

# Miura-reciprocal transformations and localizable Poisson pencils

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## Abstract

We show that the equivalence classes of deformations of localizable semisimple Poisson pencils of hydrodynamic type with respect to the action of the Miura-reciprocal group contain a local representative and are in one-to-one correspondence with the equivalence classes of deformations of local semisimple Poisson pencils of hydrodynamic type with respect to the action of the Miura group.

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## 1. Introduction

In 2001, Dubrovin and Zhang initiated a classification programme of evolutionary bi-Hamiltonian integrable systems of PDEs in two independent variables of the form [DZ01]:

$$u_t^i = \sum_{k \geq 1} \epsilon^{k-1} V_k^i(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}_{(k)}, \dots), \tag{1}$$

where  $\epsilon$  is a formal parameter and  $V_k^i$  is a differential polynomial of degree  $k$ :

$$\begin{aligned} V_1^i &= A_j^i(\mathbf{u}) u_x^j, \\ V_2^i &= B_j^i(\mathbf{u}) u_{xx}^j + B_{jk}^i(\mathbf{u}) u_x^j u_x^k, \\ V_3^i &= C_j^i(\mathbf{u}) u_{xxx}^j + C_{jk}^i(\mathbf{u}) u_{xx}^j u_x^k + C_{jkl}^i(\mathbf{u}) u_x^j u_x^k u_x^l, \dots \end{aligned}$$

Many examples of integrable (systems of) PDEs like the KdV equation and the Boussinesq system belong to this class. Other equations can be included in this class after a suitable choice of the variables. For instance the NLS equation

$$i\epsilon\psi_t + \frac{1}{2}\epsilon^2\psi_{xx} \pm |\psi|^2 = 0$$

in the variables  $u$  and  $v$  defined by

$$u = |\psi|^2, \quad v = \frac{\epsilon}{2i} \left( \frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}} \right)$$

reads

$$u_t = (-uv)_x, \quad v_t = \left( -\frac{v^2}{2} \pm u \right)_x + \epsilon^2 \left( \frac{1}{4} \frac{u_{xx}}{u} - \frac{1}{8} \frac{u_x^2}{u^2} \right)_x.$$

The sum in the right hand side of (1) is not assumed to be finite. This allows to include into the class equations like the Camassa–Holm equation and the continuous limit of the Toda lattice. In the first case the equation is not of evolutionary type

$$(1 - \epsilon^2 \partial_x^2) u_t = -3uu_x + \epsilon^2 (uu_{xx} + 2u_x u_{xx}).$$

However, formally inverting the operator  $(1 - \epsilon^2 \partial_x^2)$ :

$$(1 - \epsilon^2 \partial_x^2)^{-1} = 1 + \epsilon^2 \partial_x^2 + \epsilon^4 \partial_x^4 + \dots$$

one obtains an evolutionary PDE with infinitely many terms in the right hand side. In the second case the presence of an infinite expansion in the right hand side is the result of the continuous limit procedure. Indeed starting from the Toda lattice ( $n \in \mathbb{Z}$ )

$$\dot{u}_n = v_n - v_{n-1}, \quad \dot{v}_n = e^{u_{n+1}} - e^{u_n}$$

the result of the continuous limit

$$u_n = u(\epsilon n) = u(x), \quad v_n = v(\epsilon n) = v(x), \quad t \rightarrow \epsilon t$$

is the system

$$\begin{aligned} u_t &= \frac{1}{\epsilon} (v(x) - v(x - \epsilon)) = v_x - \epsilon \frac{v_{xx}}{2} + \mathcal{O}(\epsilon^2), \\ v_t &= \frac{1}{\epsilon} (e^{u(x+\epsilon)} - e^{u(x)}) = (e^u)_x + \epsilon \frac{(e^u)_{xx}}{2} + \mathcal{O}(\epsilon^2). \end{aligned}$$

All the above examples (KdV, Boussinesq, NLS, Camassa–Holm, Toda) are bi-Hamiltonian w.r.t. a pair of compatible Hamiltonian operators of the form

$$P = \sum_{d=1}^{\infty} \epsilon^{d-1} \left( \sum_{s=0}^d P_{d,d-s}^{ij}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots) \partial_x^s \right), \quad \det(P_{1,0}^{ij}) \neq 0, \quad \deg(P_{d,k}^{ij}) = k. \quad (2)$$

Here  $\mathbf{u} = \{u^i\}_{i=1, \dots, N}$ , where  $N$  is the dimension of the system ( $N = 1$  for KdV and Camassa–Holm,  $N = 2$  for Boussinesq, NLS, and Toda) and  $\deg$  refers to the so-called standard gradation with respect of  $\partial_x$ , we define it explicitly below. Bi-Hamiltonian structures of this form are also called *deformations of bi-Hamiltonian structures of hydrodynamic type* since their leading terms (corresponding to the dispersionless limit  $\epsilon \rightarrow 0$ ) are defined by a pair of Hamiltonian operators of hydrodynamic type

$$\lim_{\epsilon \rightarrow 0} P^{ij} = P_{1,0}^{ij}(\mathbf{u}) \partial_x + P_{1,1}^{ij}(\mathbf{u}, \mathbf{u}_x), \quad \det(P_{1,0}^{ij}) \neq 0, \quad \deg(P_{1,0}^{ij}) = 0, \quad \deg(P_{1,1}^{ij}) = 1.$$

It is well known that such operators are defined by a flat contravariant metric  $g^{ij} = P_{1,0}^{ij}(\mathbf{u})$  and by the associated Christoffel symbols:  $P_{1,1}^{ij} = -g^{il} \Gamma_{lk}^j u_x^k$ , while the compatibility between a pair of operators of this form leads to the notion of *flat pencil of metrics* [Dub98].

In this framework it is natural to consider the group of Miura transformations, i.e. transformations depending on the dependent variables and, polynomially, by their derivatives of higher order through a perturbative series:

$$w^i = \sum_{k=0}^{\infty} \epsilon^k F^i_k(\mathbf{u}), \quad \det \frac{\partial F^i_0}{\partial u^j} \neq 0, \quad \deg(F^i_k) = k. \quad (3)$$

This is the most general group of transformations of the dependent variables preserving the form of the equations (1) and of the Hamiltonian operators (2). The Miura group can be described [LZ06] as the semi-direct product of the subgroup of diffeomorphisms of the dependent variables and the subgroup of Miura transformations starting from the identity. The effect of a Miura transformation (3) on vector fields  $V$  (like the r.h.s. of (1)) and bivectors  $P$  (like the Hamiltonian operator (2)) is given by

$$\tilde{V}^i(\mathbf{w}, \mathbf{w}_x, \dots, \epsilon) = (L_k^i V^k)|_{\mathbf{u}=\mathbf{u}(\mathbf{w}, \mathbf{w}_x, \dots, \epsilon)}, \quad \tilde{P}^{ij}(\mathbf{w}, \mathbf{w}_x, \dots, \epsilon) = (L_k^i P^{ks} L_s^{*j})|_{\mathbf{u}=\mathbf{u}(\mathbf{w}, \mathbf{w}_x, \dots, \epsilon)}, \tag{4}$$

where  $L_k^i = \sum_s \frac{\partial w^i}{\partial u^{ks}} \partial_x^s$  and  $L^*$  is its adjoint. Dubrovin-Zhang classification problem consists in classifying equivalence classes of semisimple bi-Hamiltonian structure of the form (2) with respect to the action of the Miura group. It turned out (we refer to section 1.4 for more details) that fixed the dispersionless limit (which is described by a flat pencil of metrics) the equivalence classes are not empty and are labeled by  $N$  functions of a single variable called *central invariants* due to their invariance with respect to the action of the Miura transformations starting from the identity [DLZ06] (see also [DLZ08, CKS18]). Taking into account that, in the semisimple case, semisimple flat pencils of hydrodynamic type are parametrized by  $N^2 - N$  functions of one variable (see [Fer01, Mok02]) we have in total  $N^2$  functions of one variable labeling equivalence classes.

Among the questions that the above approach raises there is the issue of extending the group action to include (possibly non-local) changes of variables in one of the independent variables. Indeed, an important class of such transformations is that of reciprocal transformations, which play an important role in Mathematical Physics (see e.g. [Rog69, Rog68, Fer89, Fer91, FP03, XZ06, AG07, Abe09, BS09, LZ11, AL13] and the recent paper [Rog22] with historical notes in the Introduction).

Reciprocal transformations for the system (1) are nonlocal (or non-holonomic) transformations of the independent variables  $(t, x)$  of the form

$$\begin{aligned} dy &= A(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots) dt + B(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots) dx, \\ ds &= M(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots) dt + N(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots) dx, \end{aligned} \tag{5}$$

where the right-hand sides of the transformation are two conservation laws of the system at hand. Any reciprocal transformation can be written as a composition of two kinds of reciprocal transformations [FP03]:

- Exchange of the time and space variables. This transformation preserve the locality of Hamiltonian structures and have been extensively studied in the recent preprint [LWZ23]. According to the results of this paper the central invariants of the original and of the transformed structures are the same while the metrics defining their dispersionless limits are Combescure equivalent (i.e. they have the same rotation coefficients).
- Nonlinear reciprocal transformations preserving time:  $s = t$ . These transformations include the one relating the Camassa–Holm equation and the first negative flow of the KdV hierarchy [Fuc96], the one relating the Degasperis–Procesi equation and a negative flow in the hierarchy of the Kaup–Kupershmidt equation [HW08], the one relating Novikov equation and Sawada Kotera equation [DHH02] and the one relating the 2-component Camassa–Holm equation and the first negative flow of the AKNS hierarchy [CLZ06, AGZ06a, AGZ06b].

Since nonlinear reciprocal transformations do not preserve the locality of Hamiltonian operators in order to include such transformations in the classification scheme one has to rephrase the original Dubrovin–Zhang problem replacing Miura group with the group obtained combining Miura transformations and nonlinear reciprocal transformations (we call this group *the Miura-reciprocal group*) and local Hamiltonian operators (2) with nonlocal Hamiltonian operators of *localizable shape*

$$P = \sum_{d=1}^{\infty} \epsilon^{d-1} \left( \sum_{s=0}^d P_{d,d-s}^{ij} \partial_s + u_x^i \partial_x^{-1} V_d^j + V_d^i \partial_x^{-1} u_x^j \right), \det(P_{1,0}^{ij}) \neq 0, \deg(V_k^i) = \deg(P_{d,k}^{ij}) = k. \quad (6)$$

This paper is concerned with this problem under the assumption that the nonlocal Hamiltonian operators of hydrodynamic type defining the dispersionless limit are simultaneously localizable. This is a natural extension for the classification program in [DZ01]. We point out that also the nonlocal representatives of a class might be related to an integrable hierarchy admitting a pair of compatible Hamiltonian structures of the form (2). For instance, the reciprocal transformations relating KdV and Camassa–Holm modulation equations brings the local pair of the first ones to a nonlocal pair of the second ones and, conversely, the local pair of the second ones into a nonlocal pair of the first ones (see [AG05]).

The main result of the paper is that *equivalence classes contain a local representative and are in one-to-one correspondence with the equivalence classes of deformations of local semisimple bi-Hamiltonian structures of hydrodynamic type with respect to the action of the Miura group*.

The role of central invariants in this more general setting is played by a set of functions of a single variable that are invariant with respect to Miura-reciprocal transformations and coincide with the central invariants of the local representative of the class. To the best of our knowledge it is the first result in the literature that provides a systematic classification of orbits of the action of the group of Miura-reciprocal transformations in the bi-Hamiltonian context (for a single Poisson structure this type of result is established in [FP03, LZ11]).

The action of the Miura-reciprocal transformations on the geometric objects that we will consider, most notably Hamiltonian operators, is recomputed in section 2 using an original non-local generalization of the formulae in [Ol88], that were obtained for the special case of time-preserving differential substitutions. The interesting aspect of this formula is that it can be used to change coordinates to arbitrary weakly-nonlocal Hamiltonian operators, not just localizable ones. We will recover, among other results, the change of coordinates for non-local Hamiltonian operators of Ferapontov type (proposition 2.8) that was obtained by an indirect approach in [FP03].

Moreover, we prove in this paper (theorem 3.1) that Jacobi structures and their Schouten brackets from [LZ11] are a particular case of a more general theory of weakly nonlocal Poisson structures and their Schouten bracket, developed in [LV20]. This gives an alternative explanation and a natural introduction to the constructions of [LZ11]. Since in the approach of [LV20] the formula for the Schouten bracket is the ordinary one that is generalized in order to act on nonlocal expressions, there is no *a priori* need to use the formalism of [LZ11] with extension of Poisson structures to Jacobi structures. However, the set-up of [LZ11] still remains the most convenient way to perform computations with weakly non-local Poisson brackets of localizable shape, and we use it in the rest of the paper.

A more detailed guide to the paper and comparison between previous results and our results can be read in the following subsections.

1.1. A variety of jet space transformation groups

We consider a jet space  $J^r(1, N)$  (see [Olv93, Boc+99] for an introduction to jet spaces and Geometry of PDEs),  $r \geq 0$ , with independent variable  $x$  and dependent variables  $u^i$ ,  $i = 1, \dots, N$ , considered as coordinates on some open domain  $U \subset \mathbb{R}^N$ . Let  $u^{i,\sigma}$  denote the  $x$ -derivative of  $u^i$  taken  $\sigma$  times.

Consider the transformations (i.e. diffeomorphisms) of the jet space  $J^r(1, N)$ . We begin from the most general type of transformation: a reciprocal transformation coupled with a differential substitution. Reciprocal transformations in a modern setting were introduced in [Rog69, Rog68] in the study of gas dynamics, and later analyzed under a geometric viewpoint in [Fer89, Fer91] and many other authors (see e. g. [FP03, XZ06, AG07, Abe09, BS09, LZ11, AL13] and references therein). The class of differential substitutions was introduced in [Ibr85], although many particular differential substitutions were already present in the literature (in particular, the Miura transformations).

**Definition 1.1.** A reciprocal differential substitution is a nonlocal transformation of the independent variable  $x$  into the independent variable  $y$  of the type

$$dy = Bdx, \quad B = B(x, u^i, u^{i,\sigma}) \tag{7}$$

coupled with a differential substitution of the dependent variables of the form

$$w^i = Q^i(x, u^i, u^{i,\sigma}). \tag{8}$$

By the fact that  $dx(\partial_x) = 1 = dy(\partial_y)$  we obtain that total derivatives are related by the formula  $\partial_x = B\partial_y$ . Of course, the formula  $w^i = (1/B)\partial_x Q^i$  yields the formula for first-order derivatives, and similarly for higher-order derivatives. Note that, in general, the inversion of a differential substitution is a nonlocal operation. We will soon focus on a more restrictive class of transformations.

