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The Fractional Discrete Predator–Prey Model: Chaos, Control and Synchronization

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Abstract: This paper describes a new fractional predator–prey discrete system of the Leslie type. In addition, the non-linear dynamics of the suggested model are examined within the framework of commensurate and non-commensurate orders, using different numerical techniques such as Lyapunov exponent, phase portraits, and bifurcation diagrams. These behaviours imply that the fractional predator–prey discrete system of Leslie type has rich and complex dynamical properties that are influenced by commensurate and incommensurate orders. Moreover, the sample entropy test is carried out to measure the complexity and validate the presence of chaos. Finally, nonlinear controllers are illustrated to stabilize and synchronize the proposed model.

Keywords: fractional predator–prey discrete system; incommensurate order; chaotic dynamics; sample entropy; control; synchronization

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1. Introduction

In recent years, biological models have been receiving an increasing amount of attention from researchers. Predator–prey models play a crucial role in the study of biology. The dynamics of predator–prey interactions have been widely investigated in the fields of mathematical biology and ecology. The analyses of these models found several rich dynamics, such as bifurcations, limit cycle, and chaotic behaviors. Although numerous studies have explored the dynamic behaviors of continuous-time systems, discrete-time systems have received comparatively less attention. Discrete-time models possess their own distinct dynamical properties and a variety of practical problems can be represented through these systems in the real biological world. Owing to these characteristics, the study of discrete systems is crucial and has led to significant advancements in engineering, mathematics, ecology, biology and other fields.

In the last two decades, discrete fractional calculus has attracted the attention of numerous mathematicians, who were intrigued by its applications in different domains, such as hardware implementation, image encryption, and secure communication. Recently, a flurry of articles has been published on this hot issue, where the authors offer a variety of discrete-time fractional operators, stability analyses, and several theoretical results [1–3]. These have led to the proposal of more commensurate discrete fractional chaotic systems, such as [4–7], and more incommensurate discrete fractional chaotic systems, such as [8–12], along with a variety of control strategies and synchronization schemes between different fractional chaotic maps, such as [13–16].

In recent years, a significant number of scholars have analyzed the dynamics of discrete models; nonetheless, only a few papers have examined the dynamical behaviour of discrete-time predator–prey models of the Leslie type. For instance, the dynamics of the predator–prey discrete model of Holling and Leslie type were studied by Hu and Cao in [17]. The bifurcation analysis of the predator–prey discrete-time system with harvesting was analyzed by Lio and Jiang [18]. In [19], the authors explored the discrete prey–predator model of Leslie type with a simplified Holling type IV. The global dynamics and hybrid control in Leslie's prey–predator discrete-time model were investigated by Khan et al. [20], while Chen et al. [21] discussed the chaos in a discrete prey–predator model of Leslie type with Michaelis–Menten prey harvesting. The majority of the aforementioned discrete memristor research is of classical integer order; unfortunately, as far as we know, no research contributions have studied the dynamic behaviours of a fractional-order predator–prey discrete-time system of Leslie type. Consequently, the investigation of the chaotic dynamics of predator–prey discrete-time system of Leslie type based on fractional differences with commensurate and incommensurate order, as well as their synchronization and control, is an attractive subject.

Motivated by the aforementioned discussion, we intend, in this article, to explore and study the dynamic behaviors of the new fractional predator–prey discrete system of Leslie type using both commensurate and non-commensurate orders. The basic properties of this fractional map will be studied using certain theoretical and numerical analyses. Furthermore, we will use the sample entropy test to measure the complexity and validate the presence of chaos in the proposed system using commensurate and incommensurate orders. In addition, we propose a nonlinear controller that enables the suggested model to be stabilized and synchronized by forcing the states to asymptotically converge toward zero. Finally, we will conclude the study by summarizing the most significant findings obtained in the article.

2. Mathematical Model

In [19], the authors introduced the following predator–prey discrete system of Leslie type:

$$\begin{aligned} x_{m+1} &= x_m + \delta x_m \left[(1 - x_m) - \frac{y_m}{x_m^2 + c} \right], \\ y_{m+1} &= y_m + \delta y_m \left[a - \frac{b y_m}{x_m} \right], \end{aligned} \tag{1}$$

where a, b, c and δ are non-negative parameters. The first-order difference of the model is designed as follows:

$$\begin{aligned} \Delta x(m) &= \delta x(m) \left[(1 - x(m)) - \frac{y(m)}{x^2(m) + c} \right], \\ \Delta y(m) &= \delta y(m) \left[a - \frac{b y(m)}{x(m)} \right]. \end{aligned} \tag{2}$$

Then, the fractional version of the proposed discrete model (2) can be given as follows:

$$\begin{aligned} {}^C\Delta_d^\gamma x(m) &= \delta x(m+\gamma-1) \left[\Big(1 - x(m+\gamma-1)\Big) - \frac{y(m+\gamma-1)}{x^2(m+\gamma-1) + c} \right], \\ {}^C\Delta_d^\gamma y(m) &= \delta y(m+\gamma-1) \left[a - \frac{b y(m+\gamma-1)}{x(m+\gamma-1)} \right], \end{aligned} \tag{3}$$

where $0 < \gamma \leq 1$, $m \in \mathbb{N}_{d+1-\gamma}$, $N_d = \{d, d+1, d+2, \dots\}$ such that $d \in \mathbb{R}$. ${}^C\Delta_d^\gamma$ is the Caputo-like difference operator which is defined, according to [3], as:

$$\begin{aligned} {}^C\Delta_d^\gamma \chi(m) &= \Delta_d^{-(k-\gamma)} \Delta^k \chi(m) \\ &= \frac{1}{\Gamma(k-\gamma)} \sum_{\nu=k}^{m-(k-\gamma)} (m-\nu-1)^{(k-1-\gamma)} \Delta^k \chi(\nu), \end{aligned} \tag{4}$$

where $m \in (\mathbb{N})_{d+k-\gamma}$ and $k = \lceil \gamma \rceil + 1$. $\Delta_d^{-\gamma}$ is the $\gamma - th$ fractional sum which, according to [1], can be given by:

$$\Delta_d^{-\gamma}\chi(m) = \frac{1}{\Gamma(\gamma)}\sum_{\nu=\gamma}^{m-\gamma}(m-\nu-1)^{(\gamma-1)}\chi(\nu), \quad m \in (\mathbb{N})_{k+\gamma},\ \gamma > 0 \tag{5}$$

