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Optimal domain of Volterra operators in Korenblum spaces

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The aim of this article is to study the largest domain space $[T, X]$, whenever it exists, of a given continuous linear operator $T\colon X \to X$, where $X \subseteq H(\mathbb{D})$ is a Banach space of analytic functions on the open unit disc $\mathbb{D} \subseteq \mathbb{C}$. That is, $[T, X] \subseteq H(\mathbb{D})$ is the *largest* Banach space of analytic functions containing X to which T has a continuous, linear, X-valued extension $T\colon [T, X] \to X$. The class of operators considered consists of generalized Volterra operators T acting in the Korenblum growth Banach spaces $X := A^{-\gamma}$, for $\gamma > 0$. Previous studies dealt with the classical Cesàro operator $T := C$ acting in the Hardy spaces H^p, $1 \leq p < \infty$, [18], [19], in $A^{-\gamma}$, [3], and more recently, generalized Volterra operators T acting in $X := H^p$, [9].

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1. Introduction and preliminaries

Let X, Y be Banach spaces (of functions) and $T\colon X \to Y$ be a linear operator, perhaps given by an explicit formula or arising from an inequality which makes T continuous. Situations occur, for instance, whenever there exists $f \notin X$ such that $Tf \in Y$, when it is desirable to determine the possibility of extending T beyond its initial domain space X (but still taking its values in Y) to a larger space of functions Z containing X. Ideally, Z should be the largest or *optimal* domain for the extension of T. To make a point, no one would be content with specifying the $c_0(\mathbb{Z})$-valued Fourier transform operator on, say, $L^2([-\pi,\pi])$; its natural optimal domain (within the realm of function spaces) should be $L^1([-\pi,\pi])$. Of course, more substantial examples exist. As a sample, we point out that various classical inequalities (for example, the Sobolev inequality, [16], [17], [21], the Hausdorff-Young inequality, [26], and so on) have been shown to remain valid for larger domain spaces of functions than those in which they were initially formulated. The same is true for well known classes of operators acting between spaces of measurable functions, such as kernel operators, [15], convolution operators, [28], Fourier multiplier operators, [27], and others; see also [29] and the references there in.

The theme of this paper is analogous to that described above. However, the setting is now rather different and will require new methods and techniques. The aim is to investigate the *extension procedure* alluded to above for linear operators acting between Banach spaces of *analytic functions* on the open unit disc $\mathbb{D} \subseteq \mathbb{C}$. The ambient space involved is now the Fréchet space $H(\mathbb{D})$ consisting of all analytic functions on $\mathbb{D}$, equipped with the topology of uniform convergence on the compact subsets of $\mathbb{D}$. Given is a Banach space $X \subseteq H(\mathbb{D})$, containing the polynomials, for which the natural inclusion map is continuous and a continuous linear operator $T\colon X \to X$. The fundamental question is as follows: do there exist further Banach spaces $Z \subseteq H(\mathbb{D})$ such that $X \subseteq Z$ and $T\colon X \to X$ has a continuous, linear, X-valued extension $T\colon Z \to X$? If so, is there a *largest* such space Z, the so called *optimal domain space* of T, and can it be identified? In some cases the optimal domain Z is actually a *new* Banach space of analytic functions not available before, which generates an interest to determine structural properties of Z. Moreover, the factorization of $T\colon X \to X$ as $T_Z \circ j$, where $j\colon X \to Z$ is the inclusion map and $T_Z\colon Z \to X$ is the extension of T, provides a technique to examine operator ideal properties such as compactness, weak compactness, etc. The above question was first introduced (and solved) for the classical Cesàro operator $C := T$ acting in the family of Hardy spaces $X := H^p$, for $1 \leq p < \infty$, [18], [19], and in $X := A^{-\gamma}$, for $\gamma > 0$, [3]. In the recent article [9] a similar investigation was undertaken for the family of generalized Volterra operators $T := V_g$, with $g \in BMO \cap H^2$, acting in H^p for $1 \leq p < \infty$. If $g(z) = -\mathrm{Log}(1-z)$, for $z \in \mathbb{D}$, then $V_g(f)(z) = zC(f)(z)$, for $f \in H^p$ and $z \in \mathbb{D}$.

In this article we undertake a detailed study of the optimal domain space for the particular class of (Volterra) integral operators given by $(V_g f)(z) = \int_0^z f(\xi)g'(\xi)\,d\xi$, for $z \in \mathbb{D}$ and $f \in X$, with the function $g \in H(\mathbb{D})$ coming from the Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$, and where X varies through the family of Korenblum growth Banach spaces $A^{-\gamma}$ and

$A_0^{-\gamma}$, for $\gamma > 0$; see Section 3 for the definitions and notation. We caution the reader that the above operators V_g correspond to those which are denoted by T_g in [9] and vice versa. The notation T_g is used in this paper for another operator defined in Section 5.

It is clear from the above discussion that there is a need to clearly formulate various concepts and to identify a precise abstract setting for the "optimal extension procedure"; see Definition 2.2. This is carried out in Section 2, which not only creates the framework for this paper but, also for possible future research on this topic in general. In particular, the basic properties of the optimal domain space $[T, X]$ for a continuous linear operator $T\colon X \to X$, with $X \subseteq H(\mathbb{D})$ a Banach space of analytic functions on $\mathbb{D}$, are established in Propositions 2.4–2.7.

In Section 3 we concentrate on the *Korenblum growth spaces* $A^{-\gamma}$ and $A_0^{-\gamma}$, for $\gamma > 0$, all of which are Banach spaces of analytic functions on $\mathbb{D}$. The class of operators which we consider consists of the Volterra operators V_g, for $g \in \mathcal{B}$ (resp. $g \in \mathcal{B}_0$), defined above, which are necessarily continuous (cf. Propositions 3.1 and 3.2) and whose optimal domain space $[V_g, A^{-\gamma}]$ (resp. $[V_g, A_0^{-\gamma}]$) turns out to also be a Banach space of analytic functions on $\mathbb{D}$; see Proposition 3.3 and Corollary 3.4. An alternative description of these optimal domain spaces is given in Proposition 3.5; see also Remark 3.6. In Example 3.7 a class of functions $g \in \mathcal{B}$ is presented for which $A^{-\gamma} \subsetneqq [V_g, A^{-\gamma}]$, that is, $V_g\colon A^{-\gamma} \to A^{-\gamma}$ has a *genuine* $A^{-\gamma}$-valued extension to $[V_g, A^{-\gamma}]$.

Each function $h \in H(\mathbb{D})$ generates the continuous linear operator $M_h\colon H(\mathbb{D}) \to H(\mathbb{D})$ of pointwise multiplication by h. Given a pair of Banach spaces of analytic functions on $\mathbb{D}$, say X and Y, a function $h \in H(\mathbb{D})$ is called a *multiplier* for X, Y if $M_h(X) \subseteq Y$. The space of all such multipliers h is denoted by $\mathcal{M}(X, Y)$ or, if $X = Y$, simply by $\mathcal{M}(X)$. Since $[T, X]$ is typically a Banach space of analytic functions on $\mathbb{D}$, it is natural to study its multiplier space $\mathcal{M}([T, X])$. For the optimal domain space $[C, H^p]$ of the Cesàro operator C, for $1 \le p < \infty$, it is known that $\mathcal{M}([C, H^p]) = H^p$, [18, Theorem 3.7], and for the generalized Volterra operator T_g, whenever $g \in BMO \cap H^2$, considered in [9] that $\mathcal{M}([T_g, H^p]) = H^\infty$ for each $1 \le p < \infty$; see Theorem 4 there. In Section 4 we establish that the multiplier space $\mathcal{M}([V_g, A^{-\gamma}]) = H^\infty$ and $\mathcal{M}([V_g, A_0^{-\gamma}]) = H^\infty$ for each $\gamma > 0$ and $g \in \mathcal{B}$; see Proposition 4.2 and Corollary 4.3. For the Korenblum spaces themselves it is also shown, whenever $0 < \gamma < \delta$, that $\mathcal{M}(A^{-\gamma}, A^{-\delta}) = A^{-(\delta-\gamma)} = \mathcal{M}(A_0^{-\gamma}, A_0^{-\delta})$; see Proposition 4.5 and also [9, Proposition 5], [13, Proposition 3.1]. Moreover, Proposition 4.6 reveals that $\mathcal{M}(A^{-\gamma}, A_0^{-\delta}) = A_0^{-(\delta-\gamma)}$.

In the final Section 5 we turn our attention to the *weighted* Banach spaces of analytic functions H_v^∞ and H_v^0 on $\mathbb{D}$ for certain weight functions $v\colon [0,1) \to (0, \infty)$. The family of operators considered consists of the Cesàro operator C together with V_g for certain functions g. Aleman and Persson have made a detailed study of generalized Cesàro operators acting in various Banach spaces of analytic functions, [4], [5], [30], including the spaces $A^{-\gamma}$ and $A_0^{-\gamma}$; see also [3]. We begin by considering properties of the optimal domain spaces $[V_g, H_v^\infty]$ and $[T_g, H_v^\infty]$ and also of $[V_g, H_v^0]$ and $[T_g, H_v^0]$, where T_g is given by (5.3); see Proposition 5.3. Of particular interest is the Cesàro operator $C = T_{g_0}$ with $g_0(z) = -\mathrm{Log}(1 - z)$, for $z \in \mathbb{D}$. It turns out that the optimal domain spaces $[C, A^{-\gamma}]$

and $[V_{g_0}, A^{-\gamma}]$ with $\gamma > 0$ (resp. $[C, A_0^{-\gamma}]$ and $[V_{g_0}, A_0^{-\gamma}]$) are *genuinely larger* than their initial domain spaces $A^{-\gamma}$ (resp. $A_0^{-\gamma}$); see Proposition 5.4. Also Propositions 5.5–5.7 and Proposition 5.9 exhibit related features. In Proposition 5.12 a large class of functions g in $\mathcal{B}$ is exhibited for which both of the operators $V_g\colon A^{-\gamma} \to A^{-\gamma}$ and $V_g\colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range; see Example 5.13 for some particular functions g to which Proposition 5.12 applies.

Given Fréchet spaces X and Y, denote by $\mathcal{L}(X,Y)$ the space of all continuous linear operators from X into Y. For the case when $X = Y$, we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$. If both X and Y are Banach spaces then, for the operator norm $\|T\|_{X\to Y} := \sup_{\|x\|_X\le 1} \|Tx\|_Y$, the space $\mathcal{L}(X,Y)$ is a Banach space. Equipped with the topology of pointwise convergence on a Fréchet space X (i.e., the strong operator topology τ_s) the quasi-complete locally convex Hausdorff space $\mathcal{L}(X)$ is denoted by $\mathcal{L}_s(X)$. The range $T(X) := \{Tx :\ x \in X\}$ of $T \in \mathcal{L}(X)$ is also denoted by $\mathrm{Im}(T)$. Furthermore, $\mathrm{Ker}(T) := \{x \in X : Tx = 0\}$.

Let X be a Fréchet space. The identity operator on X is written as I. The *transpose operator* of $T \in \mathcal{L}(X)$ is denoted by T^*; it acts from the topological dual space $X^* := \mathcal{L}(X,\mathbb{C})$ of X into itself. Denote by X^*_σ (resp., by X^*_β) the space X^* equipped with the weak* topology $\sigma(X^*,X)$ (resp., with the strong topology $\beta(X^*,X)$). It is known that X^*_σ is quasicomplete with $T^* \in \mathcal{L}(X^*_\sigma)$ and that $T^* \in \mathcal{L}(X^*_\beta)$, [24, p.134]. The bi-transpose operator $(T^*)^*$ of T is simply denoted by T^{**} and belongs to $\mathcal{L}((X^*_\beta)^*_\beta)$. In the event that X is a Banach space, both X^*_β (denoted simply by X^*) and $(X^*_\beta)^*_\beta$ (denoted simply by X^{**}) are again Banach spaces. The dual norm in X^* is given by $\|x^*\| := \sup_{\|x\|\le 1} |\langle x, x^*\rangle|$, for $x^* \in X^*$.

A linear map $T\colon X \to Y$, with X and Y Banach spaces, is called *compact* if $T(B_X)$ is a relatively compact set in Y, where B_X denotes the closed unit ball of X. It is routine to show that necessarily $T \in \mathcal{L}(X,Y)$.