Reciprocal differential substitutions admit several interesting subclasses:

- A reciprocal transformation is a nonlocal transformation of the independent variables  $x$  into the independent variable  $y$  of the type

$$dy = Bdx, \quad B = B(x, u^i, u^{i,\sigma}) \tag{9}$$

coupled with the identical transformation of the dependent variables. In practical applications the functions  $u^i$  depend also on an additional parameter that plays the role of ‘time’ of the system of evolutionary PDEs

$$u_t^i = F^i(x, u^i, u^{i,\sigma}), \quad i = 1, \dots, n$$

governing their evolution. Taking into account this additional variable reciprocal transformations are often defined as

$$dy = Adt + Bdx,$$

where the function  $A, B$  are submitted to the closure condition  $B_t = A_x$ , that is,  $dy$  is a conservation law for the equation. Note that the coefficient  $A$  does not enter the transformation law for  $\partial_x$ , and thus can be disregarded throughout this paper.

- A reciprocal differential substitution is said to be *holonomic* if there exists a differentiable function  $P$  such that  $B = \partial_x P$ .

- A general differential substitution of  $(x, u^i)$  into  $(y, w^j)$ :

$$y = P(x, u^j, u^{j,\sigma}), \quad w^j = Q^j(x, u^j, u^{j,\sigma}), \tag{10}$$

yields a holonomic reciprocal differential substitution  $dy = \partial_x P dx$ ,  $w^j = Q^j$  by differentiation (in this sense the two classes of transformations coincide).

The above two categories of transformations, basically local and nonlocal differential substitutions, are, on the one hand, too wide to be used in the context of the classification programs for evolutionary PDEs and related geometric structures as, for instance, the one initiated by Dubrovin and Zhang in [DZ01], and on the other hand too restrictive since we are limited by fixing the parameter  $r \geq 0$  that controls the maximal order of jets.

For this reason, we introduce the space of differential polynomials  $\mathcal{A}$ , and the following group of transformations, which is a subclass of the reciprocal differential substitutions. Consider a jet space  $J^\infty(1, N)$  (considered as a projective limit of the jet spaces  $J^r(1, N)$ ,  $r \rightarrow \infty$ ) with independent variable  $x$  and dependent variables  $u^i$ ,  $i = 1, \dots, N$ . Denote  $u^{i,\sigma}$  the  $x$ -derivative of  $u^i$  taken  $\sigma$  times. We associate with this space the algebra of functions  $\mathcal{A} := C^\infty(U)[[u^{i,\sigma}, i = 1, \dots, N, \sigma \geq 1]]$ , where  $C^\infty(U)$  is the space of smooth functions on a domain  $U \subset \mathbb{R}^N$  in the coordinates  $u^1, \dots, u^N$ . There is a natural gradation on the algebra of densities  $\mathcal{A}$  given by  $\deg_{\partial_x} u^{i,\sigma} = \sigma$ . Let  $\mathcal{A}_d$  denote the  $\deg_{\partial_x}$ -degree  $d$  part of  $\mathcal{A}$ , which is a finite dimensional module over  $C^\infty(U)$ .

**Definition 1.2.** A Miura-type reciprocal differential substitution, or Miura-reciprocal transformation for short, is a transformation of the type

$$\begin{aligned} dy &= \left( \sum_{k=0}^{\infty} \epsilon^k H_k(u^j, u^{j,1}, \dots, u^{j,k}) \right) dx, \\ w^i &= \sum_{k=0}^{\infty} \epsilon^k K_k^i(u^j, u^{j,1}, \dots, u^{j,k}), \quad i = 1, \dots, N, \end{aligned} \tag{11}$$

with  $H_k, K_k^i \in \mathcal{A}_k$  and

$$H_0 \neq 0, \quad \det \left( \frac{\partial K_0^i(u^j)}{\partial u^k} \right) \neq 0. \tag{12}$$

The formal dispersive parameter  $\epsilon$  that we introduce here to control the  $\deg_{\partial_x}$ -degree is, in principle, not strictly necessary but it is very convenient in particular computations and applications.

The set of all Miura-reciprocal transformations is denoted by  $\mathcal{R}$ . It is a group with respect to the composition, and it has some distinguished subgroups:

- the subgroup  $\mathcal{R}_{DS}$  of Miura differential substitutions, that are Miura-type reciprocal differential substitutions which are also holonomic differential substitutions of the following type:

There exists  $P = x + P_0$ , with  $P_0 = \sum_{k=0}^{\infty} \epsilon^k F_k$  and  $F_k \in \mathcal{A}_k$ , such that

$$\partial_x P = \sum_{k=0}^{\infty} \epsilon^k H_k(u^j, u_x^j, \dots, u_\sigma^j); \tag{13}$$

- the subgroup of Miura transformations characterized by  $H_0 = 1$  and  $H_k = 0$  for all  $k \geq 1$ . This subgroup is called the *Miura group*  $\mathcal{G} \subset \mathcal{R}$  [DZ01] and bears his name from the transformation relating KdV and modified KdV equations introduced by Miura [Miu68]. Note that the Miura group is also a subgroup of the group of Miura differential substitutions:  $\mathcal{G} \subset \mathcal{R}_{DS}$ .

**Definition 1.3.** By analogy with the way the standard Miura group is typically presented, we introduce the following two subgroups.

- We define Miura-reciprocal transformations of the *1st kind* to be the Miura-reciprocal transformations of the form

$$\begin{aligned} dy &= H_0(u^j) dx, \\ w^i &= K_0^i(u^j), \end{aligned} \quad i = 1, \dots, N. \tag{14}$$

The group of all Miura RDS of the first type is denoted by  $\mathcal{R}_I$ . This group contains as a subgroup the group of Miura transformations of the 1st kind,  $\mathcal{G}_I \subset \mathcal{R}_I$ , characterized by  $H_0 = 1$ .

- We define Miura-reciprocal transformations of the *2nd kind* to be the Miura-reciprocal transformations of the form

$$\begin{aligned} dy &= \left( 1 + \sum_{k=1}^{\infty} \epsilon^k H_k(u^j, u^{j,1}, \dots, u^{j,\sigma}) \right) dx, \\ w^i &= u^i + \sum_{k=1}^{\infty} \epsilon^k K_k^i(u^j, u^{j,x}, \dots, u^{j,\sigma}), \end{aligned} \quad i = 1, \dots, N. \tag{15}$$

The group of all Miura-reciprocal transformations of the second type is denoted by  $\mathcal{R}_{II}$ . It contains as a subgroup the group of Miura transformations of the 2nd kind,  $\mathcal{G}_{II} \subset \mathcal{R}_{II}$ , characterized by  $H_k = 0$  for all  $k \geq 1$ .

**Definition 1.4.** A distinguished subgroup of  $\mathcal{R}_I$  is the group of *projective reciprocal transformations*  $\mathcal{P}$ . The group was introduced in [FPV14] as the invariance group of a class of a Hamiltonian operators. A recent instance of such a transformation in the literature can be found here [Rog22].

Projective reciprocal transformations are characterized by the requirements that  $K^i$  in equation (14) is a projective transformation (in an affine chart) and  $H_0$  is the common denominator of the projective transformation. More explicitly,

$$\begin{aligned} dy &= (a_j^0 u^j + a_0^0) dx, \\ w^i &= \frac{a_j^i u^j + a_0^i}{a_j^0 u^j + a_0^0}, \end{aligned} \quad i = 1, \dots, N. \tag{16}$$

The goal of this paper is to discuss some aspects of the actions of these groups on the natural suitable geometric structures that emerge in the study of integrable systems of evolutionary PDEs. In order to describe our results we have to recall some of these structures, which we do in the rest of the introduction.



### 1.2. Action of the transformation groups

The above group of Miura reciprocal differential substitutions act on spaces of geometric entities that play important roles in the geometric theory of integrability. In particular, it acts on:

- densities, that have the form

$$F = \int f(u^j, u^{j,\sigma}) dx \in \mathcal{F} := \mathcal{A}/\partial_x \mathcal{A}, \quad \text{with } f \in \mathcal{A}; \quad (17)$$

- variational vector fields, that include symmetries of partial differential equations, and have the form

$$\varphi = \varphi^i(u^j, u^{j,\sigma}) \delta_{u^i}, \quad \varphi^i \in \mathcal{A}; \quad (18)$$

- covector-valued densities, that include characteristic vectors of conserved quantities of differential equations, and have the form

$$\psi = \psi_i(u^j, u^{j,\sigma}) du^i \otimes dx, \quad \psi_i \in \mathcal{A}; \quad (19)$$

- the Euler–Lagrange operator, which sends densities into covector-valued densities,

$$\mathcal{E}(F) = \delta_{u^i} F du^i \otimes dx; \quad (20)$$

- variational multivectors of degree  $p$ , that include Hamiltonian operators of partial differential equations as particular bivectors. They can be regarded as maps from  $(p - 1)$ -covector-valued densities to variational vector fields.

In section 2 we will prove our change of coordinate formulae for reciprocal differential substitutions for these geometric objects. As an example, we re-derive in section 2.1 the Ferapontov–Pavlov formula for the reciprocal transformation of a Poisson bi-vector of the differential order 1 [FP03], and this brings us to the realm of weakly non-local Poisson structures of localizable shape.

### 1.3. Weakly non-local Poisson bi-vectors of localizable shape

Let dependent variables  $u^i$  also dependent on one external parameter, denoted by  $t$ . The most studied structures in geometric theory of integrability are the local Poisson structures needed for representation of equations of the form

$$u_t^i = f^i(u^j, u^{j,\sigma}) \quad (21)$$

in Hamiltonian form (note that we do not allow possible explicit dependence of  $f^i$ 's on  $x$ ), that is, in the form

$$u_t^i = \sum_{s=0}^d P_s^{ij} \partial_s \frac{\delta}{\delta u^j} \int h(u^k, u^{k,\sigma}) dx, \quad (22)$$

where  $H = \int h(u^k, u^{k,\sigma}) dx$  is the Hamiltonian functional and  $P = \sum_{s=0}^d P_s^{ij} \partial_s$ ,  $P_s^{ij} \in \mathcal{A}$  defines a bi-vector which in the language of densities can be written as

$$\{u^i(x), u^j(y)\}_P = \sum_{s=0}^d P_s^{ij} \partial_x^s \delta(x - y) \quad (23)$$

(in this paper bi-vectors and, more generally, multivectors are assumed to be skew-symmetric by default).

In addition to the language of densities, there is a very convenient formalism, the so-called  $\theta$ -formalism, to encode the variational multivectors [Get02], see also [IVV02]. Namely, extend the space  $\mathcal{A}$  to a space  $\hat{\mathcal{A}} := \mathcal{A}[[\theta_i^\sigma, i = 1, \dots, N, \sigma \geq 0]]$ , where  $\theta_i^\sigma$  are formal odd variables. We often denote  $\theta_i^0$  by  $\theta_i$ , and we extend the  $\partial_x$  operator to  $\hat{\mathcal{A}}$  as  $\partial_x := \sum_{s=0}^\infty u^{i,s+1} \partial_{u^{i,s}} + \theta_i^{s+1} \partial_{\theta_i^s}$ . The  $\deg_{\partial_x}$ -gradation is extended to  $\hat{\mathcal{A}}$  by  $\deg_{\partial_x} \theta_i^\sigma = \sigma$ , and there is a natural  $\theta$ -degree given by  $\deg_\theta u^{i,\sigma} = 0$  and  $\deg_\theta \theta_i^\sigma = 1$ . Let  $\hat{\mathcal{A}}^p$  denote the subspace  $\hat{\mathcal{A}}$  of  $\theta$ -degree  $p$ . We can consider it as a space of densities of variational  $p$ -vectors. Let  $\hat{\mathcal{A}}_d^p := \hat{\mathcal{A}}^p \cap \hat{\mathcal{A}}^d$ .

The space  $\hat{\mathcal{F}} := \hat{\mathcal{A}}/\partial_x \hat{\mathcal{A}}$  can be considered as the space of variational multivectors. It inherits under the projection  $\int: \hat{\mathcal{A}} \rightarrow \hat{\mathcal{F}}$  both gradations,  $\deg_{\partial_x}$  and  $\deg_\theta$ , and  $\mathcal{F}_d^p$  denotes its subspace of  $p$ -vectors of differential degree  $\deg_{\partial_x} = d$ . The Schouten bracket is defined as

$$\left[ \int P, \int Q \right] = \int (-1)^{\deg_\theta P} \delta_{u^i} P \delta_{\theta_i} Q + \delta_{\theta_i} P \delta_{u^i} Q \tag{24}$$

for  $P, Q \in \hat{\mathcal{A}}$ , where  $\delta_{u^i} := \sum_{s=0}^\infty (-\partial_x)^s \partial_{u^{i,s}}$  and  $\delta_{\theta_i} := \sum_{s=0}^\infty (-\partial_x)^s \partial_{\theta_i^s}$ . Various cohomological computations in terms of this space and related formalism allow to efficiently control the deformation theory of Poisson bi-vectors and their pencils, see e. g. [LZ05, LZ11, DLZ06, LZ13, CPS18, CKS18, CPS16a, CPS16b, CCS17, CCS18].

However, studying the action of the group of Miura-reciprocal transformations we can not restrict ourselves to the local Poisson bi-vectors. As we have seen, the reciprocal transformations generate non-locality of some very particular shape, and in terms of the operator  $P$  we have to extend its possible shape to

$$P = \sum_{s=0}^d P_s^{ij} \partial_s + u^{i,1} \partial_x^{-1} V^j + V^i \partial_x^{-1} u^{j,1}, \quad P_s^{ij}, V^i \in \mathcal{A}. \tag{25}$$

Hamiltonian operators of the form above with  $d = 1$ ,  $P_1^{ij} = g^{ij}(u)$  ( $\det g^{ij} \neq 0$ ),  $P_0^{ij} = -g^{il} \Gamma_{lk}^j u_x^k$  and  $V^i = V_j^i(u) u_x^j$  were studied by Ferapontov in [Fer95a]. They belong to the larger class of *weakly non-local operators*, that was introduced in [MN01]. Like in the local case the coefficients  $g^{ij}$  define a metric and the coefficients  $\Gamma_{lk}^j$  the Christoffel symbols of the associated Levi-Civita connection but unlike in the local case the metric is no longer flat. It turns out that the Riemann tensor  $R$  and the tensor field  $V$  defining the non-local tail satisfy the conditions

$$g_{is} V_j^s = g_{js} V_i^s, \quad \nabla_j V_i^k = \nabla_i V_j^k, \quad R_{kl}^{ij} = V_k^i \delta_l^j + V_l^j \delta_k^i - V_k^j \delta_l^i - V_l^i \delta_k^j.$$

These are a particular instance of Ferapontov's conditions for weakly non-local Hamiltonian operators of hydrodynamic type [Fer95b]. An algorithm to compute such conditions for general weakly non-local Hamiltonian operators has been developed in [CLV20] and implemented in three different computer algebra systems in [Cas+22].