To discuss the complex dynamics of model (3), we shall give the theorem below, which enables us to acquire the numerical formula of the new fractional predator–prey discrete system:

Theorem 1 ([22])**.** *The solution of the initial value problem* ***IVP***

$$\begin{cases} {}^{C}\Delta_d^{\gamma}\chi(m) = g(m+\gamma-1, \chi(m+\gamma-1)) \\ \Delta^k\chi(d) = \chi_k, \quad n = \lceil\gamma\rceil + 1, \quad k = 0,1,\dots,n-1, \end{cases} \tag{6}$$

is written as

$$\chi(m) = \chi_0(d) + \frac{1}{\Gamma(\gamma)}\sum_{\nu=d+n-\gamma}^{m-\gamma}(m-1-\nu)^{(\gamma-1)}g(\nu-1+\gamma, \chi(\nu-1+\gamma)), \quad m \in \mathbb{N}_{d+n}, \tag{7}$$

where

$$\chi_0(d) = \sum_{k=0}^{n-1}\frac{(m-d)^k}{\Gamma(k+1)}\Delta^k\chi(d). \tag{8}$$

Remark 1. *Take* $d = 0$*, since* $(m-1-\nu)^{(\gamma-1)} = \frac{\Gamma(m-\nu)}{\Gamma(m+1-\nu-\gamma)}$ *and for* $l = \nu+\gamma-1$ *and* $n = 1$*, the numerical formula* (7) *can be designed for* $\gamma \in (0,1]$ *as follows:*

$$\chi(m) = \chi(0) + \frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\frac{\Gamma(m-1-l+\gamma)}{\Gamma(m-l)}g(l, \chi(l)). \tag{9}$$

As a result of this theorem, the numerical formula of the fractional predator–prey discrete system (3) is given by:

$$\begin{aligned} x(m) &= x(0) + \frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\frac{\Gamma(m-l-1+\gamma)}{\Gamma(m-l)}\left(\delta x(l)\left[\left(1-x(l)\right) - \frac{y(l)}{x^2(l)+c}\right]\right), \\ y(m) &= y(0) + \frac{1}{\Gamma(\gamma)}\sum_{l=0}^{m-1}\frac{\Gamma(m-l-1+\gamma)}{\Gamma(m-l)}\left(\delta y(l)\left[a - \frac{by(l)}{x(l)}\right]\right), \quad m = 1,2,\dots. \end{aligned} \tag{10}$$

3. Commensurate Fractional Discrete System

The dynamics of the commensurate fractional predator–prey discrete system of Leslie type (3) will be investigated in this part, using numerical analysis tools such as phase portraits, bifurcation charts, and the estimation of maximum Lyapunov exponents (MLE). It should be noted that the system with commensurate order is a system of equations generated with identical orders. The maximum Lyapunov exponents of the attractors of the fractional map (3) will be determined using the Jacobian matrix approach [23].

The variation in the dynamic behaviour of the proposed predator–prey map from periodic states to chaotic states is illustrated with the help of bifurcations. The commensurate fractional order of the map (3) is regarded as the bifurcation parameter, which ranges from 0 to 1. The initial values were fixed as $(x(0), y(0)) = (0.9, 0.56)$, and the other parameters were $a = 2.5,\ b = 5,\ c = 5.5,\ \delta = 1.05$. The maximum Lyapunov exponents of the fractional parameter γ and the bifurcation chart of the state variable x were computed numerically via MATLAB, and the results are presented in Figure 1. We can observe that when the order $\gamma < 0.66$, the trajectories diverge toward infinity. When $\gamma \in [0.66, 0.904]$, chaotic behaviors can be obtained, where the values of MLE are positive. Moreover, when γ grows larger and approaches 1, the MLEs are negative or equal to zero, meaning that the commensurate fractional discrete system (3) is stable and periodic windows appear. To obtain a better understanding of these characteristics, we frew different phase attractors to suitably support the MLE and the bifurcation diagram in Figure 1. Figure 2 depicts the results for $\gamma = 0.67$, $\gamma = 0.7$, $\gamma = 0.8$, $\gamma = 0.85$, $\gamma = 0.9$ and $\gamma = 0.95$. It is clear that the suggested commensurate fractional predator–prey discrete system exhibits chaotic motions when the order γ decreases, and periodic orbits when the commensurate order γ takes higher values. These numerical simulations demonstrate that the commensurate fractional predator–prey discrete system (3) has a variety of interesting dynamical properties.

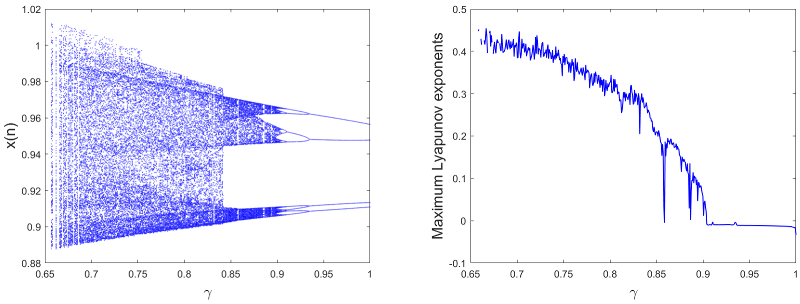


Figure 1. (**a**) Bifurcation of (3) versus γ for $a = 2.5$, $b = 5$, $c = 5.5$, $\delta = 1.05$, (**b**) The corresponding MLE.