2. General properties concerning optimal domains

Denote by $H(\mathbb{D})$ the space of all analytic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The space $H(\mathbb{D})$ is equipped with the topology τ_c of uniform convergence on the compact subsets of $\mathbb{D}$. According to [23, §27.3(3)] the space $H(\mathbb{D})$ is a Fréchet-Montel space. An increasing family of norms generating τ_c is given, for each $0 < r < 1$, by

$$q_r(f) := \sup_{|z|\le r} |f(z)|, \quad f \in H(\mathbb{D}). \tag{2.1}$$

We identify a function $f \in H(\mathbb{D})$ with its sequence of Taylor coefficients $\hat{f} := (\hat{f}(n))_{n\in\mathbb{N}_0}$ (i.e., $\hat{f}(n) := \frac{f^{(n)}(0)}{n!}$, for $n \in \mathbb{N}_0$), so that $f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n$, for $z \in \mathbb{D}$. Given $z \in \mathbb{D}$, it is clear from (2.1) that each evaluation functional $\delta_z\colon f \mapsto f(z)$, for $f \in H(\mathbb{D})$, is linear and continuous, that is, $\delta_z \in H(\mathbb{D})^*$.

A vector space $X \subseteq H(\mathbb{D})$ is called a *Banach space of analytic functions* on $\mathbb{D}$ if it is a Banach space relative to a norm $\|\cdot\|_X$ and the natural inclusion of X into $H(\mathbb{D})$ is continuous.

Lemma 2.1. *Let X be a Banach space such that $X \subseteq H(\mathbb{D})$ as linear spaces. Then the natural inclusion map of X into $H(\mathbb{D})$ is continuous if and only if $\{\delta_z : z \in \mathbb{D}\} \subseteq X^*$.*

Proof. Suppose that $X \subseteq H(\mathbb{D})$ continuously. Fix $z \in \mathbb{D}$. Let $(f_n)_{n\in\mathbb{N}} \subseteq X$ satisfy $\|f_n\|_X \to 0$ as $n \to \infty$. Then also $f_n \to 0$ in $H(\mathbb{D})$ as $n \to \infty$. Since $\delta_z \in H(\mathbb{D})^*$ it follows that $f_n(z) = \langle f_n, \delta_z\rangle \to 0$ in $\mathbb{C}$ as $n \to \infty$. This shows that $\delta_z \colon X \to \mathbb{C}$ is continuous, that is $\delta_z \in X^*$.

Suppose now that $\{\delta_z : z \in \mathbb{D}\} \subseteq X^*$. Let $(f_n)_{n\in\mathbb{N}} \subseteq X$ satisfy $f_n \to 0$ in X and $f_n \to g$ in $H(\mathbb{D})$, for some $g \in H(\mathbb{D})$, as $n \to \infty$. Fix $z \in \mathbb{D}$. Since $\delta_z \in X^*$, it follows that $f_n(z) = \langle f_n, \delta_z\rangle \to 0$ in $\mathbb{C}$ as $n \to \infty$. Also $\delta_z \in H(\mathbb{D})^*$ and so $\langle f_n, \delta_z\rangle \to \langle g, \delta_z\rangle$, that is, $f_n(z) \to g(z)$ in $\mathbb{C}$ as $n \to \infty$. Hence, $g(z) = 0$. Since $z \in \mathbb{D}$ is arbitrary, we may conclude that $g = 0$ in $H(\mathbb{D})$. Accordingly, the inclusion map $X \subseteq H(\mathbb{D})$ is a closed, linear map which implies its continuity by the closed graph theorem for Fréchet spaces. □

Let $(X, \|\cdot\|_X)$ be a Banach space of analytic functions on $\mathbb{D}$ and $T \colon H(\mathbb{D}) \to H(\mathbb{D})$ be a continuous, linear operator satisfying $T(X) \subseteq X$. By the closed graph theorem it follows that $T \in \mathcal{L}(X)$. Indeed, if $(f_n)_{n\in\mathbb{N}} \subseteq X$ satisfies both $f_n \to 0$ in X and $Tf_n \to g$ in X for some $g \in X$ as $n \to \infty$, then also $f_n \to 0$ in $H(\mathbb{D})$ and $Tf_n \to g$ in $H(\mathbb{D})$ as $n \to \infty$. But, $T \in \mathcal{L}(H(\mathbb{D}))$. This implies that $Tf_n \to 0$ in $H(\mathbb{D})$ as $n \to \infty$. So, $g = 0$. Hence, $T \colon X \to X$ is a closed operator.

The following notion first occurs in [18] for the Cesàro operator $T := C$ acting in $X = H^p$.

Definition 2.2. The *optimal domain* of an operator $T \in \mathcal{L}(H(\mathbb{D}))$ satisfying $T(X) \subseteq X$, with X a Banach space of analytic functions on $\mathbb{D}$, is the linear subspace

$$[T, X] := \{f \in H(\mathbb{D}) \colon \; Tf \in X\}$$

of $H(\mathbb{D})$ endowed with the semi-norm

$$\|f\|_{[T,X]} := \|Tf\|_X, \quad f \in [T, X].$$

Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ satisfy $T(X) \subseteq X$. A Banach space Z of analytic functions on $\mathbb{D}$ is said to be a (T, X)-*admissible space* if it satisfies

$$T(Z) \subseteq X \subseteq Z. \tag{2.2}$$

Then the map $T_Z \colon Z \to X$ defined by $T_Z f := Tf$, for $f \in Z$, is an X-valued, linear extension of $T \colon X \to X$ which, in this notation, can also be written as T_X.

Lemma 2.3. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ satisfy $T(X) \subseteq X$. Let Z be a (T, X)-admissible space.*

(i) *The inclusion $X \subseteq Z$ is continuous.*
(ii) *The linear map $T_Z \colon Z \to X$ is continuous and its restriction to X coincides with $T \colon X \to X$.*

Proof. (i) Let $\Phi \colon X \to Z$ denote the natural inclusion map. Suppose that $(f_n)_{n\in\mathbb{N}} \subseteq X$ satisfies $f_n \to 0$ in X and $\Phi f_n \to g$ in Z, for some $g \in Z$. Lemma 2.1 implies that $\{\delta_z : z \in \mathbb{D}\} \subseteq X^*$. Accordingly, for any $w \in \mathbb{D}$, we have $f_n(w) = \langle f_n, \delta_w\rangle \to 0$ in $\mathbb{C}$ as $n \to \infty$. Lemma 2.1 also implies that $\{\delta_z : z \in \mathbb{D}\} \subseteq Z^*$ and so $f_n(w) = \langle \Phi f_n, \delta_w\rangle \to \langle g, \delta_w\rangle = g(w)$ in $\mathbb{C}$ as $n \to \infty$. Accordingly, $g(w) = 0$. Since $w \in \mathbb{D}$ is arbitrary, we have $g = 0$. The closed graph theorem ensures that Φ is continuous.

(ii) It is clear from the definition of T_Z that it is linear and its restriction to X coincides with $T \colon X \to X$. Let $(p_n)_{n\in\mathbb{N}} \subseteq Z$ satisfy $p_n \to 0$ in Z and $T_Z p_n \to q$ in X, for some $q \in X$. Since $(T_Z p_n)_{n\in\mathbb{N}} = (T p_n)_{n\in\mathbb{N}} \subseteq X$, it follows from part (i) that $T p_n \to q$ in Z as $n \to \infty$. Moreover, $T(Z) \subseteq Z$ by (2.2) and so, via the discussion prior to Definition 2.2, we know that $T \in \mathcal{L}(Z)$. Hence, $p_n \to 0$ in Z implies that $T p_n \to 0$ in Z and so we can conclude that $q = 0$. By the closed graph theorem $T_Z \colon Z \to X$ is continuous. □

Lemma 2.3 states, whenever Z is a (T, X)-admissible space, that $T_Z \colon Z \to X$ is a *continuous*, X-valued, linear extension of $T \colon X \to X$ from X to Z. Our aim is to show that the choice $Z = [T, X]$ yields the *largest* (T, X)-admissible space, which thereby justifies the terminology "optimal domain of T" used in Definition 2.2. First we require some preparation.

Proposition 2.4. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ satisfy $T(X) \subseteq X$, in which case $T \in \mathcal{L}(X)$. Then the following properties are satisfied.*

(i) *$[T, X] \subseteq H(\mathbb{D})$.*
(ii) *Suppose that $T \in \mathcal{L}(H(\mathbb{D}))$ is injective. Then $([T, X], \|\cdot\|_{[T,X]})$ is a normed space.*
(iii) *$X \subseteq [T, X]$ with a continuous inclusion from X into the semi-normed space $[T, X]$.*
(iv) *The operator $T_{[T,X]} \colon [T, X] \to X$, defined by $f \mapsto Tf$, is linear and continuous from the semi-normed space $[T, X]$ into X.*
(v) *Let Y be a closed subspace of X satisfying $T(Y) \subseteq Y$. Then Y is a Banach space of analytic functions on $\mathbb{D}$ and $[T, Y]$ is a closed subspace of $[T, X]$.*
(vi) $\mathrm{Ker}(T_X) = X \cap \mathrm{Ker}(T_{[T,X]}) \subseteq \mathrm{Ker}(T_{[T,X]}) \subseteq \mathrm{Ker}(T)$ *with $T \in \mathcal{L}(H(\mathbb{D}))$.*

Proof. (i) Clear from the definition.

(ii) Let $f \in [T, X]$ satisfy $\|f\|_{[T,X]} = 0$. Then $f \in H(\mathbb{D})$ and $\|Tf\|_X = \|f\|_{[T,X]} = 0$. Since T is injective when it acts on $H(\mathbb{D})$, it follows that $f = 0$. So, $\|\cdot\|_{[T,X]}$ is a norm on $[T, X]$.

(iii) Since $T(X) \subseteq X$, it is clear that $X \subseteq [T,X]$. On the other hand, for every $f \in X$ we have that $\|f\|_{[T,X]} = \|Tf\|_X \leq \|T\|_{X\to X}\|f\|_X$. So, the inclusion $X \subseteq [T,X]$ is continuous.

(iv) Clearly, $T_{[T,X]}$ is linear. Moreover, given $f \in [T,X]$ we have that $\|T_{[T,X]}f\|_X = \|Tf\|_X = \|f\|_{[T,X]}$. So, the operator $T_{[T,X]}\colon [T,X] \to X$ is surely continuous.

(v) Clearly Y, endowed with the norm $\|\cdot\|_X$ inherited from X, is a Banach space of analytic functions on $\mathbb{D}$. Since $T(Y) \subseteq Y$, it is clear that the restriction $T\colon Y \to Y$ of $T \in \mathcal{L}(X)$ is continuous. Let $f \in [T,Y]$, that is, $f \in H(\mathbb{D})$ satisfies $Tf \in Y$. Then also $Tf \in X$ and hence, $f \in [T,X]$. So, $[T,Y] \subseteq [T,X]$.

Now, let $(f_n)_{n\in\mathbb{N}} \subseteq [T,Y]$ be a sequence convergent to some f in $[T,X]$. This means that

$$\|f_n - f\|_{[T,X]} = \|Tf_n - Tf\|_X \to 0, \quad \text{as } n \to \infty.$$

But, $(Tf_n)_{n\in\mathbb{N}} \subseteq Y$ and Y is a closed subspace of X. It follows that $Tf \in Y$ and hence, that $f \in [T,Y]$. This shows that $[T,Y]$ is a closed subspace of $[T,X]$.

(vi) Clear from the definitions involved. □

Combining Lemma 2.3 and Proposition 2.4 yields the following result.

Proposition 2.5. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ satisfy $T(X) \subseteq X$. Suppose that $[T,X]$ is a Banach space. Then, amongst all (T,X)-admissible spaces, the optimal domain space $[T,X]$ of T is the largest one.*

Proof. By hypothesis $[T,X] \subseteq H(\mathbb{D})$ is a Banach space of analytic functions on $\mathbb{D}$. Moreover, $X \subseteq [T,X]$ continuously; see Proposition 2.4(iii). Let $f \in [T,X]$. According to Definition 2.2 the function $f \in H(\mathbb{D})$ and $Tf \in X$. Hence, T maps $[T,X]$ into X. So, (2.2) is satisfied with $[T,X]$ in place of Z, that is, $[T,X]$ is a (T,X)-admissible space.

Let Z be any (T,X)-admissible space. Given $f \in Z$ we have $f \in H(\mathbb{D})$ and, via (2.2), necessarily $Tf \in X$. Hence, $f \in [T,X]$; see Definition 2.2. This shows that $Z \subseteq [T,X]$. □

Proposition 2.5 shows the importance of being able to decide when $[T,X]$ is a Banach space. The following result provides a sufficient condition.

