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Ordinal sums: From triangular norms to bi- and multivariate copulas

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This paper is dedicated to Radko Mesiar: for stimulating scientific cooperations and for a deeply felt friendship over more than three decades

Abstract

Following a historical overview of the development of ordinal sums, a presentation of the most relevant results for ordinal sums of triangular norms and copulas is given (including gluing of copulas, orthogonal grid constructions and patchwork operators). The ordinal sums of copulas considered here are constructed not only by means of the comonotonic copula, but also by using the lower Fréchet-Hoeffding bound and the independence copula. We provide alternative proofs to some results on ordinal sums, elaborate properties common to all or just some of the ordinal sums discussed. Also included are a discussion of the relationship between ordinal sums of copulas and the Markov product and an overview of ordinal sums of multivariate copulas, illustrating aspects to be considered when extending concepts for ordinal sums of bivariate copulas to the multivariate case.

Keywords: Semigroup; Triangular norm; Copula; Quasi-copula; Ordinal sum; Aggregation function; Associativity

1. Introduction

The topic of this paper is a presentation of various types of ordinal sums, particularly of triangular norms and copulas. On the one hand, ordinal sums, which have roots in algebraic structures, serve as a genuine mathematical tool, fully characterizing continuous t-norms and helping to solve a distinguished functional equation. On the other hand, ordinal sums of bi- and multivariate copulas have been of great interest for the scientific community in dependence modeling and copula theory.

First we sketch (in Section 2) the development over the years, from ordinal sums of partially ordered sets and semigroups over ordinal sums of triangular norms to the ordinal sums of bivariate copulas, not only with the minimum M as background copula, but also the lower Fréchet-Hoeffding bound W and the independence copula Π, to ordinal sums of multivariate copulas.

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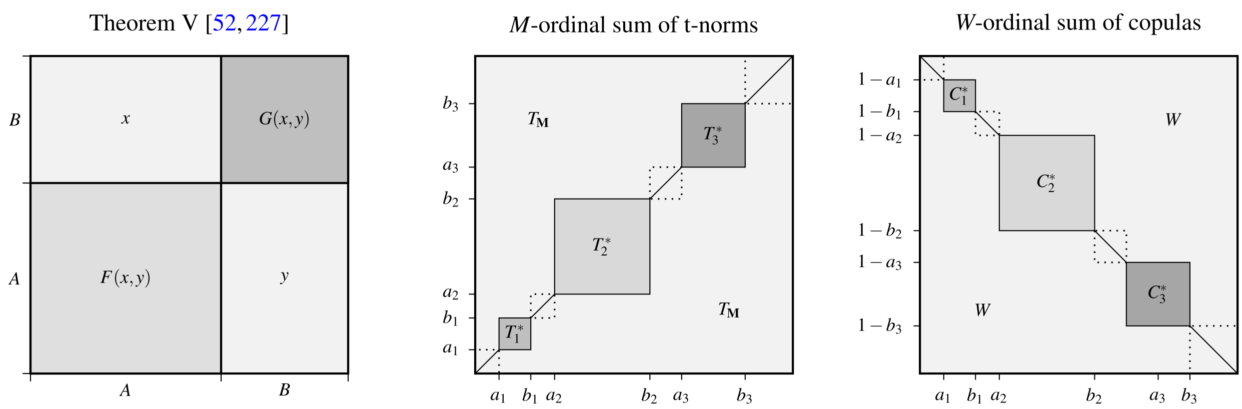

Fig. 1. Visualizations of the construction in Theorem V (left) in [52,227], and of the structure of an M-ordinal sum of triangular norms (center) M-$(\langle a_1, b_1, T_1\rangle, \langle a_2, b_2, T_2\rangle, \langle a_3, b_3, T_3\rangle)$ and of a W-ordinal sum of copulas (right) W-$(\langle a_1, b_1, C_1\rangle, \langle a_2, b_2, C_2\rangle, \langle a_3, b_3, C_3\rangle)$.

After recalling the ordinal sums of t-norms in Section 3 we discuss the various forms of ordinal sums of bivariate copulas. Here also other approaches to ordinal sums (such as orthogonal grid constructions and patchwork operators) are compared, and we pay particular attention to the geometric point of view on ordinal sums and an interpretation by patchwork operators in Subsection 4.1, and to the role of M-, W- and Π-ordinal sums as representations of copulas (Subsection 4.2).

The topic of Section 5 is the relationship between ordinal sums and the Markov product of bivariate copulas and a study of copulas which are idempotent with respect to the Markov product. Finally, several examples and results for ordinal sums of d-copulas are given in Section 6, including algorithms for the simulation of M_d- and Π_d-ordinal sums.

A considerable effort was made to include, when discussing the various aspects of ordinal sums, the original and most relevant references for each topic, leading to a list of references of a rather unusual length. On the other hand, we tried to preserve the flow of the presentations and have, therefore, collected a number of mathematical conventions, basic definitions and notations at the end of the paper in Appendix A — the readers are invited to consult them whenever there is a need.

2. Ordinal sums — a historical overview

To the best of our knowledge, the original source of ordinal sums was in algebra, as ordinal sums of partially ordered sets (posets), and the earliest traces go back at least to the 1940s. A classical reference is Birkhoff [24, Chapter VIII, 10] (compare also [242]), a more recent one is [56, Section 1, 1.24], where ordinal sums were called linear sums of posets (compare also [215, Definition 2.1] and [219, Example 6.8]).

2.1. Ordinal sums of semigroups

Climescu [52] and Clifford [46–50] worked with ordinal sums of semigroups which are special cases of the so-called strong bands of semigroups, also studied by Clifford [45]. For some related work see [70,122,148], and in [211] one finds an extensive survey describing Clifford's contributions to various fields of algebra.

Faucett [93–95], Wallace [247] and Mostert & Shields [195] proved important results for various types of continuous semigroup operations, i.e., for topological semigroups. For overviews and further details concerning these algebraic topics we recommend some standard monographs, e.g., [24,38,51,106,111,148,205].

To get started, we present a verbatim quotation of a result for the ordinal sum of two semigroups, namely, [52, Theorem V] in its translated form, as it appeared in the Appendix of [227] (the structure of this ordinal sum is visualized in Fig. 1(left)):

Theorem V [52,227] *Let (A, F) and (B, G) be semigroups. If the sets A and B are disjoint and if U is the mapping defined on $(A \cup B) \times (A \cup B)$ by*

$$U(x, y) = \begin{cases} F(x, y), & x \in A, y \in A, \\ x, & x \in A, y \in B, \\ y, & x \in B, y \in A, \\ G(x, y), & x \in B, y \in B, \end{cases}$$

then $(A \cup B, U)$ is a semigroup.

Theorem V [52,227] can be extended to the case of arbitrary many semigroups with pairwise disjoint carriers in a straightforward manner. As early as in the introductory part of [46], on page 631, CLIFFORD spoke about the ordinal sum of a totally ordered family of semigroups $(X_\tau)_{\tau \in I}$, where the index set $(I, \preceq)$ is totally (i.e., linearly) ordered and where each X_τ is a naturally totally ordered, commutative semigroup, i.e., the divisibility (compare [123]) $\leq$ on S_τ given by

$a \leq b$ if either $a = b$ or there exists c in S such that $ac = b$

is a total order, and where the semigroup operation on the ordinal sum S of the S_τ is defined for all $\sigma, \tau \in I$ with $\sigma \prec \tau$ and for all $a_\sigma \in S_\sigma$, $b_\tau \in S_\tau$ by $a_\sigma b_\tau = b_\tau a_\sigma = b_\tau$.

Calling a naturally totally ordered, commutative semigroup ordinally irreducible if and only if it contains a proper, non-empty, absorbent prime ideal $\mathfrak{a}$ (i.e., if $a \in \mathfrak{a}$ and $b \notin \mathfrak{a}$ then $ab = a$), one of the main results of [46] is the following Theorem 1, given here as a verbatim quotation:

Theorem 1 [46] *Every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups.*

2.2. Ordinal sums of triangular norms

A particularly interesting and powerful tool are ordinal sums of *triangular norms* (*t-norms* for short). Triangular norms (see Definition A.1) are binary operations on the unit interval $[0, 1]$ which were first considered by MENGER [180] in the context of suitable *triangle inequalities* in so-called *probabilistic* (or *statistical*) *metric spaces* (see, e.g., [224–226,228,229,233], for probabilistic normed spaces see, e.g., [14,167,231]). Later on, triangular norms were widely used as representations of the conjunction in *fuzzy logics* [117,252], where the set of Boolean truth values $\{0, 1\}$ is extended to the unit interval $[0, 1]$.

Algebraically speaking, for each triangular norm $T\colon [0, 1]^2 \to [0, 1]$ the pair $([0, 1], T)$ is a fully ordered, commutative semigroup with neutral element 1 and annihilator 0. A usual way to produce ordinal sums of t-norms is to start with an index set K, a family of non-empty, pairwise disjoint open subintervals $(]a_k, b_k[)_{k \in K}$ and a family of t-norms $(T_k)_{k \in K}$. Using linear isomorphisms $\varphi_k\colon\ [a_k, b_k] \to [0, 1]$ the respective isomorphic images of T_k turn each semigroup $([0, 1], T_k)$ into an isomorphic semigroup $([a_k, b_k], T_k^*)$. In order to construct a suitable ordinal sum of the semigroups $(([0, 1], T_k))_{k \in K}$ with carrier $[0, 1]$, one readily sees that the family $([a_k, b_k])_{k \in K}$, in general, is not a partition of $[0, 1]$: the intervals may have endpoints in common, and their union may be a proper subset of the unit interval.

The disjointness of the carriers of the semigroups in Theorem V in [52,227] can be somehow relaxed, as one can see from the following result in [154] which, in turn, is a consequence of [46]. Together with the technique of "filling the gaps" between $\bigcup_{k \in K}]a_k, b_k[$ and the unit interval $[0, 1]$ by means of the special t-norm $T_{\mathbf{M}}$ given by $T_{\mathbf{M}}(x, y) = \min\{x, y\}$, Theorem 2.1 makes it possible to consider ordinal sums of t-norms.

Theorem 2.1. [154, Theorem 3.42] *Let $(K, \preceq)$ with $K \neq \emptyset$ be a linearly ordered set and $(G_k)_{k \in K}$ with $G_k = (X_k, *_k)$ be a family of semigroups. Assume that for all $k, l \in K$ with $k \prec l$ the sets X_k and X_l are either disjoint or that $X_k \cap X_l = \{x_{kl}\}$, where x_{kl} is both the unit element of G_k and the annihilator of G_l, and where for each $j \in K$ with $k \prec j \prec l$ we have $X_j = \{x_{kl}\}$.*

Put $X = \bigcup_{k\in K} X_k$ *and define the binary operation* $*$ *on* X *by*

$$x * y = \begin{cases} x *_k y & \text{if } (x, y) \in X_k \times X_k, \\ x & \text{if } (x, y) \in X_k \times X_l \text{ and } k \prec l, \\ y & \text{if } (x, y) \in X_k \times X_l \text{ and } l \prec k. \end{cases}$$

Then $G = (X, *)$ *is a semigroup.*

A full proof of Theorem 2.1 can be found in [154, Theorem 3.42]. The following brief description gives a good idea how the shortcomings of the family of intervals mentioned above (the closed intervals may have endpoints in common, and the union of the intervals may be a proper subset of the unit interval) can be overcome.

Let us start with a family $(T_k)_{k\in K}$ of triangular norms and a family $(]a_k, b_k[)_{k\in K}$ of non-empty, pairwise disjoint open subintervals of $[0, 1]$. Note first that the pairwise disjointness of the non-empty subintervals $]a_k, b_k[$ of $[0, 1]$ implies that the index set K must be countable. Filling the gaps to the left and right of $\bigcup_{k\in K}]a_k, b_k[$ and between any two neighboring intervals by intervals of maximal length, we obtain a family of pairwise disjoint closed subintervals $([a_j, b_j])_{j\in J}$ (trivial, i.e., one-point intervals are possible), where a countable index set J can be chosen (implying that also $J \cup K$ is countable) so that $J \cap K = \emptyset$ and $\bigcup_{i\in J\cup K} [a_i, b_i] = [0, 1]$.

Using the function $s\colon J \cup K \to [0, 1]$ which assigns to each interval $[a_i, b_i]$ its midpoint $s(i) = \frac{1}{2}(a_i + b_i)$, we can define a linear order $\preceq$ on $J \cup K$ putting $i_1 \preceq i_2 \iff s(i_1) \le s(i_2)$. Considering for each interval $[a_i, b_i]$ with $a_i < b_i$ the linear bijection

$$\varphi_i\colon [a_i, b_i] \to [0, 1], \qquad \varphi_i(u) = \frac{u - a_i}{b_i - a_i},$$

we obtain for each $i \in J \cup K$ a semigroup $H_i = ([a_i, b_i], \circledast_i)$, where the operation $\circledast_i\colon [a_i, b_i]^2 \to [a_i, b_i]$ is given by

$$u \circledast_i v = \begin{cases} \varphi_i^{-1}\big(T_i(\varphi_i(u), \varphi_i(v))\big) & \text{if } i \in K, \\ \min\{u, v\} & \text{if } i \in J. \end{cases} \tag{2.1}$$

Observing that $u \circledast_i v = \varphi_i^{-1}(\min\{\varphi_i(u), \varphi_i(v)\})$ whenever $i \in J$ and $a_i < b_i$, the family of semigroups $(H_i, \circledast_i)_{i\in J\cup K}$ satisfies all the hypotheses of Theorem 2.1, and $([0, 1], T)$ is a semigroup, where the binary operation $T\colon [0, 1]^2 \to [0, 1]$ is a triangular norm:

$$T(x, y) = \begin{cases} a_k + (b_k - a_k)T_k\big(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\big) & \text{if } (x, y) \in [a_k, b_k]^2, \\ \min\{x, y\} & \text{otherwise.} \end{cases} \tag{2.2}$$

Moreover, note that in (2.1) one obtains, whenever $T_k = T_{\mathbf{M}}$ for some $k \in K$, the following identity for the function $\circledast_k\colon [a_k, b_k]^2 \to [a_k, b_k]$:

$$u \circledast_k v = \varphi_k^{-1}\big(\min\{\varphi_k(u), \varphi_k(v)\}\big) = \min\{u, v\} = T_{\mathbf{M}}(u, v). \tag{2.3}$$

In fact, using this line of arguments we have verified the following result which sometimes is called the *ordinal sum theorem for t-norms*.

Theorem 2.2. *Let* $(]a_k, b_k[)_{k\in K}$ *be a family of non-empty, pairwise disjoint open subintervals of* $[0, 1]$ *and let* $(T_k)_{k\in K}$ *be a family of triangular norms. Then the function* $T\colon [0, 1]^2 \to [0, 1]$ *given by* (2.2) *is a triangular norm.*

In the sequel, we shall call the t-norm T given by (2.2) the *M-ordinal sum* of the t-norms $(T_k)_{k\in K}$, and we shall write $T = M\text{-}(\langle a_k, b_k, T_k\rangle)_{k\in K}$ (compare also (4.1) below). Observe that “filling the gaps” described above has no influence on the ordinal sum considered there, i.e., we have

$$M\text{-}(\langle a_k, b_k, T_k\rangle)_{k\in K} = M\text{-}(\langle a_i, b_i, T_i\rangle)_{i\in J\cup K} \qquad \text{whenever } T_i = T_{\mathbf{M}} \text{ for all } i \in J. \tag{2.4}$$

Clearly, (2.4) means also that, in an arbitrary M-ordinal sum of triangular norms $T = M\text{-}(\langle a_k, b_k, T_k\rangle)_{k\in K}$ all the summands of the form $\langle a_i, b_i, T_\mathbf{M}\rangle$ can be omitted without changing the ordinal sum T. A visualization of the structure of an M-ordinal sum of t-norms is given in Fig. 1 (center), and the dotted squares along the main diagonal of $[0, 1]$ indicate the "gaps" which were filled with the "neutral" t-norm $T_\mathbf{M}$. More examples and results for M-ordinal sums of triangular norms will be presented and discussed in Section 3.

A well-known construction tool for important classes of triangular norms (and 2-copulas) are some types of unary functions from $[0, 1]$ to $[0, \infty]$, the so-called *additive generators*. This concept goes back to ideas by Abel [1], and it was further developed in, e.g., [2,94,195,226–228].

Taking into account the representation of continuous Archimedean t-norms by means of continuous additive generators due to Ling [171,172] (compare also [227], and for a generalization see [165]), it follows that associative bivariate copulas are exactly 1-Lipschitz triangular norms (see also [196]).

Another significant achievement in the context of triangular norms was the full characterization of the continuous t-norms as ordinal sums of continuous Archimedean t-norms. This result can be derived from [195] (see also [228, Theorem 5.38] and [154, Corollary 3.56, Theorem 5.11]).

In [132] it was pointed out that the ordinal sum theorem for t-norms (Theorem 2.2) remains valid if we replace the requirement that $(T_k)_{k\in K}$ be t-norms by the weaker requirement that they be *t-subnorms* (with some additional mild properties if necessary). Recall that a t-subnorm is a function $T\colon [0, 1]^2 \to [0, 1]$ which satisfies the properties (i)–(iii) in Definition A.1 and is bounded from above by the minimum, i.e., $T(x, y) \le \min\{x, y\}$ for all $(x, y) \in [0, 1]^2$, the latter being weaker than the boundary condition Definition A.1(iv). The t-subnorms (introduced in [128]) are slightly more general than the t-norms, and t-subnorms carry almost all properties of t-norms. They can be a powerful tool for constructing left-continuous t-norms by the so-called rotation–annihilation construction of new families of left-continuous t-norms and other aggregation operators [131,138].