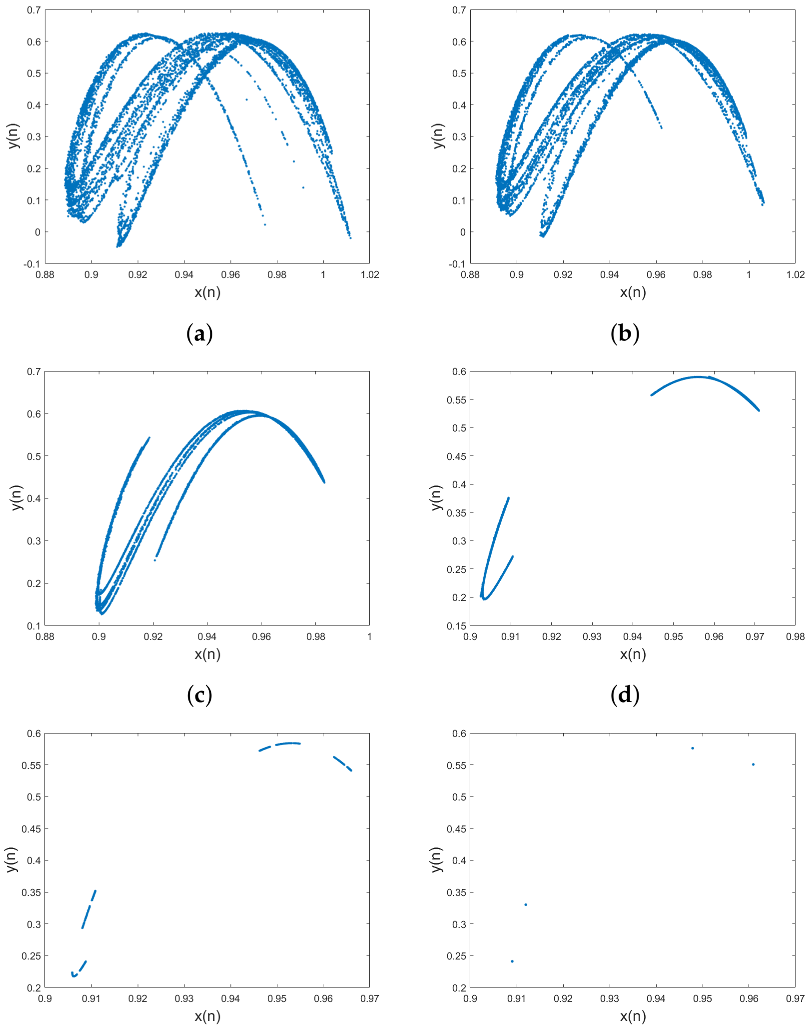


Figure 2. Phase portraits of (3) for $a = 2.5$, $b = 5$, $c = 5.5$, $\delta = 1.05$ and with (**a**) $\gamma = 0.67$, (**b**) $\gamma = 0.7$, (**c**) $\gamma = 0.8$, (**d**) $\gamma = 0.85$, (**e**) $\gamma = 0.9$, (**f**) $\gamma = 0.95$.

Considering δ as the critical parameter, we plotted the bifurcation diagrams of (3) versus $\delta \in [0.75, 1.12]$, as shown in Figure 3, which correspond to the fractional values $\gamma = 0.7$, $\gamma = 0.8$, and $\gamma = 0.9$. According to Figure 3a, the trajectories of the system gradually change from periodic to chaotic states through a period-doubling bifurcation when the parameter δ increases. When $\delta \in (0.98, 1.054)$, the system is chaotic, with a small periodic region shown in $\delta \in (1.027, 1.039)$. However, as shown in Figure 3b,c, when the commensurate order increases, the chaotic region shifts to the right. Basically, when $\gamma = 0.8$, the states are chaotic in $\delta \in (1.014, 1.09)$, while, when $\gamma = 0.9$, the chaotic region is shown in $\delta \in (1.055, 1.118)$.

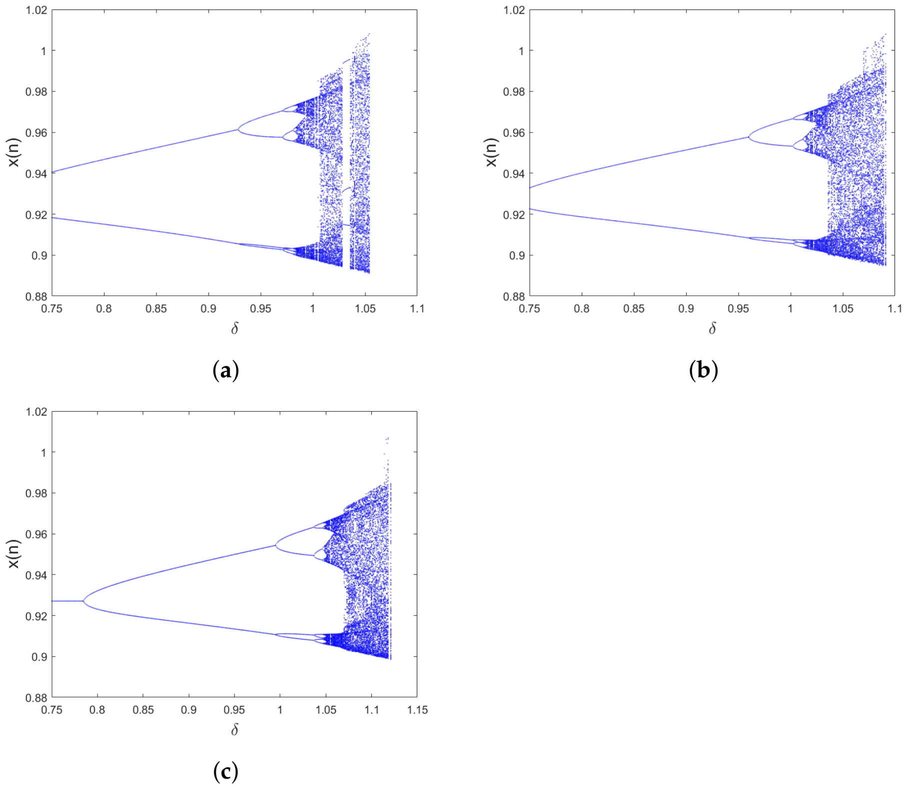


Figure 3. Bifurcation diagrams of (3) versus δ for (**a**) $\gamma = 0.7$, (**b**) $\gamma = 0.78$, (**c**) $\gamma = 0.9$.