Proposition 2.6. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ be an isomorphism of $H(\mathbb{D})$ satisfying $T(X) \subseteq X$. Then $([T,X], \|\cdot\|_{[T,X]})$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq [T,X]$ be a Cauchy sequence in $[T,X]$. Then $(Tf_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a Banach space, there exists $g \in X$ such that $Tf_n \to g$ in X as $n \to \infty$ and hence, $Tf_n \to g$ in $H(\mathbb{D})$ as $n \to \infty$. Since $T\colon H(\mathbb{D}) \to H(\mathbb{D})$ is an isomorphism, it follows that $f_n \to T^{-1}g$ in $H(\mathbb{D})$ as $n \to \infty$. Accordingly, $h :=$

$T^{-1}g \in H(\mathbb{D})$ satisfies $Th = g \in X$. So, $h \in [T, X]$. On the other hand, $\|f_n - h\|_{[T,X]} = \|Tf_n - Th\|_X = \|Tf_n - g\|_X \to 0$ as $n \to \infty$, that is, $f_n \to h$ in $[T, X]$ as $n \to \infty$. $\square$

In the following result we analyze when the inclusion $[T, X] \subseteq H(\mathbb{D})$ (see Proposition 2.4(i)) is continuous, equivalently, when $[T, X]$ is a Banach space.

Proposition 2.7. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ be an injective operator satisfying $T(X) \subseteq X$, in which case $\|\cdot\|_{[T,X]}$ is a norm. The following three properties are equivalent.*

(i) *The inclusion $[T, X] \subseteq H(\mathbb{D})$ is continuous.*
(ii) *The evaluation functionals $\{\delta_z : z \in \mathbb{D}\} \subseteq [T, X]^*$ and, for each compact subset K of $\mathbb{D}$, the set $\{\delta_w : \ w \in K\}$ is bounded in $[T, X]^*$.*
(iii) *$([T, X], \|\cdot\|_{[T,X]})$ is a Banach space.*

If any one of (i)–(iii) *is satisfied, then $[T, X]$ is a Banach space of analytic functions on $\mathbb{D}$ and $\{\delta_z : z \in \mathbb{D}\} \subseteq [T, X]^*$.*

Proof. (i)⇒(ii) Let $z \in \mathbb{D}$. The evaluation functional $\delta_z \colon H(\mathbb{D}) \to \mathbb{C}$ is known to be continuous. By (i) it follows that $\delta_z \colon [T, X] \to \mathbb{C}$ is also continuous, that is, $\delta_z \in [T, X]^*$.

For a fixed compact subset K of $\mathbb{D}$, by the assumption of (i) and the definition of the topology τ_c in $H(\mathbb{D})$ there exists $C > 0$ such that

$$\sup_{z \in K} |f(z)| \leq C\|f\|_{[T,X]}, \quad f \in [T, X].$$

Fix $w \in K$. Given $f \in [T, X]$ satisfying $\|f\|_{[T,X]} \leq 1$, it follows from the previous inequality that

$$|\delta_w(f)| = |f(w)| \leq C.$$

This implies that $\delta_w \in [T, X]^*$ and $\|\delta_w\|_{[T,X]^*} \leq C$. So, $\{\delta_w : \ w \in K\}$ is a bounded subset of $[T, X]^*$.

(ii)⇒(i) Let K be a compact subset of $\mathbb{D}$. Since $\{\delta_z : \ z \in K\}$ is a bounded subset of $[T, X]^*$, there exists $C > 0$ such that $\|\delta_w\|_{[T,X]^*} \leq C$ for every $w \in K$. It follows, for every $w \in K$, that

$$\left|\frac{f(w)}{\|f\|_{[T,X]}}\right| = \left|\delta_w\left(\frac{f}{\|f\|_{[T,X]}}\right)\right| \leq C, \quad f \in [T, X] \setminus \{0\},$$

and hence, that

$$|f(w)| \leq C\|f\|_{[T,X]}.$$

Accordingly, $\sup_{w\in K}|f(w)| \leq C\|f\|_{[T,X]}$ for every $f \in [T,X]$. Since K is an arbitrary compact subset of $\mathbb{D}$, this implies that the inclusion $[T,X] \subseteq H(\mathbb{D})$ is continuous.

(i)⇒(iii) Let $(f_n)_{n\in\mathbb{N}} \subseteq [T,X]$ be a Cauchy sequence in $[T,X]$, that is, $\|f_n - f_m\|_{[T,x]} = \|Tf_n - Tf_m\|_X \to 0$ as $n,m \to \infty$. Accordingly, $(Tf_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence in X and hence, there exists $g \in X$ such that $Tf_n \to g$ in X as $n \to \infty$. Then also $Tf_n \to g$ in $H(\mathbb{D})$ as $n \to \infty$. On the other hand, by the hypothesis of (i) the sequence $(f_n)_{n\in\mathbb{N}} \subseteq H(\mathbb{D})$ is also a Cauchy sequence in $H(\mathbb{D})$. Therefore, there exists $f \in H(\mathbb{D})$ such that $f_n \to f$ in $H(\mathbb{D})$ as $n \to \infty$. Since $T \in \mathcal{L}(H(\mathbb{D}))$, it follows that $Tf_n \to Tf$ in $H(\mathbb{D})$ as $n \to \infty$. So, $Tf = g$ in $H(\mathbb{D})$. But, $g \in X$ and so $Tf \in X$, that is, $f \in [T,X]$. Moreover, $\|f_n - f\|_{[T,X]} = \|Tf_n - Tf\|_X = \|Tf_n - g\|_X \to 0$ as $n \to \infty$, that is, $f_n \to f$ in $[T,X]$. Accordingly, $[T,X]$ is a Banach space relative to $\|\cdot\|_{[T,X]}$.

(iii)⇒(i) We first prove that the natural inclusion $J\colon [T,X] \to H(\mathbb{D})$, defined by $f \mapsto Jf := f$, has a closed graph. To see this, let $(f_n)_{n\in\mathbb{N}} \subseteq [T,X]$ satisfy $f_n \to f$ in $[T,X]$ and $Jf_n = f_n \to g$ in $H(\mathbb{D})$ for some $g \in H(\mathbb{D})$ as $n \to \infty$. The fact that $(f_n)_{n\in\mathbb{N}} \subseteq [T,X]$ converges to f in $[T,X]$ implies that $Tf_n \to Tf$ in X as $n \to \infty$, and hence, that $Tf_n \to Tf$ in $H(\mathbb{D})$ as $n \to \infty$. On the other hand, $T \in \mathcal{L}(H(\mathbb{D}))$ implies that $Tf_n \to Tg$ in $H(\mathbb{D})$ as $n \to \infty$. Therefore, $Tf = Tg$. Since T is injective, it follows that $f = g$. So, J has a closed graph.

By property (iii) the space $[T,X]$ is a Banach space. Moreover, $H(\mathbb{D})$ is a Fréchet space. So, the continuity of the inclusion $J\colon [T,X] \to H(\mathbb{D})$ follows by the closed graph theorem for Fréchet spaces.

Finally, if any one of (i)–(iii) is satisfied, then $[T,X]$ is a Banach space with $[T,X] \subseteq H(\mathbb{D})$ and the inclusion is continuous. Hence, $[T,X]$ is a Banach space of analytic functions on $\mathbb{D}$. □

We now establish further properties of the space $([T,X], \|\cdot\|_{[T,X]})$ and of the inclusion $X \subseteq [T,X]$ under suitable assumptions.

Proposition 2.8. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ be an isomorphism satisfying $T(X) \subseteq X$. Then $([T,X], \|\cdot\|_{[T,X]})$ is isometrically isomorphic to X.*

Proof. The injectivity of $T \in \mathcal{L}(H(\mathbb{D}))$ clearly implies that the operator $T\colon [T,X] \to X$, given by $f \mapsto Tf$, is injective. Moreover, the fact that $T \in \mathcal{L}(H(\mathbb{D}))$ is surjective implies that the operator $T\colon [T,X] \to X$, is surjective. Indeed, given $g \in X \subseteq H(\mathbb{D})$ the function $T^{-1}g \in H(\mathbb{D})$ exists and satisfies $T(T^{-1}g) = g$. Accordingly, $T^{-1}g \in [T,X]$ and $T(T^{-1}g) = g$. On the other hand, $\|Tf\|_X = \|f\|_{[T,X]}$ for every $f \in [T,X]$. So, $T\colon [T,X] \to X$ is a surjective isometry. In particular, $([T,X], \|\cdot\|_{[T,X]})$ is isometrically isomorphic to X. □

Proposition 2.9. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T \in \mathcal{L}(H(\mathbb{D}))$ be an injective operator such that $T(X) \subseteq X$ and $([T,X], \|\cdot\|_{[T,X]})$ is a Banach space. The following properties are equivalent.*

(i) *The continuous linear operator* $T\colon X \to X$ *has closed range in* X*.*
(ii) *The natural inclusion* $J_{[T,X]}\colon X \to [T,X]$*, defined by* $f \mapsto J_{[T,X]}f := f$*, has closed range in* $[T,X]$*.*

Proof. (i)⇒(ii) Since $T(X) \subseteq X$ is a closed subspace of X, necessarily $(T(X), \|\cdot\|_X)$ is a Banach space. So, an application of the open mapping theorem implies that the continuous, bijective operator $T\colon X \to T(X)$ is an isomorphism. Accordingly, there exists $c > 0$ such that

$$\|Tf\|_X \geq c\|f\|_X, \quad f \in X. \tag{2.3}$$

This implies that the inclusion $J_{[T,X]}\colon X \to [T,X]$ has closed range. Indeed, from (2.3) it follows that

$$\|J_{[T,X]}f\|_{[T,X]} = \|f\|_{[T,X]} = \|Tf\|_X \geq c\|f\|_X, \quad f \in X,$$

with $J_{[T,X]}$ continuous (cf. Proposition 2.4(iii)), which yields that the inclusion $J_{[T,X]}\colon X \to [T,X]$ has closed range.

(ii)⇒(i) Since the inclusion $J_{[T,X]}\colon X \to [T,X]$ has closed range, the linear subspace $J_{[T,X]}(X) = X$ is closed in the Banach space $[T,X]$. So, $(X, \|\cdot\|_{[T,X]})$ is a Banach space. In view of the continuity of $J_{[T,X]}\colon (X, \|\cdot\|_X) \to ([T,X], \|\cdot\|_{[T,X]})$, we observe that the identity operator $L\colon (X, \|\cdot\|_X) \to (X, \|\cdot\|_{[T,X]})$ is continuous and bijective. Therefore, we can apply the open mapping theorem to conclude that $L\colon (X, \|\cdot\|_X) \to (X, \|\cdot\|_{[T,X]})$ is an isomorphism. Accordingly, there exists $c > 0$ such that $\|Lf\|_{[T,X]} \geq c\|f\|_X$ for every $f \in X$, and hence,

$$\|Tf\|_X = \|f\|_{[T,X]} = \|Lf\|_{[T,X]} \geq c\|f\|_X, \quad f \in X.$$

This implies that the operator $T\colon X \to X$ has closed range. □

An immediate consequence of Proposition 2.9 is the following fact.

Corollary 2.10. *Let* X *be a Banach space of analytic functions on* $\mathbb{D}$ *and* $T \in \mathcal{L}(H(\mathbb{D}))$ *be an injective operator such that* $T(X) \subseteq X$ *and* $([T,X], \|\cdot\|_{[T,X]})$ *is a Banach space.*

(i) *Suppose that the natural inclusion* $J_{[T,X]}\colon X \to [T,X]$ *is surjective. Then the operator* $T\colon X \to X$ *has closed range in* X*.*
(ii) *Suppose that the operator* $T\colon X \to X$ *fails to have closed range in* X*. Then* X *is a proper subspace of* $[T,X]$*.*

Proof. (i) The inclusion $J_{[T,X]}\colon X \to [T,X]$ is a continuous bijection of X onto its range in $[T,X]$ and hence, it is an isomorphism by the open mapping theorem. Accordingly,

$J_{[T,X]}$ has a closed range in $[T, X]$. Then Proposition 2.9 implies that $T: X \to X$ has a closed range in X.

(ii) Suppose that $X = [T, X]$. Then the inclusion $J_{[T,X]}: X \to [T, X]$ is surjective. Since $([T, X], \|\cdot\|_{[T,X]})$ is a Banach space, the open mapping theorem again implies that $J_{[T,X]}: X \to [T, X]$ is an isomorphism and hence, it has a closed range in $[T, X]$. Therefore, by Proposition 2.9, also the operator $T: X \to X$ has closed range in X; a contradiction. So, $X \subsetneq [T, X]$. $\square$

3. Optimal domain of Volterra operators on Korenblum growth Banach spaces

Let $g \in H(\mathbb{D})$ be a non-constant function. The *Volterra operator* $V_g: H(\mathbb{D}) \to H(\mathbb{D})$ is the linear operator defined by

$$(V_g f)(z) := \int_0^z f(\xi)g'(\xi)\, d\xi, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}. \tag{3.1}$$

The operator V_g acts continuously in $H(\mathbb{D})$. Moreover, V_g is *injective* on $H(\mathbb{D})$, [6], but, not surjective because $(V_g f)(0) = 0$ for every $f \in H(\mathbb{D})$.

