_2, x_3) + f_6(x_2, x_3),$$

for an arbitrary function $f_6(x_2, x_3)$. (E_5) now reduces to $af_1(x_2, x_3) - \partial_1 f_2(x_1, x_2, x_3) = 0$ and by integrating, we obtain

$$f_2(x_1, x_2, x_3) = ax_1 f_1(x_2, x_3) + f_7(x_2, x_3),$$

where $f_7(x_2, x_3)$ is an arbitrary function. Then, one can calculate X_1 from (E_6) as follows:

$$X_1 = \frac{1}{4a}\big(2a^2 x_2 x_4 \partial_2 f_1(x_2, x_3) - 4a^2 x_4 f_1(x_2, x_3) + a x_2^2(\partial_2 f_5(x_2, x_3) + \partial_2 f_6(x_2, x_3)) \\ - a x_2(f_5(x_2, x_3) + f_6(x_2, x_3)) - 4a x_1 f_1(x_2, x_3) - 2a x_3 X_2 - 2x_2 \partial_1 X_2 e^{-2\mu x_4} \\ - 2x_2 x_4 \partial_2 f_1(x_2, x_3) - 4 f_7(x_2, x_3)\big).$$

We now substitute the above expressions into (E_9). By integrating, we find

$$X_2 = \sin(a x_1) f_9(x_2, x_3, x_4) + \cos(a x_1) f_8(x_2, x_3, x_4) \\ \frac{1}{2a}\big(2x_4(1 - a^2)\partial_3 f_1(x_2, x_3) - a x_2(\partial_3 f_6(x_2, x_3) + \partial_3 f_5(x_2, x_3))\big),$$

where $f_8(x_2, x_3, x_4)$ and $f_9(x_2, x_3, x_4)$ are arbitrary functions. Next, we substitute the above expressions into (E_2) and obtain

$$-2a\mu e^{-2\mu x_4} f_1(x_2, x_3) x_1 + H(x_2, x_3, x_4)\cos(a x_1) + K(x_2, x_3, x_4)\sin(a x_1) + T(x_2, x_3, x_4) = 0,$$

for some appropriate functions H, K and T. The above can be read as a null linear combination of the linearly independent functions x_1, $\cos(a x_1)$, $\sin(a x_1)$, and $1 = x_1^0$, where x_1 only appears in the first term. Therefore, it yields at once $f_1(x_2, x_3) = 0$. Applying a similar argument, from (E_8), we find

$$e^{-2\mu x_4}(\partial_4 f_8(x_2, x_3, x_4) - f_9(x_2, x_3, x_4))\cos(a x_1) \\ + e^{-2\mu x_4}(\partial_4 f_9(x_2, x_3, x_4) + f_8(x_2, x_3, x_4))\sin(a x_1) + \frac{a^2-1}{a}\partial_2 f_7(x_2, x_3) = 0$$

for all values of x_1, where

$$\partial_4 f_8(x_2, x_3, x_4) - f_9(x_2, x_3, x_4) = 0,\ \partial_4 f_9(x_2, x_3, x_4) + f_8(x_2, x_3, x_4) = 0,\ \partial_2 f_7(x_2, x_3) = 0.$$

By integrating, we obtain

$$f_7(x_2, x_3) = f_7(x_3), \\ f_8(x_2, x_3, x_4) = f_{10}(x_2, x_3)\sin(x_4) + f_{11}(x_2, x_3)\cos(x_4), \\ f_9(x_2, x_3, x_4) = f_{10}(x_2, x_3)\cos(x_4) - f_{11}(x_2, x_3)\sin(x_4).$$