4. Incommensurate Fractional Discrete System

This section explores the behaviour of the predator–prey discrete system of Leslie type with incommensurate fractional order values. The incommensurate order system refers to the idea of using distinct fractional orders for each equation in the system. The incommensurate fractional discrete fractional predator–prey discrete system of Leslie type is represented as follows:

$$\begin{aligned} {}^{C}\Delta_{d}^{\gamma_1} x(m) &= \delta x(m-1+\gamma_1)\left[\Big(1 - x(m-1+\gamma_1)\Big) - \frac{y(m-1+\gamma_1)}{x^2(m-1+\gamma_1)+c}\right], \quad m \in \mathbb{N}_{d+1-\gamma_1} \\ {}^{C}\Delta_{d}^{\gamma_2} y(m) &= \delta y(m-1+\gamma_2)\left[a - \frac{by(m-1+\gamma_2)}{x(m-1+\gamma_2)}\right], \quad m \in \mathbb{N}_{d+1-\gamma_2}. \end{aligned} \tag{11}$$

Employing Theorem 1, the numerical model of the incommensurate fractional discrete system (11) is signed as follows:

$$\begin{aligned} x(m) &= x(0) + \frac{1}{\Gamma(\gamma_1)} \sum_{l=0}^{m-1} \frac{\Gamma(m-l-1+\gamma_1)}{\Gamma(m-l)} \left(\delta x(l)\left[\Big(1 - x(l)\Big) - \frac{y(l)}{x^2(l)+c}\right]\right), \\ y(m) &= y(0) + \frac{1}{\Gamma(\gamma_2)} \sum_{l=0}^{m-1} \frac{\Gamma(m-l-1+\gamma_2)}{\Gamma(m-l)} \left(\delta y(l)\left[a - \frac{by(l)}{x(l)}\right]\right), \quad m = 1, 2, \ldots. \end{aligned} \tag{12}$$

The bifurcation diagrams in Figure 4 reflect the system behaviors (11) by varying $\delta \in (0.75, 1.15)$ with the values of parameters $a = 2.5, b = 5, c = 5.5$ and initial values $(x(0), y(0)) = (0.9, 0.56)$. It is plain to observe that the change in orders γ_1 and γ_2 influences the states of the incommensurate fractional predator–prey discrete system of Leslie type. For instance, when we fix $\gamma_2 = 1$ and increase γ_1 from 0.1 to 1, we can observe that the chaotic region expands, and when we fix $\gamma_1 = 0.9$ and increase γ_2 from 0.7 to 1, we see that the region in which chaos exists shrinks. Additionally, when the system parameter δ increases, the trajectories of the incommensurate map gradually evolve from a periodic states with a one-period orbit to a chaotic ones by means of period-doubling bifurcations. Furthermore, we investigate the following two cases for a more accurate illustration of the influence of incommensurate orders on the behaviours of the fractional predator–prey discrete system (11):

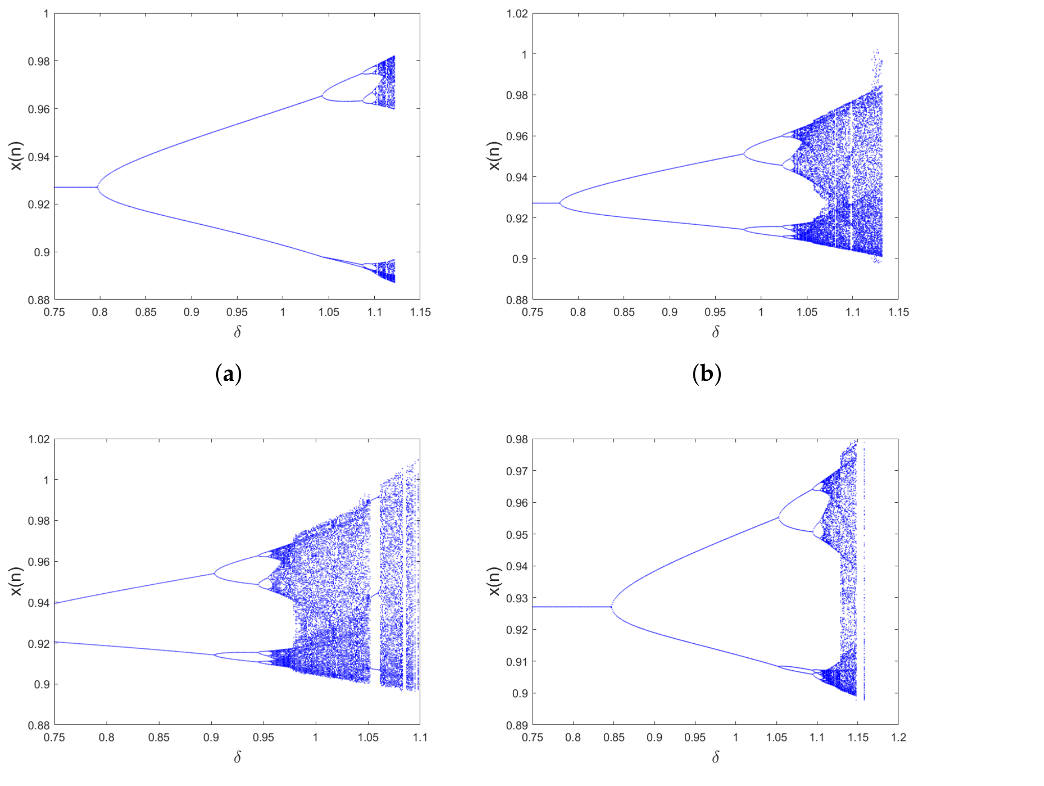
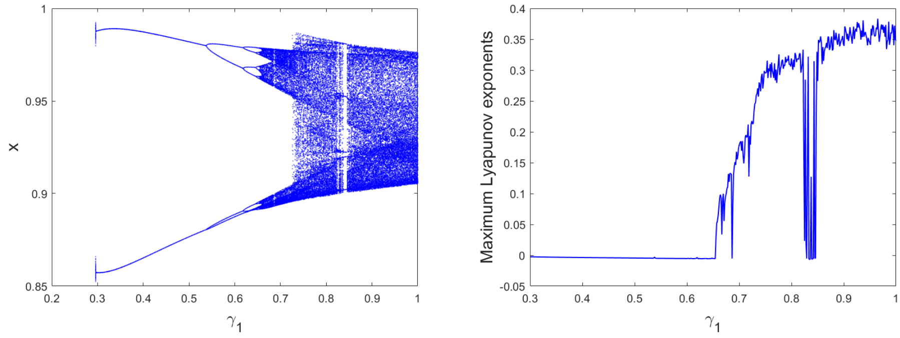


Figure 4. Bifurcations of (11) versus δ for **(a)** $(\gamma_1, \gamma_2) = (0.7, 0.9)$, **(b)** $(\gamma_1, \gamma_2) = (1, 0.9)$, **(c)** $(\gamma_1, \gamma_2) = (0.9, 0.7)$, **(d)** $(\gamma_1, \gamma_2) = (0.9, 1)$.