In the definition of the operators V_g it can be assumed, if necessary, that $g(0) = 0$. Indeed, the functions g and $G(z) := g(z) - g(0)$, for $z \in \mathbb{D}$, define the same Volterra operator in $H(\mathbb{D})$, that is, $V_G = V_g$, because $G'(z) = g'(z)$ for all $z \in \mathbb{D}$. For $g(z) := z$ the operator V_g reduces to the classical (Volterra) integral operator.

The Volterra operators V_g have been investigated on different spaces of analytic functions by many authors. We refer to [14,33], for example, and the references therein.

Let us briefly recall the definition of the relevant spaces involved. For each $\gamma > 0$ the *Korenblum growth Banach spaces* are defined by

$$A^{-\gamma} := \{f \in H(\mathbb{D}) :\ \|f\|_{-\gamma} := \sup_{z\in\mathbb{D}} (1 - |z|)^\gamma |f(z)| < \infty\}$$

and its (proper) closed subspace by

$$A_0^{-\gamma} := \{f \in H(\mathbb{D}) :\ \lim_{|z|\to 1^-} (1 - |z|)^\gamma |f(z)| = 0\}.$$

Both are Banach spaces when endowed with the norm

$$\|f\|_{-\gamma} := \sup_{z\in\mathbb{D}} (1 - |z|)^\gamma |f(z)|. \tag{3.2}$$

The space $A_0^{-\gamma}$ coincides with the closure of the polynomials in $A^{-\gamma}$, [32, Lemma 3], and point evaluations on $\mathbb{D}$ belong to both $(A_0^{-\gamma})^*$ and $(A^{-\gamma})^*$, [32, Lemma 1]. Hence, via Lemma 2.1, both $A_0^{-\gamma}$ and $A^{-\gamma}$ are Banach spaces of analytic functions on $\mathbb{D}$, for every $\gamma > 0$. Moreover, the bidual $(A_0^{-\gamma})^{**} = A^{-\gamma}$ for all $\gamma > 0$, [31], [32, Theorem 2].

Whenever $0 < \gamma < \beta$, the inclusion $A^{-\gamma} \subseteq A^{-\beta}$ is *proper*. This follows from the fact (routine to verify) that $A^{-\gamma} \subseteq A_0^{-\beta}$ and that $A_0^{-\beta}$ is a proper subspace of $A^{-\beta}$.

Related to $A^{-\gamma}$ and $A_0^{-\gamma}$ are the *Bloch spaces*. A function $f \in H(\mathbb{D})$ belongs to the *Bloch space* $\mathcal{B}$ whenever $f' \in A^{-1}$, that is, $\sup_{z\in\mathbb{D}}(1-|z|)|f'(z)| < \infty$. The Bloch space $\mathcal{B}$ is a Banach space of analytic functions on $\mathbb{D}$ when endowed with the norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|)|f'(z)|. \tag{3.3}$$

The inequalities

$$(1-|z|) \leq (1-|z|^2) = (1-|z|)(1+|z|) \leq 2(1-|z|), \quad z \in \mathbb{D},$$

show that the norm (3.3) is equivalent to the norm

$$f \mapsto |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$$

in $\mathcal{B}$, which is also commonly used. A function $f \in H(\mathbb{D})$ belongs to the *little Bloch space* $\mathcal{B}_0$ if $f \in \mathcal{B}$ and

$$\lim_{|z|\to 1^-}(1-|z|)|f'(z)| = 0. \tag{3.4}$$

The little Bloch space $\mathcal{B}_0$ is a closed subspace of the Bloch space $\mathcal{B}$ and hence, $\mathcal{B}_0$ is a Banach space when it is endowed with the norm defined in (3.3). Moreover, the bidual $\mathcal{B}_0^{**} = \mathcal{B}$. For properties of Bloch spaces, see [7], [34], for example.

The following continuity and compactness results for the operators V_g on both $A^{-\gamma}$ and $A_0^{-\gamma}$ are known; see [8, Theorems 1 and 2] or [25, Proposition 3.1].

Proposition 3.1. *Let $g \in H(\mathbb{D})$. For each $\gamma > 0$ the following three properties are equivalent.*

(i) *The operator $V_g\colon A^{-\gamma} \to A^{-\gamma}$ is continuous.*
(ii) *The operator $V_g\colon A_0^{-\gamma} \to A_0^{-\gamma}$ is continuous.*
(iii) *The function $g \in \mathcal{B}$.*

Proposition 3.2. *Let $g \in H(\mathbb{D})$. For each $\gamma > 0$ the following three properties are equivalent.*

(i) *The operator $V_g\colon A^{-\gamma} \to A^{-\gamma}$ is compact.*
(ii) *The operator $V_g\colon A_0^{-\gamma} \to A_0^{-\gamma}$ is compact.*
(iii) *The function $g \in \mathcal{B}_0$.*

Since $A^{-\gamma}$ and $A_0^{-\gamma}$ are Banach spaces of analytic functions on $\mathbb{D}$ and, for any non-constant function $g \in \mathcal{B}$, the Volterra operator $V_g \in \mathcal{L}(H(\mathbb{D}))$ is injective and satisfies both $V_g(A^{-\gamma}) \subseteq A^{-\gamma}$ and $V_g(A_0^{-\gamma}) \subseteq A_0^{-\gamma}$ (cf. Proposition 3.1), we can define the optimal domain for each of the operators $V_g\colon A^{-\gamma} \to A^{-\gamma}$ and $V_g\colon A_0^{-\gamma} \to A_0^{\cdot\gamma}$ via Definition 2.2. Namely,

$$[V_g, A^{-\gamma}] := \{f \in H(\mathbb{D}) : \; V_g f \in A^{-\gamma}\}$$

and

$$[V_g, A_0^{-\gamma}] := \{f \in H(\mathbb{D}) : \; V_g f \in A_0^{-\gamma}\},$$

where both spaces are endowed, respectively, with the semi-norm

$$\|f\|_{[V_g, A^{-\gamma}]} := \|V_g f\|_{-\gamma} \quad \text{and} \quad \|f\|_{[V_g, A_0^{-\gamma}]} := \|V_g f\|_{-\gamma}. \tag{3.5}$$

Since $V_g \in \mathcal{L}(H(\mathbb{D}))$ is injective, Proposition 2.4(ii) implies that $\|\cdot\|_{[V_g, A^{-\gamma}]}$ and $\|\cdot\|_{[V_g, A_0^{-\gamma}]}$ are actually norms. Moreover, again by Proposition 2.4(v) we have that $[V_g, A_0^{-\gamma}]$ is a closed subspace of $[V_g, A^{-\gamma}]$.

We point out, for the Hardy spaces H^p on $\mathbb{D}$, with $1 < p < \infty$, that the study of the optimal domain of $V_g\colon H^p \to H^p$ has recently been treated in [9].

Proposition 3.3. *Let $g \in \mathcal{B}$ be a non-constant function. For each $\gamma > 0$, both $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$ are Banach spaces.*

Proof. In view of Proposition 2.7 to establish that $[V_g, A^{-\gamma}]$ is a Banach space, it suffices to show that the inclusion map $J\colon [V_g, A^{-\gamma}] \to H(\mathbb{D})$, defined by $f \mapsto Jf := f$, is continuous. To this effect, note that Theorem 5.5 in [20], with $p = \infty$, implies that the differentiation operator $D \in \mathcal{L}(H(\mathbb{D}))$ given by $Dh := h'$, for $h \in H(\mathbb{D})$, maps $A^{-\gamma}$ into $A^{-(\gamma+1)}$. Since the evaluation functionals at points of $\mathbb{D}$ belong to $(A^{-\gamma})^*$ and $(A^{-(\gamma+1)})^*$, a closed graph argument shows that $D \in \mathcal{L}(A^{-\gamma}, A^{-(\gamma+1)})$; see also [22, Theorem 2.1(a)]. Hence, there exists $C > 0$ such that

$$\|h'\|_{-(\gamma+1)} = \|Dh\|_{-(\gamma+1)} \leq C\|h\|_{-\gamma}, \quad h \in A^{-\gamma}. \tag{3.6}$$

Now, fix $r \in (0,1)$ and $f \in [V_g, A^{-\gamma}]$. Since g' is not identically zero on $\mathbb{D}$ (as g is a non-constant function) and the zeros of $g' \in H(\mathbb{D})$ are isolated, there exists $s \in (r, 1)$ such that $g'(z) \neq 0$ for every $z \in \mathbb{D}$ such that $|z| = s$. Accordingly, $m := \min_{|z|=s} |g'(z)| > 0$. It follows, for every $u \in \mathbb{D}$ satisfying $|u| \leq r$, that

$$|f(u)| \leq \max_{|z|=s} |f(z)| = \max_{|z|=s} \frac{|f(z)g'(z)|}{|g'(z)|} \leq \frac{1}{m} \max_{|z|=s} |f(z)g'(z)|. \tag{3.7}$$

On the other hand, $f(z)g'(z) = (V_g f)'(z)$, for $z \in \mathbb{D}$, and hence, via (3.6), we obtain that

$$\|fg'\|_{-(\gamma+1)} = \|(V_g f)'\|_{-(\gamma+1)} \le C\|V_g f\|_{-\gamma} = C\|f\|_{[V_g, A^{-\gamma}]}. \tag{3.8}$$

Combining (3.7) and (3.8) it follows, for every $u \in \mathbb{D}$ with $|u| \le r$, that

$$\begin{aligned}
|f(u)| \le & \frac{1}{m} \max_{|z|=s} |f(z)g'(z)| = \frac{1}{m} \max_{|z|=s} |(V_g f)'(z)| \\
= & \frac{1}{m} \frac{1}{(1-s)^{\gamma+1}} (1-s)^{\gamma+1} \max_{|z|=s} |(V_g f)'(z)| \\
= & \frac{1}{m} \frac{1}{(1-s)^{\gamma+1}} \max_{|z|=s} (1-|z|)^{\gamma+1} |(V_g f)'(z)| \\
\le & \frac{1}{m} \frac{1}{(1-s)^{\gamma+1}} \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+1} |(V_g f)'(z)| \\
\le & \frac{C}{m(1-s)^{\gamma+1}} \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma} |(V_g f)(z)| \\
= & \frac{C}{m(1-s)^{\gamma+1}} \|V_g f\|_{-\gamma} = \frac{C}{m(1-s)^{\gamma+1}} \|f\|_{[V_g, A^{-\gamma}]}.
\end{aligned}$$

In view of (2.1) this implies that

$$q_r(f) = \sup_{|u|\le r} |f(u)| \le \frac{C}{m(1-s)^{\gamma+1}} \|f\|_{[V_g, A^{-\gamma}]}.$$

Since $r \in (0,1)$ and $f \in [V_g, A^{-\gamma}]$ are arbitrary, it follows that the inclusion $J\colon [V_g, A^{-\gamma}] \to H(\mathbb{D})$ is continuous. As noted above, this implies that $[V_g, A^{-\gamma}]$ is a Banach space.

Recall that $A_0^{-\gamma}$ is a closed subspace of $A^{-\gamma}$ and $V_g(A_0^{-\gamma}) \subseteq A_0^{-\gamma}$. So, Proposition 2.4(v) implies that $[V_g, A_0^{-\gamma}]$ is a closed subspace of $[V_g, A^{-\gamma}]$. Accordingly, $[V_g, A_0^{-\gamma}]$ is also a Banach space. □

Corollary 3.4. *Let $g \in \mathcal{B}$ be a non-constant function. For each $\gamma > 0$, both of the optimal domain spaces $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$ are Banach spaces of analytic functions on $\mathbb{D}$. In particular, their dual spaces contain $\{\delta_z : \ z \in \mathbb{D}\}$.*

Proof. Proposition 3.3 shows that both $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$ are Banach spaces. Since $V_g \in \mathcal{L}(H(\mathbb{D}))$ is injective, Proposition 2.7 gives the desired conclusion. □

The following result gives an alternate description of the optimal domain spaces $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$.

