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High Energy Physics – Theory

Comments on ABJM free energy on S^3 at large N and perturbative expansions in M-theory and string theory

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Abstract

We compare large N expansion of the localization result for the free energy F in the 3d $\mathcal{N} = 6$ superconformal $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory to its AdS/CFT counterpart, *i.e.* to the perturbative expansion of M-theory partition function on $AdS_4 \times S^7/\mathbb{Z}_k$ and to the weak string coupling expansion of type IIA effective action on $AdS_4 \times CP^3$. We show that the general form of the perturbative expansions of F on the two sides of the AdS/CFT duality is indeed the same. Moreover, the transcendentality properties of the coefficients in the large N, large k expansion of F match those in the corresponding M-theory or string theory expansions. To shed light on the structure of the 1-loop M-theory partition function on $AdS_4 \times S^7/\mathbb{Z}_k$ we use the expression for the 1-loop 4-graviton scattering amplitude in the 11d supergravity. We also use the known information about the transcendental coefficients of the leading curvature invariants in the low-energy effective action of type II string theory. Matching of the remaining rational factors in the coefficients requires a precise information about currently unknown RR field strength terms in the corresponding superinvariants.

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Contents

1. Introduction

Localization [1] provides a remarkable source of information about supersymmetric gauge theories beyond the standard weak-coupling perturbation theory. In the context of AdS/CFT duality [2] this information may be used to learn about the structure of string theory or M-theory corrections to the tree level or supergravity order.

Here we shall focus on the 3d $\mathcal{N}=6$ supersymmetric $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory [3] in which the free energy $F(N,k)$ on S^3 was computed by localization in [4–7] (see [8] for a review and further references). For large N and fixed k this theory is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ while for large N and large k with fixed $\lambda = \frac{N}{k}$ is dual to the 10d type IIA string theory on $AdS_4 \times CP^3$ background.[2]

Our aim will be to compare the large N expansion of F to its AdS/CFT counterpart, *i.e.* to perturbative expansion of the M-theory partition function on $AdS_4 \times S^7/\mathbb{Z}_k$ or weak string coupling expansion of string theory effective action on $AdS_4 \times CP^3$. Related work appeared in [9,10] and [11–15] and also in [16–18]. In the M-theory the expansion parameter is the inverse of the effective dimensionless M2-brane tension [3]

$$T_2 = \frac{1}{(2\pi)^2}\frac{L_{11}^3}{\ell_P^3}\,, \qquad \frac{L_{11}}{\ell_P} = \big(2^5\pi^2 N k\big)^{1/6}\,, \tag{1.1}$$

while the type IIA string coupling and effective string tension are

$$g_s = \sqrt{\pi}\,\Big(\frac{2}{k}\Big)^{5/4} N^{1/4} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N}\,, \qquad \lambda = \frac{N}{k}\,, \tag{1.2}$$

[2] This superconformal theory represents N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. The orbifold acts as $z_i \to e^{\frac{2\pi i}{k}} z_i$ where z_i, $i = 1, 2, 3, 4$ are four complex coordinates transverse to the M2-branes.

$$T = \frac{1}{8\pi}\frac{L^2}{\alpha'} = g_s^{2/3}\frac{L_{11}^2}{8\pi\alpha'} = \frac{\sqrt{\lambda}}{\sqrt{2}}\,, \qquad\qquad \frac{g_s^2}{8\pi\, T} = \frac{\lambda^2}{N^2} = \frac{1}{k^2}\,, \tag{1.3}$$

where L_{11} and L are curvature scales in the 11d and 10d metrics.

We will show that the general structure of perturbative expansions of F on the two sides of the AdS/CFT duality is indeed the same. Moreover, the transcendentality properties of the coefficients in the large N, large k expansion of the localization expression for F match those in the corresponding M-theory or string theory expansions. In particular, we will focus on the N-independent $A(k)$ part of F and show that the leading $\zeta(3)k^2$ term in its large k expansion corresponds to the $\zeta(3)$ term in the 1-loop 11d graviton amplitude on $M^{10} \times S^1$ [19,20] or the tree-level $\zeta(3)R^4$ term in the 10d string theory effective action.[3] Also, we will find that the π-dependent factors in the coefficients of subleading $\frac{1}{k^n}$ terms match those in the coefficients of the corresponding curvature invariants in the M-theory or string theory effective actions.[4] To match the remaining rational factors in the coefficients requires precise knowledge of the structure of the corresponding superinvariants (RR flux terms in them) and remains an open problem.

The order N^0 term in F should correspond to the order $(\mathrm{T}_2)^0$ or 1-loop correction in M-theory. A similar k-dependent factor in the 1/2 BPS Wilson loop expectation value in the ABJM theory was recently reproduced [22] as the 1-loop quantum M2-brane correction.[5] The $A(k)$ term in the free energy should represent the contribution of quantum M2-brane states propagating in the loop. In addition to point-like M2 branes one may need to include also contributions of BPS M2 branes wrapping 2-cycles in CP^3 part of $S^7/\mathbb{Z}_k$. As we shall discuss below, the structure of the $A(k)$ function suggests a close analogy of the present case with the Calabi-Yau compactification one in [24–26].[6]

This paper is organized as follows. In section 2 we review the structure of the large N perturbative part of the free energy as found from localization in ABJM theory on S^3. In section 3 we compare its large N, fixed k expansion to the perturbative expansion of the partition function or effective action in M-theory. In section 3 we discuss the large N, fixed λ expansion of F and show its correspondence with the perturbative expansion in type IIA string theory on $\mathrm{AdS}_4 \times \mathrm{CP}^3$.

Some basic relations and notation are summarized in Appendix A. In Appendix B we recall the matching of the leading large N term in F with the 11d supergravity action evaluated on the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background. In appendix C we present the $\mathrm{AdS}_4 \times \mathrm{CP}^3$ values of the R^4 invariants that appear in the tree level and one loop term in the type IIA string effective action. Appendix D contains a brief review of the structure of non-perturbative terms in the ABJM free energy.

[3] In addition to the M-theory and weakly coupled string theory limits one may consider a limit of large N with fixed N/k^5 that corresponds to the type IIA string at finite string coupling and thus interpolates between M-theory at strong coupling and perturbative string theory at weak coupling. Ref. [14] used that limit to compute R^4 terms at finite coupling.

[4] This is similar to what was observed [21] in the discussion of the leading strong-coupling terms in the localization result for free energy in the orbifold $\mathcal{N}=2$ gauge theory at each order in the $1/N^2$ expansion. These terms take the form of a series in $\frac{\lambda^{3/2}}{N^2} \sim \frac{g_s^2}{T}$ and can be matched (up to rational coefficients) with the contributions coming from the $D^n R^m$ terms (of lowest order in α' at each order in g_s^2) in type IIB string effective action.

[5] It was observed in [23] that this $(\sin\frac{2\pi}{k})^{-1}$ prefactor (where $\frac{1}{k^2} = \frac{g_s^2}{8\pi T}$ as in (1.3)) in the Wilson loop expectation value effectively sums up the leading large T contributions at each order in g_s^2. In [22] this prefactor was derived as a 1-loop correction in the M2-brane world-volume theory and thus it was concluded that this 1-loop M2 brane correction effectively sums up all large tension terms at all orders in the weak string coupling expansion in the dual type IIA theory.

[6] It would be interesting also to try to do a similar matching in the case of the topological indices (or special partition functions on $S^2 \times S^1$) for which localization results were discussed in [17,18].

2. Free energy of ABJM model in the large N expansion

Our starting point will be the localization result for the free energy of the ABJM theory on S^3 expanded at large N. We will consider both fixed k and large k perturbative expansions ignoring non-perturbative corrections.

The partition function of the ABJM theory on S^3 was first expressed in terms of a localization matrix model in [4]. It was later mapped to a lens space matrix model and solved in planar limit in [5]. Higher genus $1/N$ corrections were computed in [6,7] by integrating the holomorphic anomaly equation. Neglecting non-perturbative corrections (reviewed in [27,8]) the resummed partition function was determined in [28]. The same result was later rederived by Fermi gas methods in [29] and tested numerically in [30] at finite N, k. The resulting perturbative partition function reads

$$Z(N,k) \equiv e^{-F(N,k)} = (\tfrac{1}{2}\pi^2 k)^{1/3}\, e^{A(k)}\ \mathrm{Ai}(z), \qquad z = (\tfrac{1}{2}\pi^2 k)^{1/3}\Big(N - \frac{k}{24} - \frac{1}{3k}\Big)\,. \tag{2.1}$$

The presence of the function $A(k)$ was first detected in [30] and incorporated into the Fermi gas formalism of [29] that provided its small k expansion. The large k expansion of $A(k)$ was identified in [30] with a topological string "constant map" contribution [31].[7] Ref. [30] proposed a resummed integral representation for $A(k)$ (later improved in [34]) valid at both small and large k[8]

$$A(k) = -\frac{\zeta(3)}{8\pi^2}\Big(k^2 - \frac{16}{k}\Big) + \frac{k^2}{\pi^2}\int_0^\infty dx\, \frac{x}{e^{kx}-1}\log(1-e^{-2x})\,. \tag{2.2}$$

The expansions of (2.2) in the two regimes may be determined as asymptotic series. For $k \ll 1$

$$A(k) \overset{k\ll 1}{=} \frac{2\,\zeta(3)}{\pi^2}\frac{1}{k} + \sum_{n=1}^{\infty}(-1)^n \frac{\pi^{2n-2}}{(2n)!}\, B_{2n} B_{2n-2}\, k^{2n-1}\,, \tag{2.3}$$

where B_{2n} are Bernoulli numbers. At large k one finds

$$A(k) \overset{k\gg 1}{=} -\frac{\zeta(3)}{8\pi^2}k^2 + \frac{1}{6}\log\frac{4\pi}{k} + 2\zeta'(-1) + \bar{A}(k)\,, \qquad \bar{A}(k) = \sum_{h=2}^{\infty}\frac{q_h}{k^{2h-2}}\,, \tag{2.4}$$

where q_n are rational numbers expressed again in terms of the products of two Bernoulli numbers or even-argument zeta-function values $\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}$ as

$$q_h = \frac{(2\pi)^{2h-2}(-1)^{h+1}4^{h-1}}{h(2h-2)(2h-2)!}\, B_{2h}\, B_{2h-2}\,. \tag{2.5}$$

The expansion (2.4) reproduces (see below) the dominant terms in $\lambda = \frac{N}{k} \gg 1$ in the $1/N$ expansion of (2.1). For this reason the resummation proposal (2.2) is usually considered to be correct.[9]

[7] The lens space Chern-Simons matrix model partition function can be interpreted as a partition function of a large N dual of a topological string theory on a certain class of local Calabi-Yau geometries [32]. This is a generalization of the Gopakumar-Vafa duality [33].

[8] The specific values of $A(k)$ at integer k are given in Eq. (3.14) of [34]. In particular, $A(1) = -\frac{\zeta(3)}{8\pi^2} + \frac{1}{4}\log 2$.

[9] Numerical tests of (2.2) for intermediate values of k were also presented in [30].

Below we shall use (2.1), (2.2) as a starting point ignoring non-perturbative corrections (for some comments on them see Appendix D).