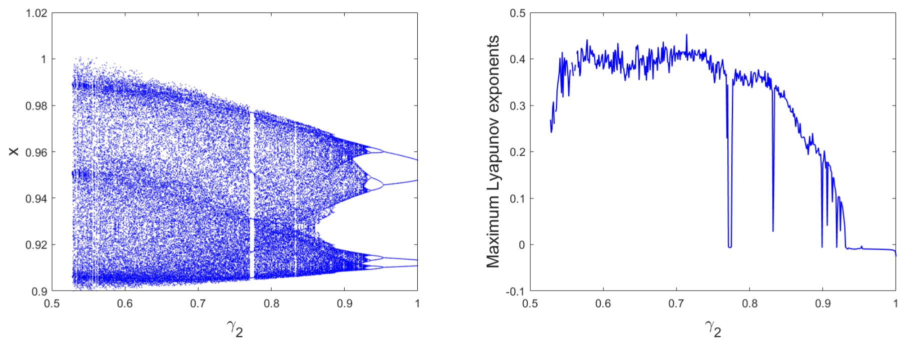
Case 1. We vary the order γ_1 from 0.3 to 1 with step size $\Delta\gamma_1 = 0.001$. Figure 5a,b illustrates the bifurcation and its corresponding MILEs for $\gamma_2 = 0.8$, the parameter values $a = 2.2, b = 55, c = 5.5, \delta = 1.05$ and initial conditions $(x(0), y(0)) = (0.9, 0.56)$. It is clear from Figure 5 that the state of the incommensurate map (11) displays chaotic behavior for larger γ_1 values, as reflected by positive Lyapunov exponents, as seen in Figure 5b. The obtained MLE is 0.383. The Lyapunov exponent shown in Figure 5b is negative for the fractional order value $\gamma_1 < 0.654$. This result means that a small periodic region is seen for $\gamma_1 \in [0.3, 0.654]$. Moreover, when γ_1 grows larger and approaches 1, the incommensurate fractional map possesses a complex chaotic attractor as its MLEs reach their maximum values.



(**a**) (**b**)

Figure 5. (**a**) Bifurcation of (11) versus γ_1 for $a = 2.5,\ b = 5,\ c = 5.5,\ \delta = 1.05$ and $\gamma_2 = 0.8$, (**b**) The corresponding MLE.

Case 2. The bifurcation and its MLE are drawn for $\gamma_1 = 1$ to examine the dynamic behaviours of the incommensurate fractional predator–prey discrete system of the Leslie type (11) when γ_2 is an adjasable parameter, as displayed in Figure 6. These results are obtained by varying γ_2 in the range $(0.5, 1)$ and with order $\gamma_1 = 1$. The initial conditions $(x(0), y(0)) = (0.9, 0.56)$, and the parameter values $a = 2.2, b = 55, c = 5.5, \delta = 1.05$ have remained unchanged. We can observe that when the order γ_2 has small values, the trajectories will diverge toward infinity. When $\gamma_2 \in [0.528, 0.93]$, chaotic behaviors can be obtained, where the MLE values are positive. A small periodic region is also seen for $\gamma_2 \in [0.7701, 0.7756]$, where the MLEs have negative values. Moreover, when γ_2 grows larger and approaches 1, the MLEs are negative, meaning that the incommensurate fractional predator–prey discrete system of Leslie type (11) is stable and periodic windows appear. According to these findings, changes in the incommensurate orders affect the dynamical properties of a fractional predator-prey discrete system of Leslie type. This also suggests that the behaviors of the system may be more accurately represented by incommensurate orders, which is supported by the phase portraits of the state variables of the incommensurate fractional system (11) seen in Figure 7.



(**a**) (**b**)

Figure 6. (**a**) Bifurcation of (11) versus γ_2 for $a = 2.5,\ b = 5,\ c = 5.5,\ \delta = 1.05$ and $\gamma_1 = 1$. (**b**) The corresponding MLE.