Proposition 3.5. *Let $g \in \mathcal{B}$ be a non-constant function and $\gamma > 0$.*

(i) *The optimal domain space* $[V_g, A^{-\gamma}] = \{f \in H(\mathbb{D}) : \ fg' \in A^{-(\gamma+1)}\}$. *Moreover, there exists* $C > 0$ *satisfying*

$$\|fg'\|_{-(\gamma+1)} \leq C\|f\|_{[V_g,A^{-\gamma}]}, \quad f \in [V_g, A^{-\gamma}].$$

(ii) *The optimal domain space* $[V_g, A_0^{-\gamma}] = \{f \in H(\mathbb{D}) : \ fg' \in A_0^{-(\gamma+1)}\}$. *Moreover, there exists* $C > 0$ *satisfying*

$$\|fg'\|_{-(\gamma+1)} \leq C\|f\|_{[V_g,A_0^{-\gamma}]}, \quad f \in [V_g, A_0^{-\gamma}].$$

Proof. (i) According to Definition 2.2 and (3.1), a function $f \in H(\mathbb{D})$ belongs to $[V_g, A^{-\gamma}]$ if and only if the function $(V_g f)(z) = \int_0^z f(\xi)g'(\xi)\, d\xi$, for $z \in \mathbb{D}$, belongs to $A^{-\gamma}$. But, $(V_g f)' = fg'$ and so [20, Theorem 5.3] implies that this is equivalent to the fact that $fg' \in A^{-(\gamma+1)}$; see also [22, Proposition 2.2(a)]. This shows that $[V_g, A^{-\gamma}] = \{f \in H(\mathbb{D}) : fg' \in A^{-(\gamma+1)}\}$ as sets.

Let $C > 0$ be a constant satisfying (3.8) for every $f \in [V_g, A^{-\gamma}]$. It follows that

$$\|fg'\|_{-(\gamma+1)} = \|(V_g f)'\|_{-(\gamma+1)} \leq C\|V_g f\|_{-\gamma} = C\|f\|_{[V_g,A^{-\gamma}]}, \quad f \in [V_g, A^{-\gamma}],$$

which is the stated inequality.

(ii) Observe that a function $f \in A_0^{-\gamma}$ if and only if $f' \in A_0^{-(\gamma+1)}$ (see the proof of Fact 2 in [3, Theorem 3.2]). So, the result follows by a similar argument as in part (i). $\square$

Remark 3.6. Let $\gamma > 0$ and $g \in \mathcal{B}$ satisfy $|g'| > 0$ on $\mathbb{D}$ and $\frac{1}{g'} \in H^\infty(\mathbb{D})$.

(i) Let the linear space $E := \{f \in H(\mathbb{D}) : \ fg' \in A^{-(\gamma+1)}\}$ be endowed with the norm

$$\|f\|_E := \|fg'\|_{-(\gamma+1)}, \quad f \in E.$$

The claim is that E is a Banach space isomorphic to $[V_g, A^{-\gamma}]$. Indeed, let $(f_n)_{n\in\mathbb{N}} \subseteq E$ be a Cauchy sequence in E, that is, $(f_n g')_{n\in\mathbb{N}}$ is a Cauchy sequence in $A^{-(\gamma+1)}$. Then there exists $h \in A^{-(\gamma+1)}$ such that $f_n g' \to h$ in $A^{-(\gamma+1)}$. But, the multiplication operator $M_{\frac{1}{g'}} : A^{-(\gamma+1)} \to A^{-(\gamma+1)}$ is continuous (as $\frac{1}{g'} \in H^\infty(\mathbb{D})$) and so $f_n \to \frac{h}{g'}$ in $A^{-(\gamma+1)}$. Since $\frac{h}{g'} \in H(\mathbb{D})$, it follows that $f_n \to \frac{h}{g'}$ in E. Hence, E is a Banach space.

Now, by Proposition 3.5(i) the operator $I\colon [V_g, A^{-\gamma}] \to E$, defined by $f \mapsto f$, is continuous. On the other hand, by Proposition 3.3, $[V_g, A^{-\gamma}]$ is a Banach space. So, by the open mapping theorem we can conclude that E is a Banach space which is isomorphic to $[V_g, A^{-\gamma}]$, that is, the norms $\|\cdot\|_{[V_g,A^{-\gamma}]}$ and $\|\cdot\|_E$ are equivalent.

(ii) Let the linear space $E_0 := \{f \in H(\mathbb{D}) : \ fg' \in A_0^{-(\gamma+1)}\}$ be endowed with the norm $\|\cdot\|_E$. Clearly, E_0 is a subspace of E. Moreover, E_0 is a Banach space. Indeed, E_0 is a closed subspace of E. This follows after observing that if $(f_n)_{n\in\mathbb{N}} \subseteq E_0$ is a sequence converging to some f in E, then $(f_n g')_{n\in\mathbb{N}} \subseteq A_0^{-(\gamma+1)}$ and $f_n g' \to fg'$ in $A^{-(\gamma+1)}$. Hence, $fg' \in A_0^{-(\gamma+1)}$ as $A_0^{-(\gamma+1)}$ is a closed subspace of $A^{-(\gamma+1)}$.

As in part (i), it can be argued that E_0 is isomorphic to $[V_g, A_0^{-\gamma}]$.

We apply Proposition 3.5 to show that the optimal domain space of V_g can be genuinely larger than $A^{-\gamma}$.

Example 3.7. Let $g \in \mathcal{B}$ satisfy $|g'| > 0$ on $\mathbb{D}$ and $\frac{1}{g'} \in H^\infty$. For example, $g'(z) = \frac{1}{1-z}$, for $z \in \mathbb{D}$. Suppose there exists $w \in \mathbb{C}$ with $|w| = 1$ such that $\lim_{r \to 1^-} |g'(rw)|(1-r) = 0$. In this case the claim is that $A^{-\gamma}$ is a *proper* subspace of $[V_g, A^{-\gamma}]$, for all $\gamma > 0$. For instance, consider $g(z) := -\mathrm{Log}(1-z)$, for $z \in \mathbb{D}$, and $w = -1$. Or, if $g \in \mathcal{B}_0$ satisfies $|g'| > 0$ on $\mathbb{D}$ and $\frac{1}{g'} \in H^\infty$, then the same features occur because $\lim_{|z| \to 1^-} (1-z)|g'(z)| = 0$ implies that $\lim_{r \to 1^-} |g'(rw)|(1-r) = 0$.

To establish the claim let $f(z) := \frac{1}{g'(z)(1-\overline{w}z)^{\gamma+1}}$, for $z \in \mathbb{D}$. Since $1 - \overline{w}z = 0$ if and only if $z = w \notin \mathbb{D}$, and $g'(z) \neq 0$ for all $z \in \mathbb{D}$, we have that $f \in H(\mathbb{D})$. On the other hand, $|1 - \overline{w}z| \geq 1 - |\overline{w}z| = 1 - |z|$ for all $z \in \mathbb{D}$, and hence, $\sup_{z \in \mathbb{D}} \frac{(1-|z|)^{\gamma+1}}{|1-\overline{w}z|^{\gamma+1}} \leq 1$. It follows that the function

$$f(z)g'(z) = \frac{1}{(1-\overline{w}z)^{\gamma+1}}, \quad z \in \mathbb{D},$$

belongs to $A^{-(\gamma+1)}$. In view of Proposition 3.5(i) we conclude that $f \in [V_g, A^{-\gamma}]$. But,

$$\begin{aligned}\sup_{z \in \mathbb{D}} (1-|z|)^\gamma |f(z)| &\geq \sup_{r \in (0,1)} (1-|rw|)^\gamma \frac{1}{|g'(rw)||1-\overline{w}rw|^{\gamma+1}} \\ &= \sup_{r \in (0,1)} \frac{1}{|g'(rw)||1-r|} = \infty\end{aligned}$$

because $\frac{1}{g'} \in H^\infty$. Accordingly, $f \notin A^{-\gamma}$.

Suppose, in addition, that the function $g \in \mathcal{B}$ satisfies $|g'(rw)|(1-r)^{1/2} \leq c$ for every $r \in (0,1)$ and some constant $c > 0$. The claim is that also $A_0^{-\gamma}$ is a *proper* subspace of $[V_g, A_0^{-\gamma}]$. Indeed, let $f_0(z) := (1-z)^{1/2} f(z)$, for $z \in \mathbb{D}$. Since $\lim_{|z| \to 1^-} (1-z)^{1/2} = 0$ and $fg' \in A^{-(\gamma+1)}$, the function

$$f_0(z)g'(z) = (1-z)^{1/2} f(z)g'(z), \quad z \in \mathbb{D},$$

belongs to $A_0^{-(\gamma+1)}$. In view of Proposition 3.5(ii) we can conclude that $f \in [V_g, A^{-\gamma}]$. But, $f \notin A_0^{-\gamma}$ because, for every $r \in (0,1)$, we have

$$\begin{aligned}(1-|rw|)^\gamma |f(rw)| &\geq \frac{(1-|rw|)^\gamma (1-|rw|)^{1/2}}{|g'(rw)|1-\overline{w}rw|^{\gamma+1}} \\ &= \frac{(1-r)^{\gamma+1/2}}{|g'(rw)|(1-r)^{\gamma+1}} = \frac{1}{|g'(rw)|(1-r)^{1/2}} \geq \frac{1}{c},\end{aligned}$$

which implies that $\lim_{|z| \to 1^-} (1-|z|)^\gamma |f(z) \neq 0$.

We now formulate useful descriptions of the Banach spaces $A^{-\gamma}$ and $A_0^{-\gamma}$ in terms of the optimal domain of Volterra operators.

Proposition 3.8. *For every $\gamma > 0$ the Korenblum growth space $A^{-\gamma}$ satisfies*

$$A^{-\gamma} = \cap_{g\in\mathcal{B}}[V_g, A^{-\gamma}]. \tag{3.9}$$

Proof. From Proposition 2.4(iii) and Proposition 3.5 it follows that $A^{-\gamma} \subseteq \cap_{g\in\mathcal{B}}[V_g, A^{-\gamma}]$.

To establish the reverse inclusion, fix $f \in \cap_{g\in\mathcal{B}}[V_g, A^{-\gamma}]$ and consider the operator $S_f\colon H(\mathbb{D}) \to H(\mathbb{D})$ defined by

$$(S_f g)(z) := \int_0^z g'(\xi)f(\xi)\,d\xi = \int_0^z f(\xi)(Dg)(\xi)\,d\xi, \quad g \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

The facts that $f \in H(\mathbb{D})$ and $D \in \mathcal{L}(H(\mathbb{D}))$ clearly imply that $S_f \in \mathcal{L}(H(\mathbb{D}))$; see (2.1) for the definition of τ_c. On the other hand, for each $g \in \mathcal{B}$, we have that $V_g f \in A^{-\gamma}$ (cf. Proposition 2.4(iii) and Proposition 3.5) and

$$(S_f g)(z) = \int_0^z g'(\xi)f(\xi)\,d\xi = (V_g f)(z), \quad z \in \mathbb{D}.$$

Therefore, $S_f(\mathcal{B}) \subseteq A^{-\gamma}$ and hence, the linear map $S_f\colon \mathcal{B} \to A^{-\gamma}$. Moreover, the operator $S_f\colon \mathcal{B} \to A^{-\gamma}$ has a closed graph and hence, it is continuous. To see this let $(g_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ be a sequence such that $g_n \to g$ in $\mathcal{B}$ and $S_f g_n \to h$ in $A^{-\gamma}$ as $n \to \infty$. Since both $\mathcal{B}$ and $A^{-\gamma}$ are Banach spaces of analytic functions on $\mathbb{D}$ we have that $g_n \to g$ and $S_f g_n \to h$ in $H(\mathbb{D})$ as $n \to \infty$. But, $S_f \in \mathcal{L}(H(\mathbb{D}))$ and so $S_f g_n \to S_f g$ in $H(\mathbb{D})$ as $n \to \infty$. Accordingly, $S_f g = h$. So, $S_f \in \mathcal{L}(\mathcal{B}, A^{-\gamma})$.

Recall from above that the differentiation operator $D\colon A^{-\gamma} \to A^{-(\gamma+1)}$ is continuous and hence, also the operator $D \circ S_f\colon \mathcal{B} \to A^{-(\gamma+1)}$, given by $g \mapsto g'f$, is continuous. Accordingly, there exists $C > 0$ such that

$$\|(D \circ S_f)g\|_{-(\gamma+1)} = \sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+1}|g'(z)f(z)| \le C\|g\|_{\mathcal{B}}, \quad g \in \mathcal{B}. \tag{3.10}$$

In particular, for $g(z) = z$, for $z \in \mathbb{D}$, it follows from (3.10) and $\|g\|_{\mathcal{B}} = 1$ that $|f(0)| \le C$.