From the exact expression of $F(N,k)$ we can work out its large N expansion at fixed k

$$F = \frac{1}{3}\sqrt{2}\pi k^{1/2}N^{3/2} - \frac{\pi}{24\sqrt{2}}\big(k^2+8\big)\,k^{-1/2}\,N^{1/2} + \frac{1}{4}\log\frac{32N}{k} - A(k) + \frac{\pi(k^2+8)^2}{2304\sqrt{2}k^{3/2}N^{1/2}} - \frac{k^2+8}{96kN} + \frac{69120k^2+\pi^2(k^2+8)^3}{331776\sqrt{2}k^{5/2}\pi N^{3/2}} - \frac{(k^2+8)^2}{4608k^2N^2} + \cdots . \tag{2.6}$$

One can then assume that k is large and isolate the leading terms in k order by order in large N

$$F^{k\gg 1} = \frac{\sqrt{2}\pi}{3}\,k^{1/2}\,N^{3/2} - \frac{\pi}{24\sqrt{2}}k^{3/2}\,N^{1/2} + \frac{1}{4}\log N + \frac{\zeta(3)}{8\pi^2}k^2 + \frac{\pi}{2304\sqrt{2}}k^{5/2}N^{-1/2} - \frac{1}{96}kN^{-1} + \frac{\pi}{331776\sqrt{2}}k^{7/2}N^{-3/2} + \cdots . \tag{2.7}$$

Here the $\zeta(3)$ term came from the first term in (2.4). As follows from (2.1), for large k at each order in $1/N$ the relevant combination should be $N - \frac{1}{24}k$ and indeed one finds that (2.7) may be rewritten as

$$F^{k\gg 1} = \frac{\pi}{3}\sqrt{2k}\Big(N - \frac{k}{24}\Big)^{3/2} + \frac{1}{4}\log\Big(N - \frac{k}{24}\Big) + \frac{\zeta(3)}{8\pi^2}k^2 + \cdots . \tag{2.8}$$

In the 't Hooft expansion, *i.e.* the expansion in $1/N$ with fixed $\lambda = \frac{N}{k}$, the resulting large N expression of F may be written as

$$F = -\log Z = -\sum_{h=0}^{\infty}(-1)^{h-1}f_h(\lambda)\Big(\frac{2\pi\lambda}{N}\Big)^{2h-2} + \frac{1}{6}\log\frac{N}{\lambda} . \tag{2.9}$$

Since we isolated in (2.9) the $1/N$ factors in the combination $\frac{\lambda}{N} = \frac{1}{k}$, it follows from (2.1) that the functions $f_h(\lambda)$ should naturally depend on the shifted coupling

$$\widehat{\lambda} \equiv \lambda - \frac{1}{24} = \frac{1}{k}(N - \frac{1}{24}k) . \tag{2.10}$$

Explicitly, one finds the following expressions for f_n [17] (A is Glaisher constant)

$$\begin{aligned}
f_0 &= \tfrac{4\sqrt{2}\pi^3}{3}\widehat{\lambda}^{3/2} + \tfrac{1}{2}\zeta(3) ,\\
f_1 &= \tfrac{\pi}{3\sqrt{2}}\widehat{\lambda}^{1/2} - \tfrac{1}{4}\log\widehat{\lambda} + \tfrac{1}{6} - \tfrac{11}{12}\log 2 + \tfrac{1}{6}\log\pi - 2\log \mathsf{A} ,\\
f_2 &= -\tfrac{1}{360} + \tfrac{1}{144\pi\sqrt{2}}\widehat{\lambda}^{-1/2} - \tfrac{1}{48\pi^2}\widehat{\lambda}^{-1} + \tfrac{5}{96\pi^3\sqrt{2}}\widehat{\lambda}^{-3/2},\\
f_3 &= -\tfrac{1}{22680} - \tfrac{1}{10368\sqrt{2}\pi^3}\widehat{\lambda}^{-3/2} + \tfrac{1}{1152\pi^4}\widehat{\lambda}^{-2} - \tfrac{5}{768\sqrt{2}\pi^5}\widehat{\lambda}^{-5/2} + \tfrac{5}{512\pi^6}\widehat{\lambda}^{-3},\\
f_4 &= -\tfrac{1}{340200} + \tfrac{1}{331776\sqrt{2}\pi^5}\widehat{\lambda}^{-5/2} - \tfrac{1}{20736\pi^6}\widehat{\lambda}^{-3} + \tfrac{25}{36864\sqrt{2}\pi^7}\widehat{\lambda}^{-7/2} - \tfrac{5}{2048\pi^8}\widehat{\lambda}^{-4}\\
&\quad + \tfrac{1105}{147456\sqrt{2}\pi^9}\widehat{\lambda}^{-9/2},\\
f_5 &= -\tfrac{1}{2494800} - \tfrac{1}{7962624\sqrt{2}\pi^7}\widehat{\lambda}^{-7/2} + \tfrac{1}{331776\pi^8}\widehat{\lambda}^{-4} - \tfrac{175}{2654208\sqrt{2}\pi^9}\widehat{\lambda}^{-9/2}\\
&\quad + \tfrac{5}{12288\pi^{10}}\widehat{\lambda}^{-5} - \tfrac{1105}{393216\sqrt{2}\pi^{11}}\widehat{\lambda}^{-11/2} + \tfrac{565}{131072\pi^{12}}\widehat{\lambda}^{-6} , \quad \ldots
\end{aligned} \tag{2.11}$$

A remarkable feature of these expressions for $f_h(\widehat{\lambda})$ is that they are given by finite sums of terms. For $h \geqslant 2$ we get

$$f_h(\widehat{\lambda}) = p_h + \sum_{s=1}^{h+1} \frac{p_{h,s}}{\big(2\pi\,\sqrt{\widehat{\lambda}}\,\big)^{s+2h+4}}\,, \qquad h \geqslant 2\,, \tag{2.12}$$

where all p_h and $p_{h,s}$ are *rational* coefficients.

As a further refinement, we may consider the $\lambda \gg 1$ expansion and isolate the leading powers of λ at each order in $1/N$ in (2.9). These special terms read (omitting $\log k = \log \frac{N}{\lambda}$ term in (2.9))

$$\widetilde{F} \equiv F^{\lambda\gg1} = N^2\Big(\frac{\pi\sqrt{2}}{3}\lambda^{-1/2} - \frac{\pi\sqrt{2}}{48}\lambda^{-3/2} + \frac{1}{8\pi^2}\zeta(3)\lambda^{-2} + ...\Big) - \frac{\pi}{3\sqrt{2}}\lambda^{1/2} \;+\; \bar{F}\,, \tag{2.13}$$

$$\bar{F} = -\bar{A}(k) = -\sum_{h=2}^{\infty}(-1)^{h-1}p_h\Big(\frac{2\pi}{k}\Big)^{2h-2} = -\sum_{h=2}^{\infty}(-1)^{h-1}p_h\Big(\frac{2\pi\lambda}{N}\Big)^{2h-2}$$
$$= -\frac{\pi^2}{90}\frac{\lambda^2}{N^2} + \frac{2\pi^4}{2835}\frac{\lambda^4}{N^4} - \frac{8\pi^6}{42525}\frac{\lambda^6}{N^6} + \frac{16\pi^8}{155925}\frac{\lambda^8}{N^8} + \cdots\,. \tag{2.14}$$

Here we kept few subleading large λ terms only in the first N^2 term. Comparing to (2.6), (2.4) we conclude that the coefficients p_h are related to q_h in $\bar{A}$ in (2.4) as (cf. (2.5))

$$p_h = \frac{(-1)^{h-1}}{(2\pi)^{2h-2}}\,q_h = \frac{4^{h-1}B_{2h}\,B_{2h-2}}{h(2h-2)(2h-2)!}\,. \tag{2.15}$$

The two expansions we have discussed (large N at fixed k and large N, large k with fixed $\lambda = \frac{N}{k}$) should correspond to the M-theory and type IIA string theory expansions. We shall discuss this connection in the next sections.

3. M-theory perturbative expansion

The large N, fixed k expansion of the ABJM theory should be dual to the perturbative expansion of M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ in which the curvature scale L_{11} is small compared to the 11d Planck length ℓ_P so that the effective dimensionless M2-brane tension T_2 is large (see Appendix A for our notation)

$$\mathrm{T}_2 \equiv L_{11}^3 T_2 = \frac{L_{11}^3}{(2\pi)^2\ell_P^3} = \frac{\mathrm{L}^3}{4\pi^2}\,, \qquad \mathrm{L} = \frac{L_{11}}{\ell_P} \gg 1\,, \tag{3.1}$$

while the parameter k of the 11d background (related to the radius of the 11d circle) is fixed. Indeed, since according to (A.9)

$$\mathrm{L}^6 = 32\pi^2 N k\,, \tag{3.2}$$

this limit is equivalent to the large N, fixed k expansion.

Thus the M-theory perturbative expansion should be in inverse powers of T_2 or in powers of L^{-3}. Expressed in terms of L and k the large N expansion of the free energy F in (2.6) is indeed

$$F = c_0\frac{1}{k}\mathrm{L}^9 + c_1\frac{8+k^2}{k}\mathrm{L}^3 + \frac{1}{2}\log\frac{\mathrm{L}^3}{\pi k} - A(k) + c_2\frac{(8+k^2)^2}{k}\mathrm{L}^{-3} + c_3\frac{8+k^2}{k}\mathrm{L}^{-6}$$

$$+\,\mathrm{O}(\mathrm{L}^{-9}),\tag{3.3}$$

$$c_0=\frac{1}{384\pi^2},\qquad c_1=-\frac{1}{192}\,,\qquad c_2=\frac{\pi^2}{576},\qquad c_3=-\frac{\pi^2}{3}\,,\ \ \ldots\tag{3.4}$$

As was suggested in [35], the presence of the $\int R^4 C_3$ term in the 11d effective action [36] implies the following shift of the M2-brane charge N

$$N\to N-\frac{1}{24}(k-k^{-1})\,.\tag{3.5}$$

This leads to the following redefinition of L in (3.6) [7]

$$\mathrm{L}^6=32\pi^2\big[Nk-\frac{1}{24}(k^2-1)\big]\,.\tag{3.6}$$

Expressing the localization result for $F(N,k)$ in (2.6) in terms of this redefined parameter L and k we find a remarkable simplification of the k-dependent coefficients of the L^3 powers

$$F=c_0\frac{1}{k}\mathrm{L}^9+c_1'\frac{1}{k}\mathrm{L}^3+\frac{1}{2}\log\frac{\mathrm{L}^3}{\pi k}-A(k)+c_2'\frac{1}{k}\mathrm{L}^{-3}+c_3'\mathrm{L}^{-6}+\mathrm{O}(\mathrm{L}^{-9}),\tag{3.7}$$

$$c_1'=-\frac{3}{64},\qquad c_2'=\frac{9\pi^2}{64}\,,\qquad c_3'=-3\pi^2\,.\tag{3.8}$$

Thus the k-dependence of the $\mathrm{L}^9,\mathrm{L}^3$ and L^{-3} terms becomes simply $\frac{1}{k}$ (though this does not apply to L^{-6} and higher order terms in the expansion).

It is natural to expect that the terms in the free energy that scale as $\frac{1}{k}$ may originate from local terms in the M-theory partition function or the effective action evaluated on the $AdS_4\times S^7/\mathbb{Z}_k$ background. Indeed, as this background is homogeneous (and its curvature does not depend on k explicitly, apart from the dependence via L_{11} or L) the integrals of curvature (and 4-form) invariants will be proportional to the factor of the radius $a=\frac{1}{k}$ of the 11d circle coming from the integration volume. Other terms that do not scale as $\frac{1}{k}$ may come from non-local contributions to the M-theory partition function.