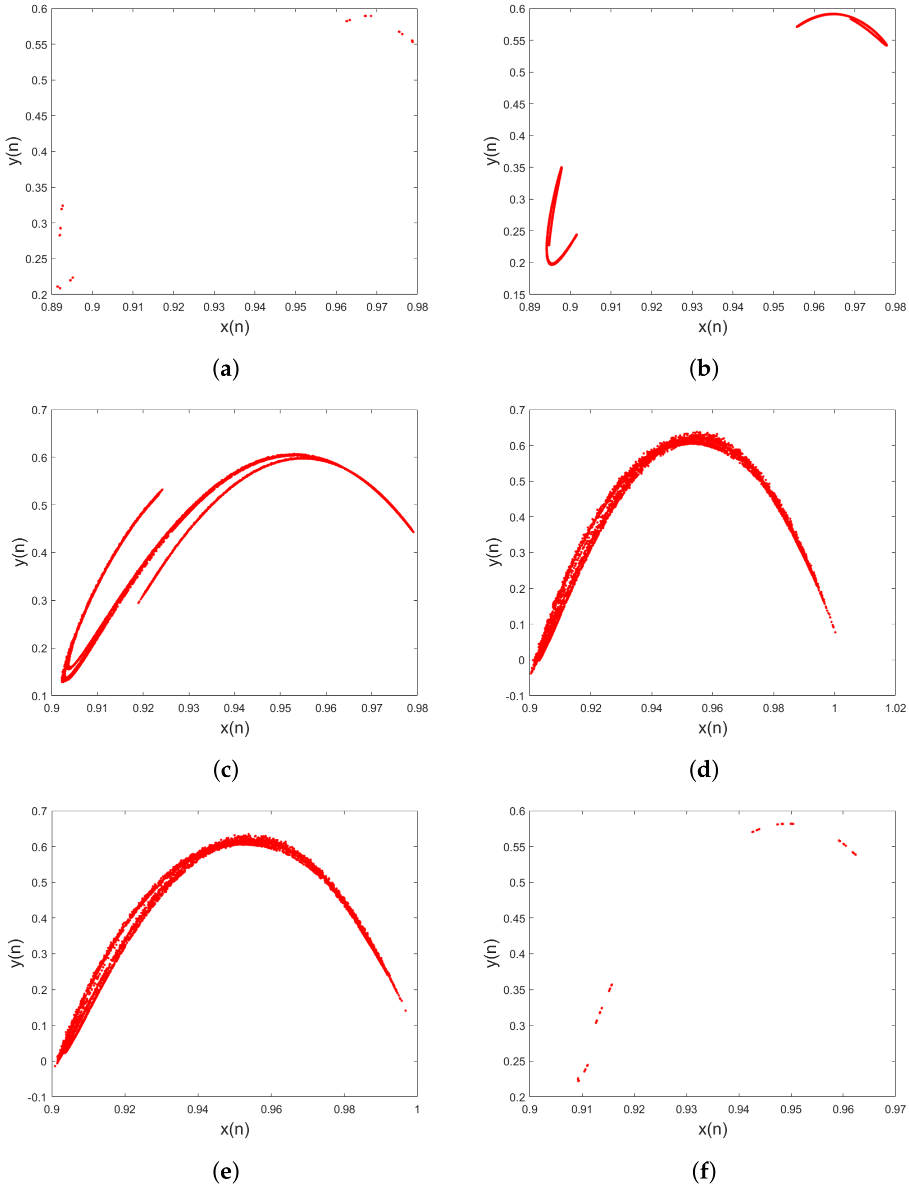


Figure 7. Phase portraits of (11) for $a = 2.5,\ b = 5,\ c = 5.5,\ \delta = 1.05$ and with different fractional orders **(a)** $(\gamma_1, \gamma_2) = (0.65, 0.8)$ **(b)** $(\gamma_1, \gamma_2) = (0.7, 0.8)$ **(c)** $(\gamma_1, \gamma_2) = (0.9, 0.8)$ **(d)** $(\gamma_1, \gamma_2) = (1, 0.55)$ **(e)** $(\gamma_1, \gamma_2) = (1, 0.6)$ **(f)** $(\gamma_1, \gamma_2) = (1, 0.931)$.

5. The Sample Entropy Test (*SampEn*)

Here, we used the sample entropy (*SampEn*) method to measure the complexity of the commensurate fractional predator–prey discrete system of Leslie type (3) and the incommensurate fractional predator–prey discrete system of Leslie type (11). Sample entropy, unlike approximate entropy, can quantify the irregularity of time series regardless of the similarity coefficient r and the embedding dimension m. As a result, *SampEn* is fairly consistent and eliminates the bias of *ApEn* [24]. The higher-complexity time series are those that have larger *SampEn* values [25]. The SampEn is calculated as follows:

$$SampEn = -\log \frac{\Psi^{j+1}(r)}{\Psi^{j}(r)}, \tag{13}$$

where $\Psi^j(r)$ is expressed as

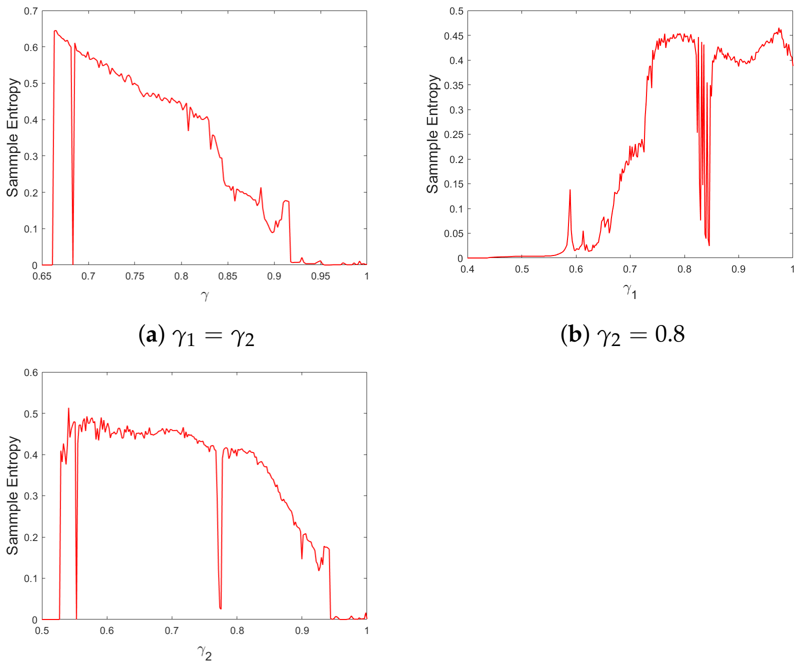
$$\Psi^j(r) = \frac{1}{m-j+1} \sum_{i=1}^{m-j+1} \log C_i^j(r). \tag{14}$$

and $r = 0.2std(C)$ is the tolerance defined and $std(C)$ represents the standard deviation.

After setting the parameters $a = 2.2, b = 5, c = 5.5, \delta = 1.05$ and the initial conditions $(x(0), y(0)) = (0.9, 0.56)$, the sample entropy results of the commensurate fractional predator–prey discrete system (3) and the incommensurate fractional predator–prey discrete system (11) are given in Table 1 and Figure 8. It is clear that, in order to acquire larger $SampEn$ values, the time series has a higher level of complexity. As a consequence, these findings are in accordance with the MLE results that were shown previously, validating the pressence of chaos in the proposed fractional system.

Table 1. The sample entropy results of the fractional predator–prey discrete system.

γ_1	γ_2	$SampEn$	γ_1	γ_2	$SampEn$
0.67	0.67	0.6179	0.8	0.8	0.4407
0.95	0.95	0.0116	1	0.6	0.4715
0.65	0.8	0.0665	0.9	0.8	0.4407
1	0.55	0.5017	1	0.8	0.3878



(a) $\gamma_1 = \gamma_2$ **(b)** $\gamma_2 = 0.8$

(c) $\gamma_1 = 1$

Figure 8. The sample entropy ($SampEn$) of the fractional predator-prey discrete system of Leslie type for $a = 2.2, b = 5, c = 5.5, \delta = 1.05$ and initial conditions $(x(0), y(0)) = (0.9, 0.56)$.

6. Control of Fractional Predator–Prey Discrete System

In the following, we suggest a stabilization controller to stabilize the chaotic trajectories of the proposed fractional predator–prey discrete system of Leslie type. The goal of the stabilization control problem is to design a good adaptive controller that makes all of the system's states tend asymptotically towards zero. To this aim, we first recall the theorem of the stability of the fractional map

Theorem 2 ([26]). *Let $z(m) = (z_1(m), \ldots, z_n(m))^T$ and $B \in \mathcal{M}_n(\mathbb{R})$. The zero fixed-point of the linear fractional order discrete system*

$$^C\Delta_d^\gamma z(m) = B\, z(m - 1 + \gamma), \tag{15}$$

$\forall\, m \in \mathbb{N}_{d+1-\gamma}$ *is asymptotically stable if*

$$\lambda_\iota \in \left\{ \vartheta \in \mathbb{C} : |\vartheta| < \left(2\cos\frac{|\arg\vartheta| - \pi}{2-\gamma} \right)^\gamma \quad \text{and} \quad |\arg\vartheta| > \frac{\gamma\,\pi}{2} \right\}, \tag{16}$$

where λ_ι are the eigenvalues of the matrix B.