A natural question here is how to extend the  $\theta$ -formalism briefly recalled above to accommodate this type of non-locality. There are two recipes in the literature given in [LZ11] (specific for this case) and [LV20] (suitable for general weakly non-local operators). The identification of the two approaches should indirectly follow from the uniqueness arguments in [LZ11], but we wanted to establish an explicit identification. We do it by an explicit computation in section 3.

**Remark 1.5.** It is important to comment on the action of the operator  $\partial_x^{-1}$ . It can be defined on  $\partial_x \mathcal{A}$  by  $\partial_x^{-1}(\partial_x(f)) = f + C$  for any  $f \in \mathcal{A}$ , here  $C$  is some constant. For a more general element  $g \in \mathcal{A}$ ,  $g \notin \partial_x \mathcal{A}$ , we can represent  $\partial_x^{-1}(g)$ , for instance, as an element of a localization of  $\mathcal{A}$  given by  $\mathcal{A}(\frac{1}{u^{i,1}})$ , that is, we can perturbatively represent it as a series  $C + \sum_{i=1}^{\infty} \frac{h_i}{(u^{i,1})^i}$  with  $h_i \in \mathcal{A}$  such that  $\partial_{u^{i,1}} h_i = 0$  (this idea is coming from [DLZ06]), here  $C$  is also a constant.

Both approaches that we compare assert that for the analysis of the weakly non-local Poisson bi-vectors of localizable shape it is sufficient to formally apply  $\partial_x^{-1}$  to just one element  $-u^{i,1} \theta_i \in \hat{\mathcal{A}}$  and denote the result by  $\zeta$ , which has different meaning in these two approaches. The subsequent usage of  $\zeta$  in computations implies that the extra constant that might occur by inverting  $\partial_x$  is uniformly set to  $C = 0$ .

### 1.4. Localizability

Consider a dispersive weakly non-local Poisson structure of localizable shape given by

$$P = \sum_{d=1}^{\infty} \epsilon^{d-1} \left( \sum_{s=0}^d P_{d,d-s}^{ij} \partial_s + u^{i,1} \partial_x^{-1} V_d^j + V_d^i \partial_x^{-1} u^{j,1} \right), \quad P_{d,k}^{ij}, V_k^i \in \mathcal{A}_k. \tag{26}$$

The leading term ( $d = 1$ ) of this structure is a Poisson structure of hydrodynamic type and thus the full Poisson structure can be thought as a deformation of a Poisson structure of hydrodynamic type. If  $\det P_{1,0}^{ij} \neq 0$ , Liu and Zhang prove in [LZ11] that there is always an element in  $\mathcal{R}$  that turns  $P$  into a constant local Poisson structure  $\eta^{ij} \partial_x$ . In the case of a purely local structure the same results is established under the action of group  $\mathcal{G}$  in [Get02] (see also [DMS05, DZ01]), and in the case  $\epsilon = 0$  (that is, a purely degree 1 case) it is established under the action of the group  $\mathcal{R}_I$  in [LZ11] for  $N = 1, 2$  and in [FP03] for  $N \geq 3$ .

Now consider a pencil  $P - \lambda Q$  of dispersive weakly non-local Poisson structure of localizable shape. Let us fix the leading term  $(P - \lambda Q)|_{\epsilon=0}$  of the pencil and assume it is semi-simple. In the purely local case (that is, under the additional assumption that both  $P$  and  $Q$  are purely local), it was suggested in [LZ05, DLZ06] (see also [Lor02] for the scalar case) and proved in [LZ13] ( $N = 1$  case) and [CPS18, CKS18] (any  $N \geq 1$ ) that the space of orbits of the action of  $\mathcal{G}_H$  on such pencils is naturally parametrized by  $N$  smooth functions of one variable, called the *central invariants*.

In section 4 we generalize these results in the following way. Let us fix the leading term  $(P - \lambda Q)|_{\epsilon=0}$  of the pencil and assume that  $P|_{\epsilon=0}$  and  $Q|_{\epsilon=0}$  are simultaneously localizable under the action of the group  $\mathcal{R}_I$ . We also still assume that  $(P - \lambda Q)|_{\epsilon=0}$  is semi-simple. In this case, we prove that the set of orbits of the action of  $\mathcal{R}_H$  on such pencils is also naturally parametrized by  $N$  smooth functions of one variable. Note that while the statement is literally the same as in the purely local case, it is quite different as both the group and the space of structures on which the group acts is much bigger. We show that it is still possible to read the central invariants from the symbol of the pencil.

This result is proved by a direct application of various techniques and results proposed in [LZ11, LZ13, CPS18, CKS18]. From the comparison with the computations in the local case, we obtain the following extra result: under the assumptions above, each orbit of  $\mathcal{R}_H$  contains a purely local representative. In other words, we prove that if the leading term of a semi-simple pencil  $P - \lambda Q$  of dispersive weakly non-local Poisson structure of localizable shape is localizable by the action of the group  $\mathcal{R}_I$ , then the whole pencil is localizable by the action of the group  $\mathcal{R}$ .

It is worth to mention that this result also generalizes and put in the right context a theorem of Liu and Zhang [LZ11, theorem 1.3] that states that if two local Poisson pencils with the

leading semi-simple hydrodynamic term are related by a reciprocal transformation, then their central invariants are the same.

1.5. Projective group and Doyle–Potëmin form

Finally, we consider the action of the group  $\mathcal{P} \subset \mathcal{R}_I$ . It is a quite small group with transparent structure, and we expect that in general the orbits of its action should have a rich geometric structure. In this paper we find a surprising connection of this group to a conjecture of Mokhov on the possible form of the local Poisson structures of differential degree  $\text{deg}_{\partial_x} \geq 2$ .

It was independently proved by Doyle [Doy93] and Potëmin [Pot91, Pot97] that homogeneous local Poisson structures of differential degree  $d = 2, 3$ , i.e. of the form

$$\sum_{s=0}^d P_{d-s}^{ij} \partial_x^s, \quad P_k^{ij} \in \mathcal{A}_k, \tag{27}$$

can always be transformed by the action of the group  $\mathcal{G}_I$  to an operator of the shape

$$\partial_x \circ \sum_{s=0}^{d-2} Q_{d-2-s}^{ij} \partial_x^s \circ \partial_x, \quad Q_k^{ij} \in \mathcal{A}_k. \tag{28}$$

Mokhov made the following interesting conjecture:

**Conjecture 1.6 (See [Mok98, proposition 2.3 and text afterwards]).** *Let  $P = \sum_{e=1}^{d+2} P_e^{ij} \partial_x^{d+2-e}$  be a local operator of homogeneous differential order  $d+2$  (that is,  $\text{deg}_{\partial_x} P_e^{ij} = e$ ),  $d \geq 0$ . Assume that  $P$  defines a Poisson bracket. Then there exists a local skew-symmetric operator  $Q^{ij}$  of homogenous differential order  $d$  such that  $P = \partial_x \circ Q^{ij} \circ \partial_x$ .*

The form (28) is called the Doyle–Potëmin form of a local homogeneous bi-vector of differential degree  $\text{deg}_{\partial_x} \geq 2$ .

It was recently proved that in the cases of homogeneous local Poisson structures of degree  $d=2$  [VV] and  $d=3$  [FPV14] the form (28) is preserved by the action of the group  $\mathcal{P}$ . In section 5, thanks to our change of coordinates formulae from section 1.2, we generalize the above results to local homogeneous bi-vectors (i.e. not necessarily Poisson structures) of degree  $d \geq 2$  and prove that the group  $\mathcal{P}$  preserves the set of local bi-vectors of Doyle–Potëmin form. A nice example of application to a Hamiltonian operator for the Dubrovin–Zhang hierarchy is pointed out.

2. Formulae for the action

The goal of this section is to compute from the scratch the effect of general reciprocal differential substitutions on variational (multi)vector fields and related geometric objects. It is clear that a reciprocal differential substitution given by  $dy = Bdx$  (or  $y = P$  in the holonomic case),  $w^i = Q^i$ , also yields a coordinate change of the  $y$ -derivative variables:

$$w^{i,\tau} = \partial_y^\tau Q^i = \left( \frac{1}{B} \partial_x \right)^\tau Q^i. \tag{29}$$

It is convenient to introduce the Fréchet derivative<sup>6</sup> of a differentiable function  $F \in \mathcal{A}$ , as

$$\ell_F(X) = (\ell_F)_i(X^i) = \sum_{\sigma=0}^{\infty} \frac{\partial F}{\partial u^{i,\sigma}} \partial_x^\sigma X^i, \tag{30}$$

where  $X = X^i \delta_{u^i}$  is a variational vector field. The formal adjoint of the above operator is

$$(\ell_F^*)_i = \sum_{\sigma=0}^{\infty} (-\partial_x)^\sigma \circ \frac{\partial F}{\partial u^{i,\sigma}} \tag{31}$$

acting on covector-valued densities.

A change of coordinates formula for Hamiltonian operators under the action of differential substitutions was already given in [Mok87, Olv88]. We rephrase the arguments of the proof in [Olv88] and obtain change of coordinates formulae for the geometric objects that we listed in section 1.2 which turn out to be valid in the more general case of reciprocal differential substitutions. We observe that also in [LZ11] there are formulae for coordinate change, but their validity is limited to the action of Miura reciprocal transformations on operators of localizable shape, while we do not have this limitation.

First of all, we provide a formula for the coordinate change of an variational vector field under a differential substitution.

**Proposition 2.1.** *Let  $X^i \delta_{u^i} = Y^i \delta_{w^i}$  be a variational vector field in the coordinate systems  $(x, u^{i,\sigma})$  and  $(y, w^{i,\sigma})$ , respectively, where the latter coordinates systems are related by a holonomic reciprocal differential substitution  $y = P$ ,  $w^i = Q^i$ . Then the following change of coordinate formula holds:*

$$Y^j = \frac{1}{\partial_x P} \mathcal{D}_i^j(X^i) \tag{32}$$

where

$$\mathcal{D}_i^j = \partial_x P (\ell_{Q^j})_i - \partial_x Q^j (\ell_P)_i. \tag{33}$$

**Proof.** The proof uses arguments that provide a change of coordinates formula for Euler–Lagrange operators in [Olv93], theorem 4.8 and exercise 5.49. Let

$$u^i = f^i(x), \quad x \in \Omega, \quad w^i = g^i(y), \quad y \in \tilde{\Omega} \tag{34}$$

be functions that are put in correspondence by a transformation. We can consider a one-parameter family of such functions defined by the variation field  $X^i \delta_{u^i}$ :

$$u_\epsilon^i = f^i(x, \epsilon) = f^i(x) + \epsilon X^i(x), \tag{35}$$

where  $X^i \delta_{u^i}$  has compact support in  $\Omega$ . Its transformed version

$$w_\epsilon^i = g^i(y, \epsilon) = g^i(y) + \epsilon Y^i(y) + \mathcal{O}(\epsilon^2). \tag{36}$$

<sup>6</sup> It should be the Gateaux derivative, but Fréchet is prevailing in the literature.

is determined by the formulae

$$y = P(x, \partial_x^\sigma (f^j(x) + \epsilon X^j(x))), \quad w_\epsilon^i = Q^i(x, \partial_x^\sigma (f^j(x) + \epsilon X^j(x))). \quad (37)$$

Since  $\eta$  has compact support on  $\Omega$ , each  $g^i(y, \epsilon)$  is defined on a common compact domain  $\tilde{\Omega} = \{x \in \Omega \mid y = P(x, \partial_x^\sigma f^j(x))\}$ . The transformed variation field is given by  $Y^i(y) = \partial_\epsilon g^i(y, \epsilon)|_{\epsilon=0}$ . As variation fields do not depend on  $\epsilon$  we have

$$\partial_\epsilon y = 0 = \partial_x P \partial_\epsilon x + \sum_{\sigma=0}^{\infty} \partial_{w^{i,\sigma}} P \partial_x^\sigma X^j, \quad (38)$$

hence

$$\partial_\epsilon x|_{\epsilon=0} = -\frac{1}{\partial_x P} \sum_{\sigma=0}^{\infty} \partial_{w^{i,\sigma}} P \partial_x^\sigma X^j. \quad (39)$$

We have:

$$\begin{aligned} Y^j &= \partial_\epsilon g^j(y, \epsilon)|_{\epsilon=0} = \sum_{\sigma=0}^{\infty} \partial_{w^{i,\sigma}} Q^j \partial_x^\sigma \partial_\epsilon f^i(x, \epsilon)|_{\epsilon=0} + \partial_x Q^j \partial_\epsilon x|_{\epsilon=0} \\ &= \frac{1}{\partial_x P} (\partial_x P (\ell_{Q^j})_i - \partial_x Q^j (\ell_P)_i) X^i. \end{aligned} \quad (40)$$

□

In the non-holonomic case, we have to regard the differential function  $P$  as the primitive of a differential function  $B$ ,  $P = \partial_x^{-1} B$ , and we obtain the following corollary.

**Corollary 2.2.** *In the non-holonomic case of the reciprocal differential substitution  $dy = Bdx$ ,  $w^i = Q^i$  the following change of coordinate formula holds for an variational vector field  $X^i \delta_{w^i} = Y^i \delta_{w^i}$ :*

$$Y^j = \frac{1}{B} \mathcal{D}_i^j (X^i), \quad (41)$$

where

$$\mathcal{D}_i^j = B (\ell_{Q^j})_i - \partial_x Q^j \circ \partial_x^{-1} \circ (\ell_B)_i. \quad (42)$$

Note that we used the property  $\ell_{\partial^{-1}B} = \partial^{-1} \circ \ell_B$ , which is very useful in computations.

Dualizing the computation above, we obtain the formulae for the change of coordinates formula for the Euler–Lagrange operator.

**Corollary 2.3.** *Let the coordinate systems  $(x, u^{i,\sigma})$  and  $(y, w^{i,\sigma})$ , respectively, where the latter coordinates systems are related by a reciprocal differential distribution  $dy = Bdx$ ,  $w^i = Q^i$ , and let  $\mathcal{E}_i^x, \mathcal{E}_i^y$  be the Euler–Lagrange operator with respect to the coordinates  $(x, u_\sigma^i), (y, w^{i,\sigma})$ . Then the change of coordinate formula is*

$$\mathcal{E}_i^x = (\mathcal{D}^*)^k_i \circ \mathcal{E}_k^y, \quad (43)$$

with  $\mathcal{D}$  given by equation (42).

In the holonomic case, the formula reduces to the known formula in [Olv93, exercise 5.49] (with  $\mathcal{D}$  given by equation (33)). In the particular case of a differential substitution of the dependent variable only we have  $\ell_B = 0$  and the above formula reduces to the well-known formula  $\mathcal{E}^x = (\ell_{Q^k}^*)_i \circ \mathcal{E}_k^y$ .