3.1. Local terms

The L^9 term in (3.7) comes from the 11d supergravity action $S_0=\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}(R+\cdots)$ in (A.1) evaluated on the $AdS_4\times S^7/\mathbb{Z}_k$ background. The value of the coefficient c_0 is reproduced after taking into account the regularized value of the volume of AdS_4 [6,37] (see Appendix B). In particular, using that $R\sim(L_{11})^{-2}$ and (extracting the overall $(\frac{1}{2}L_{11})^4(L_{11})^7$ scale factor of the 11d volume, see (B.2), (B.3))

$$\mathrm{vol}\big(AdS_4\times S^7/\mathbb{Z}_k\big)=\frac{4\pi^2}{3}\times\frac{\pi^4}{3}\frac{1}{k}\,,\tag{3.9}$$

and also that $2\kappa_{11}^2=(2\pi)^8\ell_P^9$ (see (A.1)) we conclude that the coefficient of the supergravity term should scale as $\frac{1}{\pi^2}\frac{1}{k}\mathrm{L}^9$ matching the π^{-2}-dependence of c_0 in (3.4).

Similarly, the $\frac{1}{k}\mathrm{L}^3$ term in (3.7) should come from the local 1-loop $R^4+\cdots$ term in the 11d effective action [38,19,20,39][10]

[10] Note that here our ℓ_P (see Appendix A for notation) is related to ℓ_{11} used in [20,39] as $\ell_{11}^3=2\pi\ell_P^3$ so that the values of κ_{11} and M2-brane tension T_2 are the same as in these papers.

$$S_1 = b_1 T_2 \int d^{11}x\sqrt{-G}\big(R^4+\ldots\big)\,, \qquad T_2 = \frac{1}{(2\pi)^2\ell_P^3}\,, \qquad b_1 = \frac{1}{9\cdot 2^{13}\cdot(2\pi)^4}\,. \tag{3.10}$$

Here we isolated the factor of the M2 brane tension T_2. This term may be viewed as the 1-loop 11d supergravity contribution $\Lambda^3 R^4 + \cdots$ that scales as κ_{11}^0 but is cubically divergent [40] leading to a finite term in (3.10) after assuming the M-theory UV cutoff $\Lambda \sim \ell_P^{-1}$.[11] Thus this local 1-loop R^4 term is the one that corresponds to the $k^{-1/2}N^{1/2}$ term in F in (2.6).

Let us recall that similar terms $\sim N^{3/2}$ and $\sim N^{1/2}$ appear in the finite temperature free energy of the world-volume theory of multiple M2 branes and have similar origins in the R [42] and R^4 [43] terms in the M-theory effective action.

To reproduce the value of the coefficient c_1' in (3.7), (3.8) one needs the information about the precise structure of the 4-form dependent terms in the R^4 superinvariant which is not yet known (cf. [39]).[12] Still, it is remarkable that the fact that the value of c_1' that comes from (3.10) on the $AdS_4 \times S^7/\mathbb{Z}_k$ background is rational as in (3.8) does follow from the values of b_1 in (3.10) and of the volume factor (3.9): all factors of π cancel out.

In general, on dimensional grounds, all local terms in the M-theory effective action should contain particular powers of the M2-brane tension, *i.e.* should be given by the sum of terms like [20][13]

$$S_p = T_2^{3-2p}\int d^{11}x\sqrt{-G}\Big[b_p (D^2)^{3p-3}R^4 + \tilde b_p R^{3p+1} + \cdots\Big]\,, \tag{3.11}$$

where dots stand for other possible terms (depending also on F_4) that have the same mass dimension $6p+2$. Explicitly, the S_1 in (3.3) corresponds to the $p=1$ case of (3.11), the $p=2$ case is $S_2 = T_2^{-1}\int d^{11}x\sqrt{-G}\big(b_2 D^6 R^4 + \tilde b_2 R^7 + \cdots\big)$, etc.

Evaluated on $AdS_4 \times S^7/\mathbb{Z}_k$ background (3.11) will scale as $\frac{1}{k}\mathrm{L}^{9-6p}$ and may, in principle, match some of the subleading terms in the free energy (3.7). Terms that do not scale as $\frac{1}{k}$ should come from non-local parts of the quantum M-theory effective action.

3.2. *Terms corresponding to the 1-loop M-theory contribution*

The terms $\frac{1}{2}\log\frac{\mathrm{L}^3}{\pi k}$ and $-A(k)$ in (3.3) which are of zeroth order in the effective M2-brane tension (3.1) should originate from the (UV finite part of) 1-loop contribution to the M-theory partition function. The logarithmic $\frac{3}{2}\log\mathrm{L}$ term coming from $\frac{1}{4}\log\frac{32N}{k} = \frac{1}{2}\log\frac{\mathrm{L}^3}{\pi k}$ term in (2.6) was reproduced by a 1-loop computation in 11d supergravity in [9] as a universal contribution of the zero modes of the 11d supergravity fluctuation operators on $AdS_4 \times X^7$ background (with

[11] This R^4 term should be a superpartner of the $R^4 C_3$ term. The fact that accounting for the shift (3.6) removes the "non-local" $k\mathrm{L}^3$ term in (3.3) may be viewed as a consequence of supersymmetry. Note also that in general higher loop supergravity contributions should scale as $(\kappa_{11}^2)^{L-1} \sim (T_2)^{-3(L-1)}$ but in local terms extra factors of the M-theory UV cutoff $\Lambda \sim \ell_P^{-1}$ may introduce extra positive powers of T_2, see [20,41] and Eq. (3.11) below.

[12] While matching the overall coefficient c_1' is thus an open problem, in [16] the dependence of the coefficient of the similar $N^{1/2} \sim \mathrm{L}^3$ term on extra geometric parameters (like squashing of the S^3) in the localization result for the free energy F was reproduced from the effective 4d effective action with the supersymmetric R^2 terms that should originate from the 11d R^4 superinvariant compactification to 4d.

[13] The special role of the terms (3.11) noted in [20] is that upon reduction to 10d they have perturbative dependence on the string coupling g_s. Note that some of these terms may be interpreted as higher-loop corrections in 11d supergravity proportional to $(\kappa_{11}^2)^{L-1}\Lambda^{3n} \sim (T_2)^{3-3L+n}$ ($\Lambda \sim \ell_P^{-1}$ is the 11d UV cutoff).

the dependence on ℓ_P via L coming from normalization factor related to κ_{11}). The $-\frac{1}{2}\log(\pi k)$ term should have a similar origin (being also related to the volume factor in the normalization of the supergravity modes).

The $-A(k)$ term in (3.3) (see (2.6), (2.4)) should correspond to the L-independent part of the 1-loop contribution in M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$.

In general, the 1-loop M-theory partition function should be the contribution of virtual M2-brane propagating in the loop but it is not clear how to define it precisely. In the case of a large amount of supersymmetry of the background one may conjecture that only special BPS states (*e.g.*corresponding to M2-branes wrapped on special 2-cycles of internal space) may be contributing to the 1-loop partition function, while contributions of non-BPS states may cancel due to extended supersymmetry of the background (cf. [24–26]).

One may start with the contribution of just point-like BPS states corresponding to the 11d supergravitons, *i.e.* approximate the M-theory 1-loop partition function by its 11d supergravity counterpart. To get an insight about the structure of the latter and to compare it with F in (3.7) we will be guided by the expression for the low-energy expansion of the 1-loop correction to the 4-graviton amplitude in 11d supergravity [20]. While there is no a priori reason why just the supergravity correction should be enough to capture the full M-theory result, we will show that it indeed reproduces the structure of the large k expansion of the corresponding term in F.[14]

Our strategy will be as follows. We shall consider the expression for the 1-loop 4-graviton amplitude in 11d supergravity expanded near flat space with 11d circle of radius R_{11} (found under a simplifying assumption that only 10d components of the 4 polarization tensors and external momenta are non-zero) following [20]. We shall then expand this amplitude in powers of momenta and extract its dependence on R_{11} and 11d UV cutoff $\Lambda \sim \ell_P^{-1}$. Finally, we will assume that it can be used to shed light on the structure of 11d supergravity 1-loop partition function on a curved background. Specifying to the case of the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background we shall reproduce the structure of the L^3, $\log \mathrm{L}$ and $A(k)$ terms in (3.7). Remarkably, we shall find the terms with the same transcendental coefficients $\zeta(3)$ and π^{2h-2} that appear in the large k expansion of $A(k)$ in (2.4).[15]

In order to match the remaining rational factors in the coefficients it appears that one is to include other contributions to the M-theory partition function on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background. These are presumably of other (extended) BPS M2-brane states propagating in the loop. By analogy with the case of the Calabi-Yau compactification [25] we shall then discuss how one could try to modify the supergravity-based result in order to reproduce the double-Bernoulli structure of the coefficients in $A(k)$ in (2.4), (2.5).

The 4-graviton amplitude may be written as (omitting polarization tensor and normalization factors including 10d volume and momentum delta-function) [44,19,20]

[14] Let us note that ref. [10] attempted (unsuccessfully) to reproduce the leading large k terms in the localization expression for the function $A(k)$ in (3.4), (2.4) by the 1-loop computation in the 11d supergravity on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ explicitly accounting for the contribution of the tower of all 11d supergravity KK modes on $S^7/\mathbb{Z}_k$. The computation involved several subtle points that remain to be sorted out. In particular, it is also possible that one needs a special regularization (consistent with 11d symmetry) different from the one used in [10].

[15] Let us note that in refs. [15] the values of the function $A(k)$ and its second and fourth derivatives at $k = 1$ and $k = 2$ where related to the M-theory 4-graviton 1-loop amplitude and were shown to be consistent with the coefficient of the R^4 term in the effective action.

$$\hat{\mathcal{A}}_4(\mathrm{s},\mathrm{t}) = \mathcal{A}_4(\mathrm{s},\mathrm{t}) + (\text{symm in s, t, u}), \qquad \mathcal{A}_4(\mathrm{s},\mathrm{t}) = \sum_{n=-\infty}^{\infty} \int_{\Lambda^{-2}}^{\infty} \frac{d\tau}{\tau^2}\, e^{-\frac{\tau n^2}{R_{11}^2}}\, P(\mathrm{s},\mathrm{t};\tau), \tag{3.12}$$

where s, t, u are the standard kinematic variables depending on 10d momenta, the sum is over 11d component of the virtual momentum and τ has dimension of length squared. The function P is given by[16]

$$P(\mathrm{s},\mathrm{t};\tau) = \int d^3\rho\, e^{-\tau M(\mathrm{s},\mathrm{t};\rho)}\,, \qquad \int d^3\rho \equiv \int_0^1 d\rho_3 \int_0^{\rho_3} d\rho_2 \int_0^{\rho_2} d\rho_1\,, \tag{3.13}$$

$$M(\mathrm{s},\mathrm{t};\rho) \equiv \mathrm{s}\rho_1\rho_2 + \mathrm{t}\rho_2\rho_3 + \mathrm{u}\rho_1\rho_3 + \mathrm{t}(\rho_1-\rho_2), \qquad \mathrm{s}+\mathrm{t}+\mathrm{u}=0. \tag{3.14}$$

Focussing on the first term in the sum in (3.12) and expanding $e^{-\tau M}$ in powers of momenta, or, equivalently, in powers of M we get

$$\mathcal{A}_4(\mathrm{s},\mathrm{t}) = \sum_{h=0}^{\infty} \mathcal{A}_{4,h}(\mathrm{s},\mathrm{t})\,, \qquad \mathcal{A}_{4,h} = \sum_{n=-\infty}^{\infty} \int_{\Lambda^{-2}}^{\infty} \frac{d\tau}{\tau^2}\, e^{-\frac{\tau n^2}{R_{11}^2}}\, \frac{(-1)^h}{h!}\, \tau^h\, H_h(\mathrm{s},\mathrm{t})\,, \tag{3.15}$$

$$H_h \equiv \int d^3\rho\, M^h(\mathrm{s},\mathrm{t};\rho) = s^h\, \bar{H}_h\big(\frac{\mathrm{s}}{\mathrm{t}}\big)\,. \tag{3.16}$$

The $h=0$ term in (3.15) may be written (using Poisson resummation and $H_0 = \frac{1}{6}$) as [19]

$$\mathcal{A}_{4,0} = \frac{2}{3\pi} R_{11}\Lambda^3 + \frac{\zeta(3)}{\pi^2 R_{11}^2}\,. \tag{3.17}$$

Here the first term comes effectively from the $n=0$ contribution and is thus the same as in the 1-loop contribution in 10d supergravity. The second term comes from the contribution of 11d supergravity states with non-zero 11d momentum (or, from the 10d string theory point of view, from the contribution of the massive D0-brane states in the loop [19]). The $h=1$ contribution vanishes after integrating over ρ, in agreement with the absence of 1-loop logarithmic divergences in 11d theory (and also in the 1-loop 4-graviton amplitude in 10d supergravity [45]).