The controlled fractional predator–prey discrete system is given by:

$$\begin{aligned} ^C\Delta_d^\gamma x(m) &= \delta x m + \gamma - 1) \left[\Big(1 - x(m+\gamma-1)\Big) - \frac{y(m+\gamma-1)}{x^2(m+\gamma-1)+c} \right] + C_1(m+\gamma-1), \\ ^C\Delta_d^\gamma y(m) &= \delta y(m+\gamma-1) \left[a - \frac{by(m+\gamma-1)}{x(m+\gamma-1)} \right] + C_2(m+\gamma-1), \end{aligned} \tag{17}$$

where $C = (C_1, C_2)^T$ is the adaptive controller. The next theorem describes the control law to stabilize the suggested new fractional predator–prey discrete system.

Theorem 3. *The fractional predator–prey discrete system of Leslie type is the stabilized subject to the following control law*

$$\begin{cases} C_1(m) = \delta x^2(m) - (1+\delta)x(m) + \frac{\delta x(m) y(m)}{x^2(m)+c}, \\ C_2(m) = \frac{\delta b y^2(m)}{x(m)} - (1+\delta a)y(m). \end{cases} \tag{18}$$

Proof. Substituting the control law (18) into (17) yields the following linear system

$$^C\Delta_d^\gamma Z(m) = BZ(m - 1 + \gamma), \tag{19}$$

where $Z = (x, y)^T$ and

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{20}$$

It is clear that the eigenvalues of the matrix B satisfy

$$|\lambda_j| = 1 < \left(2\cos\frac{|\arg\lambda_j| - \pi}{2-\gamma} \right)^\gamma \quad \text{and} \quad |\arg\lambda_j| = \pi > \frac{\gamma\,\pi}{2}, \quad j = 1,2,3. \tag{21}$$

Therefore, by employing Theorem 2, the system (19) is asymptotically stable. □

A numerical simulation was carried out to verify the results of Theorem 3. Figures 9 and 10 illustrate the time series of the controlled fractional predator–prey discrete system of Leslie type (17) for $\gamma = 0.8$ and $\gamma = 0.67$, respectively. We observe that the system's states asymptotically converge towards zero, which confirms the stabilization results.

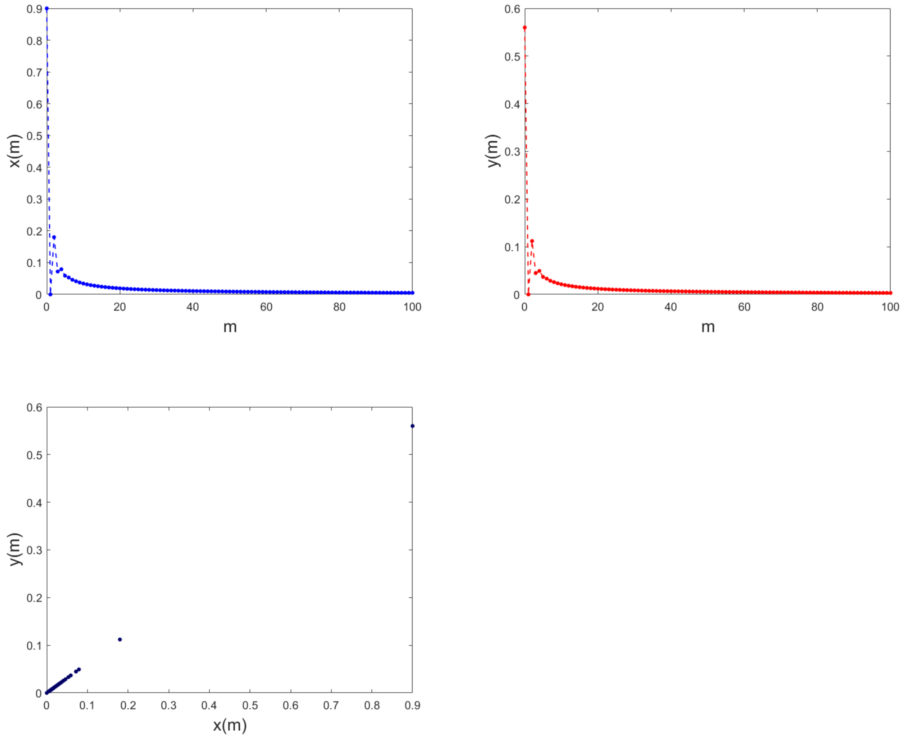


Figure 9. Time evolution of the controller fractional predator–prey discrete system of Leslie type (17) for $\gamma = 0.8$, $a = 2.2, b = 5, c = 5.5, \delta = 1.05$ and $(x(0), y(0)) = (0.9, 0.56)$.

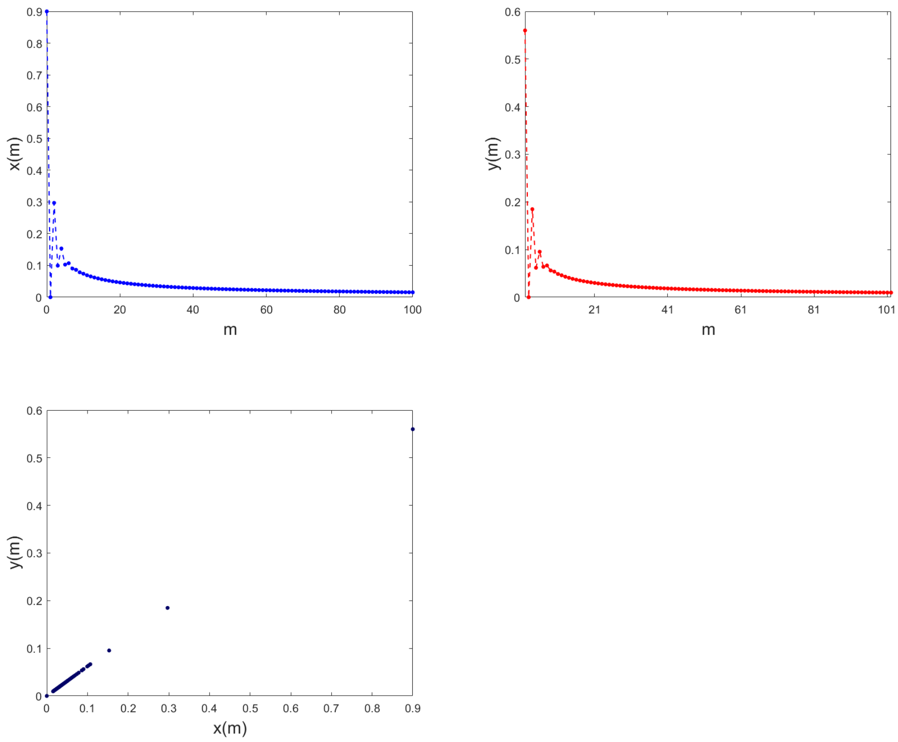


Figure 10. Time evolution of the controller fractional predator–prey discrete system of Leslie type (17) for $\gamma = 0.67$, $a = 2.2, b = 5, c = 5.5, \delta = 1.05$ and $(x(0), y(0)) = (0.9, 0.56)$.