2.3. *Ordinal sums of bivariate copulas (and other binary operations)*

When looking back, in particular at Theorem V [52,227] and Theorem 2.1, one could think that ordinal sums are applicable to partially ordered sets and to semigroups only, since in all the cases considered so far, the crucial property to be verified for the resulting ordinal sum is the associativity of the respective binary operation.

However, ordinal sums can be constructed not only in the context of posets and semigroups, but also for other rather general algebraic structures like groupoids $(X, *)$, where a non-empty set X is equipped with a binary operation $*$ (no additional properties may be required). The following result is formulated according to [154, Remark 3.45(i)] (compare also [228, Definition 5.2.4]).

Theorem 2.3. *Let $(A, \preceq)$ be a non-empty, linearly ordered index set, and $(X_\alpha, *_\alpha)_{\alpha\in A}$ a family of groupoids. If the sets X_α are pairwise disjoint and if the binary operation $*$ on $\bigcup_{\alpha\in A} X_\alpha$ is defined by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha \prec \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \beta \prec \alpha, \end{cases} \tag{2.5}$$

then $\left(\bigcup_{\alpha\in A} X_\alpha, *\right)$ is a groupoid.

This result allows us to consider families of groupoids with more additional properties. If we want to know whether the ordinal sum is also a groupoid with the same additional properties, we only have to check whether the operation $*$ given by (2.5) satisfies the additional properties, too.

Early traces of such ordinal sums can be found in Sklar [238, Definition 5] (where ordinal sums of functions $F\colon [0, 1]^2 \to \mathbb{R}$ satisfying $F(0, 0) = 0$ and $F(1, 1) = 1$ were considered), and in Definition 5.2.4 and Theorem 5.2.5 in the monograph [228] by Schweizer & Sklar, where they investigated ordinal sums of (associative) *binary systems* satisfying some compatibility conditions. Based on Theorem B in [195], ordinal sums of bivariate copulas were mentioned by Frank in [104, Section 2, (2.7)], and in Theorem 4 a representation of associative copulas was given.

In the following we mention several classes of functions from $[0,1]^2$ to $[0,1]$ which have been studied in various contexts, all of them being special cases of the functions considered in [238]. Their ordinal sums were introduced and studied under various names such as linear sums, orthogonal grid constructions, patchwork constructions, to name a few. As announced in Section 1, this paper will focus on ordinal sums of triangular norms and d-copulas with $d \geq 2$ which will be discussed in some detail in Sections 3–4 and 6. For the sake of completeness we present here also some other classes of bivariate functions whose ordinal sums have been investigated either from a theoretical point of view or in the context of applications.

Obviously, triangular norms as well as bivariate copulas are prominent examples of both *conjunctive operations* [60] and of bivariate *aggregation functions* [114]. Triangular norms [12,132,154,157,158,172,223,225–228] can be seen as symmetric, associative increasing conjunctive operations, and bivariate copulas [87,183,191,192,198,222,237–239] are 2-increasing semicopulas.

If we start with conjunctive operations on $[0,1]$, i.e., functions from $[0,1]^2$ to $[0,1]$ with annihilator 0 and neutral element 1 (which can be considered as extensions of the Boolean conjunction on $\{0,1\}$ to the unit interval $[0,1]$), then we find other interesting conjunctive operations satisfying one or more additional properties (see, e.g., [216, Definition 1.1, Remark 1.4]): *semicopulas*, i.e., increasing conjunctive operations (see [21,83,86]), *1-Lipschitz conjunctive operations* [150,160–162] (see also [173,204]), and *quasi-copulas*, i.e., 1-Lipschitz semicopulas (see [13,110,213,232]).

Some early and explicit traces of ordinal sums and orthogonal grids can be found, in the case of semicopulas in [86] and [61, Proposition 5], and in the case of quasi-copulas in [61, Proposition 6], and for W-ordinal sums in [184] and for Π-ordinal sums in [59,60].

The purpose of *aggregation functions* [23,34,114] (see also [33,35,36,66,82,156,163]) is the combination of several (numerical) data into a single representative value. A bivariate aggregation function $A\colon [0,1]^2 \to [0,1]$ is required to be increasing and to satisfy $A(0,0) = 0$ and $A(1,1) = 1$, and ordinal sums of general aggregation functions were presented and studied by De Baets & Mesiar in [66] (compare also [23,114,115,190]).

Some well-known examples of bivariate aggregation functions are, e.g., *triangular conorms*, i.e., duals of triangular norms [12,154,157,158,172,227,228], *uninorms*, i.e., symmetric, associative aggregation functions with a neutral element $e \in\,]0,1[$ [32,57,58,64,65,72,88,103,109,170,177,187,249,251] (for early traces, predecessors and special types of uninorms see [71,118,153,206,236,241,250]), *nullnorms*, i.e., symmetric associative aggregation functions with annihilator $a \in\,]0,1[$ (acting on $[0,a]$ as a triangular conorm, and on $[a,1]$ as a triangular norm) [32,72,73,97,176], *hoops*, i.e., partially ordered commutative residuated integral monoids [6,9,25,92], and *BL-chains*, i.e., totally ordered bounded hoops [5,7,8,31,116,117,119,169].

Ordinal sums of triangular conorms can be found, e.g., in [105, Section 2], and for ordinal sums of uninorms see [74,185,186,188,189,240]). In the literature one finds a unique decomposition of each BL-chain into an ordinal sum of irreducible hoops [31], a representation of BL-chains by an ordinal sum of Wajsberg hoops [7,25,31,53], as well as decompositions of saturated BL-chains as ordinal sums of MV-chains, Gödel chains and PL-chains [44].

Finally we mention that many binary operations on the unit interval $[0,1]$ have been extended to the case of bounded lattices (see, e.g., [18–20,28,40,41,54,68,120,121,125,143–146,152,164,178,219,253], and ordinal sums of these mathematical objects were studied in several papers, e.g., [90,179,215,217,218,248].

2.4. Other ordinal sum-types of copulas

In the ordinal sums of operations on the unit interval $[0,1]$ mentioned so far, the function $T_{\mathbf{M}}\colon [0,1]^2 \to [0,1]$ given by $T_{\mathbf{M}}(x,y) = \min\{x,y\}$, as we usually call it in the context of triangular norms, plays an important role. In the area of copulas, the same function is usually called M, i.e., $M = T_{\mathbf{M}}$, which is also the reason why the ordinal sums considered so far will be called M-ordinal sums from Section 4 onward, where also other types of ordinal sum constructions of copulas will be discussed which are based on the lower Fréchet-Hoeffding bound W (see (4.2)) and, later on, also on the independence copula Π (see (4.4)–(4.6)). For the definition and relevant properties of copulas see Appendix A.5–A.9.

It is well-known that $T_{\mathbf{M}}$ is the only triangular norm (and copula) whose set of idempotent elements equals $[0,1]$ (see, e.g., [154, Proposition 1.9]), and that the *lower Fréchet-Hoeffding bound* $W\colon [0,1]^2 \to [0,1]$ given by $W(x,y) = \max\{x+y-1, 0\}$ [198] is the only bivariate copula which vanishes on the opposite diagonal of $[0,1]$, i.e., $W(x, 1-x) = 0$ for all $x \in [0,1]$. Based on this observation, in [184] the concept of W-ordinal sum of copulas

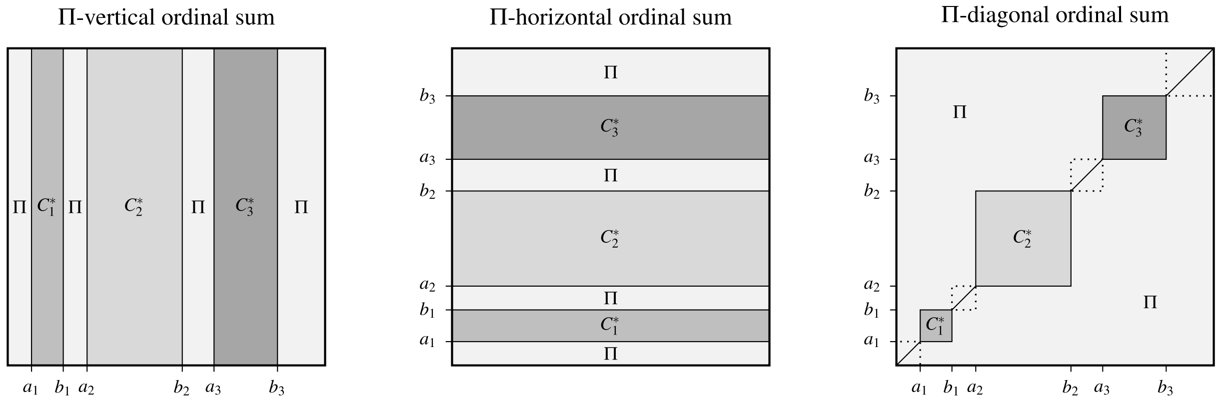

Fig. 2. Structure of the Π-vertical (left), Π-horizontal (center) and Π-diagonal (right) ordinal sum of $(\langle a_1, b_1, C_1\rangle, \langle a_2, b_2, C_2\rangle, \langle a_3, b_3, C_3\rangle)$.

(see (4.2)) was introduced in analogy to M-ordinal sum of copulas, in a way interchanging the roles of the diagonal and the opposite diagonal of the unit interval $[0, 1]$ (see Fig. 1 right).

Note, however, that there are many triangular norms which vanish on the opposite diagonal of $[0, 1]$ and that W-ordinal sums, in general, do not preserve associativity or commutativity of their summands (see, e.g., [216]). Therefore, we have no constructions and representations for t-norms by means of W-ordinal sums, although such constructions and representations exist, among others, for copulas (see, e.g., Propositions 4.27 and 4.29).

Carefully analyzing formula (4.3), one sees that, starting with a point (x, y) in an appropriate square centered around the opposite diagonal of $[0, 1]^2$, this argument (x, y) first undergoes some appropriate linear transformation yielding a transformed pair (x^*, y^*), followed by another suitable linear operation involving $W(x^*, y^*)$. The result of this linear process in two steps is exactly the value of the lower Fréchet-Hoeffding bound W at the starting point (x, y).

The similarity between (2.3) and (4.3) suggests that there is a close relationship between M-ordinal sum of copulas given in (4.1) and W-ordinal sum of copulas given in (4.2). Indeed, M- and W-ordinal sum of copulas are closely related to each other via symmetries as specified in Proposition 4.9 (i)–(ii).

Once the associativity property was not required anymore, it became clear that the ordinal sum construction may gain flexibility not only in the background function, but also in the way this latter function is specified in different subregions of its domain. Apart from M and W, it was hence clear that other constructions could be obtained as well by considering. e.g., the product t-norm $T_{\mathbf{P}}$ (or, equivalently, the independence copula Π) as proposed by De Baets & De Meyer [59,60] (for a visualization of Π-ordinal sums see Fig. 2). These constructions have hence generalized in the form of orthogonal grid representation of the aggregation function in [61], where various types of conjunctive operations like semicopulas, quasi-copulas, and copulas are also considered. Curiously, specific examples from these constructions turned out to have appeared in copula theory, like the copula of the circular uniform distribution (see [198, section 3.1.2]) and special types of shuffles (see [198, Example 5.12]).

Focusing on copulas, ordinal sum constructions may benefit of the stochastic interpretation of these functions that eases the interpretation of some concepts and their multivariate extensions. In particular, the gluing method introduced by Siburg & Stoimenov [234] has provided a flexible way to join two or more copulas starting with the independence case. It was hence naturally expressed as an ordinal sum construction in [181] and, later, formalized as Π-ordinal sums in [161] (see also some related concepts discussed in [216]). A unifying viewpoint to these constructions was hence provided by patchwork techniques, considered for 2-increasing aggregation functions in [84] and, later on, applied to copula models in [85]. Here, the main tool is to apply Sklar's representation theorem to (locally) represent a 2-increasing function in terms of a copula and some suitable marginals, that turns out to be related to the sections of the original background copula. This idea allows also a different interpretation of copulas with given horizontal or vertical sections [81,151].

Using the gluing methods [234], these copula constructions have been generalized to higher dimensions also by adopting a measure theoretic point of view to patchwork techniques [77].

Summarizing, starting with the representation of associative functions, ordinal sums of t-norms have been considered extensively in the literature. Thanks to their link to copulas, especially highlighted by Frank t-norms, they have found a way to stochastic models (i.e., copulas) and general aggregation functions. This long path is now following a way that was not expected in the early years of functional equations, but is continuing to be fruitful in many research problems where the aggregation of different structures is needed.

3. Ordinal sums of triangular norms

The history of triangular norms starts with MENGER [180] who began to construct metric spaces where the non-negative numbers were replaced by probability distributions in order to "measure" the distance between the elements of such a space. The formulation of some triangle inequality in this more general setting naturally led to triangular norms (t-norms for short). In [180] rather weak axioms for t-norms were given, as quoted below (actually, they comprised also the operations known today as triangular conorms):

> ... where $T(\alpha,\beta)$ is a function defined for $0 \le \alpha \le 1$ and $0 \le \beta \le 1$ such that
> (a) $0 \le T(\alpha,\beta) \le 1$.
> (b) T is increasing in either variable.
> (c) $T(\alpha,\beta) = T(\beta,\alpha)$.
> (d) $T(1,1) = 1$.
> (e) If $\alpha > 0$, then $T(\alpha,1) > 0$.
> [...] We shall call the function T the triangular norm of the statistical metric ...

SCHWEIZER & SKLAR [224–226] provided the axioms of t-norms as they are used today, and an adapted definition by ŠERSTNEV [233] led to a rapid development of statistical (or probabilistic, as they were called after 1964) metric spaces [12,228,229].

Mathematically speaking, (specific) functional equations and the theory of (special topological) semigroups provided two rather independent roots for (continuous) t-norms.

Looking at semigroups, FAUCETT [94] studied compact, irreducibly connected topological semigroups and gave a characterization of such semigroups, where the boundary points are the only idempotents and where no nilpotent elements exist (thus providing, in the language of t-norms, a full representation of strict t-norms). MOSTERT & SHIELDS [195] characterized all such semigroups, where the boundary points play the role of annihilator and neutral element (see also [205]). For t-norms, this gives a representation of all continuous t-norms.

Several construction methods from the theory of semigroups, such as (isomorphic) transformations (e.g., the additive generators mentioned above) and ordinal sums [45–50,52,227], have been successfully applied to construct whole families of t-norms, starting from a few given prototypical examples [227]. So three basic t-norms, the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$, allow us to construct all continuous t-norms using isomorphic transformations and ordinal sums [172].

In the context of functional equations, t-norms are closely related to the associativity equation, and the earliest source in this context seems to be ABEL [1] (see also [2,29,39,124] and the monographs by ACZÉL [3,4]). Major achievements were the full characterization of continuous Archimedean t-norms by additive generators in LING [172] (for the special case of strict t-norms see [226]) and the solution of a famous functional equation by FRANK [105].

Triangular norms (in particular, left continuous t-norms) are widely used as representations of the conjunction in *fuzzy logics* [117,252], were the classical two-point set of (Boolean) truth values "True" and "False" [26,27] is extended to the unit interval [0, 1] with the understanding, that "True" corresponds to 1 and "False" to 0.

The full definition of t-norms is presented in the Appendix A.2. In general, the properties (i)–(iv) in this definition do not imply any type of continuity of t-norms (see [154, Sections 1.3, 2.3–2.4, 3.4] and [159]).

There are even t-norms which are not Borel measurable (see [149, Example 2.2], [154, Example 3.75] and [159, Example 4.1]) and, therefore, not continuous. And KRAUSE [166] introduced a particularly interesting non-continuous triangular norm, the so-called Krause t-norm $T^{\mathbf{K}}$ which was discussed in detail in [154, Appendix B] and which is neither left- nor right-continuous, but has a continuous diagonal section, thus providing a negative answer to [228, Problem 5.8.1] by, as one can read in [12, p. 77], "... *constructing an intriguing and intricate counterexample, the so-called 'Devil's Terraces', which employs classical Cantor sets.*"

Non-continuous t-norms satisfying weaker forms of continuity are, e.g., $T_{\mathbf{D}}$ which is right-continuous, and the left-continuous nilpotent minimum $T^{\mathbf{nM}}$ [102,207,208] given by

$$T^{\mathbf{nM}}(x,y) = \mathbf{1}_{\{(u,v)\in[0,1]^2 \mid u+v>1\}}(x,y) \cdot \min\{x,y\}.$$

The t-norms T_1, T_2 given by

$$T_1(x,y) = \big(1 - \mathbf{1}_{]0,0.5[^2}(x,y)\big) \cdot \min\{x,y\} \qquad \text{and} \qquad T_2(x,y) = \mathbf{1}_{[0.5,1]^2}(x,y) \cdot \min\{x,y\}$$

are continuous at the border of $[0,1]^2$ and at the point $(1,1)$, respectively (see [154, Example 1.24]).

Left-continuous t-norms play an crucial role in many-valued logics on $[0,1]$ (fuzzy logics): they are closely related to *residual lattices* where an important type of many-valued implications exists. For the role of the left-continuous t-norms and the corresponding logics see, e.g., the monograph of HÁJEK [117] and [44,91,101,112,113,199], and for investigations and constructions of the class of left-continuous t-norms see [43,127,129–131,133–137,139–141, 174,246]. However, unlike the case of continuous t-norms (see Theorem 3.2 below), a full characterization of left-continuous t-norms does not yet exist.