The remaining $h \geq 2$ terms are UV finite. The $n=0$ term in the sum in (3.16) with $h \geqslant 2$ gives a non-analytic contribution ($\sim \mathrm{s}\log \mathrm{s}$, *etc.*) to (3.15) which is independent of R_{11} (and thus should be the same as the 1-loop amplitude in 10d supergravity)

$$\mathcal{H}(\mathrm{s},\mathrm{t}) = \sum_{h=2}^{\infty} \int_0^{\infty} \frac{d\tau}{\tau^2}\, \frac{(-1)^h}{h!}\, \tau^h\, H_h(\mathrm{s},\mathrm{t}) = \int d^3\rho\, M(\mathrm{s},\mathrm{t};\rho) \log M(\mathrm{s},\mathrm{t};\rho) \equiv \mathrm{s}\,\bar{\mathcal{H}}\big(\frac{\mathrm{s}}{\mathrm{t}}\big)\,. \tag{3.18}$$

The contribution of the $n \neq 0$ terms may be written as

16 Here we redefined τ by π compared to [20] so that it has direct proper-time interpretation. Note that the factor π in front of $M(\mathrm{s},\mathrm{t};\rho)$ was omitted in going from the first to second line of Eq. (C.7) in [44] and as a result was missing in the expression given in [20].

$$\sum_{h=2}^{\infty} \mathcal{A}'_{4,h}(\mathrm{s},\mathrm{t}) = 2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{d\tau}{\tau^2}e^{-\frac{\tau n^2}{R_{11}^2}}\sum_{h=2}^{\infty}\frac{(-1)^h}{h!}\tau^h H_h(\mathrm{s},\mathrm{t}) \equiv \sum_{h=2}^{\infty} C_h H_h(\mathrm{s},\mathrm{t}) , \tag{3.19}$$

where C_h is given by

$$C_h = 2\frac{(-1)^h}{h!}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{d\tau}{\tau^2}e^{-\frac{\tau n^2}{R_{11}^2}}\tau^h = 2\frac{(-1)^h}{h!}\sum_{n=1}^{\infty}(h-2)!\Big(\frac{n^2}{R_{11}^2}\Big)^{1-h} = \mathrm{d}_h R_{11}^{2(h-1)}, \tag{3.20}$$

$$\mathrm{d}_h = \frac{2(-1)^h}{h(h-1)}\zeta(2h-2) = \frac{(2\pi)^{2h-2}B_{2h-2}}{h(h-1)(2h-2)!} , \tag{3.21}$$

where B_{2h-2} are Bernoulli numbers. Adding to (3.19) the $h=0$ term (3.17) and the non-analytic $\mathcal{H}$ (3.18) contribution gives

$$\mathcal{A}_4(\mathrm{s},\mathrm{t}) = \frac{2}{3\pi}R_{11}\Lambda^3 + \mathrm{s}\,\bar{\mathcal{H}}\big(\frac{\mathrm{s}}{\mathrm{t}}\big) + \bar{\mathcal{A}}_4(\mathrm{s},\mathrm{t}) , \tag{3.22}$$

$$\bar{\mathcal{A}}_4(\mathrm{s},\mathrm{t}) = \frac{\zeta(3)}{\pi^2 R_{11}^2} + \sum_{h=2}^{\infty} d_h R_{11}^{2(h-1)}\mathrm{s}^h \bar{H}_h\big(\frac{\mathrm{s}}{\mathrm{t}}\big) . \tag{3.23}$$

Here $\bar{\mathcal{H}}$ contains log terms while $\bar{H}_h$ are polynomials of degree h. Note that all the terms in (3.22) have the same dimension (length)$^{-2}$. In (3.22) we separated the first term that is the only one that depends on ℓ_P via Λ.

Let us now interpret (3.22) as providing an indication about the structure of the M-theory 1-loop partition function on a curved background. Specializing to $AdS_4 \times S^7/\mathbb{Z}_k$ we will have the 11d radius $R_{11} \to \frac{1}{k}L_{11}$ and, just on dimensional grounds, the momentum variables s, t scaling as L_{11}^{-2}. Rescaling (3.22) by L_{11}^2 to get a dimensionless expression we would then get from (3.22) ($\mathrm{L} = \frac{L_{11}}{\ell_P} \sim L_{11}\Lambda$)

$$\mathcal{F} = u_0\frac{1}{k}\mathrm{L}^3 + u_1\log\mathrm{L} + u_2 + \bar{\mathcal{F}}(k) , \qquad \bar{\mathcal{F}} = \frac{\zeta(3)}{\pi^2}k^2 + \sum_{h=2}^{\infty}\frac{\mathrm{d}'_h}{k^{2h-2}} . \tag{3.24}$$

Here $u_0 = \frac{2}{3\pi}w$ where w is the coefficient of proportionality in $\Lambda^3 = w\ell_P^{-3}$ so that matching the rational c'_1 coefficient in (3.7) requires $w \sim \pi$.[17] The terms $u_1\log\mathrm{L} + u_2$ come from $s\,\bar{\mathcal{H}}(\frac{s}{t})$ term in (3.22). The coefficients d'_h are related to d_h in (3.21) by rescaling by some rational factors.

Thus (3.24) has the same structure as the sum of the L^3, $\log\mathrm{L}$ and $-A(k)$ terms in (3.7). The missing $\log k$ term should be coming from the 1-loop 11d supergravity zero mode normalization contribution mentioned above and is thus not expected to be captured by this qualitative argument based just on the structure of the 4-graviton amplitude.

Remarkably, $\bar{\mathcal{F}}$ in (3.24) has exactly the same form as the leading $\zeta(3)k^2$ term plus the sum of the subleading $\frac{1}{k^{2h-2}}$ terms in $-A(k)$ in (2.4), (2.5). Furthermore, the transcendental factors of π match between the coefficients $\mathrm{d}_h \sim \mathrm{d}'_h$ in (3.21) and q_h (2.5) in $A(k)$. To match the rational coefficient of the $\zeta(3)$ term in $-A(k)$ we would need an extra factor of $\frac{1}{8}$, *i.e.*

[17] This is indeed the right identification as follows from the discussion in [19,20] or the comparison with the coefficient of the corresponding R^4 term in 11d effective action (3.10). In the present notation $w = \frac{1}{2}\pi$, *i.e.* $\Lambda^3 = \frac{1}{2}\pi\ell_P^{-3}$.

$$\frac{1}{8}L_{11}^2\mathcal{A}_4 = \frac{1}{8}\mathcal{F} \quad \to \quad F\ . \tag{3.25}$$

The exact equality of the rational factors in d'_h (that should differ from d_h in (3.21) just by rational factors) and q_h in (2.5) may be hard to expect a priori given the crude nature of the above relation between the 1-loop supergravity amplitude and the 1-loop partition function on a curved background. But a definite mismatch in powers of the Bernoulli number factors between (3.21) and in (2.5) suggests that some other contributions (in addition to 1-loop 11d supergravity one) may be missing.

One may wonder whether to match the full expression for the $A(k)$ term in F in (3.7) one needs to include contributions of other M2-brane BPS states to the M-theory 1-loop partition function.[18] By analogy with a discussion in [24,25] one may conjecture that this may lead to a modification of the measure in the proper-time integral in (3.19) like

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{d\tau}{\tau^2}e^{-\frac{\tau n^2}{R_{11}^2}}\ \dots \quad \to \quad \sum_{n=1}^{\infty}\int_0^{\infty}\frac{d\tau}{\tau}\frac{\mu^2}{\sinh^2(\mu^2\tau)}\ e^{-\frac{\tau n^2}{R_{11}^2}}\ \dots\ , \tag{3.26}$$

where μ is a mass parameter (that may be related to L_{11}^{-1} in the present context, so that $\mu R_{11}\sim\frac{1}{k}$). Then an extra factor of the Bernoulli numbers required to match d_h in (3.21) with q_h in (2.5) may come from the expansion

$$\frac{1}{\sinh^2\tau} = -\sum_{h=0}^{\infty}\frac{2^{2h}(2h-1)}{(2h)!}B_{2h}\,\tau^{2h-2} = \frac{1}{\tau^2}-\frac{1}{3}-\sum_{h=2}^{\infty}\frac{2^{2h}(2h-1)}{(2h)!}B_{2h}\,\tau^{2h-2}\ . \tag{3.27}$$

To see at the heuristic level how that may work out we may start with the localization expression for $\bar{A}(k)$ in (2.4), (2.5) that has the following integral representation [30]

$$\bar{A}(k)\equiv\sum_{h=2}^{\infty}\frac{q_h}{k^{2h-2}} = \int_0^{\infty}\frac{dt}{t}\frac{1}{e^{kt}-1}\Big(\frac{1}{\sinh^2 t}-\frac{1}{t^2}+\frac{1}{3}\Big)\ . \tag{3.28}$$

One may also rewrite the full expression for $A(k)$ in (2.4) as[19]

$$A(k) = \int_0^{\infty}\frac{dt}{t}\,\frac{1}{e^{kt}-1}\,\frac{1}{\sinh^2 t} = \sum_{n=1}^{\infty}\int_0^{\infty}\frac{dt}{t}\,\frac{1}{\sinh^2 t}\,e^{-knt}\ . \tag{3.29}$$

In (3.29) we are assuming that the evaluation of the singular terms (corresponding to the last two terms in the bracket in (3.28)) is done using a suitable regularization.[20] Here $k=\frac{L_{11}}{R_{11}}$ and we may redefine $t\to(L_{11})^{-1}\tau$ to put the integral into a similar form as in (3.26).

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[18] In particular, one may consider contributions of M2-branes wrapped on 2-cycles of CP^3 part of $S^7/\mathbb{Z}_k$ (which, in the perturbative 10d string limit, are related to the type IIA string world-sheet instantons [46,47] but here play a role of massive modes propagating in the loop). These may be the analogs of M2-branes wrapped on 2-cycles of CY space in [25]. Note also that the field strength of the RR 1-form A in the $S^7/\mathbb{Z}_k$ metric (A.10) may be playing the role of the graviphoton strength in the discussion of [26].

[19] Note that using $\frac{1}{\sinh^2 t}=4\sum_{n=1}^{\infty}n\,e^{-2nt}$ one may also write $A(k)=4\sum_{n,m=1}^{\infty}n\int_0^{\infty}\frac{dt}{t}e^{-(mk+2n)t}$.

[20] For instance, with an analytic regularization like $\frac{dt}{t}\to\frac{dt}{t^{1+\varepsilon}}$ one has

$\int_0^{\infty}\frac{dt}{t^{1+\varepsilon}}\frac{1}{e^{kt}-1}(\frac{1}{t^2}-\frac{1}{3}) = -\frac{1}{6}\big[\frac{1}{\varepsilon}+\log k+\gamma_{\mathrm{E}}-\log(2\pi)\big]-\frac{\zeta(3)}{8\pi^2}k^2+\mathcal{O}(\varepsilon)$.