Now, (E_{10}) reads

$$2\mu(f_{10}(x_2, x_3)\cos(x_4) - f_{11}(x_2, x_3)\sin(x_4))\cos(a x_1) \\ -2\mu(f_{10}(x_2, x_3)\sin(x_4) + f_{11}(x_2, x_3)\cos(x_4))\sin(a x_1) + \frac{a^2-1}{a} f_7'(x_3) = 0,$$

for all values of x_1. In particular, $f_7(x_3)$ is a real constant; so, we set $f_7(x_3) = \frac{c_1}{\mu}$, and $f_{10}(x_2, x_3) = f_{11}(x_2, x_3) = 0$. At this point, ($E_2$) and ($E_7$), respectively, give

$$(\partial^2_{23} f_6(x_2, x_3) + \partial^2_{23} f_5(x_2, x_3))x_2 + \partial_3 f_6(x_2, x_3) + \partial_3 f_5(x_2, x_3) + 2c_1 = 0, \\ \big((\partial^2_{22} f_6(x_2, x_3) + \partial^2_{22} f_5(x_2, x_3))x_2 + 2\partial_2 f_6(x_2, x_3) + 2\partial_2 f_5(x_2, x_3)\big)e^{2\mu x_4} \\ -(\partial^2_{33} f_6(x_2, x_3) + \partial^2_{33} f_5(x_2, x_3))x_2 e^{-2\mu x_4} = 0.$$

Direct integration of the coefficients depending only on (x_2, x_3) in the above equations, gives the following solution:

$$f_6(x_2, x_3) = -f_5(x_2, x_3) + \frac{1}{x_2}(-2c_1 x_3 + k_3)x_2 + k_2 x_3 + k_4),$$

where k_2, k_3, k_4 are arbitrary real coefficients. Finally, setting $k_2 = -2c_3$, $k_3 = -2c_2$ and $k_4 = \frac{2c_1(a^2-1)}{a\mu} - 2c_4$, we conclude that for $a \neq 0$, Killing vector fields form a four-parameter family, given by

$$V = c_1\Big(-\tfrac{a}{\mu}\partial_1 + x_2\partial_2 - x_3\partial_3 + \tfrac{1}{\mu}\partial_4\Big) + c_2\big(\tfrac{x_2}{2}\partial_1 - \partial_3\big) + c_3\big(\tfrac{x_3}{2}\partial_1 + \partial_2\big) + c_4\partial_1. \tag{19}$$

where here and throughout the paper, by c_i, we denote some real constants.

Let V denote the arbitrary Killing vector field described by Equation (19). The flow $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_1(t))$ of this family of vector fields through p is given by

$$\gamma_1(t) = \tfrac{1}{2c_1^2\mu}\big(c_2c_3\mu(e^{c_1t} - e^{-c_1t}) - 2c_1t(\mu(c_2c_3 - c_1c_4) + ac_1^2)\big),$$
$$\gamma_2(t) = \tfrac{c_3}{c_1}(e^{c_1t} - 1), \quad \gamma_3(t) = \tfrac{c_2}{c_1}(e^{-c_1t} - 1), \quad \gamma_4(t) = \tfrac{c_1t}{\mu},$$

for some $c_1 \neq 0$. We now calculate $\nabla_{V_{\gamma(t)}}V$ at the origin $p = (0,0,0,0)$ (i.e., for $t = 0$) and we find

$$\nabla_{V_{\gamma(t)}}V|_{t=0} = \frac{\mu(c_2^2 - c_3^2)}{a^2-1}\partial_1 + \frac{(a^2c_2 - \mu c_3 - c_2)c_1 - a\mu c_2c_4}{\mu}\partial_2 + \frac{(a^2c_3 - \mu c_2 - c_3)c_1 - a\mu c_3c_4}{\mu}\partial_3 - \frac{a\mu(c_2^2 - c_3^2)}{a^2-1}\partial_4.$$

On the other hand,

$$V_{\gamma(t)}|_{t=0} = -\frac{2ac_1 - 2\mu c_4}{2\mu}\partial_1 + c_3\partial_2 - c_2\partial_3 + \frac{c_1}{\mu}\partial_4$$

and so, Equation (4) yields

$$\begin{cases} \mu^2(c_2^2 - c_3^2) + k_\gamma(ac_1 - c_4\mu)(a^2-1) = 0, \\ (ac_2c_4 + k_\gamma c_3 + c_1c_3)\mu - c_1c_2(a^2-1) = 0, \\ (ac_3c_4 - k_\gamma c_2 + c_1c_2)\mu - c_1c_3(a^2-1) = 0, \\ (1-a^2)k_\gamma c_1 + (c_3^2 - c_2^2)\mu^2 a = 0. \end{cases}$$