**Corollary 2.4.** Consider a covector-valued density  $\Xi_i du^i \otimes dx = \Psi_i dw^i \otimes dy$  in the coordinate systems  $(x, u^{i,\sigma})$  and  $(y, w^{i,\sigma})$  related by  $dy = Bdx$ ,  $w^i = Q^i$ . Then we have the following change of coordinates formula:

$$\Xi_i = (\mathcal{D}^*)_i^k (\Psi_k), \tag{44}$$

where  $\mathcal{D}$  is as in equation (42) (or as in equation (33) in the holonomic case  $y = P$ ).

Finally, we obtain the following:

**Proposition 2.5.** Consider a reciprocal differential substitution  $dy = Bdx$ ,  $w^i = Q^i$  and let  $P_x^{ij}$ ,  $P_y^{ij}$  be its coordinate expressions of a (possibly non-local or non-Poisson) bi-vector with respect to the coordinates  $(x, u_\sigma^i)$  and  $(y, w_\sigma^i)$ . Then we have the change of coordinate formula

$$P_y^{hk} = \frac{1}{B} (\mathcal{D})_i^h P_x^{ij} (\mathcal{D}^*)_j^k, \tag{45}$$

where  $\mathcal{D}$  is as in equation (42).

**Proof.** The proposition has already been proved in [Mok87, Olv88] for the particular case of Hamiltonian operators and differential substitutions. In our case, the proof follows from the fact that  $P^{ij}$  maps covector-valued densities into variational vector fields. So, we can use the change of coordinates formulae for these two geometric objects (independently of the Hamiltonian property) and find the above result, that holds also in the case of (nonlocal) reciprocal differential substitutions.  $\square$

**Remark 2.6.** The same argument can be applied to multivector fields considered as maps from multicovector-valued densities to variational vector fields. For instance, in the same set-up as theorem 2.5 let  $T^{ijk}$  and  $\tilde{T}^{ijk}$  be the coordinate expressions of a trivector. Then

$$\tilde{T}^{ijk} = \frac{1}{B} (\mathcal{D})_m^i T^{mnp} \left( (\mathcal{D}^*)_n^j, (\mathcal{D}^*)_p^k \right). \tag{46}$$

### 2.1. The Ferapontov–Pavlov formula

Let us apply a special case of theorem 2.5 to a local Poisson bi-vector of order  $\text{deg}_{\partial_x} = 1$  and a reciprocal transformation in  $\mathcal{R}$  that only changes the independent variable. This should reproduce the Ferapontov–Pavlov formula first derived in [FP03, section 3] (based on [Fer95a]).

Consider the change of  $x$  given by

$$\partial_x = B\partial_y, \quad \partial_y^{-1} B^{-1} = \partial_x^{-1} \tag{47}$$

as an element of  $\mathcal{R}_1$ , that is, we assume that  $B = B(u^j)$ . Let a local Poisson bracket of differential degree 1 be given by the operator

$$P^{ij} := g^{ij} \partial_x + \Gamma_k^{ij} u_x^k, \quad (P^*)^{ij} = -P^{ij} \tag{48}$$

**Convention 2.7.** Throughout the computations in this section it is important for us to distinguish between  $\partial_x u^k$  and  $\partial_y u^k$ , so we use the notation  $u_x^k$  and  $u_y^k$  rather than  $u^{k,1}$ .

**Proposition 2.8.** *The action of the reciprocal transformation (47) on the operator (48) produces a weakly non-local operator of localizable shape, whose local part is given explicitly as*

$$g^{ij}B^2\partial_y + \Gamma_k^{ij}B^2u_y^k - \frac{1}{2}g^{i\ell}B^2\left(g_{\ell m}\frac{\partial B^{-2}}{\partial u^k} + g_{km}\frac{\partial B^{-2}}{\partial u^\ell} - g_{\ell k}\frac{\partial B^{-2}}{\partial u^m}\right)g^{mj}B^2u_y^k \quad (49)$$

and the non-local part is equal to

$$\left(P^{i\ell}\left(\frac{\partial B}{\partial u^\ell}\right) - \frac{1}{2}u_y^i\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}\right)\partial_y^{-1}u_y^j + u_y^i\partial_y^{-1}\left(P^{jk}\left(\frac{\partial B}{\partial u^k}\right) - \frac{1}{2}\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}u_y^j\right). \quad (50)$$

**Remark 2.9.** Note that  $\Gamma_k^{ij}B^2 - \frac{1}{2}g^{i\ell}B^2(g_{\ell m}\partial_{u^k}B^{-2} + g_{km}\partial_{u^\ell}B^{-2} - g_{\ell k}\partial_{u^m}B^{-2})g^{mj}B^2$  is exactly the covariant Christoffel symbol for the metric  $g^{ij}B^2$ , so we indeed reproduce the Ferapontov–Pavlov formula in [FP03].

**Proof of proposition 2.8.** By theorem 2.5 the bi-vector  $P^{ij}$  is transformed under the substitution (47) to

$$B^{-1}\left(B\delta_k^i - u_x^i\partial_x^{-1}\frac{\partial B}{\partial u^k}\right)P^{k\ell}\left(B\delta_\ell^j + \frac{\partial B}{\partial u^\ell}\partial_x^{-1}u_x^j\right). \quad (51)$$

Expanding the brackets, we have the following four summands (we intentionally keep derivatives in  $x$  instead of  $y$  as long as possible):

$$-B^{-1}u_x^i\partial_x^{-1}\frac{\partial B}{\partial u^k}P^{k\ell}\frac{\partial B}{\partial u^\ell}\partial_x^{-1}u_x^j = -\frac{1}{2}B^{-1}u_x^i\partial_x^{-1}\left(\partial_x\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell} + \frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}\partial_x\right)\partial_x^{-1}u_x^j \quad (52)$$

$$\begin{aligned} &= -\frac{1}{2}B^{-1}u_x^i\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}\partial_x^{-1}u_x^j - \frac{1}{2}B^{-1}u_x^i\partial_x^{-1}\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}u_x^j \\ &= -\frac{1}{2}u_y^i\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}\partial_y^{-1}u_y^j - \frac{1}{2}u_y^i\partial_y^{-1}\frac{\partial B}{\partial u^k}g^{k\ell}\frac{\partial B}{\partial u^\ell}u_y^j; \end{aligned}$$

$$\begin{aligned} B^{-1}B\delta_k^iP^{k\ell}\frac{\partial B}{\partial u^\ell}\partial_x^{-1}u_x^j &= P^{i\ell}\left(\frac{\partial B}{\partial u^\ell}\right)\partial_x^{-1}u_x^j + g^{i\ell}\frac{\partial B}{\partial u^\ell}u_x^j \\ &= P^{i\ell}\left(\frac{\partial B}{\partial u^\ell}\right)\partial_y^{-1}u_y^j + g^{i\ell}\frac{\partial B}{\partial u^\ell}Bu_y^j; \end{aligned} \quad (53)$$

$$\begin{aligned} -B^{-1}u_x^i\partial_x^{-1}\frac{\partial B}{\partial u^k}P^{k\ell}B\delta_\ell^j &= -B^{-1}u_x^i\partial_x^{-1}(P^*)^{kj}\left(\frac{\partial B}{\partial u^k}\right)B - B^{-1}u_x^i\frac{\partial B}{\partial u^k}g^{kj}B \\ &= u_y^i\partial_y^{-1}P^{jk}\left(\frac{\partial B}{\partial u^k}\right) - u_y^i\frac{\partial B}{\partial u^k}g^{kj}B; \end{aligned} \quad (54)$$

$$B^{-1}B\delta_k^iP^{k\ell}B\delta_\ell^j = P^{ij}B. \quad (55)$$

Thus the non-local term is given by equation (50), and the local term is given by

$$\begin{aligned} &P^{ij}B + g^{i\ell}\frac{\partial B}{\partial u^\ell}Bu_y^j - u_y^i\frac{\partial B}{\partial u^k}g^{kj}B \\ &= g^{ij}B^2\partial_y + g^{ij}B\frac{\partial B}{\partial u^k}u_y^k + \Gamma_k^{ij}B^2u_y^k - \frac{1}{2}g^{i\ell}B^4\frac{\partial B^{-2}}{\partial u^\ell}u_y^j + \frac{1}{2}u_y^i\frac{\partial B^{-2}}{\partial u^k}g^{kj}B^4, \end{aligned} \quad (56)$$

where the latter expression is equal to (49). □



### 2.2. Weakly non-local bi-vectors of localizable shape

The goal of this section is to prove that the space of weakly non-local bi-vectors of localizable shape is closed under the action of reciprocal differential substitutions. We narrow the scope to the Miura-type substitutions  $\mathcal{R}$  as in definition 1.2.

Let us consider the effect of a reciprocal transformation of the form (47) on a general weakly nonlocal bi-vector of localizable shape:

$$P^{ij} = \sum_{d=1}^{\infty} \epsilon^{d-1} \left( \sum_{s=0}^d P_{d,d-s}^{ij} \partial_x^s + u^{i,1} \partial_x^{-1} V_d^j + V_d^i \partial_x^{-1} u^{j,1} \right) = P_{\text{loc}}^{ij} + P_{\text{nonloc}}^{ij}, \quad (57)$$

where  $P_{d,d-s}^{ij} \in \mathcal{A}_{d-s}$  and  $V_d^i \in \mathcal{A}_d$ . Note that both  $P_{\text{loc}}^{ij}$  and  $P_{\text{nonloc}}^{ij}$  define skew-symmetric bi-vectors, that is  $(P_{\text{loc}}^*)^{ij} = -P_{\text{loc}}^{ji}$  and  $(P_{\text{nonloc}}^*)^{ij} = -P_{\text{nonloc}}^{ji}$ .

**Proposition 2.10.** Consider a Miura-reciprocal transformation in  $\mathcal{R}$  given by  $dy = Bdx$ ,  $w^i = Q^i$ . Under this transformation any weakly non-local bi-vector  $P^{ij}$  of localizable shape (57) is transformed into a weakly non-local bi-vector of localizable shape.

**Remark 2.11.** In principle, this proposition follows from the arguments of [LZ11, FP03]. However, it can also be directly obtained using theorem 2.5.

**Proof.** Repeating *mutatis mutandis* the proof of proposition 2.8 one can check that the local part  $P_{\text{loc}}^{ij}$  produces a weakly non-local operator of localizable shape (the only thing that matters for that computation is skew-symmetry of the bi-vector defined by  $P_{\text{loc}}^{ij}$ ). So let us focus on the non-local part  $P_{\text{nonloc}}^{ij} = u_x^i \partial_x^{-1} V^j + V^i \partial_x^{-1} u_x^j$ , where  $V^i = \sum_{d=1}^{\infty} \epsilon^{d-1} V_d^i$  and we use convention 2.7 here and below in computations. We have:

$$B^{-1} \left( B \frac{\partial Q^i}{\partial u^{k,s}} \partial_x^s - Q_x^i \partial_x^{-1} \frac{\partial B}{\partial u^{k,s}} \partial_x^s \right) \circ (u_x^k \partial_x^{-1} V^t + V^k \partial_x^{-1} u_x^t) \left( (-\partial_x)^t \circ \frac{\partial Q^j}{\partial u^{l,t}} B + (-\partial_x)^t \circ \frac{\partial B}{\partial u^{l,t}} \partial_x^{-1} Q_x^j \right) \quad (58)$$

(we omit the summation over  $s$  and  $t$  for brevity). We compute (58) as follows. First, note that

$$\begin{aligned} \frac{\partial Q^i}{\partial u^{k,s}} \partial_x^s \circ u_x^k \partial_x^{-1} V^t (-\partial_x)^t \circ \frac{\partial Q^j}{\partial u^{l,t}} B &= Q_x^i \partial_x^{-1} (\ell_{Q^j})_l (V^t) B + \text{loc}; \\ \frac{\partial Q^i}{\partial u^{k,s}} \partial_x^s \circ V^k \partial_x^{-1} u_x^l (-\partial_x)^t \circ \frac{\partial Q^j}{\partial u^{l,t}} B &= (\ell_{Q^j})_k (V^k) \partial_x^{-1} Q_x^j B + \text{loc}. \end{aligned} \quad (59)$$

Here **loc** are the terms where we collect some purely local operators. Furthermore,

$$\begin{aligned} -\frac{1}{B} Q_x^i \partial_x^{-1} \frac{\partial B}{\partial u^{k,s}} \partial_x^s \circ u_x^k \partial_x^{-1} V^t (-\partial_x)^t \circ \frac{\partial Q^j}{\partial u^{l,t}} B &= -\frac{1}{B} Q_x^i \partial_x^{-1} B_x \partial_x^{-1} (\ell_{Q^j})_l (V^t) B - \frac{1}{B} Q_x^i \partial_x^{-1} O_{B u^l V^t}^j B; \\ -\frac{1}{B} Q_x^i \partial_x^{-1} \frac{\partial B}{\partial u^{k,s}} \partial_x^s \circ V^k \partial_x^{-1} u_x^l (-\partial_x)^t \circ \frac{\partial Q^j}{\partial u^{l,t}} B &= -\frac{1}{B} Q_x^i \partial_x^{-1} (\ell_B)_k (V^k) \partial_x^{-1} Q_x^j B - \frac{1}{B} Q_x^i \partial_x^{-1} O_{B V^k u^l}^j B; \\ \frac{\partial Q^i}{\partial u^{k,s}} \partial_x^s \circ u_x^k \partial_x^{-1} V^t (-\partial_x)^t \circ \frac{\partial B}{\partial u^{l,t}} \partial_x^{-1} Q_x^j &= Q_x^i \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} Q_x^j + O_{Q u^l V^t}^i \partial_x^{-1} Q_x^j; \\ \frac{\partial Q^i}{\partial u^{k,s}} \partial_x^s \circ V^k \partial_x^{-1} u_x^l (-\partial_x)^t \circ \frac{\partial B}{\partial u^{l,t}} \partial_x^{-1} Q_x^j &= (\ell_{Q^j})_l (V^l) \partial_x^{-1} B_x \partial_x^{-1} Q_x^j + O_{Q V^k u^l}^i \partial_x^{-1} Q_x^j. \end{aligned} \quad (60)$$

Here  $O_{BuVQ}^j$ ,  $O_{BVuQ}^j$ ,  $O_{QuVB}^j$ , and  $O_{QVub}^j$  are some scalar local operators, whose main property is that  $(O_{BuVQ}^j)^* = -O_{QVub}^j$  and  $(O_{BVuQ}^j)^* = -O_{QuVB}^j$ . We omit their explicit formulas. Finally,

$$\begin{aligned} -\frac{1}{B} Q_x^i \partial_x^{-1} \frac{\partial B}{\partial u^{k,s}} \partial_x^s \circ u_x^k \partial_x^{-1} V^t (-\partial_x)^t \circ \frac{\partial B}{\partial u^{l,t}} \partial_x^{-1} Q_x^j &= -\frac{1}{B} Q_x^i \partial_x^{-1} B_x \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} Q_x^j \\ &\quad - \frac{1}{B} Q_x^i \partial_x^{-1} O_{BuVB} \partial_x^{-1} Q_x^j; \\ -\frac{1}{B} Q_x^i \partial_x^{-1} \frac{\partial B}{\partial u^{k,s}} \partial_x^s \circ V^k \partial_x^{-1} u_x^l (-\partial_x)^t \circ \frac{\partial B}{\partial u^{l,t}} \partial_x^{-1} Q_x^j &= -\frac{1}{B} Q_x^i \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} B_x \partial_x^{-1} Q_x^j \\ &\quad - \frac{1}{B} Q_x^i \partial_x^{-1} O_{BVuB} \partial_x^{-1} Q_x^j, \end{aligned} \quad (61)$$

where  $O_{BuVB}$  and  $O_{BVuB}$  are scalar local operators such that  $O_{BuVB}^* = -O_{BVuB}$ . We omit their explicit formulas, but we use below that  $O_{BuVB} + O_{BVuB} = -\tilde{O}^* \partial_x - \partial_x \circ \tilde{O}$  for some local operator  $\tilde{O}$ .