Now, for each $u \in \mathbb{D} \setminus \{0\}$ let $g_u \in H(\mathbb{D})$ satisfy $g_u'(z) := \frac{1}{1-\frac{\overline{u}}{|u|}z}$, for $z \in \mathbb{D}$, and $g_u(0) = 0$ (namely, $g_u(z) = -\frac{|u|}{\overline{u}}\mathrm{Log}(1 - \frac{\overline{u}}{|u|}z)$. Then, for each $u \in \mathbb{D} \setminus \{0\}$, we have that $\sup_{z\in\mathbb{D}}(1-|z|)|g_u'(z)| \le 1$ and so $g_u \in \mathcal{B}$. Moreover, $(1-|u|)|g_u'(u)| = \frac{1-|u|}{|1-\frac{\overline{u}u}{|u|}|} = 1$ for every $u \in \mathbb{D} \setminus \{0\}$. Hence, $\|g_u\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|)|g_u'(z)| = 1$. Via (3.3) and (3.10) we can conclude that

$$(1-|u|)^{\gamma+1}|g_u'(u)||f(u)| \le C\sup_{z\in\mathbb{D}}(1-|z|)|g_u'(z)| = C, \quad u \in \mathbb{D} \setminus \{0\},$$

that is,

$$\frac{(1-|u|)^{\gamma+1}|f(u)|}{|1-\frac{\overline{u}u}{|u|}|} = (1-|u|)^{\gamma}|f(u)| \leq C, \quad u \in \mathbb{D} \setminus \{0\}.$$

It follows that $f \in A^{-\gamma}$. $\square$

The analogue of Proposition 3.8 for $A_0^{-\gamma}$ is as follows.

Proposition 3.9. *For each $\gamma > 0$, the Korenblum growth space $A_0^{-\gamma}$ satisfies*

$$A_0^{-\gamma} = \cap_{g\in\mathcal{B}}[V_g, A_0^{-\gamma}].$$

Proof. Proposition 2.4(iii) and Proposition 3.5 imply that $A_0^{-\gamma} \subseteq \cap_{g\in\mathcal{B}}[V_g, A_0^{-\gamma}]$.

For the reverse inclusion fix $f \in \cap_{g\in\mathcal{B}}[V_g, A_0^{-\gamma}]$. It follows from Proposition 3.8 (as $A_0^{-\gamma} \subseteq A^{-\gamma}$) that $f \in A^{-\gamma}$ and $f \in \cap_{g\in\mathcal{B}}[V_g, A^{-\gamma}]$. Proposition 3.1 of [13], for the choices $\varphi(z) := z$ and $\psi(z) := f(z)$, implies that the multiplication operator $M_f\colon A^{-1} \to A^{-(\gamma+1)}$, given by $h \mapsto fh$ for $h \in A^{-1}$, is continuous. Actually, $M_f(A^{-1}) \subseteq A_0^{-(\gamma+1)}$. Indeed, given $h \in A^{-1}$, define $g \in H(\mathbb{D})$ by $g(z) := \int_0^z h(\xi)\, d\xi$ for $z \in \mathbb{D}$. Then $g' = h$ and $g \in \mathcal{B}$ because

$$\sup_{z\in\mathbb{D}}(1-|z|)|g'(z)| = \|h\|_{-1} < \infty.$$

Moreover, $f \in [V_g, A_0^{-\gamma}]$ implies, via Proposition 3.5(ii), that $fg' \in A_0^{-(\gamma+1)}$, that is, $M_f(h) = fh = fg' \in A_0^{-(\gamma+1)}$. Accordingly, $M_f\colon A^{-1} \to A_0^{-(\gamma+1)}$ is continuous and hence, so is its restriction $M_f\colon A_0^{-1} \to A_0^{-(\gamma+1)}$ to the closed subspace $A_0^{-1} \subseteq A^{-1}$. Since the bidual Banach space $(A_0^{-\alpha})^{**} = A^{-\alpha}$ for all $\alpha > 0$, the bi-transpose operator of $M_f\colon A_0^{-1} \to A_0^{-(\gamma+1)}$ is precisely $M_f^{**}\colon A^{-\gamma} \to A^{-(\gamma+1)}$. But, it was shown above that $M_f^{**}((A_0^{-1})^{**}) = M_f(A^{-1}) \subseteq A_0^{-(\gamma+1)}$ and hence, $M_f\colon A_0^{-1} \to A_0^{-(\gamma+1)}$ is a weakly compact operator; see [24, §42, Proposition 2(2)], for example. According to [13, Theorem 5.1] the operator $M_f\colon A_0^{-1} \to A_0^{-(\gamma+1)}$ is then also a compact operator. So, applying Corollary 4.5 of [13] with $\varphi(z) := z$ for $z \in \mathbb{D}$, we can conclude that

$$\lim_{|z|\to 1^-} \frac{|f(z)|(1-|z|)^{\gamma+1}}{1-|z|} = 0,$$

that is, $\lim_{|z|\to 1^-} |f(z)|(1-|z|)^{\gamma} = 0$. This means precisely that $f \in A_0^{-\gamma}$. $\square$

We will require the following fact; see [9, Lemma 11].

Lemma 3.10. *Let $\delta \in (0,1)$ and $g \in A^{-\delta}$. For each $\varepsilon \in (0, \min\{\delta, 1-\delta\})$ there exists a function $\phi \in H(\mathbb{D})$ such that $\phi g \in A^{-(\delta+\varepsilon)} \setminus A^{-\delta}$.*

Proposition 3.11. *Let $g \in \mathcal{B}$ and $0 < \gamma < \beta$.*

(i) $[V_g, A^{-\gamma}]$ *is a proper subspace of* $[V_g, A^{-\beta}]$.
(ii) $[V_g, A_0^{-\gamma}]$ *is a proper subspace of* $[V_g, A_0^{-\beta}]$.

Proof. (i) Since $\gamma < \beta$, we have that $A^{-\gamma} \subseteq A^{-\beta}$. The containment $[V_g, A^{-\gamma}] \subseteq [V_g, A^{-\beta}]$ is then clear from the definition.

Suppose that $[V_g, A^{-\gamma}] = [V_g, A^{-\beta}]$. Given $h \in H(\mathbb{D})$ denote by $Z(h)$ the discrete set of zeros of h repeated with their multiplicity. Consider the subset $X := \{G \in \mathcal{B} : Z(g') \subseteq Z(G')\}$ of $\mathcal{B}$. For each $G \in X$ note that $\frac{G'}{g'} \in H(\mathbb{D})$.

Now, if $\beta \geq (\gamma + 1)$, then $[V_g, A^{-\gamma}] \subseteq [V_g, A^{-(\gamma+\frac{1}{2})}] \subseteq [V_g, A^{-\beta}]$ and hence, $[V_g, A^{-(\gamma+\frac{1}{2})}] = [V_g, A^{-\beta}]$. So, we can suppose that $\gamma < \beta < (\gamma + 1)$ in which case $\delta := 1 - (\beta - \gamma) \in (0, 1)$.

The claim is that the assumption $[V_g, A^{-\gamma}] = [V_g, A^{-\beta}]$ implies that every $G \in X$ satisfies $G' \in A^{-\delta}$. To prove the claim we proceed as follows.

Fix $G \in X$. Given $h \in A^{-\beta}$ the function $\psi := \frac{G'h}{g'} \in H(\mathbb{D})$ satisfies

$$V_g(\psi)(z) = V_g\left(\frac{G'h}{g'}\right)(z) = \int_0^z g'(\xi)\frac{G'(\xi)}{g'(\xi)}h(\xi)\,d\xi = V_G(h)(z), \quad z \in \mathbb{D}. \tag{3.11}$$

Since $G \in \mathcal{B}$, Proposition 3.1 implies that $V_G h \in A^{-\beta}$. Accordingly, via (3.11), it follows that $V_g\psi \in A^{-\beta}$, that is, $\psi \in [V_g, A^{-\beta}] = [V_g, A^{-\gamma}]$, and hence, $V_g\psi \in A^{-\gamma}$. So, $V_G h \in A^{-\gamma}$.

Since $h \in A^{-\beta}$ is arbitrary, we can conclude that $V_G(h) \in A^{-\gamma}$ for all $h \in A^{-\beta}$. Therefore, the continuous linear operator $V_G \colon H(\mathbb{D}) \to H(\mathbb{D})$ satisfies $V_G(A^{-\beta}) \subseteq A^{-\gamma}$. By the closed graph theorem it follows that $V_G \colon A^{-\beta} \to A^{-\gamma}$ is also continuous. By [8, Theorem 1] this is equivalent to the fact that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\gamma}}{(1-|z|)^{\beta}}(1-|z|)|G'(z)| < \infty.$$

Accordingly, $G' \in A^{-\delta}$. The claim is thereby proved.

Now, fix $G \in X$ such that $G' \neq 0$. By the above claim $G' \in A^{-\delta}$ with $\delta \in (0, 1)$. Hence, by Lemma 3.10(i), for any $\varepsilon \in (0, \min\{\delta, 1 - \delta\})$ there exists $\phi \in H(\mathbb{D})$ such that $\phi G' \in A^{-(\delta+\varepsilon)} \setminus A^{-\delta}$. Therefore, the function

$$H(z) := \int_0^z \phi(\xi)G'(\xi)\,d\xi, \quad z \in \mathbb{D},$$

belongs to $H(\mathbb{D})$, has derivative $H' = \phi G'$ and $H \in X$ (as $Z(g') \subseteq Z(G') \subseteq Z(H')$). But, $H' \notin A^{-\delta}$. A contradiction.

(ii) Arguing as in the proof of part (i) we can suppose that $(\beta - \gamma) < 1$ and prove that the assumption $[V_g, A_0^{-\gamma}] = [V_g, A_0^{-\beta}]$ implies that every $G \in X$ satisfies $G' \in A^{-\delta}$, where $\delta := 1 - (\beta - \gamma) \in (0, 1)$. To this effect fix $G \in X$. For every $h \in A_0^{-\beta}$ the identity

$$V_g\psi = V_G h \in A_0^{-\gamma},$$

where $\psi = \frac{G'h}{g'}$ (cf. (3.11)) implies that $V_g\psi \in A_0^{-\beta}$ and hence, $V_g\psi \in A_0^{-\gamma}$. Therefore, the continuous linear operator $V_G\colon H(\mathbb{D}) \to H(\mathbb{D})$ satisfies $V_G(A_0^{-\beta}) \subseteq A_0^{-\gamma}$. According to the closed graph theorem $V_G\colon A_0^{-\beta} \to A_0^{-\gamma}$ is actually continuous. By [8, Theorem 1] this is equivalent to the fact that $\sup_{z\in\mathbb{D}} \frac{(1-|z|)^\gamma}{(1-|z|)^\beta}(1-|z|)|G'(z)| < \infty$. Hence, $G' \in A^{-\delta}$.

At this point, the contradiction follows by proceeding as in the latter part of the proof of part (i). $\square$

Our final result in this section shows that the Banach space $[V_g, A_0^{-\gamma}]$ is separable for suitable functions $g \in \mathcal{B}$.

Proposition 3.12. *Let $\gamma > 0$ and $g \in \mathcal{B}$ satisfy both $|g'| > 0$ on $\mathbb{D}$ and $\frac{1}{g'} \in H^\infty$. Then the space $\mathcal{P}$ of all polynomials on $\mathbb{D}$ is dense in $[V_g, A_0^{-\gamma}]$.*

Proof. Remark 3.6(ii) implies that

$$[V_g, A_0^{-\gamma}] = \{f \in H(\mathbb{D}) :\ fg' \in A_0^{-(\gamma+1)}\}$$

and that $f \mapsto \|fg'\|_{-(\gamma+1)}$ is an equivalent norm in $[V_g, A_0^{-\gamma}]$. So, for a fixed $f \in [V_g, A_0^{-\gamma}]$, the function $fg' \in A_0^{-(\gamma+1)}$ and hence, there exists a sequence $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}$ such that $\|p_n - fg'\|_{-(\gamma+1)} \to 0$ as $n \to \infty$; see the discussion after (3.2). We now observe that $\frac{p_n}{g'} \in A_0^{-\gamma}$ for each $n \in \mathbb{N}$. Indeed, for $n \in \mathbb{N}$ fixed, we have

$$(1-|z|)^\gamma |p_n(z)| \frac{1}{|g'(z)|} \leq (1-|z|)^\gamma |p_n(z)| \left\| \frac{1}{g'} \right\|_{H^\infty}, \quad z \in \mathbb{D}.$$