We observe that the above equations also hold in the case $c_1 = 0$. Solving this system, we find the following sets of solutions:

$k_\gamma = 0, c_3 = \varepsilon c_2, c_4 = \frac{c_1(a^2-1-\varepsilon\mu)}{a\mu}, \varepsilon = \pm 1$ (for all $a \neq 0$),
$k_\gamma = 0, c_2 = c_3 = 0$ (for all $a \neq 0$),
$k_\gamma = arbitrary, c_1 = -k_\gamma, c_2 = 0, c_3 = \pm\frac{1}{\mu}\sqrt{\frac{1-a^2}{a}}k_\gamma, c_4 = -\frac{k_\gamma(a^2-1)}{a\mu}$ ($a < -1$ or $0 < a < 1$),
$k_\gamma = c_1 = \varepsilon\mu\sqrt{\frac{a}{1-a^2}}c_2, c_3 = 0, c_4 = \varepsilon\sqrt{\frac{1-a^2}{a}}c_2, \varepsilon = \pm 1$ ($a < -1$ or $0 < a < 1$),

where we excluded the solutions giving rise to vector fields vanishing at p. We substitute the first solution of the above set in (19) and we obtain that the integral curve through the origin p, of the Killing vector field V corresponding to $W_1 = -\frac{c_1(\varepsilon\mu+1)}{a\mu}\partial_1 + c_3\partial_2 - \varepsilon c_3\partial_3 + \frac{c_1}{\mu}\partial_4$, is homogeneous. Observe that we can rewrite W_1 in a simpler way by setting $w_1 = \frac{c_1}{\mu}$ and $w_3 = c_3$. By applying similar arguments to the other above solutions, we prove the following.

Theorem 3. *Let $G_{\tilde{S}} = H \rtimes_{\tilde{S}} \mathbb{R}$ be the hyperbolic oscillator group, corresponding to the semi-direct extension of the three-dimensional Heisenberg group with Lie algebra (A) described in Proposition 5, equipped with the family of metrics g_a ($a \neq 0$) given by (12). With respect to coordinates (x_1, x_2, x_3, x_4) described in Proposition 4, homogeneous geodesics through the base point $p(0,0,0,0)$ are the integral curves of Killing vector fields, determined by the tangent vectors $W_i \in T_pM$ listed below:*

$$\begin{aligned}
&W_1 = \left(\tfrac{w_1(\pm\mu-1)}{a}, w_3, \varepsilon w_3, w_1\right), && \textit{for all } a \neq 0, \pm 1,\\
&W_2 = (w_1, 0, 0, w_4), && \textit{for all } a \neq 0, \pm 1,\\
&W_3 = w_1\left(1, \pm\sqrt{a(1-a^2)}, 0, -a\right), && \textit{for either } a < -1 \textit{ or } 0 < a < 1,\\
&W_4 = w_1\left(1, 0, \pm\sqrt{a(1-a^2)}, -a\right), && \textit{for either } a < -1 \textit{ or } 0 < a < 1,
\end{aligned}$$

where w_i denotes some real constants.

We now determine geodesic Killing vector fields. We first calculate $\nabla_V V$ for the Killing vector fields determined above and apply Equation (5). So, for a Killing vector field as described by (19), we find that V is a geodesic Killing vector field (i.e., $\nabla_V V = k.V$) if and only if