We have already mentioned that three of the four basic t-norms are also copulas, while $T_{\mathbf{D}}$ is not. As a consequence of [197], the exact relationship between t-norms and bivariate copulas is given as follows (see [154, Corollary 9.9, Theorem 9.10]): A bivariate copula C is a triangular norm if and only if C is associative, while a triangular norm T is a copula if and only if T is 1-Lipschitz.

The Archimedean property can be defined in rather abstract algebraic structures. In the case of continuous t-norms we can simplify the definition: A continuous t-norm T is called *Archimedean* if T is either strict or nilpotent.

Recall that a continuous triangular norm is called *strict* if T is strictly increasing, i.e., if for all $x \in\,]0,1]$, $y,z \in [0,1]$ the strict inequality $y < z$ implies $T(x,y) < T(x,z)$, and T is called *nilpotent* if T has some zero divisor, i.e., if there exist an $a \in\,]0,1]$ such that $T(a,b) = 0$ for some $b \in\,]0,1]$. The existence of a zero divisor of T is equivalent to the fact that the set of nilpotent elements of T equals $[0,1[$, i.e., for each $a \in [0,1[$ there exists an $n \in \mathbb{N} \setminus \{1\}$ such that $a_T^{(n)} = 0$, where $a_T^{(n)}$ is defined inductively by $a_T^{(2)} = T(a,a)$, and $a_T^{(n)} = T\big(a_T^{(n-1)}, a\big)$ whenever $n \geq 3$.

Early traces of the following characterization of continuous Archimedean t-norms can be found in [1,2,94] in the strict case, and in [195] (in the framework of nilpotent compact topological semigroups). The result below follows from [172] (see also [228, Theorem 5.5.2] and [154, Theorem 5.1]). Observe that in [165] one can find a strengthened version of the theorem in [172].

Theorem 3.1. *For a function $T\colon [0,1]^2 \to [0,1]$ the following are equivalent:*

(i) *T is a continuous Archimedean t-norm.*
(ii) *There exists a continuous, strictly decreasing function $t\colon [0,1] \to [0,\infty]$ satisfying $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that $T(x,y) = t^{(-1)}(t(x) + t(y))$ holds for all $(x,y) \in [0,1]^2$, where $t^{(-1)}$ is the pseudo-inverse of t.*

The characterization of continuous t-norms by means of ordinal sums can be derived from [195] (compare also [205]), and for our formulation see [228, Theorem 5.38] and [154, Theorem 5.11]:

Theorem 3.2. *For a function $T\colon [0,1]^2 \to [0,1]$ the following are equivalent:*

(i) *T is a continuous t-norm.*
(ii) *T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exists a countable index set K, a unique family of pairwise disjoint open subintervals $(]a_k, b_k[)_{k \in K}$ of $[0,1]$ and a unique family of continuous Archimedean t-norms $(T_k)_{k \in K}$ such that $T = (\langle a_k, b_k, T_k \rangle)_{k \in K}$.*

Two continuous triangular norms T_1 and T_2 are called isomorphic if there exists a continuous, strictly increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that for all $(x,y) \in [0,1]^2$ we have $\varphi(T_1(x,y)) = T_2(\varphi(x), \varphi(y))$. One of the consequences of Theorem 3.2 is that (see [154, Propositions 5.9, 5.10]) a continuous t-norm T is strict if and only if it is isomorphic to $T_{\mathbf{P}}$, and that it is nilpotent if and only if T is isomorphic to $T_{\mathbf{L}}$.

Therefore, Theorems 3.1 and 3.2 imply that ordinal sums and isomorphic images of the three t-norms $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{M}}$ (the latter being needed in almost each ordinal sum) are sufficient to characterize all continuous t-norms:

Corollary 3.3. *For a function $T\colon [0,1]^2 \to [0,1]$ the following are equivalent:*

(i) *T is a continuous t-norm.*
(ii) *T is uniquely representable as an ordinal sum $(\langle a_k, b_k, T_k\rangle)_{k\in K}$ of t-norms where each T_k is isomorphic to a t-norm in $\{T_{\mathbf{P}}, T_{\mathbf{L}}\}$.*

Corollary 3.3 can be furthermore formulated in the context of 1-Lipschitz t-norms, i.e., for associative copulas, taking into account the following result.

Theorem 3.4. *For a function $T\colon [0,1]^2 \to [0,1]$ the following are equivalent:*

(i) *T is an associative copula.*
(ii) *T is uniquely representable as an ordinal sum $(\langle a_k, b_k, T_k\rangle)_{k\in K}$ of t-norms where each T_k is an Archimedean copula, i.e., it is a continuous Archimedean t-norm with a convex additive generator.*

Interestingly, ordinal sums of strict (or nilpotent) Archimedean copulas can approximate any associative copula in the topology of uniform convergence (see [155]). Moreover, in the stronger topology induced by the metric D_1 studied in [244], (finite) ordinal sums of strict Archimedean copulas can approximate any associative copula [147, Theorem 3.6].

Ordinal sum of some t-norms, in this case of the so-called Frank t-norms, play a crucial role in the solution of a famous functional equation. Recall that the family $(T_\kappa^{\mathbf{F}})_{\kappa\in[0,\infty]}$ of *Frank t-norms* was defined in [105, (1.7)] as follows:

$$T_\kappa^{\mathbf{F}}(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \kappa = 0,\\ T_{\mathbf{P}}(x,y) & \text{if } \kappa = 1,\\ T_{\mathbf{L}}(x,y) & \text{if } \kappa = \infty,\\ \log_\kappa\left(1 + \frac{(\kappa^x-1)(\kappa^y-1)}{\kappa-1}\right) & \text{otherwise.}\end{cases}$$

The Frank family $(T_\kappa^{\mathbf{F}})_{\kappa\in[0,\infty]}$ of t-norms is also a remarkable family of bivariate copulas, but for the proof of Theorem 3.5 below the fact that they turn $[0,1]$ into a topological semigroup was an important aspect.

The following functional equation was studied in [105]. Consider the set $\mathscr{N}$ of all continuous bivariate functions $F\colon [0,1]^2 \to [0,1]$ satisfying the boundary conditions $F(0,x) = F(x,0) = 0$ and $F(1,x) = F(x,1) = x$ for all $x \in [0,1]$, define for each $F \in \mathscr{N}$ the function $F\hat{}\colon [0,1]^2 \to \mathbb{R}$ via

$$F\hat{}(x,y) = x + y - F(x,y), \tag{3.1}$$

and find all functions $F \in \mathscr{N}$ such that both F and $F\hat{}$ are associative.

Using results on topological semigroups [195,205] and transforming the problem into differential equations, Frank [105] provided a solution. The following is an equivalent reformulation of the solution of this problem in terms of the unknown function $F \in \mathscr{N}$ only (the original solution given in [105, Theorem 1.1] considered pairs of functions (F,G) with $F \in \mathscr{N}$ and F and G being coupled by $F(x,y) + G(x,y) = x + y$ for all $(x,y) \in [0,1]^2$). Taking into account that each t-norm T equals the trivial ordinal sum $(\langle 0,1,T\rangle)$ we may write:

Theorem 3.5. *Let $F \in \mathscr{N}$ such that F and $F\hat{}$ as given by (3.1) are increasing in each variable. Then both F and $F\hat{}$ are associative if and only if F is representable as an ordinal sum of members of the Frank family of t-norms.*

An interesting inequality that characterizes Frank t-norms and some related ordinal sums has been recently considered in [62] (see also [220]).

4. Ordinal sums of bivariate copulas

Historically, it was clear that ordinal sums of associative copulas preserve the 2-increasing property (see, e.g., [104, Theorem 4]), the Frank t-norms being a prominent example of associative copulas. Moreover, as made explicit by Sklar in [238, Definition 5], ordinal sums of triangular norms can be naturally extended to other bivariate functions.

The need to consider more flexible classes of aggregation functions as well as new classes of copulas for the description of different features of random vectors has stimulated the investigations of several ordinal sum-type constructions, introduced under various names, like W-ordinal sums [59,60,184], orthogonal grid constructions [59–61], gluing of copulas [181,234], patchwork constructions [85], Π-ordinal sums [161], and also some related concepts discussed in [216].

In this section we focus on five different ordinal sum operations taking bivariate copulas as summand operations and leading again to a bivariate copula (see Appendix A.5–A.9 for the definition and relevant properties of copulas). In particular, we emphasize some of their common features as a result of a measure-theoretic interpretation of these concepts via patchwork techniques. It should be stressed that other similar constructions are possible as well, the ones considered being those with a closed form analytical expression.

We shall first turn to M- and W-ordinal sums. Proofs of the following proposition can be found, e.g., for the M-ordinal sums in [61, Proposition 9] and [183, Theorem 2.1], and for the W-ordinal sums in [184, Theorem 2] and [61, Proposition 11]. An alternative proof will be provided in Subsection 4.2.

Proposition 4.1. *Let $(]a_k, b_k[)_{k\in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_k)_{k\in K}$ be a family of copulas. Then the following two functions $M^{\mathrm{OSum}}, W^{\mathrm{OSum}} \colon [0,1]^2 \to [0,1]$ are copulas:*

$$M^{\mathrm{OSum}}(x,y) = \begin{cases} a_k + (b_k - a_k)\cdot C_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y)\in[a_k,b_k]^2\,,\\ M(x,y) & \text{otherwise,}\end{cases} \tag{4.1}$$

$$W^{\mathrm{OSum}}(x,y) = \begin{cases} (b_k - a_k)\cdot C_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y+b_k-1}{b_k-a_k}\right) & \text{if } (x,y)\in[a_k,b_k]\times[1-b_k,1-a_k]\,,\\ W(x,y) & \text{otherwise.}\end{cases} \tag{4.2}$$

The functions M^{OSum} and W^{OSum} are called the *M-ordinal sum* and *W-ordinal sum*, respectively, of the summands $(]a_k, b_k[\,, C_k)_{k\in K}$, and we use the notations $M^{\mathrm{OSum}} = M\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ and $W^{\mathrm{OSum}} = W\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$.

An ordinal sum $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ with $D\in\{M,W\}$ will be called *non-trivial* if the family $(]a_k, b_k[)_{k\in K}$ does not consist of $]0,1[$ only and $C_k \neq D$ for at least one $k\in K$.

Note that in case $C_k = D$ for some $k\in K$, we obtain by straightforward computations that, in case $D = M$, for each $(x,y)\in[a_k,b_k]^2$

$$M^{\mathrm{OSum}}(x,y) = a_k + (b_k-a_k)\cdot M\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) = M(x,y)$$

and, in case $D = W$, for each $(x,y)\in[a_k,b_k]\times[1-b_k,1-a_k]$

$$W^{\mathrm{OSum}}(x,y) = (b_k-a_k)\cdot W\left(\frac{x-a_k}{b_k-a_k}, \frac{y+b_k-1}{b_k-a_k}\right) = W(x,y), \tag{4.3}$$

i.e., the ordinal sums coincide with M or W on the respective subdomains. As a consequence we immediately have:

Proposition 4.2. *Let $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ with $D\in\{M,W\}$ be an M- or W-ordinal sum as given in Proposition 4.1. Then $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K} = D$ if and only if $C_k = D$ for all $k\in K$.*

On the other hand, choosing $C_k = C$ for all $k\in K$, we may interpret ordinal sums as one-place mappings from $\mathscr{C}_2$ to $\mathscr{C}_2$.

Remark 4.3. For a family $(]a_k, b_k[)_{k\in K}$ of non-empty, pairwise disjoint open subintervals of $[0,1]$ with $]a_k, b_k[\, \neq\,]0,1[$ for at least one k, the unique fixed points of the following mappings from $\mathscr{C}_2$ to $\mathscr{C}_2$

$$C \longmapsto M\text{-}(\langle a_k, b_k, C\rangle)_{k\in K} \qquad \text{and} \qquad C \longmapsto W\text{-}(\langle a_k, b_k, C\rangle)_{k\in K}$$

are M and W, respectively.

In a similar way ordinal sum-type operations can also be constructed from the independence copula Π. The proofs of the following proposition can be found, e.g., in [61, Proposition 12], [234, Theorem 2.1], [181, Proposition 2] and [216, Proposition 2.2, Proposition 4.2].

Proposition 4.4. *Let* $(]a_k, b_k[)_{k\in K}$ *be a family of non-empty, pairwise disjoint open subintervals of* $[0, 1]$ *and let* $(C_k)_{k\in K}$ *be a family of copulas. Then the following functions* $\Pi^{(\mathrm{hOSum})}, \Pi^{(\mathrm{vOSum})}, \Pi^{(\mathrm{dOSum})} \colon [0, 1]^2 \to [0, 1]$ *are copulas:*

$$\Pi^{(\mathrm{vOSum})}(x, y) = \begin{cases} a_k \cdot y + (b_k - a_k) \cdot C_k\big(\frac{x-a_k}{b_k-a_k}, y\big) & \text{if } x \in [a_k, b_k]\,, \\ \Pi(x, y) & \text{otherwise,} \end{cases} \tag{4.4}$$

$$\Pi^{(\mathrm{hOSum})}(x, y) = \begin{cases} a_k \cdot x + (b_k - a_k) \cdot C_k\big(x, \frac{y-a_k}{b_k-a_k}\big) & \text{if } y \in [a_k, b_k]\,, \\ \Pi(x, y) & \text{otherwise,} \end{cases} \tag{4.5}$$

$$\Pi^{(\mathrm{dOSum})}(x, y) = \begin{cases} x \cdot y - (x - a_k) \cdot (y - a_k) + (b_k - a_k)^2 \cdot C_k\big(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\big) & \text{if } (x, y) \in [a_k, b_k]^2\,, \\ \Pi(x, y) & \text{otherwise.} \end{cases} \tag{4.6}$$

The functions $\Pi^{(\mathrm{vOSum})}$, $\Pi^{(\mathrm{hOSum})}$ and $\Pi^{(\mathrm{dOSum})}$ are called the Π-*vertical*, Π-*horizontal* and Π-*diagonal ordinal sum*, respectively, of the summands $(]a_k, b_k[\,, C_k)_{k\in K}$, and we will denote them by $\Pi^{(\mathrm{vOSum})} = \Pi^{(\mathbf{v})}\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$, $\Pi^{(\mathrm{hOSum})} = \Pi^{(\mathbf{h})}\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$, and $\Pi^{(\mathrm{dOSum})} = \Pi^{(\mathbf{d})}\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$, respectively.

A Π-ordinal sum $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ with $D \in \{\Pi^{(\mathbf{v})}, \Pi^{(\mathbf{h})}, \Pi^{(\mathbf{d})}\}$ will be called *non-trivial* if the family $(]a_k, b_k[)_{k\in K}$ does not consist of $]0, 1[$ only and if $C_k \neq \Pi$ for at least one $k \in K$.

Similarly to the case of M- and W-ordinal sums (see Remark 4.3), straightforward computations show that whenever $C_k = \Pi$ for some $k \in K$ then the Π-ordinal sum coincides with Π on the respective subdomain. Thus, we have the following:

Proposition 4.5. *Let* $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ *with* $D \in \{\Pi^{(\mathbf{v})}, \Pi^{(\mathbf{h})}, \Pi^{(\mathbf{d})}\}$ *be a* Π*-ordinal sum as given in Proposition 4.4. Then* $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K} = \Pi$ *if and only if* $C_k = \Pi$ *for all* $k \in K$.

Also, for a family $(]a_k, b_k[)_{k\in K}$ of non-empty, pairwise disjoint open subintervals of $[0, 1]$ with $]a_k, b_k[\neq]0, 1[$ for at least one k, Π is the unique fixed point of each of the following mappings from $\mathscr{C}_2$ to $\mathscr{C}_2$

$$C \longmapsto \Pi^{(\mathbf{v})}\text{-}(\langle a_k, b_k, C\rangle)_{k\in K}\,, \qquad C \longmapsto \Pi^{(\mathbf{h})}\text{-}(\langle a_k, b_k, C\rangle)_{k\in K}\,, \qquad \text{and} \qquad C \longmapsto \Pi^{(\mathbf{d})}\text{-}(\langle a_k, b_k, C\rangle)_{k\in K}\,.$$

Remark 4.6. Note that in a similar way as explained in Subsection 2.2, we may "fill the gaps" between the non-empty, pairwise disjoint open summand carriers $]a_k, b_k[$ with pairwise disjoint closed intervals $[a_i, b_i]$ for some additional countable index set J (trivial, one-point intervals are possible) in such a way that the index sets K and J are disjoint and we obtain $\bigcup_{i\in J\cup K} [a_i, b_i] = [0, 1]$. As a consequence we obtain the following:

(i) If, in case $D \in \{M, W\}$, $C_i = D$ for all $i \in J$ or, in case $D \in \{\Pi^{(\mathbf{v})}, \Pi^{(\mathbf{h})}, \Pi^{(\mathbf{d})}\}$, $C_i = \Pi$ for all $i \in J$ (compare also (2.4)) then

$$D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K} = D\text{-}(\langle a_i, b_i, C_i\rangle)_{i\in J\cup K}\,.$$

Thus the representation of a D-ordinal sum with $D \in \{M, W, \Pi^{(\mathbf{v})}, \Pi^{(\mathbf{h})}, \Pi^{(\mathbf{d})}\}$ is, in general, not unique.