Now we collect the terms together. Firstly, we list all terms with  $B_x$  that emerged in (60) and (61):

$$\begin{aligned} -\frac{1}{B} Q_x^i \partial_x^{-1} B_x \partial_x^{-1} (\ell_{Q^i})_l (V^t) B &= -Q_x^i \partial_x^{-1} (\ell_{Q^i})_l (V^t) B + \frac{1}{B} Q_x^i \partial_x^{-1} (\ell_{Q^i})_l (V^t) B^2 \\ (\ell_{Q^i})_l (V^t) \partial_x^{-1} B_x \partial_x^{-1} Q_x^j &= (\ell_{Q^i})_l (V^t) B \partial_x^{-1} Q_x^j - (\ell_{Q^i})_l (V^t) \partial_x^{-1} Q_x^j B \\ -\frac{1}{B} Q_x^i \partial_x^{-1} B_x \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} Q_x^j &= -Q_x^i \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} Q_x^j + \frac{1}{B} Q_x^i \partial_x^{-1} B (\ell_B)_l (V^t) \partial_x^{-1} Q_x^j \\ -\frac{1}{B} Q_x^i \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} B_x \partial_x^{-1} Q_x^j &= -\frac{1}{B} Q_x^i \partial_x^{-1} (\ell_B)_l (V^t) B \partial_x^{-1} Q_x^j + \frac{1}{B} Q_x^i \partial_x^{-1} (\ell_B)_l (V^t) \partial_x^{-1} Q_x^j B \end{aligned} \quad (62)$$

Note some cancellations: the non-local terms in (59) cancel with the corresponding summands in the first and the second line of (62), two non-local terms in the second and third line of (60) cancel with the two terms in the third and fourth line of (62), are there are two terms in the latter lines that cancel each other. So, modulo the purely local terms, (58) is equal to the sum of the following four expressions:

$$\begin{aligned} \frac{1}{B} Q_x^i \partial_x^{-1} (\ell_{Q^i})_l (V^t) B^2 + (\ell_{Q^i})_l (V^t) B \partial_x^{-1} Q_x^j &= w_y^i \partial_y^{-1} (\ell_{Q^i})_l (V^t) B + (\ell_{Q^i})_l (V^t) B \partial_y^{-1} w_y^j; \\ \frac{1}{B} Q_x^i \partial_x^{-1} (O_{QVub}^j)^* B + O_{QVub}^j \partial_x^{-1} Q_x^j &= \frac{1}{B} Q_x^i \partial_x^{-1} O_{QVub}^j (1) B + O_{QVub}^j (1) \partial_x^{-1} Q_x^j + \text{loc} \\ &= w_y^i \partial_y^{-1} O_{QVub}^j (1) + O_{QVub}^j (1) \partial_y^{-1} w_y^j + \text{loc}; \\ \frac{1}{B} Q_x^i \partial_x^{-1} (O_{QuVB}^j)^* B + O_{QuVB}^j \partial_x^{-1} Q_x^j &= \frac{1}{B} Q_x^i \partial_x^{-1} O_{QuVB}^j (1) B + O_{QuVB}^j (1) \partial_x^{-1} Q_x^j + \text{loc} \\ &= w_y^i \partial_y^{-1} O_{QuVB}^j (1) + O_{QuVB}^j (1) \partial_y^{-1} w_y^j + \text{loc}; \\ -\frac{1}{B} Q_x^i \partial_x^{-1} O_{BuVB} \partial_x^{-1} Q_x^j - \frac{1}{B} Q_x^i \partial_x^{-1} O_{BVuB} \partial_x^{-1} Q_x^j &= \frac{1}{B} Q_x^i \partial_x^{-1} (\tilde{O}^* \partial_x + \partial_x \circ \tilde{O}) \partial_x^{-1} Q_x^j \\ &= \frac{1}{B} Q_x^i \partial_x^{-1} \tilde{O} (1) Q_x^j + \frac{1}{B} Q_x^i \tilde{O} (1) \partial_x^{-1} Q_x^j + \text{loc} \\ &= w_y^i \partial_y^{-1} \frac{1}{B} \tilde{O} (1) Q_x^j + \frac{1}{B} Q_x^i \tilde{O} (1) \partial_y^{-1} w_y^j + \text{loc}, \end{aligned} \quad (63)$$

which is manifestly a weakly non-local operator of localizable shape.  $\square$

### 3. Schouten bracket for weakly non-local operators of localizable shape

The goal of this section is to compare two ways to encode weakly non-local Poisson structures of localizable shape: the one given in [LZ11] (by design only working for the localizable shape case) and [LV20] (it is working for general weakly non-local case, but we specialize it for the localizable shape). In principle, the identification of these two approaches follows from the uniqueness property of the bracket, see [LZ11, theorem 2.4.1], but we want to present an explicit computation for this identification.

#### 3.1. The two approaches

In both approaches the weakly non-local  $p$ -vectors of localizable shape are encoded as

$$\int P = \int P_L + \zeta P_N, \tag{64}$$

where  $P_L \in \hat{\mathcal{A}}^p, P_N \in \hat{\mathcal{A}}^{p-1}$ , and

$$\partial_x \zeta = -u^{i,1} \theta_i. \tag{65}$$

The difference in two approaches is the meaning of  $\zeta$ . In the approach of [LZ11],  $\zeta$  is a new dependent variable such that  $\text{deg}_{\partial_x} \zeta = 0$  and  $\text{deg}_{\theta} \zeta = 1$ . The new space of multivector densities is defined as  $\mathcal{S} := \hat{\mathcal{A}}[\zeta]$ , equipped with the operator

$$\partial_x = -u^{i,1} \theta_i \partial_\zeta + \sum u^{i,d+1} \partial_{u^{i,d}} + \theta_i^{d+1} \partial_{\theta_i^d}, \tag{66}$$

and the space of weakly non-local multivectors of localizable shape is defined as  $\mathcal{E} := \mathcal{S} / \partial_x \mathcal{S}$ .

In the approach of [LV20],  $\zeta$  is not a new dependent variable, but rather an expression in the existing dependent variables (still of differential degree  $\text{deg}_{\partial_x} \zeta = 0$  and multivector degree  $\text{deg}_{\theta} \zeta = 1$ ), such that equation (65) is satisfied for the standard operator

$$\tilde{\partial}_x = \sum u^{i,d+1} \partial_{u^{i,d}} + \theta_i^{d+1} \partial_{\theta_i^d}. \tag{67}$$

For instance, one can find such a function in  $\hat{\mathcal{A}}((\frac{1}{u^{1,1}}))$ , see [DLZ06]. To this end, one looks for a unique solution  $\tilde{\partial}_x \zeta = -u^{i,1} \theta_i$  of the form  $\zeta = \sum_{i=1}^\infty \frac{f_i}{(u^{1,1})^i}$ , with  $f_i \in \hat{\mathcal{A}}$  such that  $\partial_{u^{1,1}} f_i = 0$ .

Once the objects are defined, we have two different formulae for the Schouten bracket in these two approaches:

- The formula in the approach of [LV20] is

$$\left[ \int P, \int Q \right] = \int (-1)^{\text{deg}_\theta P} \tilde{\delta}_{u^i} P \tilde{\delta}_{\theta_i} Q + \tilde{\delta}_{\theta_i} P \tilde{\delta}_{u^i} Q. \tag{68}$$

Recall that  $\zeta$  is regarded as a function of  $(u^i_\sigma, \theta_i^\sigma)$  in the variational derivatives (which are denoted by  $\tilde{\delta}_{u^i}$  and  $\tilde{\delta}_{\theta_i}$  for that reason).

- The formula in the approach of [LZ11] is

$$\left[ \int P, \int Q \right] = \int (-1)^{\deg_\theta P} \delta_{u^i} P \delta_{\theta_i} Q + \delta_{\theta_i} P \delta_{u^i} Q + (-1)^{\deg_\theta P} \hat{E}(P) \partial_\zeta Q + \partial_\zeta P \hat{E}(Q) \quad (69)$$

Here  $\zeta$  is regarded as an extra dependent variable, and the operator  $\hat{E}$  is defined as

$$\hat{E} = \sum_{\substack{s \geq 1 \\ t \geq 0}} \left( u^{i,s} (-\partial_x)^t \partial_{u^{i,s+t}} + \theta_i^s (-\partial_x)^t \partial_{\theta_i^{s+t}} \right) - 1 + \theta_i \delta_{\theta_i}. \quad (70)$$

### 3.2. Identification of the two approaches

We prove the following:

**Theorem 3.1.** *The identity map  $\hat{\mathcal{A}}[\zeta] \rightarrow \hat{\mathcal{A}}[\zeta]$  induces the isomorphism of the Lie algebras of local multivector fields defined by the Schouten brackets in these two approaches.*

**Proof.** We represent any density  $P \in \hat{\mathcal{A}}[\zeta]$  as  $P = P_L + \zeta P_N$  and consider  $\zeta$  to be a nonlocal function. Note that

$$\begin{aligned} \tilde{\delta}_{u^i} P &= \delta_{u^i} P + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) \\ &= \delta_{u^i} P_L + (-\partial_x)^\sigma (\zeta \partial_{u^i, \sigma} P_N) + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N), \end{aligned} \quad (71)$$

$$\begin{aligned} \tilde{\delta}_{\theta_i} P &= \delta_{\theta_i} P + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) \\ &= \delta_{\theta_i} P_L - (-\partial_x)^\sigma (\zeta \partial_{\theta_i^\sigma} P_N) + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N), \end{aligned} \quad (72)$$

where we used that

$$\delta_{u^i} P = \delta_{u^i} P_L + (-\partial_x)^\sigma (\zeta \partial_{u^i, \sigma} P_N), \quad (73)$$

$$\delta_{\theta_i} P = \delta_{\theta_i} P_L - (-\partial_x)^\sigma (\zeta \partial_{\theta_i^\sigma} P_N). \quad (74)$$

Using these formulas, we obtain

$$\begin{aligned} \tilde{\delta}_{u^i} P \tilde{\delta}_{\theta_i} Q &= (\delta_{u^i} P + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N)) (\delta_{\theta_i} Q + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N)) \\ &= \delta_{u^i} P \delta_{\theta_i} Q + \delta_{u^i} P (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N) + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) \delta_{\theta_i} Q \\ &\quad + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N); \end{aligned} \quad (75)$$

$$\begin{aligned} \tilde{\delta}_{\theta_i} P \tilde{\delta}_{u^i} Q &= (\delta_{\theta_i} P + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N)) (\delta_{u^i} Q + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N)) \\ &= \delta_{\theta_i} P \delta_{u^i} Q + \delta_{\theta_i} P (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N) + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) \delta_{u^i} Q \\ &\quad + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N). \end{aligned} \quad (76)$$

If we want to treat  $\zeta$  as a new dependent variable, we have  $\hat{E}(P) \partial_\zeta Q = \hat{E}(P) Q_N$  (and similarly for the other summand in the formula), so we have to prove that

$$\begin{aligned}
 & \int (-1)^{\deg_\theta P} \hat{E}(P) Q_N + P_N \hat{E}(Q) \\
 &= \int (-1)^{\deg_\theta P} \left( \delta_{u^i} P (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N) + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) \delta_{\theta_i} Q \right. \\
 &\quad \left. + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N) \right) \\
 &\quad + \left( \delta_{\theta_i} P (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N) + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) \delta_{u^i} Q \right. \\
 &\quad \left. + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N) \right). \tag{77}
 \end{aligned}$$

Let us use the following property of the operator  $\hat{E}$ :

$$\partial_x \hat{E} = -u^{i,1} \delta_{u^i} + \theta_i \partial_x \delta_{\theta_i} + u^{i,1} \theta_i \delta_\zeta; \tag{78}$$

So, we obtain

$$\begin{aligned}
 \int \hat{E}(P) Q_N &= \int \partial_x^{-1} (-u^{i,1} \delta_{u^i} P + \theta_i \partial_x \delta_{\theta_i} P + u^{i,1} \theta_i P_N) Q_N \\
 &= \int -(-u^{i,1} \delta_{u^i} P + \theta_i \partial_x \delta_{\theta_i} P + u^{i,1} \theta_i P_N) \partial_x^{-1} (Q_N), \tag{79}
 \end{aligned}$$

$$\begin{aligned}
 \int P_N \hat{E}(Q) &= \int P_N \partial_x^{-1} (-u^{i,1} \delta_{u^i} Q + \theta_i \partial_x \delta_{\theta_i} Q + u^{i,1} \theta_i Q_N) \\
 &= \int -\partial_x^{-1} (P_N) (-u^{i,1} \delta_{u^i} Q + \theta_i \partial_x \delta_{\theta_i} Q + u^{i,1} \theta_i Q_N). \tag{80}
 \end{aligned}$$

Substituting equations (79) and (80) into (77), we see that the statement of the theorem reduces to the following equality:

$$\begin{aligned}
 & \int (-1)^{\deg_\theta P+1} (-u^{i,1} \delta_{u^i} P + \theta_i \partial_x \delta_{\theta_i} P + u^{i,1} \theta_i P_N) \partial_x^{-1} (Q_N) \\
 & - \partial_x^{-1} (P_N) (-u^{i,1} \delta_{u^i} Q + \theta_i \partial_x \delta_{\theta_i} Q + u^{i,1} \theta_i Q_N) \\
 &= \int (-1)^{\deg_\theta P} \left( \delta_{u^i} P (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N) + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) \delta_{\theta_i} Q \right. \\
 &\quad \left. + (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta P_N) (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta Q_N) \right) \\
 &\quad + \left( \delta_{\theta_i} P (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N) + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) \delta_{u^i} Q \right. \\
 &\quad \left. + (-\partial_x)^\sigma (\partial_{\theta_i^\sigma} \zeta P_N) (-\partial_x)^\sigma (\partial_{u^i, \sigma} \zeta Q_N) \right). \tag{81}
 \end{aligned}$$