$$\begin{cases}
\left((2c_1x_2+2c_3)\mu^2 + ax_2(a^2-1)(c_1x_2x_3+c_2x_2+c_3x_3+c_4)\mu - c_1x_2(a^2-1)^2\right)(c_1x_2+c_3)e^{-2\mu x_4}\\
-\left((2c_1x_3+2c_2)\mu^2 + ax_3(a^2-1)(c_1x_2x_3+c_2x_2+c_3x_3+c_4)\mu - c_1x_3(a^2-1)^2\right)(c_1x_3+c_2)e^{2\mu x_4}\\
+\left(\left((c_2x_2+c_3x_3+2c_4)\mu - 2ac_1\right)k + \mu c_1(c_2x_2-c_3x_3)\right)(a^2-1) = 0,\\
(c_1x_3+c_2)(-a(c_1x_2x_3+c_2x_2+c_3x_3+c_4)\mu + c_1(a^2-1))e^{2\mu x_4} - \mu(c_1x_2+c_3)(k+c_1) = 0,\\
(c_1x_2+c_3)(-a(c_1x_2x_3+c_2x_2+c_3x_3+c_4)\mu + c_1(a^2-1))e^{-2\mu x_4} + \mu(c_1x_3+c_2)(k-c_1) = 0,\\
a\mu^2(c_1x_2+c_3)^2e^{-2\mu x_4} - a\mu^2(c_1x_3+c_2)^2e^{2\mu x_4} + c_1(1-a^2)k = 0.
\end{cases}$$

Solving the above system, we find $k = 0$ and $c_1 = c_2 = c_3 = 0$, that is, $V = c_4\partial_1$. Hence, we proved the following.

Theorem 4. *The only geodesic Killing vector fields V of the hyperbolic oscillator group $(G_{\tilde{S}}, g_a)$ ($a \neq 0$) are given by $V = c_4\partial_1$.*

5. The Remaining Model of Semi-Direct Extensions of *H*

We now consider the semi-direct extension $(G_{\tilde{S}}, g_a)$ of the Heisenberg group as described in Case (B) of Proposition 5, i.e., the remaining model with non-isomorphic Lie algebras for the semi-direct extension of the Heisenberg group. In this case, functions described in Proposition 4 become

$$u(x_4) = 1, \quad v(x_4) = 0, \quad w(x_4) = \mu x_4, \quad z(x_4) = 1. \tag{20}$$

From now on, we shall exclude the case where $\mu = 0$ (i.e., $\tilde{S} = 0$), as in this case, the manifold is locally symmetric ([7], Theorem 3.4) and so, it is naturally reductive.

The arguments follow the same ideas we illustrated in detail for the hyperbolic oscillator group. In this case, the Levi-Civita connection, Riemann curvature, and Ricci tensor are, respectively, described as follows:

$$
\begin{aligned}
\nabla_{\partial_1}\partial_2 &= -\tfrac{a}{4}(x_2(\mu^2x_4^2+1)-\mu x_3x_4)\partial_1 - \tfrac{a}{2}\mu x_4\partial_2 - \tfrac{a}{2}(\mu^2x_4^2+1)\partial_3,\\
\nabla_{\partial_1}\partial_3 &= \tfrac{a}{4}(\mu x_2x_4 - x_3)\partial_1 + \tfrac{a}{2}\partial_2 + \tfrac{a}{2}\mu x_4\partial_3,\\
\nabla_{\partial_2}\partial_2 &= \tfrac{a\mu x_3^2x_4(a^2-1)-ax_2x_3(\mu^2x_4^2+1)(a^2-1)+4\mu^2x_4}{4(a^2-1)}\partial_1 - \tfrac{a}{2}\mu x_3x_4\partial_2\\
&\quad -\tfrac{a}{2}x_3(\mu^2x_4^2+1)\partial_3 - \tfrac{a}{a^2-1}\mu^2x_4\partial_4,\\
\nabla_{\partial_2}\partial_3 &= \tfrac{((\mu^2x_4^2+1)x_2^2-x_3^2)(a^3-a)-4\mu}{8(a^2-1)}\partial_1\\
&\quad +\tfrac{a}{4}(\mu x_2x_4+x_3)\partial_2 + \tfrac{a}{4}(\mu^2x_2x_4^2+\mu x_3x_4+x_2)\partial_3 + \tfrac{a}{2(a^2-1)}\mu\partial_4,\\
\nabla_{\partial_2}\partial_4 &= \tfrac{1}{4}((\mu^2(\mu-1)x_4^2-\mu-1)x_2-\mu x_3x_4(\mu-1))\partial_1 + \tfrac{1}{2}\mu x_4(\mu-1)\partial_2\\
&\quad -\tfrac{1}{2}(\mu^2x_4^2+1-\mu^3x_4^2+\mu)\partial_3,\\
\nabla_{\partial_3}\partial_3 &= -\tfrac{a}{4}x_2(\mu x_2x_4-x_3)\partial_1 - \tfrac{a}{2}x_2\partial_2 - \tfrac{a}{2}\mu x_2x_4\partial_3,\\
\nabla_{\partial_3}\partial_4 &= -\tfrac{1}{4}(\mu x_2x_4-x_3)(\mu-1)\partial_1 + \tfrac{1}{2}(1-\mu)\partial_2 - \tfrac{1}{2}\mu x_4(\mu-1)\partial_3.
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
R_{1212} &= \tfrac{a^2}{4}(\mu^2x_4^2+1), & R_{1213} &= -\tfrac{a^2}{4}\mu x_4,\\
R_{1223} &= \tfrac{a^2}{8}(x_2(\mu^2x_4^2+1)-\mu x_3x_4), & R_{1224} &= \tfrac{a}{4}(x_4^2\mu^2(\mu-1)-\mu-1),\\
R_{1234} &= -\tfrac{a}{4}\mu x_4(\mu-1), & R_{1313} &= \tfrac{a^2}{4},\\
R_{1323} &= -\tfrac{a^2}{8}(\mu x_2x_4-x_3), & R_{1324} &= -\tfrac{a}{4}\mu x_4(\mu-1),\\
R_{1334} &= \tfrac{a}{4}(\mu-1), & &\\
R_{2323} &= \tfrac{a}{16(a^2-1)}(((\mu^2x_4^2+1)x_2^2-2\mu x_4x_3x_2+x_3^2)(a^3-a)-12(a^2-1)+4\mu^2), & &\\
R_{2324} &= \tfrac{a}{8}((\mu^3-\mu^2)x_2x_4^2+(\mu-\mu^2)x_3x_4-\mu x_2-x_2), & &\\
R_{2334} &= -\tfrac{a}{8}(\mu-1)(\mu x_2x_4-x_3), & R_{2424} &= \tfrac{1}{4}(\mu-1)((\mu^3-\mu^2)x_4^2-3\mu-1),\\
R_{2434} &= -\tfrac{1}{4}\mu x_4(\mu-1)^2, & R_{3434} &= \tfrac{1}{4}(\mu-1)^2,
\end{aligned}
$$

$$
\begin{aligned}
\varrho = {} & \tfrac{a^2}{2}dx_1^2 + \tfrac{a^2}{2}x_3dx_1dx_2 - \tfrac{a^2}{2}x_2dx_1dx_3 + adx_1dx_4\\
& + \tfrac{a}{8(a^2-1)}((4\mu^2x_4^2-ax_3^2+4)(1-a^2)+4\mu^4x_4^2-4\mu^2)dx_2^2\\
& - \tfrac{a}{4(a^2-1)}((ax_2x_3-4\mu x_4)(a^2-1)+4\mu^3x_4)dx_2dx_3 + \tfrac{a}{2}x_3dx_2dx_4\\
& + \tfrac{a}{8(a^2-1)}((ax_2^2-4)(a^2-1)+4\mu^2)dx_3^2 - \tfrac{a}{2}x_2dx_3dx_4 - \tfrac{1}{2}(\mu^2-1)dx_4^2.
\end{aligned}
$$

The analogue of Theorem 2 is the following result.

Theorem 5. *The nilpotent semi-direct extension of the three-dimensional Heisenberg group with Lie algebra* (B) *described in Proposition 5 (with* $\mu \neq 0$*), equipped with the family of metrics* g_a *given by (12), is naturally reductive if and only if* $a = 0$*. In fact, the non-zero components* T_{ijk}*,* $j < k$ *given by*

$$T_{234} = -T_{324} = T_{423} = -\frac{1}{2}(1-\mu),$$

describe a homogeneous structure belonging to class $\mathcal{S}_3$ *in the case where* $a = 0$*. On the other hand, for* $a \neq 0$*, the following non-zero components describe the unique homogeneous structure of* g_a *(which does not belong to class* $\mathcal{S}_3$*):*

$$
\begin{aligned}
T_{312} &= -T_{213} = -T_{123} = \tfrac{a}{2}, & T_{223} &= -\tfrac{ax_3}{2}, & T_{323} &= \tfrac{ax_2}{2},\\
T_{423} &= T_{324} = -\tfrac{1}{2}(1-\mu), & T_{224} &= -\mu^2x_4, & T_{234} &= \tfrac{1}{2}(1+\mu).
\end{aligned}
$$