(ii) On the other hand, if, in case $D \in \{M, W\}$, $C_i = D$ for all $i \in J$ or, in case $D \in \{\Pi^{(\mathbf{v})}, \Pi^{(\mathbf{h})}, \Pi^{(\mathbf{d})}\}$, $C_i = \Pi$ for all $i \in J$, we may remove the summand C_i without changing the copula, i.e.,

$$D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K} = D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K\setminus J}\,.$$

Remark 4.7. The demand for a linearly ordered index set has been essential in the introduction of ordinal sums of semigroups (see Section 2). The linear order can be recovered from the ordinal sums discussed here: as outlined in Remark 4.6, we may extend the family $(]a_k, b_k[)_{k\in K}$ of non-empty, pairwise disjoint subintervals of $[0, 1]$ to a family $(]a_i, b_i[)_{i\in J\cup K}$ with $\bigcup_{i\in J\cup K} [a_i, b_i] = [0, 1]$. Since both J as well as K are countable, so is their union, and hence $J \cup K$ turns into a linearly ordered index set $(J \cup K, \preceq)$ where $\preceq$ is compatible with the usual linear order on $\mathbb{N}$ (compare also Subsection 2.2).

Moreover, the linear order on the index set $J \cup K$ is in accordance with a suitable linear order on the set of 2-boxes R_i considered in each ordinal sum in the following way: For an ordinal sum $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ with index set K we put, for each $k \in K$,

$$R_k = \begin{cases} [a_k, b_k]^2 & \text{if } D \in \{M, \Pi^{(\mathbf{d})}\}, \\ [a_k, b_k] \times [1-b_k, 1-a_k] & \text{if } D = W, \\ [a_k, b_k] \times [0,1] & \text{if } D = \Pi^{(\mathbf{v})}, \\ [0,1] \times [a_k, b_k] & \text{if } D = \Pi^{(\mathbf{h})}, \end{cases} \tag{4.7}$$

and let $\mathscr{R} = \{R_k \mid k \in K\}$ be the set of all 2-boxes. For each $R \in \mathscr{R}$, we denote by $\operatorname{cent}(R)$ its center. It is immediate that $(\mathscr{R}, \preceq_{\mathscr{R}})$ with

$$R_i \preceq_{\mathscr{R}} R_j \quad \Longleftrightarrow \quad \operatorname{cent}(R_i) \leq_\ell \operatorname{cent}(R_j),$$

where $\leq_\ell$ is the lexicographic order on $\mathbb{R}^2$, is a linearly ordered set. Clearly, we also have $R_i \preceq_{\mathscr{R}} R_j$ if and only if $i \preceq j$. Note that for the ordinal sums as introduced in Propositions 4.1 and 4.4 we additionally get

$$R_i \cap R_j = \partial R_i \cap \partial R_j$$

for all $R_i, R_j \in \mathscr{R}$ with $R_i \neq R_j$, where ∂R denotes the boundary of the set R (see Appendix A.1).

Remark 4.8. From the viewpoint of the induced probability measures, the ordinal sums as introduced in Proposition 4.4 differ from the ones introduced in Proposition 4.1 in the following way:

If every C_k of the D-ordinal sum $D\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ is absolutely continuous, then so are the D-ordinal sum copulas $\Pi^{(\mathrm{hOSum})}$, $\Pi^{(\mathrm{vOSum})}$ and $\Pi^{(\mathrm{dOSum})}$, i.e., copulas of the type (4.4), (4.5) and (4.6) (see [181, Corollary 2]).

However, in case $D \in \{M, W\}$, the D-ordinal sums M^{OSum} and W^{OSum}, i.e., copulas of the type (4.1) and (4.2), are absolutely continuous if every C_k is absolutely continuous and $\sum_{k\in K}(b_k - a_k) = 1$. In case $\sum_{k\in K}(b_k - a_k) < 1$, the supports of the singular components of the copulas M^{OSum} and W^{OSum} lie on the main and opposite diagonal of $[0,1]^2$, respectively.

We shall further mention that the ordinal sums presented here are linked to each other via suitable transformations in the space of the copulas, sometimes called rotation [142] or flipping [63], which are closely related to the symmetries of the random vector associated to a copula [107,108].

The following result proves that M- and W-ordinal sums as introduced in Proposition 4.1 are connected to each other by the symmetries σ_1 and σ_2, also termed reflections (for more details on symmetries see also Appendix A.9). Note that the copula C^{σ_2} is sometimes referred to as the y-flipping of C, whereas C^{σ_1} is called the x-flipping of C (compare also [63,198] and Appendix A.9).

The proof of the following result, which can be obtained by tedious computations, can also be found in [221, Lemma 5.1].

Proposition 4.9. *Let $(]a_k, b_k[)_{k\in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_k)_{k\in K}$ be a family of copulas. Then the following hold:*

(i) $\big(W\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}\big)^{\sigma_1} = M\text{-}(\langle 1-b_k, 1-a_k, C_k^{\sigma_1}\rangle)_{k\in K}$,
(ii) $\big(W\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}\big)^{\sigma_2} = M\text{-}(\langle a_k, b_k, C_k^{\sigma_2}\rangle)_{k\in K}$.

The Π-horizontal and Π-vertical ordinal sums as given in Proposition 4.4 can be connected to each other by the reflections σ_1 and σ_2 and the permutation η (compare also [221, Lemma 5.1]).

Proposition 4.10. *Let $(]a_k, b_k[)_{k\in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_k)_{k\in K}$ be a family of copulas. Then the following hold:*

(i) $\big(\Pi^{(\mathbf{v})}\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}\big)^{\eta} = \Pi^{(\mathbf{h})}\text{-}(\langle a_k, b_k, C_k^{\eta}\rangle)_{k\in K}$,
(ii) $\big(\Pi^{(\mathbf{v})}\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}\big)^{\sigma_1} = \Pi^{(\mathbf{h})}\text{-}(\langle 1-b_k, 1-a_k, C_k^{\sigma_1}\rangle)_{k\in K}$,
(iii) $\big(\Pi^{(\mathbf{v})}\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}\big)^{\sigma_2} = \Pi^{(\mathbf{h})}\text{-}(\langle a_k, b_k, C_k^{\sigma_2}\rangle)_{k\in K}$.

Example 4.11. Starting with the Π-diagonal ordinal sum $\Pi^{(\mathbf{d})}$-$(\langle 0, a_1, C_1\rangle, \langle a_1, 1, C_2\rangle)$ and applying the transformation induced by σ_2 to it we obtain the copula

$$C(x,y)=\begin{cases} x-a_1^2\cdot C_1\big(\frac{x}{a_1},\frac{1-y}{a_1}\big) & \text{if } (x,y)\in[0,a_1]\times[1-a_1,1],\\ x-a_1x-a_1(1-y)+a_1^2-(1-a_1^2)\cdot C_2\big(\frac{x-a_1}{1-a_1},\frac{1-y-a_1}{1-a_1}\big) & \text{if } (x,y)\in[a_1,1]\times[0,1-a_1],\\ xy & \text{otherwise,}\end{cases}$$

which is an ordinal sum (with respect to the background copula Π) defined on 2-boxes whose centers belong to the opposite diagonal of $[0,1]^2$.

Remark 4.12. The ordinal sums in Propositions 4.1 and 4.4 can also be used to generate copulas which, for instance, cover a larger range of the values of the associated measures of concordance (see, e.g., [198]). To this end, it is helpful to remind that a formula for calculating the Kendall's τ and Spearman's ρ for M-ordinal sum is given in [198, Exercise 5.14]. It can be easily adapted to the case of W-ordinal sums via properties of concordance measures (see [198, Definition 5.1.7 (5)]) thanks to the representation in Proposition 4.9. For the Π-ordinal sums, similar formulas are obtained in [234, Theorem 3.2], [181, Proposition 3] and [80, Theorem 4.1].

We shall finally mention, and it has already been outlined also in Subsection 2.3, that ordinal sums have been considered for the more general classes of (aggregation) functions leading to characterizations for copulas, quasi-copulas and semicopulas.

Proposition 4.13. *Let $(]a_k,b_k[)_{k\in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_k)_{k\in K}$ be a family of binary aggregation functions, i.e., mappings from $[0,1]^2$ into $[0,1]$, increasing in each coordinate and fulfilling $C_k(0,0)=0$ and $C_k(1,1)=1$. Let C be one of the functions given by (4.1), (4.2) or (4.4)–(4.6). Then the following are equivalent:*

(i) *C is a bivariate copula.*
(ii) *C_k is a bivariate copula for every $k\in K$.*

Proof. The implication (ii) $\Longrightarrow$ (i) follows from Propositions 4.1 and 4.4. For proving (i) $\Longrightarrow$ (ii), assume that $C=D$-$(\langle a_k,b_k,C_k\rangle)_{k\in K}$ defined by (4.1), (4.2) or (4.4)–(4.6) is a copula. Then, for every $k\in K$, it holds:

$$C_k(x,y)=\begin{cases} \frac{1}{b_k-a_k}\left(C(a_k+(b_k-a_k)x,a_k+(b_k-a_k)y)-a_k\right) & \text{if } D=M,\\ \frac{1}{b_k-a_k}\left(C(a_k+(b_k-a_k)x,(1-b_k)+(b_k-a_k)y)\right) & \text{if } D=W,\\ \frac{1}{b_k-a_k}\left(C(a_k+(b_k-a_k)x,y)-a_ky\right) & \text{if } D=\Pi^{(\mathbf{v})},\\ \frac{1}{b_k-a_k}\left(C(x,a_k+(b_k-a_k)y)-a_kx\right) & \text{if } D=\Pi^{(\mathbf{h})},\\ \frac{1}{(b_k-a_k)^2}\big(C(a_k+(b_k-a_k)x,a_k+(b_k-a_k)y)-a_k\cdot(b_k-a_k)(x+y)-a_k^2\big) & \text{if } D=\Pi^{(\mathbf{d})}.\end{cases} \tag{4.8}$$

From (4.8) it follows that C_k is a copula, i.e., is 2-increasing and fulfills the boundary conditions of a copula (see, for instance, [183, Theorem 3.3] for $D=M$, [184, Proposition 2] for $D=W$, [181, Theorem 1] for $D\in\{\Pi^{(\mathbf{h})},\Pi^{(\mathbf{v})}\}$, and [216, Proposition 4.2] for $D=\Pi^{(\mathbf{d})}$). □

Remark 4.14. Note that Proposition 4.13 holds analogously for semicopulas and quasi-copulas in case of M- and W-ordinal sums. For Π-ordinal sums, instead, the implication (i) $\Longrightarrow$ (ii) in Proposition 4.13 may not hold since the monotonicity or the 1-Lipschitz property may be violated (compare [61, Proposition 5 and 6] and [216, Example 2.4]).

Note also that results similar to Proposition 4.13 hold in the case of M-ordinal sums also for d-copulas [183] with $d\geq 3$ (see also Section 6), and for continuous t-norms (compare Corollaries 2.8 and 3.56 and Theorem 5.11 in [154], for the original proof see [228, Theorem 5.38]).

4.1. Ordinal sums — the geometric view

Different to the early definitions of ordinal sum operations whose aim has been to construct operations on some union of linearly ordered subdomains based on the operations on the subdomains, compare Section 2, Propositions 4.1 and 4.4 allow a different view: defining functions by means of determining or adopting their behavior on subdomains $R_k \subset [0,1]^2$ and a "default behavior" on the complementary domain.

Thus, copulas are defined with some information of a *geometric nature* (see [198, Section 3.2]), such as the description of the behavior of a copula in 2-boxes R_k at particular places within $[0,1]^2$, i.e., along the main or the opposite diagonal, or on horizontal or vertical stripes (see also Figs. 1 and 2).

In order to illustrate this viewpoint, let us consider the case when the copula is modified in one single box $R \subseteq [0,1]^2$. The following result clarifies some basic conditions that are needed in order to redefine the values assumed by a copula in a specific subdomain.

Proposition 4.15. [61, Proposition 7] *Let $R = [a_1, b_1] \times [a_2, b_2] \subseteq [0,1]^2$ be a 2-box in $[0,1]^2$. Let C be a copula and let $D\colon R \to [0,1]$ be a function. Then*

$$\widetilde{C}(x,y) = \begin{cases} D(x,y) & \text{if } (x,y) \in R, \\ C(x,y) & \text{otherwise,} \end{cases} \tag{4.9}$$

is a copula if and only if C and D coincide on the boundaries of R and D is 2-increasing on R.

Thus, the choice of the function D is crucial in order to define a *bona fide* copula $\widetilde{C}$. Different constructions have provided possible answers in a specific context. For instance, [89, Theorem 2.1] uses a similar reasoning in order to construct different copulas with the same diagonal section. The following result (slightly reformulated from [61, Theorem 2]) gives another general way of selecting an appropriate function D.

Proposition 4.16. [61, Theorem 2] *Let $R = [a_1, b_1] \times [a_2, b_2] \subseteq [0,1]^2$ be a 2-box in $[0,1]^2$. Let C and C^f be copulas and let $\beta_R \geq 0$. Define the function $D_R\colon R \to \mathbb{R}$ by*

$$D_R(x,y) = C(x,y) - \beta_R \cdot C^f\left(\tfrac{x-a_1}{b_1-a_1}, \tfrac{y-a_2}{b_2-a_2}\right).$$

Then

$$\widetilde{C}(x,y) = \begin{cases} D_R(x,y) + \beta_R \cdot C\left(\tfrac{x-a_1}{b_1-a_1}, \tfrac{y-a_2}{b_2-a_2}\right) & \text{if } (x,y) \in R, \\ C(x,y) & \text{otherwise,} \end{cases} \tag{4.10}$$

is a copula if the function D_R is 2-increasing on R.

Note that the condition that D_R be 2-increasing on R for $\widetilde{C}$ to become a copula imposes restrictions on the possible choices of R and β_R in (4.10). In the cases $C = M$ and $C = W$, this condition leads to R being a square centered around the main and opposite diagonal, respectively, only. In the case $C = \Pi$, for any 2-box R, β_R can be chosen to be positive. From (4.10) it is clear that, if $C^f = C$, then also $C \restriction_R = C$, i.e., the copula C will not change its behavior on R.

Another way of determining the function D in (4.9) is provided by rectangular patchwork techniques, as introduced in [84] for bivariate copulas and later discussed in [77] for multivariate copulas.

Proposition 4.17. [84, Theorem 2.2] *Let $R = [a_1, b_1] \times [a_2, b_2] \subseteq [0,1]^2$ be a 2-box. Let C and C_R be bivariate copulas, put $\alpha_R = V_C(R)$, and define the function $\widetilde{C}\colon [0,1]^2 \to [0,1]$ by*

$$\widetilde{C}(x,y) = \begin{cases} \alpha_R\, C_R\left(F_R^1(x), F_R^2(y)\right) + C(x,a_2) + C(a_1,y) - C(a_1,a_2) & \text{if } (x,y) \in R, \\ C(x,y) & \text{otherwise,} \end{cases} \tag{4.11}$$

where the functions $F_R^1, F_R^2\colon [0,1] \to [0,1]$ are given by

$$F_R^1(x) = \begin{cases} \frac{1}{\alpha_R} V_C\left([a_1, x] \times [a_2, b_2]\right) & \textit{if } \alpha_R \neq 0, \\ 0 & \textit{if } \alpha_R = 0, \end{cases} \qquad F_R^2(y) = \begin{cases} \frac{1}{\alpha_R} V_C\left([a_1, b_1] \times [a_2, y]\right) & \textit{if } \alpha_R \neq 0, \\ 0 & \textit{if } \alpha_R = 0. \end{cases}$$

Then the function $\widetilde{C}$ is a copula.

The copula $\widetilde{C}$ is called the *rectangular patchwork* of (R, C_R) into the *background copula* C and is denoted by $\widetilde{C} = C\text{-}\langle R, C_R\rangle$. Note that $\widetilde{C}$ will be different from C if and only if $C_R \neq C$ and $\alpha_R \neq 0$ for the particular choice of R and background copula C.

Applying a measure-theoretic view on the copulas involved, $\widetilde{C}$ can be equivalently expressed by

$$\widetilde{C}(x, y) = \mu_C\left(([0, x] \times [0, y]) \setminus R\right) + \alpha_R\, C_R\left(\tfrac{1}{\alpha_R}\mu_C\left([a_1, x] \times [a_2, b_2]\right), \tfrac{1}{\alpha_R}\mu_C\left([a_1, b_1] \times [a_2, y]\right)\right),$$

where μ_C is the measure induced by C on the Borel sets of $[0, 1]^2$ (see also Appendix A.8 for additional details).

Remark 4.18. It is a matter of direct verification that M- and W-ordinal sums with one single summand (a_1, b_1, C_1) are represented via patchwork constructions of the type $M\text{-}\langle [a_1, b_1]^2, C_1\rangle$ and $W\text{-}\langle [a_1, b_1] \times [1 - b_1, 1 - a_1], C_1\rangle$. Analogous results can be formulated for the Π-ordinal sums.

Remark 4.19. By using similar arguments as in [61, Proposition 8], it can be shown that, for each copula C_1 and for every 2-box $R = [a_1, b_1] \times [a_2, b_2] \subseteq [0, 1]^2$,

$$M\text{-}\langle R, C_1\rangle = \begin{cases} M\text{-}\langle [\max\{a_1, a_2\}, \min\{b_1, b_2\}]^2, C_1\rangle & \text{if } R \cap \{(x, x) \mid x \in [0, 1]\} \neq \emptyset, \\ M & \text{if } R \cap \{(x, x) \mid x \in [0, 1]\} = \emptyset, \end{cases}$$

i.e., the only 2-boxes that can be considered in a rectangular patchwork with the background copula M are squares centered around the main diagonal. Thus, in some sense, the classical ordinal sum construction is the only patchwork construction whose background copula is M.