It follows that

$$\lim_{|z|\to 1^-} (1-|z|)^\gamma |p_n(z)| \frac{1}{|g'(z)|} = 0,$$

that is, $\frac{p_n}{g'} \in A_0^{-\gamma}$. Since the space $\mathcal{P}$ is dense in $A_0^{-\gamma}$ and $\left(\frac{p_n}{g'}\right)_{n\in\mathbb{N}} \subseteq A_0^{-\gamma}$, for every $n \in \mathbb{N}$ there exists $q_n \in \mathcal{P}$ such that

$$\left\| \frac{p_n}{g'} - q_n \right\|_{-\gamma} < \frac{1}{n}.$$

Recall that $\sup_{z\in\mathbb{D}}(1-|z|)|g'(z)| \leq \|g\|_{\mathcal{B}}$. So, for each $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+1} |q_n(z) - f(z)| \cdot |g'(z)| \\
&\leq \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+1} \left| q_n(z) - \frac{p_n(z)}{g'(z)} \right| |g'(z)| + \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+1} \left| \frac{p_n(z)}{g'(z)} - f(z) \right| |g'(z)|
\end{aligned}$$

$$\leq \|g\|_{\mathcal{B}} \left\| \frac{p_n}{g'} - q_n \right\|_{-\gamma} + \|p_n - fg'\|_{-(\gamma+1)} \leq \frac{1}{n}\|g\|_{\mathcal{B}} + \|p_n - fg'\|_{-(\gamma+1)}.$$

This implies that

$$\sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+1}|q_n(z) - f(z)||g'(z)| \to 0$$

as $n \to \infty$, that is, $q_n \to f$ in $[V_g, A_0^{-\gamma}]$. This completes the proof. □

4. Multipliers of optimal domain spaces

Let X, Y be Banach spaces of analytic functions on $\mathbb{D}$. A function $h \in H(\mathbb{D})$ is called a *multiplier* for X, Y if the multiplication operator $M_h \in \mathcal{L}(H(\mathbb{D}))$ given by $f \mapsto hf$, for $f \in H(\mathbb{D})$, satisfies $M_h(X) \subseteq Y$. We also say that $M_h\colon X \to Y$ exists as a linear map. Since both $X \subseteq H(\mathbb{D})$ and $Y \subseteq H(\mathbb{D})$ continuously, a closed graph argument shows that necessarily $M_h \in \mathcal{L}(X, Y)$. The space of all multipliers for X, Y is denoted by $\mathcal{M}(X, Y)$ or, if $X = Y$, simply by $\mathcal{M}(X)$. The aim of this section is to identify the multiplier spaces of the optimal domain spaces $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$ for $g \in \mathcal{B}$.

For each $\gamma > 0$ the following operators are known to be continuous.

(i) The differentiation operator $D\colon A^{-\gamma} \to A^{-(\gamma+1)}$, defined by $h \mapsto Dh := h'$; see Section 3.
(ii) The integration operator $J\colon A^{-(\gamma+1)} \to A^{-\gamma}$, defined by $h \mapsto (Jh)(z) := \int_0^z h(\xi)\,d\xi$, for $z \in \mathbb{D}$, which satisfies $\|J\|_{A^{-(\gamma+1)}\to A^{-\gamma}} \leq \frac{1}{\gamma}$; see [22, pp. 236-237 & Proposition 2.2(a)], [2, Proposition 3.12, Corollary 3.2].
(iii) The multiplication operator $M_h\colon A^{-\gamma} \to A^{-\gamma}$, defined by $f \mapsto hf$, is continuous if and only if $f \in H^\infty$. In this case, $\|M_h\|_{A^{-\gamma}\to A^{-\gamma}} = \|h\|_\infty$, [12, Proposition 2.1].

Remark 4.1. The fact in (ii) above that $\|J\|_{A^{-(\gamma+1)}\to A^{-\gamma}} \leq \frac{1}{\gamma}$ can be established as follows. Fix $f \in A^{-(\gamma+1)}$ and observe, for each $z \in \mathbb{D}$, that

$$\begin{aligned}(1-|z|)^\gamma \left|\int_0^z f(\xi)\,d\xi\right| &= (1-|z|)^\gamma \left|\int_0^1 zf(tz)\,dt\right| \leq (1-|z|)^\gamma \int_0^1 |z|\cdot|f(tz)|\,dt \\ &= (1-|z|)^\gamma \int_0^1 \frac{|z|}{(1-t|z|)^{\gamma+1}}(1-t|z|)^{\gamma+1}|f(tz)|\,dt \\ &\leq \|f\|_{-(\gamma+1)}(1-|z|)^\gamma \left[\frac{1}{\gamma(1-t|z|)^\gamma}\right]_{t=0}^{t=1} \\ &= \|f\|_{-(\gamma+1)}(1-|z|)^\gamma \left(\frac{1}{\gamma(1-|z|)^\gamma} - \frac{1}{\gamma}\right)\end{aligned}$$

$$= \|f\|_{-(\gamma+1)} \left(\frac{1}{\gamma} - \frac{(1-|z|)^\gamma}{\gamma} \right) \leq \frac{1}{\gamma} \|f\|_{-(\gamma+1)}.$$

The statements (i)–(iii) above also hold when these operators act on $A_0^{-\gamma}$.

The next result characterizes the action of multiplication operators on optimal domain spaces.

Proposition 4.2. *Let $g \in \mathcal{B}$ and $h \in H(\mathbb{D})$. For each $\gamma > 0$ the following three properties are equivalent.*

(i) *The multiplication operator $M_h \colon [V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]$ exists as a linear map and is continuous.*
(ii) *The multiplication operator $M_h \colon [V_g, A_0^{-\gamma}] \to [V_g, A_0^{-\gamma}]$ exists as a linear map and is continuous.*
(iii) *The function $h \in H^\infty$.*

In this case, the norms $\|M_h\|_{[V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]}$ and $\|h\|_\infty$ are equivalent.

Proof. (iii)⇒(i) Let $f \in [V_g, A^{-\gamma}]$ be fixed. Then $V_g f \in A^{-\gamma}$. The claim is that $V_g(hf) \in A^{-\gamma}$. To establish this claim, observe that $V_g f \in A^{-\gamma}$ implies that $(V_g f)' = g' f \in A^{-(\gamma+1)}$ (as the operator D maps $A^{-\gamma}$ into $A^{-(\gamma+1)}$). Since $h \in H^\infty$, the operator $M_h \in \mathcal{L}(A^{-\gamma})$ and so $hg'f \in A^{-(\gamma+1)}$. Hence, $V_g(hf) = J(hg'f) \in A^{-\gamma}$, as the operator J maps $A^{-(\gamma+1)}$ into $A^{-\gamma}$. The claim is thereby established.

From the previous paragraph we can conclude that $M_h f \in [V_g, A^{-\gamma}]$. Moreover,

$$\begin{aligned}
\|M_h f\|_{[V_g, A^{-\gamma}]} &= \|V_g(hf)\|_{-\gamma} = \|(J M_h D V_g)(f)\|_{-\gamma} \\
&\leq \|J\|_{A^{-(\gamma+1)} \to A^{-\gamma}} \|M_h\|_{A^{-(\gamma+1)} \to A^{-(\gamma+1)}} \|D\|_{A^{-\gamma} \to A^{-(\gamma+1)}} \|V_g f\|_{-\gamma} \\
&\leq \frac{1}{\gamma} \|D\|_{A^{-\gamma} \to A^{-(\gamma+1)}} \|h\|_\infty \|f\|_{[V_g, A^{-\gamma}]}.
\end{aligned}$$

This means that $M_h \colon [V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]$ is continuous with $\|M_h\|_{[V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]} \leq \frac{1}{\gamma} \|D\|_{A^{-\gamma} \to A^{-(\gamma+1)}} \|h\|_\infty$.

(i)⇒(iii) Let $f \in [V_g, A^{-\gamma}]$ satisfy $\|f\|_{[V_g, A^{-\gamma}]} = 1$. For fixed $z \in \mathbb{D}$ observe, as $[V_g, A^{-\gamma}]$ is a Banach space of analytic functions on $\mathbb{D}$ (cf. Corollary 3.4), that

$$|h(z) f(z)| = |\langle M_h f, \delta_z \rangle| \leq \|M_h\|_{[V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]} \|\delta_z\|_{[V_g, A^{-\gamma}]^*}.$$

Taking the supremum over all such functions f yields

$$|h(z)| \cdot \|\delta_z\|_{[V_g, A^{-\gamma}]^*} \leq \|M_h\|_{[V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]} \|\delta_z\|_{[V_g, A^{-\gamma}]^*}.$$

Since $z \in \mathbb{D}$ is arbitrary, it follows that $h \in H^\infty$ and $\|h\|_\infty \leq \|M_h\|_{[V_g, A^{-\gamma}] \to [V_g, A^{-\gamma}]}$.

(iii)⇔(ii) Since each of the linear maps $D\colon A_0^{-\gamma} \to A_0^{-(\gamma+1)}$, $J\colon A_0^{-(\gamma+1)} \to A_0^{-\gamma}$ and $M_h\colon A_0^{-\gamma} \to A_0^{-\gamma}$ is defined and continuous, the result follows by arguing as in the proof of (iii)⇔(i). □

As an immediate consequence we have the following result.

Corollary 4.3. *For each $g \in \mathcal{B}$ and each $\gamma > 0$ the multiplier spaces of the optimal domain spaces $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$ are given by*

$$\mathcal{M}([V_g, A^{-\gamma}]) = \mathcal{M}([V_g, A_0^{-\gamma}]) = H^\infty.$$

Certain multiplier spaces for the Korenblum spaces are known. The following result is Proposition 5 in [9]; see also [13, Proposition 3.1].

Proposition 4.4. *Let $0 < \gamma < \delta$ and $h \in H(\mathbb{D})$. The multiplication operator $M_h\colon A^{-\gamma} \to A^{-\delta}$ exists if and only if $h \in A^{-(\delta-\gamma)}$. That is,*

$$\mathcal{M}(A^{-\gamma}, A^{-\delta}) = A^{-(\delta-\gamma)}.$$

Two further results in this direction are the following Propositions 4.5 and 4.6.

Proposition 4.5. *Let $0 < \gamma < \delta$ and $h \in H(\mathbb{D})$. The multiplication operator $M_h\colon A_0^{-\gamma} \to A_0^{-\delta}$ exists if and only if $h \in A^{-(\delta-\gamma)}$. That is,*

$$\mathcal{M}(A_0^{-\gamma}, A_0^{-\delta}) = A^{-(\delta-\gamma)}.$$

Proof. Suppose that the operator $M_h\colon A_0^{-\gamma} \to A_0^{-\delta}$ exists and is continuous. Then its bi–transpose operator $M_h^{**}\colon (A_0^{-\gamma})^{**} \to (A_0^{-\delta})^{**}$ is continuous. Since $(A_0^{-\gamma})^{**} = A^{-\gamma}$ and $(A_0^{-\delta})^{**} = A^{-\delta}$ (and so $M_h^{**} = M_h$), Proposition 4.4 implies that $h \in A^{-(\delta-\gamma)}$.

Conversely, suppose that $h \in A^{-(\delta-\gamma)}$. Fix $f \in A_0^{-\gamma}$. For each $z \in \mathbb{D}$, we have that

$$|(M_h f)(z)|(1-|z|)^\delta = |h(z)|(1-|z|)^{\delta-\gamma}|f(z)|(1-|z|)^\gamma \le \|h\|_{-(\delta-\gamma)}|f(z)|(1-|z|)^\gamma.$$

Since $f \in A^{-\gamma}$, this implies that $\lim_{|z|\to 1^-} |(M_h f)(z)|(1-|z|)^\delta = 0$ and hence, $M_h f \in A_0^{-\delta}$. So, $M_h\colon A_0^{-\gamma} \to A_0^{-\delta}$ exists as a linear map. The continuity follows from the closed graph theorem. □

A more general result than Proposition 4.5, concerning weighted composition operators, occurs in [13, Proposition 3.2].

An analogue of the above characterization is the following result.

Proposition 4.6. *Let $0 < \gamma < \delta$ and $h \in H(\mathbb{D})$. The multiplication operator $M_h\colon A^{-\gamma} \to A_0^{-\delta}$ exists if and only if $h \in A_0^{-(\delta-\gamma)}$. That is,*

$$\mathcal{M}(A^{-\gamma}, A_0^{-\delta}) = A_0^{-(\delta-\gamma)}.$$

Proof. Suppose that the operator $M_h\colon A^{-\gamma} \to A_0^{-\delta}$ exists and is continuous. According to [1, Theorem 1.2] there exist functions $f_1, f_2 \in H(\mathbb{D})$ and positive constants c_1, c_2 such that $c_1(1-|z|)^{-\gamma} \leq |f_1(z)|+|f_2(z)| \leq c_2(1-|z|)^{-\gamma}$ for all $z \in \mathbb{D}$. Then both $f_1, f_2 \in A^{-\gamma}$ and hence, $M_h f_1, M_h f_2 \in A_0^{-\delta}$. Moreover,

$$|h(z)|(1-|z|)^{\delta-\gamma} \leq c_1^{-1}(1-|z|)^{\delta}(|f_1(z)|+|f_2(z)|)|h(z)|, \quad z \in \mathbb{D}.$$

Since both $M_h f_1, M_h f_2 \in A_0^{-\delta}$, this implies that $\lim_{|z|\to 1^-} |h(z)|(1-|z|)^{\delta-\gamma} = 0$, that is, $h \in A_0^{-(\delta-\gamma)}$.