The $\log k$ and k^2 terms here agree with (2.4). The pole term (plus regularization-dependent transcendental constants) should be discarded as part of the regularization prescription.

Eq. (3.29) closely resembles the expression in [25,26] used to reproduce the coefficients of the special protected $R^2\mathrm{F}^{2h-2}$ terms [48] in the 4d effective action of type II string (compactified on a CY space) from a conjectured 1-loop M-theory correction coming from M2-brane BPS states. Indeed, we may compare the summand in (3.29) with the 4d effective action of a charged scalar of mass m (representing an M2-brane wrapped on a 2-cycle in CY space) in a constant self-dual gauge field background

$$\Gamma(\mathrm{m}, e\mathrm{F}) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{tr}\, e^{-t\,(\Delta+\mathrm{m}^2)} \sim \int_0^\infty \frac{dt}{t}\frac{1}{\sinh^2(\frac{1}{2}e\mathrm{F}\,t)}\, e^{-t\mathrm{m}^2}\,. \tag{3.30}$$

Here F is the gauge field strength and the UV divergent term is assumed to be subtracted out. Specializing to a BPS state with $\mathrm{m} = e$ and rescaling t one gets the integrand as $\frac{1}{\sinh^2 t'}\, e^{-2\mathrm{m}\mathrm{F}^{-1}\,t'}$. Accounting for multiple wrappings corresponds to $\mathrm{m} \to n\mathrm{m}$ and summing over n so that we get

$$\sum_{n=1}^\infty \Gamma(n\mathrm{m}, n\mathrm{m}\mathrm{F}) \sim \sum_{n=1}^\infty \int_0^\infty \frac{dt'}{t'}\frac{1}{\sinh^2 t'}\, e^{-2n\,\mathrm{m}\mathrm{F}^{-1}\,t'}\,. \tag{3.31}$$

This matches (3.29) if $2\mathrm{m}\mathrm{F}^{-1}$ is identified with k.[21] Then the coefficients in the $\frac{1}{k}$ expansion of (3.29) are directly related to the coefficients in the expansion of (3.31) in powers of F. In the present case $\frac{1}{k} = \frac{R_{11}}{L_{11}}$ scales as the square root of the effective curvature of $AdS_4 \times S^7/\mathbb{Z}_k$ and is thus analogous to F.[22]

4. Type IIA string perturbative expansion

Let us now compare the expansion of the localization result for free energy in the 't Hooft limit (2.9) with the perturbative expansion of the effective action in type IIA string theory in $AdS_4 \times CP^3$ background.

Let us start with the free energy expanded in large N and large k with fixed $\lambda = \frac{N}{k}$ (2.9) and then expanded further in large λ (see (2.13), (2.14)). Expressing F in terms of the type IIA string parameters g_s and T in (A.14), (A.15) we will attempt to match the result to the perturbative low-energy or α' expansion of type IIA string effective action in the corresponding $AdS_4 \times CP^3$ background (A.12) order by order in small g_s.

Using the original relations between the parameters (A.14), (A.15) [3]

$$N = 4\sqrt{2\pi}\, T^{5/2}\, g_s^{-1}, \qquad \lambda = 2T^2, \qquad T = \frac{1}{8\pi}\frac{L^2}{\alpha'}\,, \tag{4.1}$$

where L is the scale in 10d metric in (A.12) we get from (2.13)

$$\widetilde{F} = \frac{1}{384\pi^2}\frac{L^8}{\alpha'^4 g_s^2} - \frac{1}{192}\frac{L^4}{\alpha'^2 g_s^2} + \frac{1}{8\pi^2}\Big[\frac{\zeta(3)}{g_s^2} - 2\zeta(2)\Big]\frac{L^2}{\alpha'} + \cdots\,. \tag{4.2}$$

[21] This matching is not totally unexpected given that both functions were noticed to be related to the topological string amplitudes (cf. [49,48] and [6,50]).

[22] The difference between (3.29) or (3.31) and the attempted modification (3.26) is that the sum over the 11d KK modes in (3.26) involves n^2 rather than n, but in going from (3.30) to (3.31) this is taken care of by a rescaling of t. For this to be possible requires the measure in (3.26) to depend on n.

Here the first term $\frac{L^8}{\alpha'^4}$ scales as the contribution of the R term in the type IIA effective action while the third $\frac{L^2}{\alpha'}$ term – as the contribution of the R^4 term. The second $\frac{L^4}{\alpha'^2}$ contribution could come from the $\frac{1}{g_s^2\alpha'^2}\int d^{10}x\sqrt{G}\,R^3$ term in tree-level string effective action but such term is absent in type IIA 10d string theory (on supersymmetry grounds).

This problematic term is eliminated if one takes into account the shift of N in (3.5), (A.18) implying that the relations between the gauge theory (N,λ) and string theory (g_s, T) parameters take the modified form (A.20), (A.19). Note that if we shift N in (3.5) as $N \to N - \frac{k}{24} + \frac{a}{24}k^{-1}$ with $a \neq 1$ then this term will not be eliminated.[23] As a result, (2.13) then gives (cf. (3.7), (3.8))

$$\widetilde{F} = c_0\,\frac{L^8}{\alpha'^4 g_s^2} + \Big(\tilde{c}_0\,\frac{\zeta(3)}{g_s^2} + c_1'\Big)\frac{L^2}{\alpha'} - \sum_{h=2}^{\infty} q_h\,\Big(g_s^2\,\frac{\alpha'}{L^2}\Big)^{h-1}, \tag{4.3}$$

$$\begin{aligned} &c_0 = \frac{1}{384\pi^2}\,, \quad \tilde{c}_0 = \frac{1}{8\pi^2}\,, \quad c_1' = -\frac{3}{64}\,, \\ &q_2 = -\frac{\pi^2}{90}\,, \quad q_3 = \frac{2\pi^4}{2835}\,, \quad q_4 = -\frac{8\pi^6}{42525}\,, \dots \end{aligned} \tag{4.4}$$

where we kept only the leading at large tension (large λ or small $\frac{\alpha'}{L^2}$) contribution at each order in g_s^2 expansion (apart from the $\zeta(3)g_s^{-2}$ term). Since according to (A.16)

$$\frac{g_s^2}{8\pi T} = g_s^2\,\frac{\alpha'}{L^2} = \frac{1}{k^2}\,, \tag{4.5}$$

the q_h coefficients in (4.4) are the same as (2.5), (2.15) appearing in the large k expansion of $A(k)$ term in (2.6) or in (3.7).

4.1. Transcendentality structure of the coefficients

Similarly to the discussion in section 3.1 above, the first term in (4.3) originates from the supergravity part $\frac{1}{g_s^2\alpha'^4}\int d^{10}x\sqrt{-G}\,(R+\cdots)$ of the tree-level 10d superstring effective action evaluated on the $AdS_4 \times CP^3$ background (see (B.2)).

The second $\frac{L^2}{\alpha'}$ term in (4.3) has the structure that corresponds to the contribution of the sum of the tree level $\frac{1}{(2\pi)^7 g_s^2\alpha'}\frac{1}{8}\zeta(3)\int d^{10}x\sqrt{-G}\,R^4$ term and 1-loop term $\frac{1}{(2\pi)^7\alpha'}\frac{1}{4}\zeta(2)\int d^{10}x\sqrt{-G}\,\times$ $(R^4+\cdots)$. The factors of π in the coefficients match perfectly after we account for π^5 coming from the volume of $AdS_4 \times CP^3$ (see (B.3)). As discussed in more detail in the next subsection, fixing the remaining rational coefficients requires the information about the RR field strength dependent terms in the corresponding superinvariants which is not available at the moment.

The higher order $h \geqslant 2$ terms in (4.4) may originate from local terms in type IIA effective action of the form (note that $R \sim L^{-2}$ and the L^{10} factor comes from the 10d volume)

[23] Let us mention that instead of expanding in large N one may expand in the effective CFT central charge parameter (the coefficient in the 2-point function of the stress tensor) $c_T \sim \sqrt{k}N + \dots$ [11,12,14] which is naturally related to the definition of the Newton's constant in the gravitational dual (the coefficient in the graviton kinetic term). This then leads [12,14] to the expected higher derivative terms avoiding spurious terms like R^3 (we thank S. Chester for pointing this out). Also, an alternative shift of N in relation to 4d Newton's constant was used in [17] and shown to lead to simplification of perturbative expansion. In general, the relation between string/M-theory parameters and dual gauge theory ones is effectively a scheme choice required to make the duality manifest; unambiguous relations are found only when expressing one observable in terms of the others.

$$g_s^{2h-2}\,\alpha'^{\,h-1}\int d^{10}x\sqrt{-G}\,\mathcal{L}_h\ ,\qquad \mathcal{L}_h = e_{h,1}R^{h+4}+\cdots+e_{h,r}D^{2r}R^{h-r+4}+\cdots+e_{h,h}D^{2h}R^4\ . \tag{4.6}$$

Here $\mathcal{L}_h$ may contain several terms of the same dimension (depending on curvature and other fields) required on supersymmetry grounds. The structure of these invariants is not known but as the relative coefficients of the terms in (4.6) should be rational, we may get some information about their transcendentality properties by looking at the particular terms $D^{2h}R^4$. Like for the tree-level and 1-loop R^4 terms, the coefficients of these terms may, in principle, be fixed using the type II string 4-graviton scattering amplitude.

In fact, one may follow [20] and conjecture that in the perturbative string theory limit ($g_s \ll 1$) the structure of the 11d supergravity amplitude (3.22) implies the presence of special 10d-local terms $g_s^{2h-2}\alpha'^{\,h-1}\int d^{10}x\sqrt{-G}\,D^{2h}R^4$ in the type IIA string effective action. These should correspond to local s^h terms in (3.23), *i.e.* should have the coefficients proportional to d_h in (3.21) or $(2\pi)^{2h}$ (after including an overall normalization factor $\sim\pi^2$ as implied by (3.17)) and should thus match the π-dependence of the q_h coefficients in (4.4). As was already pointed out in the previous section, the matching of the rational factors (proportional to $B_{2h}B_{2h-2}$ in (2.5), (2.15) instead of just B_{2h-2} in d_h in (3.21)) implies the need to account also for the contributions of other terms of the same dimension in the corresponding superinvariants in (4.6).

The same conclusion about the structure of the relevant coefficients can be reached also from the leading $D^{2n}R^4$ terms in the effective action reconstructed directly from the type II 4-graviton 10d superstring amplitudes. This applies also to the type IIB effective action (for a related discussion in connection with free energy of $\mathcal{N}=2$ 4d gauge theory models see [21]). The leading $D^{2n}R^4$ terms are the same (at least at 1-loop and 2-loop orders) in both type IIA and type IIB theories [41,51]. In the type IIB case one finds

$$S=\frac{1}{(2\pi)^7}\int d^{10}x\sqrt{-G}\Big[\alpha'^{-4}g_s^{-2}R+\alpha'^{-1}\mathrm{f}_0(g_s)R^4+\sum_{n=1}^{\infty}\alpha'^{n}\mathrm{f}_n(g_s)D^{2n+2}R^4\Big]\ . \tag{4.7}$$

The functions $\mathrm{f}_0, \mathrm{f}_1, \mathrm{f}_2$ contain a finite number of perturbative contributions plus non-perturbative $O(e^{-1/g_s^2})$ corrections that we shall omit (see, *e.g.*, [41,52])[24]

$$\mathrm{f}_0=\tfrac{1}{8}\Big(\zeta(3)g_s^{-2}+2\zeta(2)\Big)\ ,\qquad \mathrm{f}_1=\tfrac{1}{16}\Big(\zeta(5)g_s^{-2}+\tfrac{4}{3}\zeta(4)g_s^{2}\Big)\ , \tag{4.8}$$

$$\mathrm{f}_2=\tfrac{1}{48}\Big([\zeta(3)]^2g_s^{-2}+\zeta(3)\zeta(2)+6\zeta(4)g_s^2+\tfrac{2}{9}\zeta(6)g_s^4\Big)\ ,\ \ldots \tag{4.9}$$

The leading α' terms at each order in g_s^2 in (4.7) correspond to the last perturbative terms in $\mathrm{f}_0, \mathrm{f}_1, \mathrm{f}_2$ in (4.9) and their coefficients are expected to be protected by supersymmetry.