Analogously, the only 2-boxes that can be considered for a rectangular patchwork construction with the background copula W are squares centered along the opposite diagonal of $[0, 1]^2$.

Interestingly, it can be proved that *any* copula $\widetilde{C}$ of type (4.9) can be represented in terms of a suitable rectangular patchwork, as the following result shows (compare also [84]).

Proposition 4.20. *Let $R = [a_1, b_1] \times [a_2, b_2] \subseteq [0, 1]^2$ be a 2-box. Let C be a copula and let $D\colon R \to [0, 1]$ be a function. Then $\widetilde{C}$ in (4.9) is a copula if and only if there exists a copula C_R such that $\widetilde{C} = C\text{-}\langle R, C_R\rangle$.*

Proof. Let $\widetilde{C}$ be a copula. We distinguish two cases.

$V_C(R) = 0$: Then $V_D(R) = 0$ (since C and D coincide on the boundaries of R). Moreover, in view of the monotonicity of the C-volume with respect to the inclusion of sets, also $V_D([a_1, x] \times [a_2, y]) = 0$ for every $(x, y) \in R$. By the definition of the D-volume of a 2-box, it follows that

$$D(x, y) = D(x, a_2) + D(a_1, y) - D(a_1, a_2) = C(x, a_2) + C(a_1, y) - C(a_1, a_2).$$

$\alpha_R = V_C(R) > 0$: Since $\widetilde{C}$ is a copula, D is continuous. Thus, $\widetilde{D}\colon R \to [0, 1]$ defined by

$$\widetilde{D}(x, y) = \frac{V_D([a_1, x] \times [a_2, y])}{\alpha_R} \tag{4.12}$$

is a continuous distribution function supported on R. Thus, in view of Sklar's Theorem (see Theorem A.8 in Appendix A.7), it follows that a unique copula C_R exists so that

$$\widetilde{D}(x, y) = C_R\left(\tfrac{V_D([a_1, x] \times [a_2, b_2])}{\alpha_R}, \tfrac{V_D([a_1, b_1] \times [a_2, y])}{\alpha_R}\right).$$

Now, from the equality

$$D(x, y) - D(x, a_2) - D(a_1, y) + D(a_1, a_2) = V_D([a_1, x] \times [a_2, y]) = \alpha_R \widetilde{D}(x, y)$$

it follows that

$$D(x, y) = \alpha_R \widetilde{D}(x, y) + C(x, a_2) + C(a_1, y) - C(a_1, a_2).$$

The reverse implication coincides with Proposition 4.17. □

Remark 4.21. Let (U, V) be a random vector distributed according to $\widetilde{C}$ of Proposition 4.20. Assume $V_C(R) > 0$. Then $\widetilde{D}$ of (4.12) is the conditional distribution function of (U, V) given $(U, V) \in R$, while C_R represents its unique copula. Thus, C_R provides information about the probability mass distribution of $\widetilde{C}$ in the 2-box R.

The rectangular patchwork construction can also be interpreted as a patchwork operator, i.e., as a mapping that, starting with a background copula C, a copula C_R and a 2-box $R \subseteq [0, 1]^2$, produces another copula $\widetilde{C}$ that coincides with C on the whole copula domain but in the fixed 2-box R. As such, the method allows us to change the behavior of C in a specific part of its domain, a feature that is particularly helpful to introduce asymmetries and fat tails (see, for instance, [85]).

Specifically, let $\mathscr{R}$ denote the set of all 2-boxes $R \subseteq [0, 1]^2$. The mapping

$$\Psi \colon \mathscr{C}_2 \times \mathscr{R} \times \mathscr{C}_2 \to \mathscr{C}_2, \qquad (C, R, C_R) \longmapsto \widetilde{C},$$

where $\widetilde{C}$ is given by (4.11), is called the *patchwork operator* associated with the background copula C. For a fixed C and R, the mapping $(C, R, \cdot) \longmapsto \widetilde{C}$ verifies the following properties:

(i) the mapping $(C, R, \cdot) \longmapsto \widetilde{C}$ is continuous with respect to the L^∞ norm in the space $\mathscr{C}_2$ of copulas (see [77, Theorem 6]);
(ii) if $V_C(R) > 0$ then C is the unique fixed point of the mapping $(C, R, \cdot) \longmapsto \widetilde{C}$.

For a fixed background copula C, sequences of 2-boxes $(R_k)_{k\in\{1,\dots,N\}}$ and corresponding copulas $(C_k)_{k\in\{1,\dots,N\}}$ with $N \in \mathbb{N}$, the patchwork operator ψ can be applied recursively, generating a sequence of copulas $(\widetilde{C}_k)_{k\in\{0,1,\dots,N\}}$ such that for all $k \in \{2, \dots, N\}$

$$\begin{aligned}
\widetilde{C}_0 &= C,\\
\widetilde{C}_1 &= \Psi(C, R_1, C_1) = C\text{-}\langle R_1, C_1\rangle,\\
\widetilde{C}_k &= \Psi(\widetilde{C}_{k-1}, R_k, C_k) = \widetilde{C}_{k-1}\text{-}\langle R_k, C_k\rangle.
\end{aligned}$$

Remark 4.22. If two 2-boxes R_i and R_j, with $i, j \in \{1, \dots, N\}$ and $i \neq j$ are such that $R_i \cap R_j = \partial R_i \cap \partial R_j$, then for every background copula C we have

$$\Psi(C\text{-}\langle R_i, C_i\rangle, R_j, C_j) = \Psi(C\text{-}\langle R_j, C_j\rangle, R_i, C_i) = C\text{-}\langle (R_k, C_k)_{k\in\{i,j\}}.$$

In other words, one can transform the values assumed by the background copula C, first in the subdomain R_i, and then in R_j, or the other way around, without changing the final output.

Assuming that $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all $i, j \in \{1, \dots, N\}$ with $i \neq j$ and following the notation for rectangular patchworks as introduced in [85], it follows that $\widetilde{C}_N = C\text{-}\langle (R_k, C_k)_{k\in\{1,\dots,N\}}\rangle$. Such a copula $\widetilde{C}_N$ is obtained by changing the values assumed by the background copula C in each of the 2-boxes R_k by using the information provided by C_k.

Note that in case of 2-boxes with overlapping interiors the order in which the patchwork operator is applied may lead to different results as the following example illustrates:

Example 4.23. Choose $C = \Pi$ as background copula. Consider the two 2-boxes $R_1 = \left[0, \frac{1}{2}\right]^2$ and $R_2 = \left[\frac{1}{4}, \frac{3}{4}\right]^2$ with corresponding copulas $C_1 = M$ and $C_2 = W$. Clearly, $R_1 \cap R_2 = \left[\frac{1}{4}, \frac{1}{2}\right]^2 \neq \partial R_1 \cap \partial R_2$.

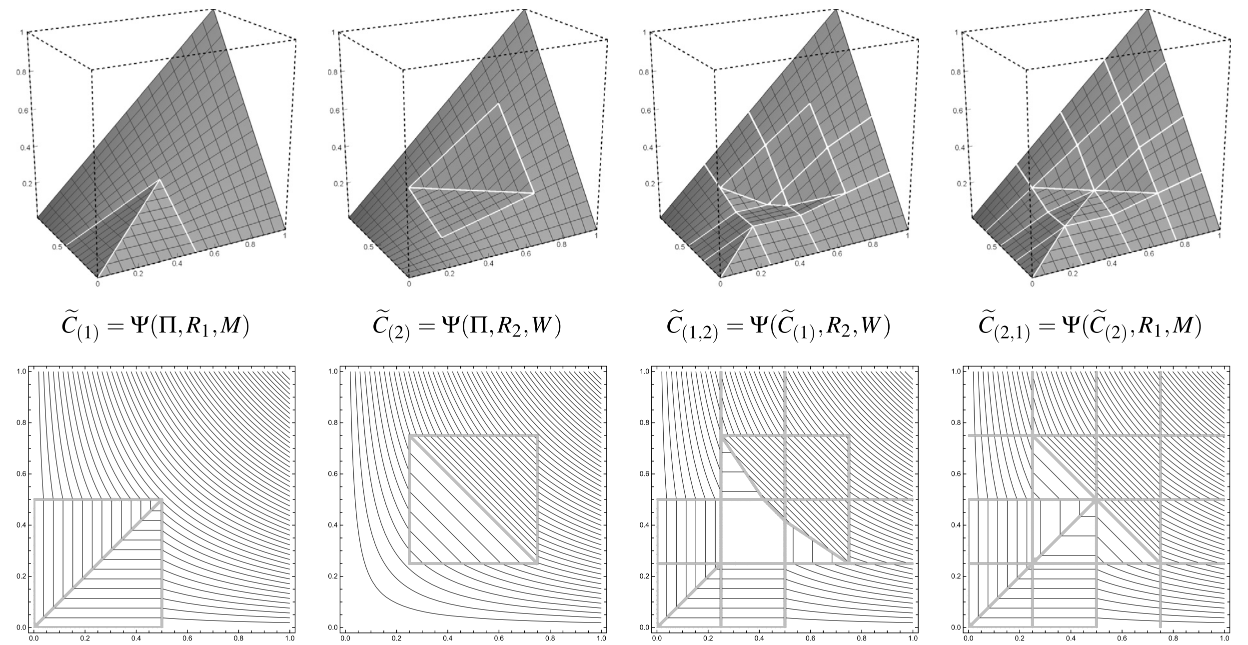

Fig. 3. Applying the patchwork operator consecutively in the case of overlapping 2-boxes, leading to $\Psi(\widetilde{C}_{(1)}, R_2, W) \neq \Psi(\widetilde{C}_{(2)}, R_1, M)$ (see Example 4.23).

We will apply the patchwork operator in two different orders. For better readability we will denote the resulting copula $\widetilde{C}$ with a subscript tuple indicating the sequence of 2-boxes (and corresponding copulas) used.

Applying first the patchwork operator Ψ to Π on R_1 by means of M and second on R_2 by W leads to the copula $\widetilde{C}_{(1,2)} = \Psi(\Pi\text{-}\langle R_1, M\rangle, R_2, W)$.

Applying first the patchwork operator Ψ to Π on R_2 by means of W and second on R_1 by M leads to the copula $\widetilde{C}_{(2,1)} = \Psi(\Pi\text{-}\langle R_2, W\rangle, R_1, M)$. The copulas $\widetilde{C}_{(1)}$, $\widetilde{C}_{(2)}$, $\widetilde{C}_{(1,2)}$ and $\widetilde{C}_{(2,1)}$ are depicted along with their contour plots in Fig. 3. As can be seen immediately, the resulting copulas are different, i.e., $\widetilde{C}_{(1,2)} \neq \widetilde{C}_{(2,1)}$ (even outside of $R_1 \cap R_2$).

Ordinal sums of Propositions 4.1 and 4.4 with countable index set K can be obtained by applying the mapping $\Psi(C, R_k, C_k)$ (for some suitable R_k) iteratively by starting with C being one of the basic copulas M, W and Π as a background copula. Here, the key condition is that $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all $i, j \in K$, $i \neq j$. In particular, if R_k is, for every $k \in K$, defined by (4.7), then the patchworks $M\text{-}(\langle R_k, C_k\rangle)_{k\in K}$, $W\text{-}(\langle R_k, C_k\rangle)_{k\in K}$ and $\Pi\text{-}(\langle R_k, C_k\rangle)_{k\in K}$ will correspond to the ordinal sums as introduced in Propositions 4.1 and 4.4.

Remark 4.24. The bounds for copulas expressed via patchwork representation are obtained from the Fréchet-Hoeffding bounds for copulas. In fact, the following inequalities hold:

$$\begin{aligned} W \leq W\text{-}(\langle R_k, C_k\rangle)_{k\in K} &\leq W\text{-}(\langle R_k, M\rangle)_{k\in K}, \\ M\text{-}(\langle R_k, W\rangle)_{k\in K} \leq M\text{-}(\langle R_k, C_k\rangle)_{k\in K} &\leq M, \\ \Pi\text{-}(\langle R_k, W\rangle)_{k\in K} \leq \Pi\text{-}(\langle R_k, C_k\rangle)_{k\in K} &\leq \Pi\text{-}(\langle R_k, M\rangle)_{k\in K}. \end{aligned}$$

4.2. Ordinal sum representations

Obviously, every copula C can be represented by means of the trivial ordinal sum $C = (\langle 0, 1, C\rangle)$ for each type of ordinal sums introduced in Propositions 4.1 and 4.4. Here we revisit some (known) representations by providing alternative proofs based on the patchwork techniques developed in [85].

4.2.1. Special case: M-ordinal sum

The equivalence between the properties (i) and (iii) in the following Proposition was established, albeit with a different proof in [198, Theorem 3.2.1] and in [183, Theorem 3.3]. The proof below uses a characterization of 2-increasing functions given in [85].

Proposition 4.25. *Consider a copula $C \in \mathscr{C}_2$. Then the following are equivalent:*

(i) *There exists some $a \in]0, 1[$ such that $C(a, a) = M(a, a) = a$.*
(ii) *There exists some $a \in]0, 1[$ with $C(x, y) = M(x, y)$ whenever $a \in \{x, y\}$.*
(iii) *C is a non-trivial M-ordinal sum, i.e., there exist at least one $a \in]0, 1[$ and some copulas C_1, C_2 such that*

$$C = M\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle).$$

Proof. (i) $\Longrightarrow$ (ii). Assume that there exists some $a \in]0, 1[$ which is an idempotent element of C, i.e., for which $C(a, a) = a = M(a, a)$. Then, for every $z \in [a, 1]$, the following hold because of the monotonicity of C:

$$\begin{aligned} a = C(a, a) \le C(a, z) \le C(a, 1) = a \quad &\Longrightarrow \quad C(a, z) = a = M(a, z),\\ a = C(a, a) \le C(z, a) \le C(1, a) = a \quad &\Longrightarrow \quad C(z, a) = a = M(z, a). \end{aligned}$$

On the other hand, the 1-Lipschitz property of copulas leads, for each $(x, y) \in [0, a]^2$, to

$$|C(a, a) - C(a, y)| \le |a - y| \qquad \text{and} \qquad |C(a, a) - C(x, a)| \le |a - x|$$

so that

$$y \le C(a, y) \le M(a, y) = y \qquad \text{and} \qquad x \le C(x, a) \le M(x, a) = x.$$

Summarizing, we obtain $C(x, y) = M(x, y)$ whenever $a \in \{x, y\}$.
(ii) $\Longrightarrow$ (iii). Assume that, for some $a \in]0, 1[$, C coincides with M whenever a appears among its arguments. By $a \in]0, 1[$, the domain $[0, 1]^2$ is divided into four different subdomains $R_1 = [0, a]^2$, $R_2 = [a, 1]^2$ and $R_3 = [0, a] \times [a, 1]$, $R_4 = [a, 1] \times [0, a]$. Each restriction $C \restriction_{R_k}$ leads to a continuous, increasing and 2-increasing function with given margins determined by M. Note that for each of the subdomains we obtain:

$$\alpha_1 = V_C(R_1) = a, \qquad \alpha_2 = V_C(R_2) = 1 - a, \qquad \alpha_3 = V_C(R_3) = 0, \qquad \alpha_4 = V_C(R_4) = 0.$$

Following [85, Theorem 2.1], the restrictions of C to R_3 and R_4 is given by, respectively,

$$\begin{aligned} C \restriction_{R_3} (x, y) &= C(x, a) + C(0, y) - C(0, a) = C(x, a) = M(x, a) = x = M(x, y),\\ C \restriction_{R_4} (x, y) &= C(x, 0) + C(a, y) - C(a, 0) = C(a, y) = M(a, y) = y = M(x, y). \end{aligned}$$

Following again [85, Theorem 2.1], for C restricted to R_1 there exists a unique copula C_1 such that

$$C \restriction_{R_1} (x, y) = a \cdot C_1\big(\tfrac{x}{a}, \tfrac{y}{a}\big).$$

In an analogous way one can argue that there exists a unique copula C_2 such that

$$C \restriction_{R_2} (x, y) C_2\left(\tfrac{V_C([a,x]\times[a,1]}{\alpha_2}, \tfrac{V_C([a,1]\times[a,y]}{\alpha_2}\right) + C(x, a) + C(a, y) - C(a, a) = a + (1 - a) \cdot C_2\big(\tfrac{x-a}{1-a}, \tfrac{y-a}{1-a}\big),$$

showing that indeed $C = M\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)$ with the copulas C_1 and C_2 given by

$$C_1(x, y) = \tfrac{1}{a} C(a \cdot x, a \cdot y) \qquad \text{and} \qquad C_2(x, y) = \tfrac{1}{1-a} \cdot (C(x(1 - a) + a, y(1 - a) + a) - a)\,.$$

Finally, (iii) immediately implies (i), since $C(a, a) = a$ for any M-ordinal sum $C = M\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)$. □

Proposition 4.25 means that identifying a single idempotent element, i.e., a single point a on the diagonal where C acts like M, suffices for describing the copula C as a non-trivial ordinal sum of some copulas C_1 and C_2 which themselves may well be M-ordinal sums again. Note that it is essential that a is an element of the diagonal as the following example illustrates (compare also [198, Theorem 3.2.3]).