In order to prove this equality, our strategy is move  $\partial_x^{-1}$  in  $\zeta = \partial_x^{-1}(-u^{i,1}\theta_i)$  to the other factor ( $P_N$  or  $Q_N$ ) using integration by parts. We have:

$$\delta_{u^i}P(-\partial_x)^\sigma(\partial_{\theta_i^\sigma}\zeta Q_N) = \delta_{u^i}P u^{i,1}\partial_x^{-1}(Q_N), \tag{82}$$

$$(-\partial_x)^\sigma(\partial_{u_i^\sigma}\zeta P_N)\delta_{\theta_i}Q = -\partial_x(\theta_i\partial_x^{-1}(P_N))\delta_{\theta_i}Q \tag{83}$$

$$(-\partial_x)^\sigma(\partial_{u_i^\sigma}\zeta P_N)(-\partial_x)^\sigma(\partial_{\theta_i^\sigma}\zeta Q_N) = -\partial_x(\theta_i\partial_x^{-1}(P_N))u^{i,1}\partial_x^{-1}(Q_N) \tag{84}$$

$$(-\partial_x)^\sigma(\partial_{\theta_i^\sigma}\zeta P_N)\delta_{u^i}Q = u^{i,1}\partial_x^{-1}(P_N)\delta_{u^i}Q \tag{85}$$

$$\delta_{\theta_i}P(-\partial_x)^\sigma(\partial_{u_i^\sigma}\zeta Q_N) = -\delta_{\theta_i}P\partial_x(\theta_i\partial_x^{-1}(Q_N)) \tag{86}$$

$$(-\partial_x)^\sigma(\partial_{\theta_i^\sigma}\zeta P_N)(-\partial_x)^\sigma(\partial_{u_i^\sigma}\zeta Q_N) = -(u^{i,1}\partial_x^{-1}(P_N))\partial_x(\theta_i\partial_x^{-1}(Q_N)) \tag{87}$$

Substituting the above expressions into the equality (81) that we shall prove, we are led to the simplified equality:

$$\begin{aligned} & \int (-1)^{\deg_\theta P+1}(\theta_i\partial_x\delta_{\theta_i}P + u^{i,1}\theta_i P_N)\partial_x^{-1}(Q_N) - \partial_x^{-1}(P_N)(\theta_i\partial_x\delta_{\theta_i}Q + u^{i,1}\theta_i Q_N) \\ &= \int (-1)^{\deg_\theta P+1}(\partial_x(\theta_i\partial_x^{-1}(P_N))\delta_{\theta_i}Q + \partial_x(\theta_i\partial_x^{-1}(P_N))u^{i,1}\partial_x^{-1}(Q_N)) \\ & \quad - (\delta_{\theta_i}P\partial_x(\theta_i\partial_x^{-1}(Q_N)) + u^{i,1}\partial_x^{-1}(P_N)\partial_x(\theta_i\partial_x^{-1}(Q_N))). \end{aligned} \tag{88}$$

Integrating by parts the summands containing  $\partial_x\delta_{\theta_i}P$ ,  $\partial_x\delta_{\theta_i}Q$  we obtain the further simplification of the equality (81) (note that  $\deg_\theta P_N = \deg_\theta P - 1$ ):

$$\begin{aligned} & \int (-1)^{\deg_\theta P+1}u^{i,1}\theta_i P_N\partial_x^{-1}(Q_N) - \partial_x^{-1}(P_N)u^{i,1}\theta_i Q_N \\ &= \int (-1)^{\deg_\theta P+1}\partial_x(\theta_i\partial_x^{-1}(P_N))u^{i,1}\partial_x^{-1}(Q_N) - u^{i,1}\partial_x^{-1}(P_N)\partial_x(\theta_i\partial_x^{-1}(Q_N)). \end{aligned} \tag{89}$$

Expanding the total derivatives on the right-hand side we easily see that the above equality is an identity. This completes the proof of the theorem.  $\square$

#### 4. Pencils of weakly non-local bi-vectors of localizable shape

In this section we compute the bi-Hamiltonian cohomology for a semi-simple pencil of weakly non-local Poisson bi-vectors of localizable shape of differential order  $\deg_{\partial_x} = 1$  satisfying the extra condition: the pencil of these bi-vectors should be localizable (or, equivalently, they should be simultaneously localizable) with respect to the Miura-reciprocal group. As a result of this computation and some further arguments we prove the following theorem:

**Theorem 4.1.** *Let  $P_1$  and  $P_2$  be two of commuting non-local Poisson bi-vectors of localizable shape. We assume that  $P_1$  and  $P_2$  have dispersive expansion given by  $P_a = \sum_{i=1}^\infty \epsilon^{i-1}P_{a,i}$ ,  $\deg_{\partial_x} P_{a,i} = i$ ,  $a = 1, 2$ ,  $i = 1, 2, \dots$*

*If the leading terms of degree  $\deg_{\partial_x} = 1$ ,  $P_{1,1}$  and  $P_{2,1}$ , are simultaneously localizable under the action of the Miura-reciprocal group and form a semi-simple Poisson pencil, then the full dispersive brackets  $P_1$  and  $P_2$  are simultaneously localizable under the action of the Miura-reciprocal group.*

In order to prove this theorem, we have to make a few preliminary computations with bi-Hamiltonian cohomology, following the ideas in [LZ11, LZ13] and subsequent steps in [CPS18, CKS18].

4.1. Bi-Hamiltonian cohomology

4.1.1. Setup for a deformation problem. Recall that following Liu and Zhang [LZ11] we denote  $\mathcal{S} := \hat{\mathcal{A}}[\zeta]$ , with  $\partial_x: \mathcal{S} \rightarrow \mathcal{S}$  given by  $\partial_x = -u^{i,1}\theta_i\partial_\zeta + \sum u^{i,d+1}\partial_{u^{i,d}} + \theta_i^{d+1}\partial_{\theta_i^d}$ , and  $\mathcal{E} := \mathcal{S}/\partial_x\mathcal{S}$ .

Let  $P_1, P_2 \in \mathcal{S}_1^2$  such that  $\int P_1$  and  $\int P_2$  form a pencil of Poisson structures (possibly non-local, but then they are automatically weakly non-local of localizable shape, since it is the only type of non-locality accommodated in the space  $\mathcal{E}$ ), that is, we assume that

$$\left[ \int P_2 - \lambda P_1, \int P_2 - \lambda P_1 \right] = 0. \tag{90}$$

Recall that there is a group  $\mathcal{R}_I$  of the Miura-reciprocal transformations of the 1st kind acting on them, see equation (14). We assume that the pencil  $\int P_2 - \lambda P_1$  is localizable under the action of  $\mathcal{R}_I$ . We also assume that the pencil formed by  $P_1$  and  $P_2$  is semi-simple, which together with the assumption of localizability implies that we can choose the coordinates  $x, u^1, \dots, u^N$  such that the densities  $P_1$  and  $P_2$  of the bivectors  $\int P_1$  and  $\int P_2$  take the form

$$P_1 = \left( \sum_{i=1}^N f^i \theta_i \theta_i^1 \right) + \Gamma_{1,k}^{ij} u^{k,1} \theta_i \theta_j; \tag{91}$$

$$P_2 = \left( \sum_{i=1}^N u^i f^i \theta_i \theta_i^1 \right) + \Gamma_{2,k}^{ij} u^{k,1} \theta_i \theta_j. \tag{92}$$

We are interested to classify the equivalence classes of the higher order dispersive deformations of the Poisson pencil  $\int P_2 - \lambda P_1$  in  $\mathcal{E}$  with respect to the Miura-reciprocal transformations of the 2nd kind,  $\mathcal{R}_{II}$ . Let  $d_i := \text{ad}_{p_i}: \mathcal{E} \rightarrow \mathcal{E}$ ,  $i = 1, 2$ . Then the deformation problem is controlled by the bi-Hamiltonian cohomology  $BH_d^p(\mathcal{E}, d_1, d_2)$  of cohomological degree  $p = 2$  and  $p = 3$  and of differential degrees  $d \geq 2$  and  $d \geq 4$ , respectively. It is a rather standard argument, see e. g. [LZ11, proposition 3.3.5]. The only extra bit that one needs in our case, that is, the space  $\mathcal{E}$  and the group  $\mathcal{R}_{II}$  of Miura-reciprocal transformations of the 2nd kind, in comparison with the usual local case, that is, the space  $\hat{\mathcal{F}}$  and the group  $\mathcal{G}_{II}$  of Miura transformations of the 2nd kind, is the identification of the action of the Lie algebra of  $\mathcal{R}_{II}$  on weakly non-local bi-vectors (or, more generally, multivectors) of localizable shape with the adjoint action of  $\mathcal{E}^1$  on  $\mathcal{E}^2$  (resp.,  $\mathcal{E}$ ). This is established in [LZ11, theorems 2.5.7 and 2.6.5]

4.1.2. Bi-Hamiltonian cohomology computation. We prove the following

**Theorem 4.2.** We have:

$$BH_d^2(\mathcal{E}, d_1, d_2) \cong \begin{cases} 0, & d = 2 \text{ and } d \geq 4; \\ \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i), & d = 3. \end{cases} \tag{93}$$

$$BH_d^3(\mathcal{E}, d_1, d_2) \cong 0, \quad d \geq 4. \tag{94}$$

**Proof.** For the proof we use that for  $d \geq 2$  we have [LZ13, lemma 4.4]:

$$BH_d^p(\mathcal{E}, d_1, d_2) \cong H_d^p(\mathcal{E}[\lambda], d_2 - \lambda d_1). \tag{95}$$

In order to compute  $H_d^p(\mathcal{E}[\lambda], d_2 - \lambda d_1)$ , we recall the definition of  $D_i := D_{P_i} : \mathcal{S} \rightarrow \mathcal{S}$  from [LZ11]:

$$D_{P_i} := \hat{E}(P_i) \partial_\zeta + \sum_{s=0}^{\infty} \partial_x^s (\delta_{w^i} P_i) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P_i) \partial_{w^{i,s}}, \quad i = 1, 2. \quad (96)$$

Note that  $[\partial_x, D_i] = 0$  (by direct computation). We prove that it is a homological vector field (which is not true in general, for a non-local bi-vector  $P_i$ ):

**Lemma 4.3.** *For a purely local bivector  $\int P$  the operator  $D_P$  does not depend on the choice of a purely local density  $P$ . Moreover, for purely local densities of the bivectors  $P, Q \in \hat{\mathcal{A}}^2$  and for any  $T \in \mathcal{S}$  we have:*

$$\int D_P(T) = \left[ \int P, \int T \right] \quad (97)$$

and

$$[D_P, D_Q] = D_{[P, Q]}, \quad (98)$$

where  $[P, Q] = \delta_{\theta_j} P \delta_{w^i} Q + \delta_{w^i} P \delta_{\theta_j} Q$ .

In particular, for a purely local density  $P$  of a Poisson bivector  $\int P$  we have  $D_P^2 = 0$  on  $\mathcal{S}$ .

**Remark 4.4.** The statements of lemma 4.3 do not hold for not purely local densities.

**Proof of lemma 4.3.** Firstly, we check that  $D_P$  does not depend on the choice of a local density  $P$ . To this end, we remind the definitions and basic properties of  $\hat{E}$  and  $\partial_x$ . We have:

$$\partial_x = -u^{i,1} \theta_i \partial_\zeta + \sum u^{i,d+1} \partial_{w^{i,d}} + \theta_i^{d+1} \partial_{\theta_j^d}; \quad (99)$$

$$\hat{E} = \sum_{\substack{s \geq 1 \\ i \geq 0}} \left( u^{i,s} (-\partial_x)^t \partial_{w^{i,s+t}} + \theta_i^s (-\partial_x)^t \partial_{\theta_j^{s+t}} \right) - 1 + \theta_i \delta_{\theta_j}; \quad (100)$$

$$\partial_x \hat{E} = -u^{i,1} \delta_{w^i} + \theta_i \partial_x \delta_{\theta_j} + u^{i,1} \theta_i \delta_\zeta; \quad (101)$$

$$\hat{E} \partial_x = -u^{i,1} \theta_i \partial_\zeta; \quad (102)$$

$$\delta_{w^i} \partial_x = \partial_x \theta_i \partial_\zeta, \quad i = 1, \dots, N; \quad (103)$$

$$\delta_{\theta_j} \partial_x = -u^{i,1} \partial_\zeta, \quad i = 1, \dots, N. \quad (104)$$

With the last three equations we immediately see that for any local  $X \in \hat{\mathcal{A}}$

$$\begin{aligned} & \hat{E}(\partial_x X) \partial_\zeta + \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{w^i} \partial_x X) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} \partial_x X) \partial_{w^{i,s}} \right) \\ &= -u^{i,1} \theta_i \partial_\zeta X \partial_\zeta + \sum_{s=0}^{\infty} \left( \partial_x^s (\partial_x (\theta_i \partial_\zeta X)) \partial_{\theta_j^s} + \partial_x^s (-u^{i,1} \partial_\zeta X) \partial_{w^{i,s}} \right) = 0, \end{aligned} \quad (105)$$

since  $\partial_\zeta = 0$ , which implies the first assertion of the lemma.