7. Synchronization Scheme

This section illustrates nonlinear controllers that enable synchronization of the fractional predator–prey discrete system of Leslie type. The aim of the synchronization process is to force the error between the master system and slave system to converge toward zero. Let us define the master system as the commensurate fractional predator–prey discrete system (3), and define the slave system as

$$\begin{aligned} {}^{C}\Delta_d^{\gamma} x_s(m) &= \delta x_s(m+\gamma-1)\left[\Big(1-x_s(m+\gamma-1)\Big) - \frac{y_s(m+\gamma-1)}{x_s^2(m+\gamma-1)+c}\right] + U_1(m+\gamma-1), \\ {}^{C}\Delta_d^{\gamma} y_s(m) &= \delta y_s(m+\gamma-1)\left[a - \frac{b y_s(m+\gamma-1)}{x_s(m+\gamma-1)}\right] + U_2(m+\gamma-1), \end{aligned} \tag{22}$$

where the functions U_1 and U_2 are synchronization controllers. The fractional error system is given by

$$\begin{aligned} {}^{C}\Delta_d^{\gamma} e_1(m) =& \delta x_s(m+\gamma-1)\left[\Big(1-x_s(m+\gamma-1)\Big) - \frac{y_s(m+\gamma-1)}{x_s^2(m+\gamma-1)+c}\right] + U_1(m+\gamma-1) \\ & - \delta x(m+\gamma-1)\left[\Big(1-x(m+\gamma-1)\Big) - \frac{y(m+\gamma-1)}{x^2(m+\gamma-1)+c}\right], \\ {}^{C}\Delta_d^{\gamma} e_2(m) =& \delta y_s(m+\gamma-1)\left[a - \frac{b y_s(m+\gamma-1)}{x_s(m+\gamma-1)}\right] + U_2(m+\gamma-1) \\ & - \delta y(m+\gamma-1)\left[a - \frac{b y(m+\gamma-1)}{x(m+\gamma-1)}\right]. \end{aligned} \tag{23}$$

To establish the synchronization scheme, the theorem presented below outlines the proposed control law.

Theorem 4. *Subject to*

$$\begin{cases} U_1(m) = -\lambda_1 e_1(m) + \delta(x_s^2(m) - x^2(m)) + \delta\left(\frac{x_s(m)y_s(m)}{x_s^2(m)+c} - \frac{x(m)y(m)}{x^2(m)+c}\right), \\ U_2(m) = -\lambda_2 e_2(m) + \delta b\left(\frac{y_s^2(m)}{x_s(m)} - \frac{y^2(m)}{x(m)}\right), \end{cases} \tag{24}$$

Where $\lambda_1 = 2$ and $\lambda_2 = 3$. Then, the master system (3) and slave system (22) are synchronized.

Proof. By substituting the control law (24) into the fractional error system (23), we obtain

$${}^{C}\Delta_d^{\gamma}\big(e_1(m), e_2(m)\big)^T = B \times \big(e_1(m+\gamma-1), e_2(m+\gamma-1)\big)^T, \tag{25}$$

where

$$B = \begin{pmatrix} -0.95 & 0 \\ 0 & -0.69 \end{pmatrix} \tag{26}$$

It is not difficult to show that the eigenvalues of B fulfill the stability condition in Theorem 2. Hence, the zero solution of the fractional error system (23) is asymptotically stable and, consequently, the master system (3) and the slave system (22) are synchronized. □

In order to verify this result, numerical simulations are performed using MATLAB. We chose $\gamma = 0.9$, $a = 2.2, b = 5, c = 5.5, \delta = 1.05$ and the initial values $(e_1(0), e_2(0)) = (0.5, 0.2)$. Figure 11 reports the time evolution of states of the fractional error system (23). It is evident that the errors approach zero, which proves that the synchronization discussed earlier is effective.

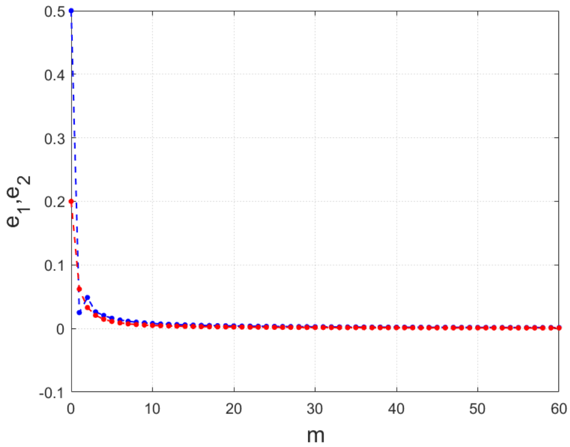


Figure 11. Time evolution of the fractional error system (23) (e_1 (blue) and e_2 (red)).

8. Conclusions

In this article, we described a new fractional predator–prey discrete system of Leslie type depending on commensurate and incommensurate orders. The model revealed that there are different and rich dynamical characteristics. The behaviours of the suggested fractional predator–prey discrete system of the Leslie type for commensurate and incommensurate orders were discussed by calculating the Lyapunov exponent and plotting phase portraits and bifurcation diagrams. Furthermore, the sample entropy algorithm was applied to estimate the complexity of the proposed model. The results show that when the incommensurate orders are varied, the fractional predator–prey discrete system of Leslie type produces chaotic behavior with a higher degree of complexity and a broader range of chaotic regions. Finally, successful control laws were suggested that enable the suggested model to be stabilized and synchronized by compelling the states to asymptotically converge toward zero. Numerical simulations using MATLAB were used to validate our results.

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