Example 4.26. Let C be a copula and suppose that $C(a,b) = M(a,b)$ with $(a,b) \in]0,1[^2$ and $a \neq b$. Then the function $D\colon [0,1]^2 \to [0,1]$ defined by

$$D(x,y) = \max\left\{W(x,y), M(a,b) - (a-x)^+ - (b-y)^+\right\}$$

with $t^+ = \max\{0,t\}$ is a copula fulfilling $D(a,b) = M(a,b) = C(a,b)$. However, D has no non-trivial idempotent element, i.e., $D(x,x) \neq x$ for all $x \in]0,1[$: In case $D(x,x) = W(x,x)$, $D(x,x) = x$ holds for $x \in \{0,1\}$ only. In case $D(x,x) > W(x,x)$, we obtain

$$M(a,b) - (a-x)^+ - (b-x)^+ = \begin{cases} 2x - \max\{a,b\} & \text{if } x \in [0, \min\{a,b\}[\,, \\ M(a,b) - \max\{a,b\} + x & \text{if } x \in [\min\{a,b\}, \max\{a,b\}]\,, \\ M(a,b) & \text{if } x \in]\max\{a,b\}, 1]\,. \end{cases}$$

Each of these cases leads to a contradiction when $D(x,x) = x$ is assumed. Therefore, D is not an M-ordinal sum.

4.2.2. Special case: W-ordinal sum

When considering now W-ordinal sums note that the equivalence between the properties (i) and (iii) in the following proposition was already established in [184, Proposition 2]. Though by applying the results on symmetries for M- and W-ordinal sums as outlined in Proposition 4.9 the following three equivalent statements may be obtained immediately:

Proposition 4.27. *Consider a copula $C \in \mathscr{C}_2$. Then the following are equivalent:*

(i) *There exists some $a \in]0,1[$ such that $C(a, 1-a) = W(a, 1-a)$.*
(ii) *There exists some $a \in]0,1[$ with $C(a,y) = W(a,y)$ and $C(x, 1-a) = W(x, 1-a)$ for all $x, y \in [0,1]$.*
(iii) *C is a non-trivial W-ordinal sum, i.e., there exist at least one $a \in]0,1[$ and some copulas C_1, C_2 such that*

$$C = W\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle).$$

Proof. Consider a copula C and its transformation induced by the σ_2 symmetry given by $C^{\sigma_2}(x,y) = x - C(x, 1-y)$ (compare Appendix A.9). Assume that there exists some $a \in]0,1[$ with $C(a, 1-a) = W(a, 1-a) = 0$. Then

$$C^{\sigma_2}(a,a) = a - C(a, 1-a) = a - W(a, 1-a) = a,$$

i.e., C^{σ_2} possesses a non-trivial idempotent element a and, according to Proposition 4.25, $C^{\sigma_2}(x,y) = M(x,y)$ whenever $a \in \{x,y\}$ and $C^{\sigma_2} = M\text{-}(\langle 0, a, \tilde{C}_1\rangle, \langle a, 1, \tilde{C}_2\rangle)$ for some copulas $\tilde{C}_1$ and $\tilde{C}_2$. As a consequence,

$$\begin{aligned} M(a, 1-y) = C^{\sigma_2}(a, 1-y) = a - C(a,y) &\iff C(a,y) = a - M(a, 1-y) = \max\{a + y - 1, 0\} = W(a,y), \\ M(x,a) = C^{\sigma_2}(x,a) = x - C(x, 1-a) &\iff C(x, 1-a) = x - M(x,a) = \max\{x - a, 0\} = W(x, 1-a). \end{aligned}$$

By $C(x,y) = x - C^{\sigma_2}(x, 1-y)$ we obtain from Proposition 4.1

$$C(x,y) = \begin{cases} x - a \cdot \tilde{C}_1\left(\frac{x}{a}, \frac{1-y}{a}\right) & \text{if } (x,y) \in [0,a] \times [1-a, 1]\,, \\ x - a + (1-a) \cdot \tilde{C}_2\left(\frac{x-a}{1-a}, \frac{1-y-a}{1-a}\right) & \text{if } (x,y) \in [a,1] \times [0, 1-a]\,, \\ x - M(x, 1-y) = W(x,y) & \text{otherwise,} \end{cases}$$

illustrating that C follows the required W-ordinal sum structure. Defining $C_1 = \tilde{C}_1^{\;\sigma_2}$ and $C_2 = \tilde{C}_2^{\;\sigma_2}$, we obtain

$$\begin{aligned} x - a \cdot \tilde{C}_1\big(\tfrac{x}{a}, \tfrac{1-y}{a}\big) &= a \cdot C_1\big(\tfrac{x}{a}, \tfrac{a+y-1}{a}\big), \\ x - a + (1-a) \cdot \tilde{C}_2\big(\tfrac{x-a}{1-a}, \tfrac{1-y-a}{1-a}\big) &= (1-a) \cdot C_2\big(\tfrac{x-a}{1-a}, \tfrac{y}{1-a}\big), \end{aligned}$$

showing that indeed $C = W\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)$. □

Similar to the case of M-ordinal sums, the property $C(a,b) = W(a,b)$ for some $b \neq 1 - a$ is not sufficient for proving that C is indeed a W-ordinal sum.

Example 4.28 *(Example 4.26 continued).* Note that the copula D defined in Example 4.26 is a W-ordinal sum, since we have $D(1 + a - b, b - a) = 0$ whenever $M(a,b) = a$, and $D(a - b, 1 - a + b) = 0$ whenever $M(a,b) = b$.

4.2.3. Special case: Π-ordinal sums

Finally, we consider the case of Π-horizontal and Π-vertical ordinal sums. Different to M- and W-ordinal sums no single point on some specific subdomain of $[0, 1]^2$ can be identified which induces the ordinal sum structure, but horizontal or vertical sections where the ordinal sum coincides with Π are needed. Note that a proof of the following equivalent statements has been established already in [216, Proposition 2.2], we here provide an alternative proof based on the patchwork techniques discussed also in Subsection 4.1.

Proposition 4.29. *Consider a copula $C \in \mathscr{C}_2$. Then the following are equivalent:*

(i) *There exists some $a \in]0, 1[$ with $C(x, a) = \Pi(x, a)$ for all $x \in [0, 1]$.*
(ii) *C is a non-trivial Π-horizontal ordinal sum, i.e., there exist some $a \in]0, 1[$ and some copulas C_1, C_2 such that*

$$C = \Pi^{(\mathbf{h})}\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle).$$

Proof. It is obvious that any Π-horizontal ordinal sum $C = \Pi^{(\mathbf{h})}\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)$ also fulfills both

$$C(x, a) = a \cdot C_1(x, 1) = a \cdot x = \Pi(x, a) \qquad \text{and} \qquad C(x, a) = a \cdot x + (1 - a) \cdot C_2(x, 0) = a \cdot x = \Pi(x, a).$$

On the other hand, assume that $C(x, a) = \Pi(x, a)$ for some $a \in]0, 1[$, i.e., the horizontal section of C at $y = a$ coincides with Π and splits $[0, 1]^2$ into the two 2-boxes $R_1 = [0, 1] \times [0, a]$ and $R_2 = [0, 1] \times [a, 1]$ with

$$\alpha_1 = V_C(R_1) = a \qquad \text{and} \qquad \alpha_2 = V_C(R_2) = 1 - a.$$

Again, according to [85, Theorem 2.1] and Proposition 4.20, there exist unique copulas C_1 and C_2 such that

$$C\restriction_{R_1}(x, y) = a \cdot C_1\big(x, \tfrac{y}{a}\big) \qquad \text{for all } (x, y) \in R_1,$$
$$C\restriction_{R_2}(x, y) = a \cdot x + (1 - a) \cdot C_2\big(x, \tfrac{y-a}{1-a}\big) \qquad \text{for all } (x, y) \in R_2,$$

showing that $C = \Pi^{(\mathbf{h})}\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle)$. □

Using analogous arguments as in the proof of Proposition 4.29, the representation as a vertical ordinal sum can be shown.

Corollary 4.30. *Consider a copula $C \in \mathscr{C}_2$. Then the following are equivalent:*

(i) *There exists some $a \in]0, 1[$ with $C(a, y) = \Pi(a, y)$ for all $y \in [0, 1]$.*
(ii) *C is a non-trivial Π-vertical ordinal sum, i.e., there exist some $a \in]0, 1[$ and some copulas C_1, C_2 such that*

$$C = \Pi^{(\mathbf{v})}\text{-}(\langle 0, a, C_1\rangle, \langle a, 1, C_2\rangle).$$

Obviously, any copula C with a non-trivial ordinal sum representation can either be a M-, a W-, or a Π-ordinal sum, so these ordinal sum representations mutually exclude each other. As a consequence, Π- and W-ordinal sums never have non-trivial idempotent elements; on the other hand Π- and M-ordinal sums C fulfill $C(x, 1-x) \neq W(x, 1-x) = 0$ for all $x \in]0, 1[$. M- and W-ordinal sums act like M and W, respectively, on some non-trivial subdomains of $[0, 1]^2$, though in case of the Π-ordinal sum the set on which the ordinal sum coincides with Π may consist of horizontal or vertical sections only.

If a copula C coincides with Π on some vertical as well as on some horizontal section it may be represented as a Π-vertical ordinal sum of two Π-horizontal ordinal sum copulas or vice-versa (compare with [151] and [85]).

5. Ordinal sums and the Markov product

In this section, we present an exemplary case where ordinal sums of copulas have appeared naturally in a purely probabilistic context, namely the study of Markov chains (see [168]).

Darsow et al. [55] introduced the following binary operation on $\mathscr{C}_2$: if A and B are copulas, a new copula $A * B$ is defined via

$$(A * B)(x, y) = \int_0^1 \partial_2 A(x, t)\, \partial_1 B(t, y)\, dt,$$

which is called the *Markov product* of A and B. This operation has the following interpretation: let X_0, X_1, X_2 be a Markov chain of continuous random variables and let A be the copula of (X_0, X_1) and let B be the copula of (X_1, X_2). Then $A * B$ is the copula of (X_0, X_2). Thus, the Markov product is able to describe the dependence among the first and last element of the random sequence given some partial information.

In particular, a copula C is said to be *idempotent* with respect to the Markov product if $C * C = C$. The set of idempotent copulas will be denoted by $\mathscr{C}_{\text{id}}$.[1] Idempotent copulas are particularly important in the study of (homogeneous) Markov chains since, if C is idempotent, then the n-fold Markov product of C with itself remains equal to C and, hence, the Markov chain in question is stationary.

In [55] it was recognized that the bivariate copulas Π and M are idempotent, and in the same paper a further example was provided, for $\alpha \in]0, 1[$, by the ordinal sum

$$C_\alpha(x, y) = \frac{\Pi(x, y)}{\alpha} \mathbf{1}_{[0,\alpha]^2}(x, y) + M(x, y)\mathbf{1}_{[0,1]^2 \setminus [0,\alpha]^2}(x, y). \tag{5.1}$$

These were the only examples in the literature of copulas belonging to $\mathscr{C}_{\text{id}}$ until countably generated idempotent copulas were introduced in [10] by exploiting the one-to-one correspondence between copulas and Markov operators.

We recall that a *Markov operator* in the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a linear operator $T : L^\infty \to L^\infty$ that is (a) positive, i.e., $Tf \geq 0$ whenever $f \geq 0$, and satisfies (b) $T\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes the constant function $f \equiv 1$, and (c) $\mathbb{E}(Tf) = \mathbb{E}(f)$ for every $f \in L^\infty$, where $\mathbb{E}(f) = \int f\, d\mathbb{P}$. The L^∞ operator norm $\|T\|_\infty$ of T is equal to 1; moreover T can be extended to L^1 and its L^1-norm still equals 1, i.e., $\|T\|_1 = 1$.

In [201] it was shown that, when the probability space is $([0, 1], \mathscr{B}([0, 1]), \lambda)$, where λ is the Lebesgue measure on the Borel sets $\mathscr{B}([0, 1])$ of $[0, 1]$, the following one-to-one correspondence exists between copulas and Markov operators. Given a copula C, the operator T_C defined on $L^1 = L^1([0, 1], \mathscr{B}([0, 1]), \lambda)$ via

$$(T_C f)(x) = \frac{d}{dx} \int_0^1 \partial_2 C(x, t) f(t)\, dt$$

is a Markov operator. Conversely, if T is a Markov operator on $([0, 1], \mathscr{B}([0, 1]), \lambda)$, the function $C_T : [0, 1]^2 \to [0, 1]$ defined by

$$C_T(x, y) = \int_0^x \left(T \mathbf{1}_{[0,y]}\right)(s)\, ds \tag{5.2}$$

is a copula. If A and B are copulas and T_A and T_B are the corresponding Markov operators, then

$$T_A \circ T_B = T_{A*B}, \tag{5.3}$$

which implies that the operation $*$ on $\mathscr{C}_2$ is associative. By recourse to Pfanzagl's characterization of conditional expectations [209] one can prove, see [230], that a Markov operator is a conditional expectation if and only if it is idempotent with respect to the composition of operators, i.e., if and only if $T^2 = T \circ T = T$. It follows from (5.3) that the Markov operator T_C is a conditional expectation if and only if the copula C is in $\mathscr{C}_{\text{id}}$.

As a consequence, there is also a one-to-one correspondence between the set $\mathscr{C}_{\text{id}} \subset \mathscr{C}_2$ and the sub-σ-algebras of $\mathscr{B}([0, 1])$ or, equivalently, between $\mathscr{C}_{\text{id}}$ and the family of conditional expectations on $([0, 1], \mathscr{B}([0, 1]), \lambda)$ with respect to sub-σ-algebras of $\mathscr{B}([0, 1])$. The conditional expectation with respect to a given σ-algebra can be used to replace the Markov operator T in (5.2) and then to generate a copula that automatically belongs to $\mathscr{C}_{\text{id}}$. In the following, only conditional expectations with respect to sub-σ-algebras of $\mathscr{B}([0, 1])$ generated by a countable partition of the unit

[1] In this section the term *idempotent* refers to the property just introduced, while in the remainder of the paper the same term denotes a property of elements of $[0, 1]$ or of $[0, 1]^d$.

interval $[0, 1]$ will be considered. The points $0 = a_0 < a_1 < \cdots < a_n$ in $[0, 1]$, with either $a_n = 1$ or $(a_n)_{n\geq 1} \to 1$, give rise to the partition of $[0, 1]$

$$[0,1] = [a_0, a_1] \cup \left(\bigcup_{k>1} \left]a_k, a_{k+1}\right] \right) \cup \{1\}. \tag{5.4}$$

The last element in this latter union, the singleton $\{1\}$, is missing when there are only a finite number of points a_k. Write (a_k, a_{k+1}) to denote the interval $\left]a_k, a_{k+1}\right]$ for $k \geq 1$ and $[a_0, a_1]$ for $k = 0$, and $\mathscr{F}(\{a_k\})$ to denote the sub-σ-algebra generated by the partition (5.4). Then, if f is in L^1, the conditional expectation of f with respect to $\mathscr{F}(\{a_k\})$ is given by

$$\mathbb{E}_{\mathscr{F}(\{a_k\})} f = \sum_{k\geq 0} \frac{\mathbf{1}_{(a_k, a_{k+1})}}{a_{k+1} - a_k} \int_{a_k}^{a_{k+1}} f \, d\lambda.$$

If $y \in (a_k, a_{k+1})$, one easily has

$$\mathbb{E}_{\mathscr{F}(\{a_k\})} \mathbf{1}_{[0,y]} = \sum_{j=0}^{k-1} \mathbf{1}_{(a_j, a_{j+1})} + \mathbf{1}_{(a_k, a_{k+1})} \frac{y - a_k}{a_{k+1} - a_k} = \mathbf{1}_{(0, a_k)} + \mathbf{1}_{(a_k, a_{k+1})} \frac{y - a_k}{a_{k+1} - a_k}.$$

After a few easy calculations the corresponding copula is

$$C_{\mathscr{F}(\{a_k\})}(x, y) = \begin{cases} a_k + \frac{(x-a_k)(y-a_k)}{a_{k+1}-a_k} & \text{if } (x, y) \in \left[a_k, a_{k+1}\right]^2, \\ M(x, y) & \text{otherwise,} \end{cases} \tag{5.5}$$

which is an ordinal sum of independence copulas, i.e., $C_{\mathscr{F}(\{a_k\})} = M\text{-}(\langle a_0, a_1, \Pi\rangle, \ldots, \langle a_k, a_{k+1}, \Pi\rangle, \ldots)$ (see also [182, Section 5]). The following result establishes that ordinal sums of this latter type play an important role in $\mathscr{C}_{\mathrm{id}}$.

Theorem 5.1. [10, Theorem 2] *The set of all M-ordinal sums of type $M\text{-}(\langle a_k, b_k, \Pi\rangle)_{k\in K}$ is dense in the set of idempotent copulas $\mathscr{C}_{\mathrm{id}}$ with respect to the topology of uniform convergence in* $[0, 1]^2$.

Interestingly, under some additional conditions on the copula, M-ordinal sums of type $M\text{-}(\langle a_k, b_k, \Pi\rangle)_{k\in K}$ can be characterized in the following way.

Theorem 5.2. [235, Theorem 5.1] *Let C be stochastically increasing in the first variable, i.e., $u_1 \mapsto \partial_1 C(u_1, u_2)$ is decreasing for almost all $u_1 \in [0, 1]$. Then C is in $\mathscr{C}_{\mathrm{id}}$ if and only if $C = M\text{-}(\langle a_k, b_k, \Pi\rangle)_{k\in K}$.*

Clearly, Theorem 5.2 also holds if C be stochastically increasing in the second variable, i.e., $u_2 \mapsto \partial_2 C(u_1, u_2)$ is decreasing for almost all $u_2 \in [0, 1]$.