Now, equation (97) is obvious from the definition of the Schouten bracket. So we focus on equation (98). Let us compute the coefficient of  $\partial_\zeta$  on the left hand side. Using the vanishing of  $\partial_\zeta$  derivatives, we have:

$$\begin{aligned} & \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} \right) \hat{E}(Q) \\ &= \partial_x^{-1} \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} \right) (-u^{i,1} \delta_{u^i} + \theta_i \partial_x \delta_{\theta_i})(Q) \\ &= \partial_x^{-1} (\delta_{u^i} P \partial_x (\delta_{\theta_j} Q) - \partial_x (\delta_{\theta_j} P) \delta_{u^i} Q) \\ &\quad + \partial_x^{-1} (-u^{i,1}) \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} \right) \delta_{u^i} Q \\ &\quad + \partial_x^{-1} (-\theta_i \partial_x) \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} \right) \delta_{\theta_i} Q. \end{aligned} \tag{106}$$

Adding to the latter expression the same one with interchanged  $P$  and  $Q$  and using that for purely local densities

$$\begin{aligned} & \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} \right) \delta_{u^i} Q + \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} Q) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} Q) \partial_{u^{i,s}} \right) \delta_{u^i} P \\ &= \delta_{u^i} \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} Q + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} Q \right) = \delta_{u^i} \sum_{s=0}^{\infty} (\delta_{u^i} P \delta_{\theta_j} Q + \delta_{\theta_j} P \delta_{u^i} Q) \end{aligned} \tag{107}$$

and

$$\begin{aligned} & \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} \right) \delta_{\theta_i} Q + \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} Q) \partial_{\theta_j^s} + \partial_x^s (\delta_{\theta_j} Q) \partial_{u^{i,s}} \right) \delta_{\theta_i} P \\ &= -\delta_{\theta_i} \sum_{s=0}^{\infty} \left( \partial_x^s (\delta_{u^i} P) \partial_{\theta_j^s} Q + \partial_x^s (\delta_{\theta_j} P) \partial_{u^{i,s}} Q \right) = -\delta_{\theta_i} \sum_{s=0}^{\infty} (\delta_{u^i} P \delta_{\theta_j} Q + \delta_{\theta_j} P \delta_{u^i} Q), \end{aligned} \tag{108}$$

we obtain that the coefficient of  $\partial_\zeta$  on the left hand side of equation (98) is equal to

$$\partial_x^{-1} (-u^{i,1} \delta_{u^i} + \theta_i \partial_x \delta_{\theta_i}) [P, Q] = \hat{E}([P, Q]), \tag{109}$$

which is the coefficient of  $\partial_\zeta$  on the right hand side of equation (98). The coefficients of all other components of the vector fields on the left hand side of equation (98) are computed in a very similar way.  $\square$

Lemma 4.3 implies that  $D_2 - \lambda D_1$  is a differential on  $\mathcal{S}[\lambda]$ , and we have a short exact sequence

$$0 \longrightarrow \frac{\mathcal{S}[\lambda]}{\mathbb{R}} \xrightarrow{\partial_x} \mathcal{S}[\lambda] \xrightarrow{f} \mathcal{E}[\lambda] \longrightarrow 0 \tag{110}$$

$\begin{array}{ccccc} & & D_2 - \lambda D_1 & & D_2 - \lambda D_1 & & d_2 - \lambda d_1 \\ & & \curvearrowright & & \curvearrowright & & \\ & & \mathcal{S}[\lambda] & & \mathcal{S}[\lambda] & & \mathcal{E}[\lambda] \end{array}$

and it implies a long exact sequence in the cohomology which reads

$$\begin{array}{c}
 H_{d-1}^p(\mathcal{S}[\lambda]/\mathbb{R}, D_2 - \lambda D_1) \longrightarrow H_d^p(\mathcal{S}[\lambda], D_2 - \lambda D_1) \longrightarrow H_d^p(\mathcal{E}[\lambda], d_2 - \lambda d_1) \quad (111) \\
 \longleftarrow \\
 H_d^{p+1}(\mathcal{S}[\lambda]/\mathbb{R}, D_2 - \lambda D_1) \longrightarrow H_{d+1}^{p+1}(\mathcal{S}[\lambda], D_2 - \lambda D_1)
 \end{array}$$

**Lemma 4.5.** *We have*

$$H_d^p(\mathcal{S}[\lambda], D_2 - \lambda D_1) \cong \begin{cases} 0 & p \leq d \text{ and } (p, d) \neq (3, 3), (0, 0) \\ \mathbb{R}[\lambda] & p = 0, d = 0; \\ \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i) & p = 3, d = 3. \end{cases} \quad (112)$$

Also,  $H_2^3(\mathcal{S}[\lambda], D_2 - \lambda D_1) \cong 0$ .

**Proof.** This lemma can be derived from [CKS18, theorems 2.12 and 2.13]. We observe that we have a bicomplex  $(\mathcal{S}[\lambda], D^{\text{loc}}, D^\zeta)$  with the differentials given by  $D^\zeta := (\hat{E}(P_2) - \lambda \hat{E}(P_1))\partial_\zeta$  and  $D^{\text{loc}} := D_2 - \lambda D_1 - D^\zeta$ . Indeed,  $(D^{\text{loc}} + D^\zeta)^2 = (D_2 - \lambda D_1)^2 = 0$  and  $(D^{\text{loc}})^2 = 0$ ; from the proof of lemma 4.3 it follows that  $D^{\text{loc}}D^\zeta + D^\zeta D^{\text{loc}} = 0$ , which implies that  $(D^\zeta)^2 = 0$ .

We start a spectral sequence associated with this bicomplex. Obviously, it converges on the second page. The computation of the first page splits as

$$\begin{aligned}
 H_d^p(\mathcal{S}[\lambda], D^{\text{loc}}) &\cong H_d^p(\mathcal{A}[\lambda], D^{\text{loc}}) \oplus H_d^p(\mathcal{A}[\lambda]\zeta, D^{\text{loc}}) \\
 &\cong H_d^p(\mathcal{A}[\lambda], D^{\text{loc}}) \oplus H_d^{p-1}(\mathcal{A}[\lambda], D^{\text{loc}}), \quad (113)
 \end{aligned}$$

which implies all desired vanishings (for  $p \leq d$  the only non-trivial cohomology groups are  $H_0^0(\mathcal{A}[\lambda], D^{\text{loc}}) \cong \mathbb{R}[\lambda]$  and  $H_3^3(\mathcal{A}[\lambda], D^{\text{loc}}) \cong \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i)$  [CKS18, theorems 2.12 and 2.13]).

Since both  $H_i^i(\mathcal{S}[\lambda], D^{\text{loc}}) = 0$  for  $i = 1, 2, 4$ , and the induced differential on the first page has the  $(p, d)$ -degree  $(1, 1)$ , we conclude that

$$H_0^0(\mathcal{S}[\lambda], D_2 - \lambda D_1) \cong H_0^0(\mathcal{S}[\lambda], D^{\text{loc}}) \cong \mathbb{R}[\lambda]; \quad (114)$$

$$H_3^3(\mathcal{S}[\lambda], D_2 - \lambda D_1) \cong H_3^3(\mathcal{S}[\lambda], D^{\text{loc}}) \cong \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i). \quad (115)$$

□

**Remark 4.6.** Almost the same statement holds for the cohomology of  $\mathcal{S}[\lambda]/\mathbb{R}$ , the only difference is  $H_0^0(\mathcal{S}[\lambda]/\mathbb{R}, D_2 - \lambda D_1) \cong 0$ .

Now we can complete the computation of the cohomology  $H_d^p(\mathcal{E}[\lambda], d_2 - \lambda d_1)$  for  $p < d$  and  $p = 2, d = 2$  using the long exact sequence (111). The relevant pieces of this long exact sequence are

$$\begin{array}{c}
 0 = H_d^p(\mathcal{S}[\lambda], D_2 - \lambda D_1) \longrightarrow H_d^p(\mathcal{E}[\lambda], d_2^{loc} - \lambda d_1^{loc}), \quad (116) \\
 \longleftarrow \\
 H_d^{p+1}(\mathcal{S}[\lambda]/\mathbb{R}, D_2 - \lambda D_1) = 0
 \end{array}$$

for  $p < d$  and  $(p + 1, d) \neq (3, 3)$ , which implies the vanishing for  $p < d$ ,  $(p, d) \neq (2, 3)$ , and  $p = 2, d = 2$ . Moreover, we have

$$\begin{aligned}
 0 = H_3^2(\mathcal{S}[\lambda], D_2 - \lambda D_1) &\longrightarrow H_3^2(\mathcal{E}[\lambda], d_2^{loc} - \lambda d_1^{loc}) \quad , \quad (117) \\
 &\longleftarrow \\
 H_3^3(\mathcal{S}[\lambda]/\mathbb{R}, D_2 - \lambda D_1) \cong \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i) &\longrightarrow H_4^3(\mathcal{S}[\lambda], D_2 - \lambda D_1) = 0
 \end{aligned}$$

which gives the answer for  $(p, d) = (2, 3)$ . Now, the special cases of these computations for  $p = 2, d \geq 2$  and  $p = 3, d \geq 4$  imply all statements of theorem 4.2.  $\square$

An immediate corollary of theorem 4.2 is the following:

**Corollary 4.7.** *Let  $\int P_2 - \lambda P_1$  be a semi-simple pencil of local Poisson bivectors of differential order 1. We consider the higher order dispersive extensions of  $\int P_2 - \lambda P_1$  in the realm of weakly non-local Poisson pencils of localizable shape, that is, we consider Poisson pencils  $\int \sum_{d=1}^\infty \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in \mathcal{E}$  such that  $\text{deg}_{\partial_x} (P_{2,d} - \lambda P_{1,d}) = d$  and  $\int P_{2,1} - \lambda P_{1,1} = \int P_2 - \lambda P_1$ .*

*The space of orbits of the action of the group  $\mathcal{R}_H$  (the group of Miura-reciprocal transformation of the 2nd kind) onto the set of these dispersive extensions is isomorphic to the space  $\bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i)$ .*

This result is strikingly similar to the corresponding statement in the local case, see [CPS18, theorem 1], see also [LZ05, LZ13, DLZ06]. However, in the local case both the space  $\hat{\mathcal{F}}$  where the deformations of  $\int P_2 - \lambda P_1$  are allowed as well as the group  $\mathcal{G}_H$  acting on them are much smaller than in corollary 4.7. Our next goal is to compare these two situations.

#### 4.2. Comparison with the purely local deformations

Within this section it is important to have a notation that distinguishes between the operator  $\partial_x$  as given by equation (99) on the space  $\mathcal{S} = \hat{\mathcal{A}}[\zeta]$  and its purely local version  $\tilde{\partial}_x := \partial_x + u^{i,1} \theta_i \partial_\zeta$  defined both on  $\mathcal{S}$  and on  $\mathcal{A}$ . Note that on  $\mathcal{S}$  the operator  $\tilde{\partial}_x$  commutes with multiplication by  $\zeta$ .

Let  $T^{nl}$  denote the space of dispersive weakly non-local Poisson pencils of localizable shape  $\int \sum_{d=1}^\infty \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in \mathcal{E}$  with the fixed leading term  $\int P_{2,1} - \lambda P_{1,1} = \int P_2 - \lambda P_1$  that is purely local and semi-simple. Let  $T^{loc}$  denote the space of dispersive local Poisson pencils  $\int \sum_{d=1}^\infty \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in \hat{\mathcal{F}}$  with the same fixed leading term  $\int P_{2,1} - \lambda P_{1,1} = \int P_2 - \lambda P_1$ .

The group  $\mathcal{R}_H$  acts on  $T^{nl}$  and the group  $\mathcal{G}_H$  acts on  $T^{loc}$ . Moreover, there is a natural embedding  $I: T^{loc} \rightarrow T^{nl}$  that is  $\mathcal{G}_H$ -equivariant ( $\mathcal{G}_H$  acts on  $T^{nl}$  as a subgroup of  $\mathcal{R}_H$ ). The map  $I$  induces a map of the sets of orbits  $\iota: T^{loc}/\mathcal{G}_H \rightarrow T^{nl}/\mathcal{R}_H$ .

**Proposition 4.8.** *The map  $\iota$  is injective.*

**Proof.** This proposition immediately follows from [LZ11, theorem 1.3] and [CPS18, theorem 2]. By the latter result in the local case, we have an isomorphism of sets  $\tilde{c}: T^{loc}/\mathcal{G}_H \rightarrow \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i)$  (these are the so-called central invariants in the local case). On the other hand, [LZ11, theorem 1.3] states that for any  $x, y \in T^{loc}/\mathcal{G}_H$  such that  $\iota(x) = \iota(y)$  we have  $\tilde{c}(x) = \tilde{c}(y)$ . Hence,  $x = y$ , and  $\iota$  is surjective.  $\square$

Corollary 4.7 implies that there is a  $\mathcal{R}_{II}$  invariant map  $C: T^{nl} \rightarrow \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i)$  that descends to a bijection  $c: T^{nl}/\mathcal{R}_{II} \rightarrow \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i)$ . We have the following

**Proposition 4.9.** *The composition  $c \circ \iota: T^{loc}/\mathcal{G}_{II} \rightarrow \bigoplus_{i=1}^N C^\infty(\mathbb{R}, u^i)$  is surjective.*

**Proof.** Basically, we want to show that any cohomology class in  $H_3^2(\mathcal{E})$  has a representative with a purely local density. Let  $\int a^2 + a^1 \zeta$  represent a class in  $H_3^2(\mathcal{E})$ ,  $a^2 \in \hat{\mathcal{A}}_3^2[\lambda]$  and  $a^1 \in \hat{\mathcal{A}}_3^1[\lambda]$ . This means that

$$(D_2 - \lambda D_1)(a^2 + a^1 \zeta) \in \partial_x(S_3^3[\lambda]), \tag{118}$$

or, in other words, that there exist  $b^3 \in \hat{\mathcal{A}}_3^3[\lambda]$  and  $b^2 \in \hat{\mathcal{A}}_3^2[\lambda]$  such that

$$D^{loc}(a^2) - (\hat{E}(P_2) - \lambda \hat{E}(P_1))(a^1) + D^{loc}(a^1) \zeta = \tilde{\partial}_x(b^3) + u^{i,1} \theta_i b^2 + \tilde{\partial}_x(b^2) \zeta. \tag{119}$$

Since  $H_3^1(\hat{\mathcal{A}}[\lambda], D^{loc}) = 0$  [CKS18, theorem 2.13], there exist  $e^0 \in \hat{\mathcal{A}}_2^0[\lambda]$  and  $f^1 \in \hat{\mathcal{A}}_2^1[\lambda]$  such that  $D^{loc}e^0 = a^1 + \tilde{\partial}_x(f^1)$ . Then,

$$\begin{aligned} (D_2 - \lambda D_1)(e^0 \zeta) &= a^1 \zeta + \tilde{\partial}_x(f^1) \zeta - (\hat{E}(P_2) - \lambda \hat{E}(P_1))(e^0) \\ &= a^1 \zeta - (\hat{E}(P_2) - \lambda \hat{E}(P_1))(e^0) - u^{i,1} \theta_i f^1 + \partial_x(f^1 \zeta), \end{aligned} \tag{120}$$

which implies that

$$(d_2 - \lambda d_1) \int e^0 \zeta = \int a^1 \zeta - (\hat{E}(P_2) - \lambda \hat{E}(P_1))(e^0) - u^{i,1} \theta_i f^1. \tag{121}$$

Thus, the cocycle  $\int a^2 + a^1 \zeta$  is cohomologous to  $\int a^2 + (\hat{E}(P_2) - \lambda \hat{E}(P_1))(e^0) + u^{i,1} \theta_i f^1$ , which gives a pure local deformation for  $\int P_2 - \lambda P_1$ .  $\square$

Taking into account that that  $c$  is a bijection, an immediate corollary of proposition 4.9 is the following:

**Corollary 4.10.** *The map  $\iota$  is surjective (and hence a bijection). In particular, every orbit of the action of  $\mathcal{R}_{II}$  on  $T^{nl}$  contains a purely local representative.*

It is just a different way to state theorem 4.1, so this corollary also completes the proof of theorem 4.1.