This suggests that the coefficients of the terms $g_s^{2h-2}\alpha'^{\,h-1}\int d^{10}x\sqrt{-G}\,D^{2h}R^4$ with $h\geqslant 2$ we are interested in are proportional to $\zeta(2h)=(-1)^{h+1}(2\pi)^{2h}\frac{B_{2h}}{2(2h)!}\sim(2\pi)^{2h}$. This is the same conclusion that follows from the above conjectured relation to the 11d supergravity 1-loop amplitude. Once again, to match the remaining rational factors in the coefficients against those in the free energy (2.15) would require the precise knowledge of the superinvariants that have the same dimension as $D^{2h}R^4$ terms.

[24] Here $\mathrm{f}_3=\frac{1}{64}\zeta(9)g_s^{-2}+k_0\zeta(3)\log(-\alpha' D^2)+O(g_s^2)$ and may contain an infinite series of terms in g_s^2 (though their presence appears to remain an open question). The logarithmic term is associated with a non-local term $p^{16}\log p^2$ in the 4-graviton amplitude on a flat background.

4.2. Contributions from tree level and 1-loop R^4 invariants

To illustrate this point let us go back to the discussion of the contribution of the tree-level and 1-loop R^4 terms in type IIA theory. They may be written as (assuming the dilaton is constant and ignoring dependence on B_2 field, see, *e.g.*, [39] for a review)

$$S = \frac{1}{(2\pi)^7}\int d^{10}x\,\sqrt{-G}\Big[\frac{1}{\alpha'^4 g_s^2}(R+\cdots)+\frac{1}{\alpha'}\Big(\frac{1}{g_s^2}r_0 J_0 + r_1 J_1\Big)+\cdots\Big]\,, \tag{4.10}$$

$$J_0 = t_8t_8RRRR+\tfrac{1}{8}\varepsilon_{10}\varepsilon_{10}RRRR+\cdots\,,\qquad J_1 = t_8t_8RRRR-\tfrac{1}{8}\varepsilon_{10}\varepsilon_{10}RRRR+\cdots\,, \tag{4.11}$$

$$r_0 = \tfrac{1}{3\cdot 2^{11}}\,\zeta(3)\,,\qquad r_1 = \tfrac{\pi^2}{3\cdot 2^{11}}\,2\zeta(2) = \tfrac{\pi^2}{3^2\cdot 2^{11}}\,. \tag{4.12}$$

Dots in J_0 and J_1 stand for other terms of the same dimension depending on RR fields.[25] In type IIB theory J_1 is replaced by J_0 so that $\frac{\zeta(3)}{g_s^2}+\frac{\pi^2}{3}$ is the total coefficient of the R^4 terms as was already indicated in (4.8). In type IIA theory J_0 and J_1 should correspond to separate superinvariants.

Explicitly, the contribution of the R^4 terms is then

$$\Delta S = \frac{1}{(2\pi)^7\,3\cdot 2^{11}\alpha'}\int d^{10}x\,\sqrt{-G}\Big[\Big(\frac{\zeta(3)}{g_s^2}+\frac{\pi^2}{3}\Big)(t_8t_8RRRR+\cdots) + \Big(\frac{\zeta(3)}{g_s^2}-\frac{\pi^2}{3}\Big)(\tfrac{1}{8}\varepsilon_{10}\varepsilon_{10}RRRR+\cdots)\Big]\,. \tag{4.13}$$

To compare to the corresponding $\frac{L^2}{\alpha'}$ term in the free energy (4.3) we are to evaluate (4.13) on the $AdS_4\times CP^3$ background.[26] As was already mentioned above, since the background is homogeneous with $R\sim F_4^2\sim F_2^2\sim L^{-2}$ and the volume of the $AdS_4\times CP^3$ given by (B.3), *i.e.*

$$\int d^{10}x\,\sqrt{-G}\,\Big|_{AdS_4\times CP^3} = \tfrac{4\pi^2}{3}(\tfrac{1}{2}L)^4\cdot\tfrac{\pi^3}{6}L^6 = \tfrac{\pi^5}{3^2\cdot 2^3}L^{10}\,, \tag{4.14}$$

the coefficient of the $\zeta(3)$ term in (4.13) scales as $\frac{1}{(2\pi)^7\cdot 3\cdot 2^{11}}\frac{\pi^5}{3^2\cdot 2^3}\frac{L^2}{\alpha'}=\frac{1}{3^3\cdot 2^{16}}\frac{1}{\pi^2}\frac{L^2}{\alpha'}$. To match $\tilde c_0=\frac{1}{8\pi^2}$ in (4.3), (4.4) thus requires an extra rational factor $3^3\cdot 2^{13}$ that should presumably come from the curvature contractions and other terms in $J_0,\ J_1$ depending on fluxes.

A factor of the same order does come from the Weyl-tensor dependent part of J_0[27]:

$$J_0=\bar J_0+\cdots\,,\qquad \bar J_0 = 3\cdot 2^8\,(C^{hmnk}C_{pmnq}C_h{}^{rsp}C^q{}_{rsk}+\tfrac{1}{2}C^{hkmn}C_{pqmn}C_h{}^{rsp}C^q{}_{rsk})\,, \tag{4.15}$$

---

25 Terms with Ricci tensor can be expressed in terms of flux-dependent terms using equations of motion (or field redefinitions). In $J_0,\ J_1$ we use Minkowski signature so that $\varepsilon_{10}\varepsilon_{10}=-10!$ and after reduction to 8 spatial dimensions $\varepsilon_{mn\ldots}\varepsilon_{mn\ldots}\to -2\varepsilon_8\varepsilon_8$. t_8 is the 10-dimensional extension of the 8-dimensional light-cone gauge involving $G^{\mu\nu}$ (see, *e.g.*, [53]). Explicitly,
$t_8t_8RRRR = t^{\mu_1\nu_1\ldots\mu_4\nu_4}t_{\mu_1'\nu_1'\ldots\mu_4'\nu_4'}R^{\mu_1'\nu_1'}_{\mu_1\nu_1}\cdots R^{\mu_4'\nu_4'}_{\mu_4\nu_4}$, $\varepsilon_{10}\varepsilon_{10}RRRR = \varepsilon^{\alpha\beta\mu_1\nu_1\ldots\mu_4\nu_4}\varepsilon_{\alpha\beta\mu_1'\nu_1'\ldots\mu_4'\nu_4'}R^{\mu_1'\nu_1'}_{\mu_1\nu_1}\cdots R^{\mu_4'\nu_4'}_{\mu_4\nu_4}$.

26 It is curious to note that if we did not apply the redefinition in (A.18) then the value of the coefficient b_1 in (4.3), (4.4) would be changed to $b_1'=-\frac{1}{24}$ so that the coefficient of the $\frac{L^2}{\alpha'}$ term in (4.3) would become $\frac{1}{8\pi^2}(\frac{\zeta(3)}{g_s^2}-\frac{\pi^2}{3})$ as in (4.2) and is thus exactly proportional to the coefficient of the second term in (4.13).

27 Note that $\bar J_0$ vanishes in the case of undeformed $AdS_5\times S^5$ background (implying, in particular, no correction to the radius or free energy, cf. *e.g.*[54,43,55]) but does not vanish on the $AdS_4\times CP^3$ one.

$$\bar{J}_0\Big|_{\mathrm{AdS}_4\times \mathrm{CP}^3} = 3^4 \cdot 2^{14} L^{-8} \ . \tag{4.16}$$

The difference from the required $3^3 \cdot 2^{13}$ factor may be attributed to the contributions of other Ricci tensor dependent terms in $t_8 t_8 RRRR$ and $\varepsilon_{10}\varepsilon_{10} RRRR$ (discussed in Appendix C) and other RR field strength dependent terms in the invariants J_0, J_1 (cf. [56]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Notation and basic relations

Here we review the relations between M-theory and type IIA string theory parameters in general and also in the specific case of the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background when they are expressed in terms of N and k of ABJM theory [3].

The action of the 11d supergravity is

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\,\sqrt{-G}\,\Big(R - \frac{1}{2\cdot 4!}F_{mnk\ell}F^{mnk\ell} + \cdots\Big), \qquad 2\kappa_{11}^2 = (2\pi)^8\,\ell_P^9\ , \tag{A.1}$$

where our normalization of 11d Planck length ℓ_P here is the same as in, *e.g.*, [3,57]. The M2-brane tension is then [42]

$$T_2 = \frac{1}{(2\pi)^2\,\ell_P^3}\ . \tag{A.2}$$

Assuming compact x^{11} direction the 11d metric may be written as

$$ds_{11}^2 = e^{-\frac{2}{3}\phi}\,ds_{10}^2 + e^{\frac{4}{3}\phi}(dx^{11} + e^{-\phi}A_m dx^m)^2, \qquad x^{11} \sim x^{11} + 2\pi \mathrm{R}_{11}, \tag{A.3}$$

where, upon reduction to 10d, ds_{10}^2 will be the string frame metric and ϕ the dilaton. The constant part of the dilaton is related to string coupling as $g_s = e^{\phi}$, so that (A.1) reduces to the standard 10d type IIA supergravity action[28] with

[28] It reads $\frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}\,[e^{-2\tilde{\phi}}(R + ...) + ...]$ where $\tilde{\phi}$ is non-constant part of the dilaton, with constant part included in κ_{10}.

$$2\pi \mathrm{R}_{11}\,\frac{1}{g_s^2}\,\frac{1}{2\kappa_{11}^2}=\frac{1}{2\kappa_{10}^2}\,,\qquad\qquad 2\kappa_{10}^2=(2\pi)^7 g_s^2\alpha'^4\,. \tag{A.4}$$

The M2 brane wrapped on the x^{11} circle gives the fundamental string action with the standard tension[29]

$$2\pi\,\mathrm{R}_{11}\,T_2=T_1\,,\qquad\qquad T_1=\frac{1}{2\pi\alpha'}\,. \tag{A.5}$$

From (A.4) and (A.5), we then learn that in the above notation[30]

$$\mathrm{R}_{11}=\ell_P=\ell_s\,,\qquad \ell_s\equiv\sqrt{\alpha'}\,. \tag{A.6}$$

For a constant ϕ the effective radius of the 11-th direction is (as in [59])

$$R_{11}=e^{\frac{2}{3}\phi}\mathrm{R}_{11}=g_s^{2/3}\mathrm{R}_{11}\,. \tag{A.7}$$

Let us now specialize to the $\mathrm{AdS}_4\times S^7$ space supported by the 4-form flux with $\hat N$ units of charge (which is the near-horizon limit of the background sourced by multiple M2-branes [60])

$$ds_{11}^2=L_{11}^2\Big(\frac{1}{4}ds^2_{\mathrm{AdS}_4}+ds^2_{S^7}\Big),\qquad ds^2_{\mathrm{AdS}_4}=dr^2+\sinh^2 r\,d\Omega_3^2,\qquad F_4=dC_3\sim\hat N\,\varepsilon_4. \tag{A.8}$$

The flux quantization condition implies that

$$\frac{L_{11}}{\ell_P}=\big(2^5\pi^2\hat N\big)^{1/6}. \tag{A.9}$$

Considering $\mathbb{Z}_k$ quotient of S^7 we get [3]

$$\begin{aligned}
&ds^2_{S^7/\mathbb{Z}_k}=ds^2_{\mathrm{CP}^3}+\frac{1}{k^2}(d\varphi+kA)^2\,,\qquad\qquad \varphi\equiv\varphi+2\pi,\\
&ds^2_{\mathrm{CP}^3}=\frac{dw^s d\bar w^s}{1+|w|^2}-\frac{w_r\bar w_s}{(1+|w|^2)^2}dw^s d\bar w^r,\\
&dA=i\Big[\frac{\delta_{sr}}{1+|w|^2}-\frac{w_s\bar w_r}{(1+|w|^2)^2}\Big]dw^r\wedge d\bar w^s,
\end{aligned} \tag{A.10}$$

and thus

$$R_{11}=g_s^{2/3}\,\mathrm{R}_{11}=\frac{L_{11}}{k}\,,\qquad\qquad \hat N=Nk\,,\qquad\qquad \frac{L_{11}}{\ell_P}=\big(2^5\pi^2Nk\big)^{1/6}\,. \tag{A.11}$$

Here L_{11} and k are the parameters of the 11d M-theory background.