We now focus on the remaining metrics g_a with $a \neq 0$. Killing vector fields again form a four-parameter family, given by

$$V = c_1\left(\tfrac{x_3}{2}\partial_1 + \partial_2\right) + c_2\left(\tfrac{a}{\mu}\partial_1 - x_2\partial_3 - \tfrac{1}{\mu}\partial_4\right) + c_3\left(\tfrac{x_2}{2}\partial_1 - \partial_3\right) + c_4(\partial_1). \quad (22)$$

The descriptions of $\nabla_{V_{\gamma(t)}} V|_{t=0}$ and $V_{\gamma(t)}|_{t=0}$ are now given by

$$\begin{aligned}\nabla_{V_{\gamma(t)}} V|_{t=0} =& \frac{\mu c_1 c_3}{a^2-1}\partial_1 - \frac{((a^2+\mu-1)c_2 + a\mu c_4)c_3}{\mu}\partial_2 \\ & - \frac{c_1(a^2 c_2 + a\mu c_4 - c_2)}{\mu}\partial_3 - \frac{a\mu c_1 c_3}{a^2-1}\partial_4, \\ V_{\gamma(t)}|_{t=0} =& \frac{1}{\mu}(ac_2 + \mu c_4)\partial_1 + c_1\partial_2 - c_3\partial_3 - \frac{c_2}{\mu}\partial_4.\end{aligned}$$

Applying Equation (4) and $\nabla_V V = k.V$, we then obtain the following analogues of Theorems 3 and 4.

Theorem 6. *Let $G_{\mathfrak{F}} = H \rtimes_{\mathfrak{F}} \mathbb{R}$ be the nilpotent semi-direct extension of the three-dimensional Heisenberg group with Lie algebra (B) described in Proposition 5 ($\mu \neq 0$), equipped with the family of metrics g_a given by (12). With respect to coordinates (x_1, x_2, x_3, x_4) described in Proposition 4, homogeneous geodesics through the base point $p(0,0,0,0)$ are the integral curves of Killing vector fields, determined by the tangent vectors $W_i \in T_p M$ listed below:*

$$W_1 = (w_1, 0, 0, w_4), \qquad W_2 = (w_1, w_2, 0, -aw_1), \qquad W_3 = \left(w_1, 0, w_3, \tfrac{aw_1}{\mu-1}\right).$$

Theorem 7. *Consider $G_{\mathfrak{F}} = H \rtimes_{\mathfrak{F}} \mathbb{R}$ to be the nilpotent semi-direct extension of the three-dimensional Heisenberg group with Lie algebra (B) described in Proposition 5 ($\mu \neq 0$), equipped with the family of metrics g_a given by (12). The geodesic Killing vector fields V of $(G_{\mathfrak{F}}, g_a)$ ($a \neq 0$) are given by*

$$V = c_4\partial_1,$$

where c_4 is a real constant.

6. Final Remarks and Comments

This paper is a contribution to the general problem of understanding which of the known properties of the oscillator group extend to other relevant semi-direct extensions of the Heisenberg group. The present study focuses on homogeneity properties, namely, pseudo-Riemannian homogeneous structures (and the geometry determined by them: in particular, natural reductivity) and homogeneous geodesics. Compared with the results in [10] for the oscillator group, our study shows that some properties are really specific for the case of the Lorentzian oscillator group and are not shared by other semi-direct extensions. Moreover, the obtained classification and explicit description of the homogeneous geodesics and geodesic vector fields for models (A) and (B) could also be useful for further investigations and applications of these models in the framework of Theoretical Physics.

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