### 4.3. Roots of the characteristic polynomial of the symbol

In the purely local case the central invariants, besides a purely cohomological definition, can be computed directly from a representative of a deformation (see [DLZ06] for details). More precisely, one has to compute the eigenvalues of the symbol of a representative of a deformation, which behave as scalars with respect to the Miura group action. In this section we extend this viewpoint to the invariants of the Miura-reciprocal group.

First, we recall the construction from [DLZ06]. Let  $\int \sum_{d=1}^\infty \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in T^{loc}$ , and the densities are expanded as  $\sum_{s=0}^d (P_{2,d,s} - \lambda P_{1,d,s})^{ij} \theta_i \theta_j^{d-s}$ ,  $d \geq 1$ , such that

$$\sum_{s=0}^d (P_{2,d,s} - \lambda P_{1,d,s})^{ij} \partial_x^{d-s} = - \sum_{s=0}^d (-\partial_x)^{d-s} \circ (P_{2,d,s} - \lambda P_{1,d,s})^{ji} \tag{122}$$

Consider the symbol of the densities of the bi-vector  $\int \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d})$ , that is, the sum  $\sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{ij} = \sum_{d=1}^{\infty} (-\epsilon)^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{ji}$ . The construction of the Miura group invariants from the eigenvalues of the symbol is based on the following lemma:

**Lemma 4.11.** *Under the group of Miura transformations  $\mathcal{G}$  the symbol transforms linearly as a pencil of bi-linear forms:*

$$\sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{ij} \mapsto \sum_{d=0}^{\infty} \epsilon^d \frac{\partial w_d^i}{\partial u^{k,d}} \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{k\ell} \sum_{d=1}^{\infty} (-\epsilon)^d \frac{\partial w_d^j}{\partial u^{\ell,d}} \tag{123}$$

(here  $w^i = \sum_{d=0}^{\infty} \epsilon^d w_d^i$ ,  $w_d^i \in \mathcal{A}_d$ ,  $i = 1, \dots, N$ , are the new coordinates). Hence, the eigenvalues of this pencil behave as scalar with respect to the action of the Miura group.

There are  $N$  roots  $\lambda_i$ ,  $i = 1, \dots, N$ , of the  $\lambda$ -polynomial

$$\det \left( \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0}) \right) \tag{124}$$

which are the formal power series in  $\epsilon$  with the coefficients given by smooth functions in  $u^1, \dots, u^N$ , with the leading terms in  $\epsilon$  given by  $m_i = u^i + O(\epsilon)$ . These eigenvalues are further used to derive the closed formulas for the central invariants of a pencil  $\int \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in T^{\text{loc}}$ .

In the weakly non-local case of localizable shape, the densities of  $\int \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in T^{\text{nl}}$  can be uniquely expanded as  $\sum_{s=0}^d (P_{2,d,s} - \lambda P_{1,d,s})^{ij} \theta_i \theta_j^{d-s} + (Q_{2,d} - \lambda Q_{1,d})^i \theta_i \zeta$ ,  $d \geq 1$ , such that

$$\sum_{s=0}^d (P_{2,d,s} - \lambda P_{1,d,s})^{ij} \partial_x^{d-s} = - \sum_{s=0}^d (-\partial_x)^{d-s} \circ (P_{2,d,s} - \lambda P_{1,d,s})^{ji}. \tag{125}$$

(this expansion we call the ‘normal form’ below).

**Proposition 4.12.** *Let  $\int \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d} - \lambda P_{1,d}) \in T^{\text{nl}}$  and let*

$$\lambda^i = r^i + \epsilon^2 \lambda_2^i + \epsilon^4 \lambda_4^i + \dots$$

*be the  $\lambda$ -roots of the characteristic polynomial (124). The quantities*

$$c_{2k}^i = \frac{\lambda_{2k}^i}{(f^i)^k}, \quad k = 0, 1, 2, \dots \tag{126}$$

(where  $f^i$  are the diagonal entries of the first metric in canonical coordinates) are invariant under the action of  $\mathcal{R}$ .

**Proof.** Taking into account the above lemma we focus on pure reciprocal transformations. The action of reciprocal transformations of 1st kind on the coefficients of the symbols can be easily obtained using the same arguments used in the proof of proposition (2.8). Indeed

$$\begin{aligned} \tilde{P}_\lambda^{ij} &= B^{-1} \left( B\delta_k^i - u_x^i \partial_x^{-1} \frac{\partial B}{\partial u^k} \right) P_\lambda^{k\ell} \left( B\delta_\ell^j + \frac{\partial B}{\partial u^\ell} \partial_x^{-1} u_x^j \right) \\ &= B P_\lambda^{ij} + P_\lambda^{il} \frac{\partial B}{\partial u^\ell} \partial_x^{-1} u_x^j - \frac{1}{B} u_x^i \partial_x^{-1} \frac{\partial B}{\partial u^k} P_\lambda^{kj} - u_x^i \partial_x^{-1} \frac{\partial B}{\partial u^k} P_\lambda^{k\ell} \frac{\partial B}{\partial u^\ell} \partial_x^{-1} u_x^j. \end{aligned}$$

The second, the third and the fourth terms cannot contribute to the symbol of  $\tilde{P}_\lambda^{ij}$ , while in the first term the only contributions come from

$$\begin{aligned} B \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0}) \partial_x^d &= B \sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0}) B^d \partial_y^d \\ &= B^2 \sum_{d=1}^{\infty} (B\epsilon)^{d-1} (P_{2,d,0} - \lambda P_{1,d,0}) \partial_y^d \end{aligned}$$

that implies

$$\sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{ij} \rightarrow B^2 \sum_{d=1}^{\infty} (B\epsilon)^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{ij}.$$

This means that

$$\lambda^i \rightarrow \lambda^i + (B\epsilon)^2 \lambda_2^i + (B\epsilon)^4 \lambda_4^i + \dots$$

or, equivalently, that

$$\lambda_{2k}^i \rightarrow B^{2k} \lambda_{2k}^i.$$

The result then follows from the transformation rule for the contravariant metric (see (49)):  $f^i \rightarrow B^2 f^i$ . In the case of reciprocal transformation of 2nd kind we observe that they do not affect the symbol of the pencil. Indeed a bivector transforms according to the following rule

$$\tilde{P}^{ij} := B^{-1} \left( B\delta_k^i - u_x^i \partial_x^{-1} \frac{\partial B}{\partial u_\sigma^k} \partial_x^\sigma \right) P^{k\ell} \left( B\delta_\ell^j + (-\partial_x)^\tau \frac{\partial B}{\partial u_\tau^\ell} \partial_x^{-1} u_x^j \right). \quad (127)$$

where

$$B = 1 + H = 1 + \sum_{k=1}^{\infty} \epsilon^k H_k(u^j, u_x^j, \dots, u_\sigma^j), \quad H_k \in \mathcal{A}_k.$$

Thus we have

$$\tilde{P}^{ij} := P^{ij} - \frac{1}{B} u_x^i \partial_x^{-1} \frac{\partial H}{\partial u_\sigma^k} \partial_x^\sigma P^{kj} + \left( \delta_k^i - \frac{1}{B} u_x^i \partial_x^{-1} \frac{\partial H}{\partial u_\sigma^k} \partial_x^\sigma \right) P^{k\ell} \left( H\delta_\ell^j + (-\partial_x)^\tau \frac{\partial H}{\partial u_\tau^\ell} \partial_x^{-1} u_x^j \right). \quad (128)$$

Since the symbol of the bivector contains only the subset of the coefficients which depend only on the  $u$ 's but not on their  $x$ -derivatives the second term and the third terms above cannot contribute to it. This implies that the symbol of each bivector defining the pencil is unaffected by these transformations. For Miura reciprocal transformations (11) the transformation rule

for the symbol of the pencil is obtained combining the lemma 4.11 with the above rule. It turns out that the symbol of the pencil transforms in the following way

$$\sum_{d=1}^{\infty} \epsilon^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{ij} \mapsto B^2 \sum_{d=0}^{\infty} \epsilon^d \frac{\partial w_d^i}{\partial u^{k,d}} \sum_{d=1}^{\infty} (B\epsilon)^{d-1} (P_{2,d,0} - \lambda P_{1,d,0})^{k\ell} \sum_{d=1}^{\infty} (-\epsilon)^d \frac{\partial w_d^j}{\partial u^{\ell,d}}. \tag{129}$$

□

**Remark 4.13.** Notice that in the local case the quantities (126) for  $k = 1$  (up to a constant) coincide with the standard central invariants. However, we point out that while in the standard setting the division by  $f^i$  allows to get functions of a single variable but is not important in order to get invariant quantities (the functions  $\lambda_2^i$  already define a complete set of invariants), in this more general setting the division by  $f^i$  becomes crucial for this purpose.

### 5. Projective-reciprocal invariance of the Doyle–Potěmin form

In this section we make a first step towards the study of the projective-reciprocal group action. Consider a local operator of homogeneous differential order  $d + 2$ ,  $d \geq 2$  of the form  $P^{ij} = \partial_x \circ Q^{ij} \circ \partial_x$ . We call this presentation of an operator the *Doyle–Potěmin form* (see section 1.5).

We prove the following theorem:

**Theorem 5.1.** *The projective group preserves the Doyle–Potěmin form of an operator. More precisely, the image of a homogeneous skew-symmetric operator of the form  $\partial_x \circ Q^{ij} \circ \partial_x$ ,  $\text{deg}_{\partial_x} Q^{ij} = d \geq 0$  under the action of an element of  $\mathcal{P}$  is a homogeneous skew-symmetric operator of the form  $\partial_x \circ \tilde{Q}^{ij} \circ \partial_x$ ,  $\text{deg}_{\partial_x} \tilde{Q}^{ij} = d \geq 0$*

**Proof.** Consider an element of the group  $\mathcal{P}$  given by

$$\begin{aligned} dy &= A^0 dx, \\ w^i &= A^i / A^0, \end{aligned} \quad i = 1, \dots, N, \tag{130}$$

where  $A^i := a_l^i u^l + a_0^i$ ,  $i = 0, 1, \dots, N$ . Since the functions  $A^i$  and  $A^0$  do not depend on the higher jet variables, theorem 2.5 implies that the operator  $P^{ij} = \partial_x \circ Q^{ij} \circ \partial_x$  in the coordinates  $y, w^1, \dots, w^N$  is represented as

$$\begin{aligned} \tilde{P}^{ij} &= \frac{1}{A^0} \left( A^0 \partial_{u^k} \left( \frac{A^i}{A^0} \right) - \partial_x \left( \frac{A^i}{A^0} \right) \partial_x^{-1} \circ \partial_{u^k} A^0 \right) \partial_x \circ Q^{kl} \circ \partial_x \\ &\quad \circ \left( \partial_{u^l} \left( \frac{A^j}{A^0} \right) A^0 + \partial_{u^l} A^0 \partial_x^{-1} \circ \partial_x \left( \frac{A^j}{A^0} \right) \right). \end{aligned} \tag{131}$$

Now we see that

$$\begin{aligned} \partial_x \circ \left( \partial_{u^l} \left( \frac{A^j}{A^0} \right) A^0 + \partial_{u^l} A^0 \partial_x^{-1} \circ \partial_x \left( \frac{A^j}{A^0} \right) \right) &= \partial_x \circ \left( a_l^j - a_l^0 \left( \frac{A^j}{A^0} \right) + a_l^0 \partial_x^{-1} \circ \partial_x \left( \frac{A^j}{A^0} \right) \right) \\ &= \left( a_l^j - a_l^0 \left( \frac{A^j}{A^0} \right) \right) \partial_x = (A^0)^2 \partial_{u^l} w^j \partial_y \end{aligned} \tag{132}$$

and analogously

$$\begin{aligned} \frac{1}{A^0} \left( A^0 \partial_{u^k} \left( \frac{A^i}{A^0} \right) - \partial_x \left( \frac{A^i}{A^0} \right) \partial_x^{-1} \circ \partial_{u^k} A^0 \right) \partial_x &= \frac{1}{A^0} \left( a_k^i - \left( \frac{A^i}{A^0} \right) a_k^0 - \partial_x \left( \frac{A^i}{A^0} \right) \partial_x^{-1} \circ a_k^0 \right) \partial_x \\ &= \frac{1}{A^0} \partial_x \circ \left( a_k^i - \left( \frac{A^i}{A^0} \right) a_k^0 \right) = \partial_y \circ \frac{1}{A^0} \partial_{u^k} w^i (A^0)^2. \end{aligned} \tag{133}$$

Thus we see that  $\tilde{P}^{ij}$  takes the form  $\partial_y \circ \tilde{Q}^{ij} \circ \partial_y$ , where the operator  $\tilde{Q}^{ij}$  is equal to

$$\tilde{Q}^{ij} = \frac{1}{A^0} \partial_{u^k} w^i (A^0)^2 Q^{kl} (A^0)^2 \partial_{u^l} w^j \tag{134}$$

after the substitution  $w^i(u^1, \dots, u^N) = A^i/A^0$  and  $\partial_y = (A^0)^{-1} \partial_x$ , which makes it manifestly skew-symmetric and homogeneous of the same degree.  $\square$

**Remark 5.2.** Note that we do not use the Poisson property in the proof (and we do not have it in the statement of the theorem). This allows us to apply the projective-reciprocal transformation to any homogeneous skew-symmetric operators of the Doyle–Potěmin form, and the action would preserve the form.

**Remark 5.3.** Interesting examples of skew-symmetric operators in the Doyle–Potěmin form are coming from the theory of Dubrovin–Zhang hierarchies [DZ01]. Is it proved in [BPS12a, BPS12b] that Dubrovin–Zhang hierarchies possess a Poisson bracket given by an operator of the shape  $\sum_{p=0}^{\infty} \epsilon^{2p} P_{2p+1}^{ij}$ , where  $P_1^{ij} = \eta^{ij} \partial_x$  for some constant inner product  $\eta^{ij}$ , and for  $p \geq 1$  the operators  $P_{2p+1}^{ij}$  are homogeneous skew-symmetric operators of the shape  $\sum_{e=0}^{2p+1} P_{2p+1,e}^{ij} \partial_x^{2p+1-e}$ , where  $\text{deg}_{\partial_x} P_{2p+1,e}^{ij} = e$ , such that  $P_{2p+1,0}^{ij} = 0$ . Using that the operators  $P_{2p+1}^{ij}$ ,  $p \geq 1$ , are skew-symmetric, it is easy to show that each of them is of the Doyle–Potěmin form.

### Data availability statement

No new data were created or analysed in this study.

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