Moreover, notice that the Markov product of $C = M\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ by itself is given (see [11]) by

$$(C * C)(x, y) = \begin{cases} a_k + (b_k - a_k)\,(C_k * C_k)\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x, y) \in [a_k, b_k]^2, \\ M(x, y) & \text{otherwise.} \end{cases} \tag{5.6}$$

Therefore, the ordinal sum C is in $\mathscr{C}_{\mathrm{id}}$ if and only if so is the copula C_k for every $k \in K$. Notice that the copula (5.1) is a special case of (5.6) obtained when $n = 2$, $a_1 = 0$, $a_2 = 1$, $C_1 = \Pi$ and $C_2 = M$. Moreover, since Π is in $\mathscr{C}_{\mathrm{id}}$, also the copulas in (5.5), which are in $\mathscr{C}_{\mathrm{id}}$, are obtained as a special case of (5.6) by taking Π for every $k \in K$.

From these aspects, the following result can be also derived by using the study about idempotent Archimedean copulas in [99].

Theorem 5.3. [99, Theorem 5.9] *Let C be an associative copula which is represented as $C = M\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ via Theorem 3.4. Then C is idempotent if and only if $C_k = \Pi$ for every $k \in K$.*

6. Ordinal sums of multivariate copulas

In order to provide a suitable definition of the M-ordinal sum construction in any dimension $d \geq 2$, from a purely algebraic viewpoint the expression similar to (4.1) would be

$$C(x_1, \ldots, x_d) = \begin{cases} a_k + (b_k - a_k) C_k\left(\frac{x_1 - a_k}{b_k - a_k}, \ldots, \frac{x_d - a_k}{b_k - a_k}\right) & \text{if } (x_1, \ldots, x_d) \in [a_k, b_k]^d, \\ \min\{x_1, \ldots, x_d\} & \text{otherwise.} \end{cases}$$

But this latter function need not be a copula, as can be easily seen.

Example 6.1. Let the function $C : [0,1]^3 \to [0,1]$ be defined by

$$C(x_1, x_2, x_3) = \begin{cases} \frac{1}{3} \Pi_3(3x_1, 3x_2, 3x_3) & \text{if } (x_1, x_2, x_3) \in \left[0, \frac{1}{3}\right]^3, \\ \min\{x_1, x_2, x_3\} & \text{otherwise.} \end{cases}$$

At the point $(\frac{1}{3}, \frac{1}{4}, \frac{1}{4})$, which lies on the boundary of the box $\left[0, \frac{1}{3}\right]^3$, one has

$$\frac{1}{3} \Pi_3\left(1, \frac{3}{4}, \frac{3}{4}\right) = \frac{3}{16} \neq \frac{1}{4} = \min\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{4}\right\},$$

thus C is not continuous at that point and therefore cannot be a copula.

In the absence of the associativity property, which would allow to extend a bivariate operation to higher dimensions, an alternative idea to define ordinal sums for d-copulas ($d \geq 3$) is to use some measure-theoretic arguments.

To this end, consider an M-ordinal sum copula C of type (4.1). Let denote by $V_C(R)$ the C-volume of any 2-box $R \subseteq [0,1]^2$ and by μ_C the probability measure that extends V_C to any Borel set of $[0,1]^2$ via Carathéodory's extension procedure [22,212]. By using the σ-additivity of the measure μ_C, for every $(x, y) \in [0,1]^2$ such that $\min\{x, y\} \in]a_k, b_k[$ for some $k \in K$ it holds

$$\begin{aligned} C(x, y) &= \mu_C([0, x] \times [0, y]) \\ &= \sum_{i \in K, i<k} \mu_C(]a_i, b_i[^2) + \mu_C([a_k, \min\{x, b_k\}] \times [a_k, \min\{y, b_k\}]) \\ &\quad + \mu_C\left(([0, x] \times [0, y]) \setminus \bigcup_{i \in K, i<k}]a_i, b_i[^2\right) \\ &= \sum_{i \in K, i<k} (b_i - a_i) + (b_k - a_k) C_k\left(\tfrac{\min\{x, b_k\} - a_k}{b_k - a_k}, \tfrac{\min\{y, b_k\} - a_k}{b_k - a_k}\right) + \lambda\left([0, \min\{x, y\}] \setminus \bigcup_{i \in K, i<k}]a_i, b_i[\right) \\ &= a_k + (b_k - a_k) C_k\left(\tfrac{\min\{x, b_k\} - a_k}{b_k - a_k}, \tfrac{\min\{y, b_k\} - a_k}{b_k - a_k}\right), \end{aligned}$$

where λ is the Lebesgue measure on $[0,1]$; while $C(x, y) = M(x, y)$, otherwise. The previous equality suggests the following expression for a d-dimensional version of M-ordinal sums.

Theorem 6.2. [183, Theorem 2.1] *Let $(]a_k, b_k[)_{k \in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_k)_{k \in K}$ be a family of d-dimensional copulas. Then the following function $M_d^{\mathrm{OSum}} \colon [0,1]^d \to [0,1]$ given by*

$$M_d^{\mathrm{OSum}}(x_1, \ldots, x_d) = \begin{cases} a_k + (b_k - a_k) \cdot C_k\left(\frac{\min\{x_1, b_k\} - a_k}{b_k - a_k}, \ldots, \frac{\min\{x_d, b_k\} - a_k}{b_k - a_k}\right) & \textit{if } \min\{x_1, \ldots, x_d\} \in [a_k, b_k], \\ M_d(x_1, \ldots, x_d) & \textit{otherwise,} \end{cases} \tag{6.1}$$

is a d-copula.

Concerning the definition of W-ordinal sums for d-copulas ($d \geq 3$), it is worth noticing that W_d is not a copula for $d \geq 3$. Thus, a definition of W_d-ordinal sums cannot be obtained via algebraic similarity. However, in analogy with Proposition 4.9, one could work with the transformations on $\mathscr{C}_d$ that are induced by the symmetries of $[0,1]^d$ (see also Appendix A.9). We start with the following preliminary example about the transformations of the copula M_d.

Example 6.3. Let $\mathbf{U} = (U_1, \dots, U_d)$ be a random vector distributed according to M_d. Consider the symmetry ξ of $[0,1]^d$ given by $\xi = \sigma_{k_1} \circ \sigma_{k_2} \circ \cdots \circ \sigma_{k_r}$ for $\{k_1, \dots, k_r\} \subseteq \{1, 2, \dots, d\}$ with $1 \leq r \leq d-1$. Then $(M_d)^\xi$ is the distribution function of the random vector $\mathbf{V} = (V_1, \dots, V_d)$ such that $V_i = 1 - U_i$ for every $i \in \{k_1, \dots, k_r\}$, while $V_i = U_i$, otherwise. The copula $(M_d)^\xi$ is given by

$$(M_d)^\xi(x_1, \dots, x_d) = \max\bigl\{\min\{x_i \mid i \in \{k_1, \dots, k_r\}\} + \min\{x_i \mid i \notin \{k_1, \dots, k_r\}\}\bigr\}$$

and corresponds to the quasi-extremal copula given in [243, Definition 4]. Such copulas are associated to random vectors whose pairs are either comonotonic or countermonotonic.

Definition 6.4. Consider the symmetry ξ of $[0,1]^d$ given by $\xi = \sigma_{k_1} \circ \sigma_{k_2} \circ \cdots \circ \sigma_{k_r}$ for $\{k_1, \dots, k_r\} \subseteq \{1, 2, \dots, d\}$ with $1 \leq r \leq d-1$. The copula obtained as a ξ-transformation of the copula M_d^{OSum} by (6.1) will be called an *$(M_d)^\xi$-ordinal sum.*

Clearly, $(M_d)^\xi$ is the background copula associated with the $(M_d)^\xi$-ordinal sum.

Example 6.5. Let us consider two special cases of (6.1): If $K = \{1\}$ and $]a_1, b_1[=]0, b[$, then

$$C(x_1, \dots, x_d) = \begin{cases} b \cdot C_1\bigl(\frac{\min\{x_1, b\}}{b}, \dots, \frac{\min\{x_d, b\}}{b}\bigr) & \text{if } \min\{x_1, \dots, x_d\} \in [0, b], \\ M_d(x_1, \dots, x_d) & \text{otherwise,} \end{cases}$$

which coincides with the upper comonotonic copula of [42, Proposition 3]. If $K = \{1\}$ and $]a_1, b_1[=]a, 1[$, then

$$C(x_1, \dots, x_d) = \begin{cases} a + (1-a) C_1\bigl(\frac{x_1 - a}{1-a}, \dots, \frac{x_d - a}{1-a}\bigr) & \text{if } \min\{x_1, \dots, x_d\} \in [a, 1], \\ M_d(x_1, \dots, x_d) & \text{otherwise,} \end{cases}$$

which can be analogously used to describe a random vector that is comonotonic in the lower tail. As shown in [210], such cases of ordinal sum constructions may serve to change the tail of a multivariate distribution and, hence, are helpful in determining the unfavorable scenarios from a risk management perspective.

Remark 6.6. M-ordinal sums of copulas can be naturally extended to quasi-copulas in the sense that, if all C_k in formula (6.1) are quasi-copulas, but not necessarily copulas, then the function M_d^{OSum} is a quasi-copula too (see [98, Definition 4] and also [17]). Moreover, if each C_k is a supermodular quasi-copula, then M_d^{OSum} remains supermodular in view of [79, Proposition 5]. Notice that we cannot directly extend Definition 6.4 to d-quasi-copulas since the ξ-transformation of a quasi-copula (also known as flipping) may not be a d-quasi-copula (see [75]).

The concept of Π-horizontal and Π-vertical ordinal sums presented in Proposition 4.4, can instead be introduced in any dimension via the gluing methods in [234]. This leads to the following result.

Proposition 6.7. *Let $i \in \{1, \dots, d\}$. Let $(]a_k, b_k[)_{k \in K}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_k)_{k \in K}$ be a family of d-copulas. Suppose $\bigcup_{k \in K} [a_k, b_k] = [0,1]$. Then the following function $\Pi_d^{(i)}$, defined, for every $\mathbf{x} \in [0,1]^d$ with $x_i \in [a_k, b_k]$, by*

$$\Pi_d^{(i)}(x_1, \dots, x_d) = a_k \cdot C_{k-1}(x_1, \dots, x_{i-1}, 1, \dots, x_d) + (b_k - a_k) \cdot C_k\bigl(x_1, \dots, \tfrac{x_i - a_k}{b_k - a_k}, \dots, x_d\bigr) \tag{6.2}$$

is a d-copula.

We denote copulas of type (6.2) as $\Pi_d^{(i)}$-ordinal sum (with respect to the i-th variable). Notice that the condition $\bigcup_{k \in K} [a_k, b_k] = [0,1]$ in Proposition 6.7 is not really restrictive compared to the bivariate case since, for $d = 2$, one can always assume that some summands fulfill $C_k = \Pi_k$.

By using the symmetries of $[0,1]^d$, various results in the spirit of Proposition 4.10 can be obtained as well.

Remark 6.8. Notice that Π-diagonal ordinal sums can be obtained in the multivariate case via patchwork techniques following [77].

In the study of high-dimensional copulas, it is often crucial for applications that some suitable algorithms do exist to generate random points in $[0,1]^d$ according to the probability law induced by the copula (see, e.g., [175]).

The M_d- and Π_d-ordinal sum constructions introduced above also admit a sampling algorithm that depends on the fact that they can be represented in terms of mixtures (i.e., convex combinations) of suitable probability distribution functions. To this end, it is convenient to extend the domain of a copula from $[0,1]^d$ to $\mathbb{R}^d$. Specifically, given a copula C, we denote by C^{ext} the probability distribution function of the random vector $\mathbf{U} = (U_1, \dots, U_d)$ such that the restriction of C^{ext} to $[0,1]^d$ is equal to C. The existence of such a $\mathbf{U}$ in a suitable probability space is guaranteed by Kolmogorov's Extension Theorem.

For the M_d-ordinal sums, it holds (see, e.g., [126, (4.31)]) that (6.1) can be rewritten as

$$C(x_1, \dots, x_d) = \sum_{k\in K} (b_k - a_k) C_k^{\text{ext}}\Big(\frac{x_1 - a_1}{b_1 - a_1}, \dots, \frac{x_d - a_d}{b_d - a_d}\Big) + \lambda\Big([0, \min\{x_1, \dots, x_d\}] \setminus \bigcup_{k\in K}]a_k, b_k[\Big), \tag{6.3}$$

where λ is the Lebesgue measure. Thus, the copula C in (6.3) can be considered as a mixture (i.e., a convex sum) of the distribution functions

$$F_k(x, y) = C_k^{\text{ext}}\Big(\frac{x_1 - a_k}{b_k - a_k}, \dots, \frac{x_d - a_k}{b_k - a_k}\Big)$$

with weight $b_k - a_k$ and the copula M_d with weight $1 - \sum_{k\in k}(b_k - a_k)$. This can be translated into the following procedure.

Algorithm 1 *(Simulation of an M_d-ordinal sum).* To generate a random number $\mathbf{v} = (v_1, \dots, v_d)$ from the copula C in (6.3), proceed as follows:

1. Generate a random number v from the uniform distribution on $[0,1]$.
2. If $v \in]a_k, b_k[$ for some $k \in K$, then
   (a) generate $(u_1, \dots, u_d)$ from the copula C_k;
   (b) set $v_i = a_i + (b_i - a_i)u_i$ for $i = 1, 2, \dots, d$;
   else set $v_1 = v_2 = \cdots = v_d = v$.
3. Return $\mathbf{v} = (v_1, v_2, \dots, v_d)$.

Notice that, with probability zero, v can be equal to a_k or b_k for some $i \in K$.

Analogously, for the function Π_d^C from Proposition 6.7 it follows from [69] that

$$\Pi_d^C(x_1, \dots, x_d) = \sum_{k\in K} (b_k - a_k) C_k^{\text{ext}}\Big(x_1, \dots, \frac{x_i - a_i}{b_i - a_i}, \dots, x_d\Big).$$

Thus we obtain the following procedure.

Algorithm 2 *(Simulation of a Π_d-ordinal sum).* To generate a random number $\mathbf{v} = (v_1, \dots, v_d)$ from the copula in (6.2), proceed as follows:

1. Generate a random number v from the uniform distribution on $[0,1]$.
2. If $v \in]a_k, b_k[$, then
   (a) generate $(u_1, \dots, u_d)$ from the copula C_k;
   (b) set $v_i = a_i + (b_i - a_i)u_i$;
   (c) set $v_k = u_i$ for every $k \neq i$.
3. Return $\mathbf{v} = (v_1, \dots, v_d)$.

7. Concluding remarks

For a considerable amount of time (in one case even through thirty years) the authors of the present article have closely cooperated with Radko Mesiar on some of the many topics to which he has made significant contributions. We first mention triangular norms, copulas, and aggregation functions at large, where he addressed many of their algebraic, analytic and probabilistic facets, then also extensions of classical measures and non-standard integrals related to them, important aspects of many-valued and, particularly, fuzzy logics, as well as dependence modeling.

Dedicating this paper to him, we here have focused on ordinal sums, especially of triangular norms and copulas, in this way recognizing some of the major achievements of Radko Mesiar and his co-authors to these fields.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Conventions, definitions and notations used in this paper

A.1. Some conventions

Many mathematical objects appear in the literature under different names, and often different symbols are used, too. For some mathematical terms and notations in this paper we have fixed the way how we use them.

Countable set: The term *countable* for the cardinality of a set will always be used in the sense of "finite or countably infinite".

Characteristic function of a set: For each set $A \subseteq \Omega$ the *characteristic function of* A is the function $\mathbf{1}_A \colon \Omega \to \mathbb{R}$ defined by $\mathbf{1}_A(x) = 1$ if $x \in A$, and $\mathbf{1}_A(x) = 0$ if $x \notin A$.

Dimension: The canonical symbol for an arbitrary dimension will be d, for example in $\mathbb{R}^d$, $[0, 1]^d$, d-boxes, d-copulas, $\mathscr{C}_d$, and so on. In the case $d = 2$, the subscript 2 is often omitted (the only exception being $\mathscr{C}_2$).

Restriction of functions: The *restriction* of a function $f \colon \Omega_1 \to \Omega_2$ to a subset $A \subseteq \Omega_1$ is the function $f \restriction_A \colon A \to \Omega_2$ which is defined by $f \restriction_A (u) = f(u)$ for all $u \in A$.

Monotonic functions: A function $f \colon A \to B$, where A and B are partially ordered sets, will be called *increasing* if the weak inequality is preserved by f (i.e., if $u \leq v$ implies $f(u) \leq f(v)$), and *strictly increasing* if the strict inequality is preserved by f (i.e., if $u < v$ implies $f(u) < f(v)$). In analogy, f will be called *decreasing* if it reverses the weak inequality, and *strictly decreasing* if it reverses the strong inequality.

Boundary of a set: If $(\Omega, \mathscr{O})$ is a topological space and $A \subseteq \Omega$ then $\partial A \subseteq \Omega$ denotes the *boundary* of the set A, i.e., $\partial A = \overline{A} \setminus \operatorname{int}(A)$, where $\overline{A}$ denotes the (topological) closure and $\operatorname{int}(A)$ the (topological) interior of the set A.

Ordinal sums — index set and summands: The canonical index set of the family of summands of an ordinal sum will be K (in some cases, e.g., when working on the real line, K has to be countable or, equivalently, a subset of $\mathbb{N}$). An ordinal sum D-$(\langle a_k, b_k, F_k \rangle)_{k \in K}$ with $D \in \{M, W, \Pi^{(\mathbf{v})}, \Pi^{(\mathbf{h})}, \Pi^{(\mathbf{d})}\}$ will be said to be an *ordinal sum of (the summands)* $(]a_k, b_k[\,, F_k)_{k \in K}$ or an *ordinal sum of (the summands)* $(\langle a_k, b_k, F_k \rangle)_{k \in K}$ or, if no confusion is possible, an *ordinal sum of (the summands)* $(F_k)_{k \in K}$ or, simply, an *ordinal sum of (the summands)* F_k.