[29] In relating M2-brane action and the fundamental string action by this double dimensional reduction the dilaton factors cancel [58].

[30] These relations follow from natural assumption that 11d action does not know about g_s which enters only via the dilaton. The resulting identifications are also consistent with relations for D-brane tensions as D-brane actions contain $e^{-\phi}$ factor, *i.e.* scale as $\frac{1}{g_s}$ for constant dilaton. A different option is to assume that relation between 11d and 10d actions involves only non-constant part of the dilaton. Then (A.4) would not have the $\frac{1}{g_s^2}$ factor in the l.h.s. and (A.6) would then take the form (see, *e.g.*, [57]) $\ell'_P=g_s^{1/3}\ell_s$, $\mathrm{R}'_{11}=g_s\,\ell_s$.

Upon dimensional reduction we then get the metric and parameters of 10d string theory

$$ds^2_{10} = L^2\Big(\frac{1}{4}ds^2_{\mathrm{AdS}_4} + ds^2_{\mathrm{CP}^3}\Big), \qquad L_{\mathrm{AdS}_4} = \frac{1}{2}L \ , \tag{A.12}$$

$$L = g_s^{1/3}\, L_{11} \ , \qquad g_s = \big(\frac{L_{11}}{k\,\ell_P}\big)^{3/2} \ . \tag{A.13}$$

Expressed in terms of the dual gauge-theory parameters N and k the string coupling and the effective dimensionless string tension are

$$g_s \equiv \sqrt{\pi}\,\big(\frac{2}{k}\big)^{5/4} N^{1/4} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N} \ , \qquad \lambda = \frac{N}{k} \ , \tag{A.14}$$

$$T \equiv L^2_{\mathrm{AdS}_4} T_1 = \frac{L^2}{8\pi\alpha'} = g_s^{2/3}\,\frac{L^2_{11}}{8\pi\alpha'} = \frac{\sqrt{\lambda}}{\sqrt{2}} \ , \tag{A.15}$$

$$\frac{g_s^2}{8\pi\, T} = \frac{\lambda^2}{N^2} = \frac{1}{k^2} \ . \tag{A.16}$$

The M-theory perturbative expansion corresponds to large curvature scale or large effective M2 brane tension for fixed parameter k of the background

$$\mathrm{L} \equiv \frac{L_{11}}{\ell_P} \gg 1 \ , \qquad \mathrm{T}_2 \equiv T_2 L^3_{11} \gg 1 \ , \qquad k = \text{fixed} \ , \tag{A.17}$$

i.e. to the large N limit with fixed k. The 10d string perturbative expansion corresponds to $g_s \ll 1$ for fixed T, *i.e.* to the 't Hooft expansion in the large N, large k limit with fixed $\lambda = \frac{N}{k}$.

As was argued in [35], the presence of the M-theory correction $R^4 C_3$ implies the shift

$$N \to N - \frac{1}{24}\big(k - \frac{1}{k}\big), \tag{A.18}$$

which modifies the relation between L_{11} and N in (A.11). This leads also to a modification of the expressions for the 10d string parameters g_s and T in (A.14), (A.15)

$$g_s = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N}\Big(1 - \frac{1}{24\lambda} + \frac{\lambda}{24N^2}\Big)^{1/4}, \qquad T \equiv \frac{L^2}{8\pi\alpha'} = \frac{\sqrt{2\lambda}}{2}\Big(1 - \frac{1}{24\lambda} + \frac{\lambda}{24N^2}\Big)^{1/2}, \tag{A.19}$$

or, equivalently, of how N and λ are expressed in terms of them:

$$N = 4\sqrt{2\pi}\,\frac{T^{5/2}}{g_s}\Big(1 + \frac{1}{48T^2} - \frac{1}{384\pi}\frac{g_s^2}{T^3}\Big), \qquad \lambda = 2T^2\Big(1 + \frac{1}{48T^2} - \frac{1}{384\pi}\frac{g_s^2}{T^3}\Big) \ . \tag{A.20}$$

Note that the useful relation (A.16) remains unchanged.

Appendix B. Supergravity contribution to the free energy

To find the leading term in the M-theory effective action one is to evaluate the 11d supergravity action (A.1) on the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background. We may first reduce to 10d and evaluate the resulting type IIA 10d supergravity action on the corresponding $\mathrm{AdS}_4 \times \mathrm{CP}^3$ background. Compactifying on CP^3 we get the 4d Einstein action with a cosmological constant that admits the AdS_4 solution with the radius $L_{\mathrm{AdS}_4} = \frac{1}{2}L$ (cf. (A.12)). Explicitly, one finds (using the negative overall sign for the action corresponding to the Euclidean signature)

$$S_0 = -\frac{1}{2\kappa_{10}^2} L^6 \,\mathrm{vol}(\mathrm{CP}^3) \int d^4x \sqrt{g}\Big(R + 6L_{\mathrm{AdS}_4}^{-2}\Big) \,. \tag{B.1}$$

Here κ_{10} is given in (A.4) and on the AdS$_4$ solution $R = -12L_{\mathrm{AdS}_4}^{-2}$ so that

$$\begin{aligned} S_0 &= -\frac{1}{2\kappa_{10}^2} L^6 \,\mathrm{vol}(\mathrm{CP}^3)\, L_{\mathrm{AdS}_4}^4 \mathrm{vol}(\mathrm{AdS}_4)\,(-6) L_{\mathrm{AdS}_4}^{-2} \\ &= \frac{1}{3\cdot 2^7\pi^2}\,\frac{1}{g_s^2}\,\frac{L^8}{\alpha'^4} = \frac{\pi\sqrt{2}}{3}\lambda^{-1/2}N^2 \,. \end{aligned} \tag{B.2}$$

Here we used that the volumes for the unit-radius spaces are[31]

$$\mathrm{vol}(\mathrm{AdS}_4) = \frac{4\pi^2}{3} \,, \qquad \mathrm{vol}(\mathrm{CP}^3) = \frac{1}{2\pi}\mathrm{vol}(S^3) = \frac{\pi^3}{6} \,, \tag{B.3}$$

and also that $L_{\mathrm{AdS}_4} = \frac{1}{2}L$ and (A.13), (A.15), (A.14). Thus (B.2) matches the first term in F in (2.14) [6].

The same value is found of course directly from the 11d supergravity action evaluated on the on AdS$_4 \times S^7/\mathbb{Z}_k$ background

$$\begin{aligned} S_0 &= -\frac{1}{2\kappa_{11}^2}\,\frac{1}{k} L_{11}^7 \,\mathrm{vol}(S^7)\,(\tfrac{1}{2}L_{11})^4 \,\mathrm{vol}(\mathrm{AdS}_4)\,(-6)(\tfrac{1}{2}L_{11})^{-2} \\ &= \frac{1}{3\cdot 2^7\pi^2}\,\frac{1}{k}\,\frac{L_{11}^9}{\ell_P^9} = \frac{\pi\sqrt{2}}{3}k^{1/2}N^{3/2} \,, \end{aligned} \tag{B.4}$$

where we used (A.11). This matches the first term in the large N expansion of free energy (2.6).

Appendix C. Values of R^4 invariants on AdS$_4$ × CP^3

In (4.16) we presented the value of the Weyl-tensor part (4.15) of the J_0 invariant in (4.11) on AdS$_4 \times$ CP^3 background. Here we shall present the values of the full curvature-dependent parts of the invariants J_0 and J_1 keeping also the Ricci tensor dependent contributions.

Using the explicit form of the t_8 tensor [61] one finds

$$\begin{aligned} t_8t_8RRRR = &\, t^{mnlrstpq}t_{abcdefgh}R_{mn}{}^{ab}R_{lr}{}^{cd}R_{st}{}^{ef}R_{pq}{}^{gh} \\ = &-96R_{ab}{}^{ef}R^{abcd}R_{ce}{}^{gh}R_{dfgh} + 384R_a{}^e{}_c{}^f R^{abcd}R_b{}^g{}_f{}^h R_{dheg} \\ &+ 24R_{ab}{}^{ef}R^{abcd}R_{cd}{}^{gh}R_{efgh} - 192R_{abc}{}^e R^{abcd}R_d{}^{fgh}R_{efgh} \\ &+ 192R_a{}^e{}_c{}^f R^{abcd}R_b{}^g{}_d{}^h R_{egfh} + 12(R_{mnkl}R^{mnkl})^2. \end{aligned} \tag{C.1}$$

In (4.11) one has[32]

$$J_0 = t_8t_8RRRR - \frac{1}{4}E_8 + \dots \,, \tag{C.2}$$

$$\begin{aligned} E_8 \equiv &\,\delta^{n_1\cdots n_8}_{m_1\cdots m_8}R^{m_1m_2}{}_{n_1n_2}\cdots R^{m_7m_8}{}_{n_7n_8} \\ = &-384R^2R_{kl}R^{kl} + 768(R_{kl}R^{kl})^2 + 96R^2R_{klmn}R^{klmn} - 384R_{pq}R^{pq}R_{klmn}R^{klmn} \end{aligned} \tag{C.3}$$

---

31 In general, $\mathrm{vol}(\mathrm{AdS}_{2n}) = \frac{(-2\pi)^n}{(2n-1)!!}$, $\quad \mathrm{vol}(\mathrm{CP}^n) = \frac{1}{2\pi}\mathrm{vol}(S^{2n+1}) = \frac{\pi^n}{n!}$.

32 In Minkowski signature metric used in (4.11) $\frac{1}{2}\varepsilon_{10}\varepsilon_{10}RRRR = -E_8$, cf. [39].

$$
\begin{aligned}
&+48(R_{klmn}R^{klmn})^2+16R^4+R\big(1024R_a{}^cR^{ab}R_{bc}+1536R^{ab}R^{cd}R_{acbd}\\
&-1536R^{ab}R_a{}^{cde}R_{bcde}-512R_a{}^e{}_c{}^fR^{abcd}R_{bfde}+128R_{ab}{}^{ef}R^{abcd}R_{cdef}\big)\\
&-1536R_a{}^cR^{ab}R_b{}^dR_{cd}-6144R_a{}^cR^{ab}R^{de}R_{bdce}+1536R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef}\\
&+3072R^{ab}R^{cd}R_a{}^e{}_c{}^fR_{bedf}+3072R_a{}^cR^{ab}R_b{}^{def}R_{cdef}\\
&-3072R^{ab}R^{cd}R_a{}^e{}_b{}^fR_{cedf}+6144R^{ab}R_a{}^{cde}R_b{}^f{}_d{}^gR_{cgef}\\
&-1536R^{ab}R_a{}^{cde}R_{bc}{}^{fg}R_{defg}+3072R^{ab}R_a{}^c{}_b{}^dR_c{}^{efg}R_{defg}\\
&-1536R_a{}^e{}_c{}^fR^{abcd}R_b{}^g{}_e{}^hR_{dgfh}-1536R_{ab}{}^{ef}R^{abcd}R_c{}^g{}_e{}^hR_{dhfg}\\
&+96R_{ab}{}^{ef}R^{abcd}R_{cd}{}^{gh}R_{efgh}-768R_{abc}{}^eR^{abcd}R_d{}^{fgh}R_{efgh}\\
&+768R_a{}^e{}_c{}^fR^{abcd}R_b{}^g{}_d{}^hR_{egfh}.
\end{aligned}
$$

Note that $(R_{klmn}R^{klmn})^2$ terms cancel in the combination of (C.1) and (C.3) that enters J_0 (cf. [62]).