Partial derivatives for functions in several variables: For the *partial derivative* of a function $F \colon \mathbb{R}^d \to \mathbb{R}$ with respect to the k-th component ($k \in \{1, \dots, d\}$) we will write briefly $\partial_k F(\mathbf{x})$, i.e.,

$$\partial_k F(\mathbf{x}) = \frac{\partial F(\mathbf{x})}{\partial x_k}.$$

A.2. Triangular norms

The concept of triangular norms goes back to [180], and in [224–226] the axioms of t-norms as they are used today were provided.

Definition A.1. A function $T\colon [0,1]^2 \to [0,1]$ is called a *triangular norm* (*t-norm*) if the following properties hold:

(i) T is commutative, i.e., $T(x,y) = T(y,x)$ for all $x, y \in [0,1]$,
(ii) T is associative, i.e., $T(x, T(y,z)) = T(T(x,y), z)$ for all $x, y, z \in [0,1]$,
(iii) T is increasing in the second component, i.e., for all $x, y, z \in [0,1]$ the inequality $y \leq z$ implies $T(x,y) \leq T(x,z)$,
(iv) T satisfies the boundary condition $T(x,1) = x$ for all $x \in [0,1]$.

The set of triangular norms will be denoted by $\mathscr{T}$. There are infinitely many t-norms, four of them are often called the *basic t-norms*:

- the *drastic product* $T_{\mathbf{D}}$ given by $T_{\mathbf{D}}(x,y) = \bigl(1 - \mathbf{1}_{[0,1[^2}(x,y)\bigr) \cdot \min\{x,y\}$,
- the *Łukasiewicz t-norm* $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max\{x+y-1, 0\}$,
- the *product* $T_{\mathbf{P}}$ given by $T_{\mathbf{P}}(x,y) = xy$,
- the *minimum* $T_{\mathbf{M}}$ given by $T_{\mathbf{M}}(x,y) = \min\{x,y\}$.

Observe that $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{M}}$ are also bivariate copulas: $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$ coincide with the lower and upper Fréchet-Hoeffding bounds W and M, respectively, and the product $T_{\mathbf{P}}$ is usually denoted by Π in the context of copulas.

One immediately sees that each t-norm $T \in \mathscr{T}$ satisfies the additional boundary conditions $T(1,x) = x$ and $T(x,0) = T(0,x) = 0$ for each $x \in [0,1]$, and that it is increasing in both components, i.e., for all $x_1, x_2, y_1, y_2 \in [0,1]$ we have $T(x_1,y_1) \leq T(x_2,y_2)$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$. Since the comparison of t-norms is done componentwise, i.e., $T_1 \leq T_2$ if and only if $T_1(x,y) \leq T_2(x,y)$ for all $(x,y) \in [0,1]^2$, we obtain $T_{\mathbf{D}} \leq T \leq T_{\mathbf{M}}$ for each t-norm T and, in particular, $T_{\mathbf{D}} < T_{\mathbf{L}} < T_{\mathbf{P}} < T_{\mathbf{M}}$.

A.3. Pseudo-inverse

For continuous monotonic functions (which are not bijective, in general) their so-called pseudo-inverse plays an important role, e.g., in the context of additive or multiplicative generators of (Archimedean) t-norms and copulas.

Definition A.2. Let $t\colon [0,1] \to [0,\infty]$ be continuous and strictly decreasing with $t(1) = 0$. The *pseudo-inverse* of t is the function $t^{(-1)}\colon [0,\infty] \to [0,1]$ defined by

$$t^{(-1)}(u) = \begin{cases} t^{-1}(u) & \text{if } u \in [0, t(0)]\,, \\ 0, & \text{if } u \in \,]t(0), \infty]\,. \end{cases}$$

The pseudo-inverse $t^{(-1)}$ is continuous and decreasing (and strictly decreasing on $[0, t(0)]$). Moreover, we have $t^{(-1)} \circ t(u) = u$ for all $u \in [0,1]$, while

$$t \circ t^{(-1)}(u) = \begin{cases} u & \text{if } u \in [0, t(0)]\,, \\ t(0) & \text{if } u \in \,]t(0), \infty]\,, \end{cases}$$

i.e., $t \circ t^{(-1)}(u) = \min\{u, t(0)\}$ for all $u \in [0,\infty]$. If $t(0) = \infty$ then $t^{(-1)} = t^{-1}$.

A.4. H-volume

For functions H which are closely related to (probability) distributions (such as, e.g., copulas or quasi-copulas) one has to compute the H-volume of a d-box:

Definition A.3. Let the function $H\colon \mathbb{R}^d \to \mathbb{R}$ be increasing in each argument and let $]\mathbf{a}, \mathbf{b}] =]a_1, b_1] \times \cdots \times]a_d, b_d]$ be a d-box. Then the H-volume V_H of $]\mathbf{a}, \mathbf{b}]$ is defined by

$$V_H(]\mathbf{a}, \mathbf{b}]) = \sum_{\mathbf{v} \in \operatorname{ver}(]\mathbf{a}, \mathbf{b}])} \operatorname{sign}(\mathbf{v}) H(\mathbf{v}),$$

where $\operatorname{ver}(]\mathbf{a}, \mathbf{b}]) = \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}$ denotes the set of vertices of $]\mathbf{a}, \mathbf{b}]$ and the function $\operatorname{sign}\colon \mathbb{R}^d \to \mathbb{R}$ is given by

$$\operatorname{sign}(\mathbf{v}) = \begin{cases} 1 & \text{if } v_k = a_k \text{ for an even number of indices,} \\ -1 & \text{if } v_k = a_k \text{ for an odd number of indices.} \end{cases}$$

A.5. Copulas

The concept of copulas, which link a d-dimensional distribution function with its one-dimensional margins, goes back to [237]:

Definition A.4. A function $C\colon [0, 1]^d \to [0, 1]$ is called a *d-copula* if it satisfies the following conditions:

(i) C is *grounded*, i.e., $C(u_1, u_2, \ldots, u_d) = 0$ whenever $u_i = 0$ for some $i \in \{1, \ldots, d\}$;
(ii) C has *uniform marginals*, i.e., $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $u_i \in [0, 1]$ and all $i = 1, \ldots, d$;
(iii) C is *d-increasing*, i.e., $V_C(R) \geq 0$ for every d-box $R =]\mathbf{a},\ \mathbf{b}] = \prod_{i=1}^d]a_i, b_i] \subseteq [0, 1]^d$.

The set of d-copulas will be denoted by $\mathscr{C}_d$.

When $d = 2$ we shall speak of a *bivariate copula*; in this case condition (iii) in Definition A.4 reads as follows:

$$V_C(]\mathbf{a}, \mathbf{b}]) = C(a_1, a_2) - C(a_1, b_2) - C(b_1, a_2) + C(b_1, b_2) \geq 0$$

for all a_1, a_2, b_1, b_2 in $[0, 1]$ with $a_1 \leq b_1$ and $a_2 \leq b_2$.

Remark A.5. The following are properties of a copula C:

- Every d-copula C is increasing in each place.
- Every d-copula C fulfills, for all $\mathbf{u}, \mathbf{v} \in [0, 1]^d$, the 1-*Lipschitz*-condition

$$|C(\mathbf{u}) - C(\mathbf{v})| \leq \sum_{i=1}^{d} |u_i - v_i| = \|\mathbf{u} - \mathbf{v}\|_1$$

  and is therefore uniformly continuous on $[0, 1]^d$.
- Copulas can be compared with respect to the usual pointwise order, i.e., for $C_1, C_2 \in \mathscr{C}_d$ we have $C_1 \leq C_2$ if and only if $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$ for all $\mathbf{u} \in [0, 1]^d$.
- *Fréchet-Hoeffding bounds*: For all copulas $C \in \mathscr{C}_d$ and all $\mathbf{u} \in [0, 1]^d$

$$W_d(\mathbf{u}) = \max\left\{\sum_{i=1}^{d} u_i - (d-1), 0\right\} \leq C(\mathbf{u}) \leq \min\{u_1, \ldots, u_d\} = M_d(\mathbf{u}).$$

  Note that W_d is a copula for $d = 2$ only, whereas $M_d \in \mathscr{C}_d$ holds for all $d \geq 2$.
- In the bivariate case (i.e., $d = 2$) we usually will write M rather than M_2, W rather than W_2, and also for the *independence* or *product copula* Π_d given by $\Pi_d(u_1, \ldots, u_d) = \prod_{i=1}^d u_i$ we usually write Π rather than Π_2, in particular in Sections 1–5.

Remark A.6 *(Alternative definition of a copula).* A d-dimensional copula C_d is the restriction/concentration of a multivariate distribution function to/on the unit cube $[0, 1]^d$ with uniform univariate marginals on $[0, 1]$.

A.6. Random variables and distribution functions

Let $X_1, \dots, X_d$ be random variables on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e., measurable functions from Ω into $\overline{\mathbb{R}} = [-\infty, \infty]$, and let $\mathbf{X} = (X_1, \dots, X_d)$ be a random vector. Its distribution function $F_{\mathbf{X}}$ is, for all $\mathbf{x} \in \overline{\mathbb{R}}^d$, defined by

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(X_1 \le x_1, \dots, X_d \le x_d).$$

We shall write $\mathbf{X} \sim F_{\mathbf{X}}$. The one-dimensional marginal F_i with $i \in \{1, \dots, d\}$, of $F_{\mathbf{X}}$ is the distribution function of X_i, i.e., $F_i(t) = F_{\mathbf{X}}(\infty, \dots, \infty, t, \infty, \dots, \infty)$. If $\mathbf{X}$ is a random variable (on a suitable probability space) with $\mathbf{X} \sim F$, then $V_F(]\mathbf{a}, \mathbf{b}]) = \mathbb{P}(\mathbf{X} \in]\mathbf{a}, \mathbf{b}])$. If F is continuous, $V_F(]\mathbf{a}, \mathbf{b}]) = V_F([\mathbf{a}, \mathbf{b}])$ for every $\mathbf{a} \le \mathbf{b}$ (see also [87, Remark 1.2.14]).

Theorem A.7 *(Invariance principle as provided in [87]). Let $X_1, \dots, X_d$ be continuous random variables on $(\Omega, \mathscr{F}, \mathbb{P})$.*
Let φ_i, $i = 1, \dots, d$, be continuous, increasing functions on Ran X_i. *Then the random vectors $\mathbf{X} = (X_1, \dots, X_d)$ and $\mathbf{Y} = (\varphi_1 \circ X_1, \dots, \varphi_d \circ X_d)$ have the same copula, i.e.,*

$$C_{(X_1,\dots,X_d)} = C_{(\varphi_1 \circ X_1,\dots,\varphi_n \circ X_d)}.$$

Any scale-invariant dependence between the random variables X_i is therefore captured by the copula $C_{(X_1,\dots,X_d)}$. They are independent if and only if $C_{(X_1,\dots,X_d)} = \Pi$.

A.7. Sklar's theorem

A fundamental result in the theory of copulas is Sklar's theorem [237]:

Theorem A.8 *(Sklar's theorem as provided in [87]). Let $\mathbf{X}$ be a random vector on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $\mathbf{X} \sim H$, i.e.,*

$$H(\mathbf{x}) = \mathbb{P}(\mathbf{X} \le \mathbf{x}) = \mathbb{P}(X_1 \le x_1, \dots, X_d \le x_d)$$

and let $F_1, \dots, F_d$ be its marginal distribution functions. Then there exists a copula $C_{\mathbf{X}}$ such that for all $\mathbf{x} \in \overline{\mathbb{R}}^d$

$$H(\mathbf{x}) = C_{\mathbf{X}}(F_1(x_1), \dots F_d(x_d)).$$

In case of continuous marginal distributions, the copula is unique; otherwise it is unique on $\prod_{i=1}^d \mathrm{Ran}(F_i)$.

As is not unusual for important results, Sklar's theorem was rediscovered several times (see [67,194]), sometimes extended to a more general setting (see [193,202,203]), and given different proofs ([15,16,30,37,76,78,96,100,200, 214]), but see also the survey paper [245].

A.8. Stochastic measures

The concepts of d-fold stochastic measures and d-copulas are closely related:

Definition A.9. A measure μ on $([0,1]^d, \mathscr{B}([0,1]^d))$ is called *d-fold stochastic* if, for all $A \in \mathscr{B}([0,1])$ and for every $i \in \{1, \dots, d\}$

$$\mu(\pi_i^{-1}(A)) = \lambda(A)$$

where $\pi_i \colon [0,1]^n \to [0,1]$ is the i-th canonical projection $\pi_i(\mathbf{u}) = u_i$, $i \in \{1, \dots, d\}$ and λ the Lebesgue measure.

Corollary A.10 *(One-to-one correspondence). Every d-copula C induces a d-fold stochastic measure μ_C on the measurable space $([0,1]^d, \mathscr{B}([0,1]^d))$ defined, for all $R =]\mathbf{a}, \mathbf{b}] \in [0,1]^d$, by $\mu_C(R) = V_C(]\mathbf{a}, \mathbf{b}])$.*

Conversely, to any d-fold stochastic measure μ there corresponds a unique d-copula C_μ given by $C_\mu(\mathbf{u}) = \mu(]0, \mathbf{u}])$.

Since every copula is continuous, it holds that $\mu_C(]\mathbf{a}, \mathbf{b}]) = \mu_C([\mathbf{a}, \mathbf{b}])$.

Definition A.11. The *support* $\operatorname{supp}(C)$ of a d-copula C is the support of the d-fold stochastic measure μ_C, i.e., the complement of the union of all open subsets of $([0,1]^d, \mathscr{B}([0,1]^d))$ with μ_C-measure zero.

Example A.12. Examples for the support of some copulas:

$$\operatorname{supp}(\Pi_d) = [0,1]^d, \quad \operatorname{supp}(M_d) = \{\mathbf{u} \in [0,1]^n \mid u_1 = \ldots = u_d\}, \quad \operatorname{supp}(W_2) = \{(u,v) \in [0,1]^2 \mid u = 1 - v\}.$$

A.9. Symmetries

We start by recalling that a *symmetry* of $[0,1]^d$ is a bijection $\xi\colon [0,1]^d \to [0,1]^d$ of the form

$$\xi(u_1, \ldots, u_d) = (v_1, \ldots, v_d),$$

where, for each i and for every permutation $(k_1, \ldots, k_d)$ of $(1, \ldots, d)$, either $v_i = u_{k_i}$ or $v_i = 1 - u_{k_i}$.

The set of all symmetries of $[0,1]^d$ is a group under the operation of composition. Moreover, each symmetry can be represented as composition of the following mappings:

(i) permutations $\eta\colon [0,1]^d \to [0,1]^d$ given by $\eta(x_1, \ldots, x_d) = (x_{k_1}, \ldots, x_{k_d})$ for some permutation $(k_1, \ldots, k_d)$ of $(1, \ldots, d)$,
(ii) reflections $\sigma_i\colon [0,1]^d \to [0,1]^d$ given by $\sigma_i(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_d)$ for some $i \in \{1, \ldots, d\}$.

A symmetry ξ can be used to transform a random vector $\mathbf{U}$ with $[0,1]$-uniform marginals into another random vector $\mathbf{U}' = \xi(\mathbf{U})$, while preserving the marginal distributions. As a byproduct, if $\mathbf{U}$ is distributed according to $C \in \mathscr{C}_d$, then $\xi(\mathbf{U})$ is also distributed according to a copula, denoted by C^ξ (see, for instance, [87]). In dimension 2, the usefulness of this transformation for copulas is summarized in the following result, whose proof can be done by probabilistic arguments and is contained, for instance, in [87, Corollary 2.4.4].

Theorem A.13. *For a given bivariate copula C, let (U,V) be a random vector whose distribution function is given by C, and let ξ be a symmetry in $[0,1]^2$. Then the distribution function C^ξ of $\xi(U,V)$ is again a copula. In particular, the following equalities hold for all $x, y \in [0,1]$:*

(i) $C^\eta(x,y) = C(y,x)$,
(ii) $C^{\sigma_2}(x,y) = x - C(x, 1-y)$,
(iii) $C^{\sigma_1}(x,y) = y - C(1-x, y)$,
(iv) $C^{\sigma_1 \circ \sigma_2}(x,y) = x + y - 1 + C(1-x, 1-y)$.

Note that the copula C^{σ_2} is sometimes also referred to as y-flipping of C; whereas C^{σ_1} is called the x-flipping of C (compare also [63,198]). The copula $C^{\sigma_1 \circ \sigma_2}$ is known as the *survival copula* associated with C [198].

Example A.14. For each of the basic bivariate copulas M, Π and W the permutation and basic reflections lead to:

$$\begin{aligned}
M^\eta &= M, & \Pi^\eta &= \Pi, & W^\eta &= W,\\
M^{\sigma_1} = M^{\sigma_2} &= W, & \Pi^{\sigma_1} = \Pi^{\sigma_2} &= \Pi, & W^{\sigma_1} = W^{\sigma_2} &= M,\\
M^{\sigma_1 \circ \sigma_2} = M^{\sigma_2 \circ \sigma_1} &= M, & \Pi^{\sigma_1 \circ \sigma_2} = \Pi^{\sigma_2 \circ \sigma_1} &= \Pi, & W^{\sigma_1 \circ \sigma_2} = W^{\sigma_2 \circ \sigma_1} &= W.
\end{aligned}$$

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