Evaluating these two invariants on $AdS_4\times CP^3$ with the metric (A.12) introducing for generality $\gamma=(\frac{L_{CP3}}{L_{AdS_4}})^2$ as the ratio of the squares of the radii we find[33]

$$
t_8t_8RRRR=2^9\cdot 3^2\,(3\gamma^4+48\gamma^2+512)L^{-8}\,,\qquad E_8=2^{15}\cdot 3^3\,\gamma\,(3\gamma-8)L^{-8}\,. \tag{C.4}
$$

For $\gamma=4$ corresponding to the metric in (A.12) we get

$$
t_8t_8RRRR=2^{20}\cdot 3^2L^{-8}\,,\qquad E_8=2^{19}\cdot 3^3L^{-8}\,. \tag{C.5}
$$

Thus if we would keep only these curvature-dependent terms in J_0, J_1 in (4.10) we would get from (4.13) using (4.14)

$$
\overline{\Delta S}=\frac{1}{8\pi^2}\frac{L^2}{\alpha'}\Big[\tfrac{4}{3}\Big(\tfrac{\zeta(3)}{g_s^2}+\tfrac{\pi^2}{3}\Big)-\tfrac{1}{2}\Big(\tfrac{\zeta(3)}{g_s^2}-\tfrac{\pi^2}{3}\Big)\Big]\,. \tag{C.6}
$$

This is of the same order as just the Weyl-tensor part contribution in (4.16) but does not match the precise rational coefficients in the $\frac{L^2}{\alpha'}$ term in $\widetilde{F}$ in (4.3). This suggests that it is important to include also the contributions of the RR field strength dependent terms in J_0 and J_1 to get the matching.

As an aside, let us note that E_8 has an interpretation of an Euler density in 8 dimensions. In general, for a d dimensional space $\mathcal{M}^d$ with Euclidean signature[34]

$$
E_{2n}(\mathcal{M}^d)=\frac{1}{(d-2n)!}\,\varepsilon_d\varepsilon_dR^n=\delta^{a_1\cdots a_{2n}}_{b_1\cdots b_{2n}}R^{b_1b_2}{}_{a_1a_2}\cdots R^{b_{2n-1}b_{2n}}{}_{a_{2n-1}a_{2n}}\,, \tag{C.7}
$$

which vanishes for $2n>d$. For example, for a sphere S^d we have $R^{ab}{}_{ce}=\frac{1}{r_d^2}\delta^{ab}_{ce}$, and thus $E_{2n}(S^d)=2^n\frac{d!}{(d-2n)!}\big(\frac{1}{r_d^2}\big)^n$. For a product manifold $\mathcal{M}^m\times S^n$, with $d=m+n$, we have [63]

$$
E_{2p}(\mathcal{M}^m\times S^n)=\sum_{t=0}^{[m/2]}\binom{p}{t}\frac{n!}{(n-2(p-t))!}\Big(\frac{1}{r_n^2}\Big)^{p-t}2^{t-p}E_{2t}(\mathcal{M}^m)\,. \tag{C.8}
$$

---

33 In particular, $R_{kl}R^{kl}=(384+36\gamma^2)L^{-4}$, $R_{klmn}R^{klmn}=(384+24\gamma^2)L^{-4}$, $R^{ab}R_{bc}R^{cd}R_{ca}=(24576+324\gamma^4)L^{-4}$.

34 Recall that in d dimensions $\varepsilon_{i_1\cdots i_n}\varepsilon^{j_1\cdots j_n}=\delta^{j_1\cdots j_n}_{i_1\cdots i_n}=\sum_\sigma(-1)^\sigma\delta^{j_1}_{i_{\sigma_1}}\cdots\delta^{j_n}_{i_{\sigma_n}}$, $\delta^{j_1\cdots j_sj_{s+1}\cdots j_p}_{i_1\cdots i_si_{s+1}\cdots i_p}=\frac{(d-s)!}{(d-p)!}\delta^{j_1\cdots j_s}_{i_1\cdots i_s}$.

This is a special case of the general relation[35]

$$E_{2p}(\mathcal{M}^m \times \mathcal{M}^n) = \sum_{t=0}^{[m/2]} \binom{p}{t} E_{2t}(\mathcal{M}^m) E_{2(p-t)}(\mathcal{M}^n) \,, \tag{C.9}$$

implying, in particular, that $E_8(\mathcal{M}^4 \times \mathcal{M}^6) = 4E_2(\mathcal{M}^4)\, E_6(\mathcal{M}^6) + 6E_4(\mathcal{M}^4)\, E_4(\mathcal{M}^6)$. Indeed, one can check that

$$E_8(\mathrm{AdS}_4 \times \mathrm{CP}^3) = 4E_2(\mathrm{AdS}_4)\, E_6(\mathrm{CP}^3) + 6E_4(\mathrm{AdS}_4)\, E_4(\mathrm{CP}^3), \tag{C.10}$$

in agreement with the value of E_8 in (C.4).

Appendix D. Non-perturbative corrections to the ABJM free energy

Here, for completeness, we recall some facts about non-perturbative corrections to free energy of the ABJM theory.

In M-theory one may expect non-perturbative contributions to the M2-brane partition function related to membranes wrapping a 3-cycle $\mathcal{C}_3$ of 11d space and thus producing a factor $\sim \exp(-T_2 \mathrm{vol}(\mathcal{C}_3))$ where T_2 is M2-brane tension in (A.2). If $\mathcal{C}_3$ wraps 11d circle then this contribution may be interpreted as the 10d fundamental string instanton with $T_2\mathrm{vol}(\mathcal{C}_3) \to T_1\mathrm{vol}(\mathcal{C}_2)$ where T_1 is the string tension (cf. (A.5)) and $\mathcal{C}_2$ is a 2-cycle in 10d space on which the string worldsheet is wrapped. If $\mathcal{C}_3$ lies in 10d subspace then the corresponding contribution is that of the D2-brane instanton.

In the context of the ABJM theory one may thus expect two types of non-perturbative contributions to F proportional to ($n = 1, 2, ...$)

$$e^{-2\pi n\sqrt{2\lambda}} = e^{-2\pi n\sqrt{2}\sqrt{\frac{N}{k}}} \,, \tag{D.1}$$

related to $\mathrm{CP}^1 \subset \mathrm{CP}^3$ world-sheet instantons [46] (cf. (A.15)), and to ($\ell = 1, 2, ...$)

$$e^{-(\pi^6 2^7 \lambda^3)^{1/4}\frac{\ell}{g_s}} = e^{-\pi\ell\sqrt{2}\frac{N}{\sqrt{\lambda}}} = e^{-\pi\ell\sqrt{2}\sqrt{kN}} \,, \tag{D.2}$$

due to D2-brane instantons on generalized Lagrangian submanifolds with topology of $\mathrm{RP}^3 \subset \mathrm{CP}^3$ [7] (cf. (A.14)).

In the Fermi gas approach of [29] the exact localization expression for the ABJM partition function is expressed in terms of the grand potential $\mathrm{J}(\mu, k)$ of a non-trivial fermionic system as

$$Z(N,k) = \frac{1}{2\pi i}\int d\mu\, e^{\mathrm{J}(\mu,k)-\mu N}, \tag{D.3}$$

that may be evaluated at large N by a saddle point method. The grand potential is given by the sum of the perturbative and non-perturbative parts

$$\mathrm{J}(\mu, k) = \mathrm{J}^{\mathrm{p}}(\mu, k) + \mathrm{J}^{\mathrm{np}}(\mu, k), \tag{D.4}$$

where the perturbative one is

[35] We use this opportunity to point out a misprint in eq. (4.1) in [39]: the coefficient of the second term in
$E_8(\mathcal{M}^4 \times \mathcal{M}^7) = 4E_2(\mathcal{M}^4)\, E_6(\mathcal{M}^7) + 6E_4(\mathcal{M}^4)\, E_4(\mathcal{M}^7)$
is 6 not 12. The value of this coefficient was not, actually used in [39].

$$J^{p}(\mu,k)=\frac{1}{3}C(k)\,\mu^3+B(k)\,\mu+A(k),\qquad C(k)=\frac{2}{\pi^2 k},\quad B(k)=\frac{k}{24}+\frac{1}{3k}.\tag{D.5}$$

Evaluating (D.3) with J^p part only gives the function partition function in (2.1) given by the Airy function and the $e^{A(k)}$ factor.

The non-perturbative part is expected to have the form

$$J^{np}(\mu,k)=\sum_{n,\ell}u_{n,\ell}(k,\mu)\exp\Big[-\Big(\frac{4}{k}n+2\ell\Big)\mu\Big],\tag{D.6}$$

where the two sums may be interpreted as accounting for the contributions of the two types of instantons mentioned above. Isolating the terms with $\ell=0$ and $n=0$ we may write

$$J^{np}(\mu,k)=J_{I}(\mu,k)+J_{II}(\mu,k)+\delta J^{np}(\mu,k)\ .\tag{D.7}$$

Terms with both $\ell>0$ and $n>0$ in (D.6) (or "bound state" corrections) given by $\delta J^{np}(\mu,k)$ were discussed in [64]. Here $J_I(\mu,k)$ is given by

$$J_{I}(\mu,k)=\sum_{n=1}^{\infty}d_n(k)\,e^{-\frac{4n\mu}{k}}\ ,\tag{D.8}$$

where d_n may be determined using that the ABJM matrix integral is dual to the partition function of topological string theory on $P_1\times P_1$. $J_{II}(\mu,k)$ has the following structure for $\mu\gg 1$

$$J_{II}(\mu,k)=\sum_{\ell=1}^{\infty}\big[a_\ell(k)\,\mu^2+b_\ell(k)\,\mu+c_\ell(k)\big]\,e^{-2\ell\mu}\ ,\tag{D.9}$$

where the expansion of the coefficients a_ℓ, b_ℓ, c_ℓ for small k follows from the WKB expansion of the Fermi gas representation [29] $a_\ell(k)=\frac{1}{k}\sum_{m=0}^{\infty}a_{\ell,m}k^{2m}$, etc. Conjectures for the closed form of some of these coefficients were suggested in [65] and a unifying picture were all (ℓ,n) terms in (D.6) arise from a refined topological string representation was presented in [66]. The saddle point evaluation of (D.3) sets $\mu\simeq\sqrt{N/C(k)}$. Then, the exponent in (D.6) reproduces the expected weights in (D.1) and (D.2)

$$F^{np}=-\log Z^{np}=\sum_{n,\ell}f_{\ell,n}(N,\lambda)\,\exp\Big[-\pi\sqrt{2}\Big(2n\sqrt{\lambda}+\ell\frac{N}{\sqrt{\lambda}}\Big)\Big].\tag{D.10}$$

Recently, the prefactor of the leading worldsheet instanton correction to the free energy [6] was directly computed on the 10d string theory side in [47].

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