Conversely, suppose that $h \in A_0^{-(\delta-\gamma)}$. Fix $f \in A^{-\gamma}$. For each $z \in \mathbb{D}$, we have that

$$|(M_h f)(z)|(1-|z|)^{\delta} = |h(z)|(1-|z|)^{\delta-\gamma}|f(z)|(1-|z|)^{\gamma} \leq \|f\|_{-\gamma}|h(z)|(1-|z|)^{\delta-\gamma}.$$

Since $h \in A^{-(\delta-\gamma)}$, this implies that $\lim_{|z|\to 1^-} |(M_h f)(z)|(1-|z|)^{\delta} = 0$ and hence, that $M_h f \in A_0^{-\delta}$. So, $M_h\colon A^{-\gamma} \to A_0^{-\delta}$ exists as a linear map. The continuity now follows from the closed graph theorem. $\square$

5. The range of V_g and related properties

Let X be a Banach space of analytic functions on $\mathbb{D}$. Whenever $T\colon X \to X$ is an injective continuous operator such that $([T,X], \|\cdot\|_{[T,X]})$ is a Banach space, Proposition 2.9 and Corollary 2.10 reveal an intimate connection between the three properties of the operator T having closed range, of the natural inclusion map $X \subseteq [T,X]$ having closed range, and of X being a proper subspace of $[T,X]$. In the latter part of this section we investigate these connections for the Cesàro operator C and for the operators V_g, when they act in Korenblum growth spaces. This requires various preliminary results concerning the auxiliary operators S and T defined in (5.1) and (5.2), respectively, which act in certain weighted Banach spaces of analytic functions on $\mathbb{D}$ which we now introduce.

A *weight* v is a continuous, non-increasing function $v\colon [0,1) \to (0,\infty)$. We extend v to $\mathbb{D}$ by setting $v(z) := v(|z|)$, for $z \in \mathbb{D}$. Note that $v(z) \leq v(0)$ for all $z \in \mathbb{D}$. It is assumed throughout this section that $\lim_{r\to 1^-} v(r) = 0$. Given a weight v on $[0,1)$, define the corresponding *weighted Banach spaces of analytic functions* on $\mathbb{D}$ by

$$H_v^\infty := \{f \in H(\mathbb{D}) : \ \|f\|_{\infty,v} := \sup_{z\in\mathbb{D}} |f(z)|v(z) < \infty\},$$

and

$$H_v^0 := \{f \in H(\mathbb{D}) : \ \lim_{|z|\to 1^-} |f(z)|v(z) = 0\},$$

both endowed with the norm $\|\cdot\|_{\infty,v}$. For details concerning these spaces, see [11] and the extensive list of references given there.

Observe that the Korenblum growth spaces $A^{-\gamma}$ and $A_0^{-\gamma}$, for $\gamma > 0$, correspond to H_v^∞ and H_v^0, respectively, for the weight function $v(r) := (1-r)^\gamma$, for $r \in [0, 1)$.

Consider now the forward shift operator $S\colon H(\mathbb{D}) \to H(\mathbb{D})$ defined to be the multiplication operator

$$S(f)(z) := zf(z), \quad f \in H(\mathbb{D}), z \in \mathbb{D}. \tag{5.1}$$

Clearly, $S \in \mathcal{L}(H(\mathbb{D}))$. Let the linear subspace $H_0(\mathbb{D}) := \{f \in H(\mathbb{D}) : \; f(0) = 0\} \subseteq H(\mathbb{D})$ be equipped with the topology induced by $H(\mathbb{D})$. Clearly, $H_0(\mathbb{D})$ is a closed subspace of $H(\mathbb{D})$ and $S(H(\mathbb{D})) \subseteq H_0(\mathbb{D})$. Therefore, $S \in \mathcal{L}(H(\mathbb{D}), H_0(\mathbb{D}))$. Moreover, $S\colon H(\mathbb{D}) \to H_0(\mathbb{D})$ is surjective. To see this, we consider the operator $T\colon H_0(\mathbb{D}) \to H(\mathbb{D})$ defined, for each $f \in H_0(\mathbb{D})$, by $(Tf)(0) := f'(0)$ and

$$(Tf)(z) := \frac{1}{z} f(z), \quad z \in \mathbb{D} \setminus \{0\}. \tag{5.2}$$

Since $(Tf)(z) := \frac{f(z)-f(0)}{z-0}$, for $z \in \mathbb{D} \setminus \{0\}$, it is clear that $Tf \in H(\mathbb{D})$ for all $f \in H_0(\mathbb{D})$. Moreover, $T \in \mathcal{L}(H_0(\mathbb{D}), H(\mathbb{D}))$. Indeed, for $r \in (0, 1)$ and $f \in H_0(\mathbb{D})$ fixed, note that

$$\sup_{|z| \le r} |(Tf)(z)| = \max_{|z|=r} |(Tf)(z)| = \frac{1}{r} \max_{|z|=r} |f(z)|.$$

Now, given $h \in H_0(\mathbb{D})$, recall that $Th \in H(\mathbb{D})$ and $(STh)(z) = z\frac{1}{z}h(z) = h(z)$ for all $z \in \mathbb{D} \setminus \{0\}$. So, STh and h are analytic functions on $\mathbb{D}$ which coincide except possibly at 0. Therefore, $STh = h$ on $\mathbb{D}$. This proves that $S\colon H(\mathbb{D}) \to H_0(\mathbb{D})$ is surjective.

Since the evaluation functional δ_0 belongs to both $(H_v^\infty)^*$ and $(H_v^0)^*$ we can define the Banach spaces

$$\mathcal{E}_v^\infty := \{f \in H_v^\infty : \; f(0) = 0\} = \mathrm{Ker}(\delta_0)$$

and

$$\mathcal{E}_v^0 := \{f \in H_v^0 : \; f(0) = 0\} = \mathrm{Ker}(\delta_0),$$

both endowed with the norm $\| \cdot \|_{\infty,v}$. Of course, $\mathcal{E}_v^\infty$ (resp., $\mathcal{E}_v^0$) is a closed subspace of H_v^∞ (resp., H_v^0).

In the notation of Proposition 3.12 we have the following fact.

Lemma 5.1. *Let $\mathcal{P}_0 := \{p \in \mathcal{P} : \; p(0) = 0\}$. Then the closure of $\mathcal{P}_0$ in H_v^∞ is equal to $\mathcal{E}_v^0$.*

Proof. Let $g \in H_v^\infty$. If there exists $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_0$ such that $p_n \to g$ in H_v^∞ as $n \to \infty$, then necessarily $g \in H_v^0$ because $\mathcal{P}_0 \subseteq H_v^0$ and H_v^0 is closed in H_v^∞. Moreover, $g(0) = \lim_{n\to\infty} p_n(0) = 0$. Accordingly, $g \in \mathcal{E}_v^\infty$. Since $g \in H_v^0$, we can conclude that $g \in \mathcal{E}_v^0$.

Let $h \in \mathcal{E}_v^0 \subseteq H_v^0$ be fixed. Then there exists a sequence $(q_n)_{n\in\mathbb{N}}$ of polynomials such that $q_n \to h$ in H_v^0, [10]. Accordingly, $\lim_{n\to\infty} q_n(0) = h(0) = 0$. This implies that $p_n := q_n - q_n(0) \to h$ in H_v^0 as $n \to \infty$. Since $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_0$, this completes the proof. $\square$

Proposition 5.2. *Let S and T be the operators given by (5.1) and (5.2), respectively.*

(i) *The operator S satisfies $S \in \mathcal{L}(H_v^\infty)$ with $S(H_v^\infty) \subseteq \mathcal{E}_v^\infty$ and T satisfies $T \in \mathcal{L}(\mathcal{E}_v^\infty, H_v^\infty)$. Moreover, $STh = h$, for all $h \in \mathcal{E}_v^\infty$.*
(ii) *The operator S satisfies $S \in \mathcal{L}(H_v^0)$ with $S(H_v^0) \subseteq \mathcal{E}_v^0$ and T satisfies $T \in \mathcal{L}(\mathcal{E}_v^0, H_v^0)$. Moreover, $STh = h$, for all $h \in \mathcal{E}_v^0$.*

Proof. (i) Let $f \in H_v^\infty$. Then

$$\|Sf\|_{\infty,v} = \sup_{z\in\mathbb{D}} v(z)|z| \cdot |f(z)| \le \sup_{z\in\mathbb{D}} v(z)|f(z)| = \|f\|_{\infty,v},$$

which implies that $S \in \mathcal{L}(H_v^\infty)$ with $\|S\|_{H_v^\infty \to H_v^\infty} \le 1$. Clearly, $S(H_v^\infty) \subseteq \mathcal{E}_v^\infty$.

Next, since $\mathcal{E}_v^\infty \subseteq H_0(\mathbb{D})$, observe that $T(\mathcal{E}_v^\infty) \subseteq H(\mathbb{D})$. Moreover, given $f \in \mathcal{E}_v^\infty$, it follows from (5.2) that

$$\begin{aligned}\sup_{|z|\le 1/2} v(z)|(Tf)(z)| &\le v(0) \max_{|z|=1/2} |(Tf)(z)| = v(0) \max_{|z|=1/2} \frac{1}{|z|}|f(z)| \\ &= \frac{2v(0)}{v(1/2)} \max_{|z|=1/2} v(1/2)|f(z)| \le \frac{2v(0)}{v(1/2)} \|f\|_{\infty,v},\end{aligned}$$

and that

$$\sup_{1/2<|z|<1} v(z)|(Tf)(z)| = \sup_{1/2<|z|<1} v(z)\frac{1}{|z|}|f(z)| \le 2 \sup_{1/2<|z|<1} v(z)|f(z)| \le 2\|f\|_{\infty,v}.$$

Therefore,

$$\|Tf\|_{\infty,v} \le \max\left\{2, \frac{2v(0)}{v(1/2)}\right\} \|f\|_{\infty,v} = \frac{2v(0)}{v(1/2)} \|f\|_{\infty,v}.$$

This implies that $T \in \mathcal{L}(\mathcal{E}_v^\infty, H_v^\infty)$.

Finally, for every $h \in \mathcal{E}_v^\infty$ we have that $Th \in H_v^\infty$ and that $STh = h$; see the discussion after (5.2).

(ii) By part (i) we have that $S \in \mathcal{L}(H_v^\infty)$. To conclude that $S \in \mathcal{L}(H_v^0)$, it therefore suffices to establish that $S(H_v^0) \subseteq H_v^0$. So, fix $f \in H_v^0$, in which case $\lim_{|z|\to 1^-} v(z)|f(z)| = 0$. Since $v(z)|(Sf)(z)| \le v(z)|f(z)|$ for all $z \in \mathbb{D}$, it follows that $\lim_{|z|\to 1^-} v(z)|(Sf)(z)| = 0$. Accordingly, $Sf \in H_v^0$.

Since $(Sf)(0) = 0$ for all $f \in H_v^0$, it is clear that $S(H_v^0) \subseteq \mathcal{E}_v^0$.

By part (i) we also have that $T \in \mathcal{L}(\mathcal{E}_v^\infty, H_v^\infty)$. So, to complete the proof it remains to show that $T(\mathcal{E}_v^0) \subseteq H_v^0$. To see this, observe that Tp is a polynomial for every $p \in \mathcal{P}_0$;

see (5.2). Then $T(\mathcal{P}_0) \subseteq H_v^0$ (as $\lim_{r\to 1^-} v(r) = 0$). Since $T \in \mathcal{L}(\mathcal{E}_v^\infty, H_v^\infty)$ and H_v^0 is closed in H_v^∞, it follows via Lemma 5.1 that

$$T(\mathcal{E}_v^0) = T(\overline{\mathcal{P}_0}^{H_v^\infty}) \subseteq \overline{T(\mathcal{P}_0)}^{H_v^\infty} \subseteq \overline{H_v^0}^{H_v^\infty} = H_v^0.$$

So, $T \in \mathcal{L}(\mathcal{E}_v^0, H_v^0)$. Finally, $STh = h$ for each $h \in \mathcal{E}_v^0$, as follows from part (i). $\square$

Related to the Volterra operator $V_g\colon H(\mathbb{D}) \to H(\mathbb{D})$, where $g \in H(\mathbb{D})$, is the operator $T_g\colon H(\mathbb{D}) \to H(\mathbb{D})$ defined by

$$(T_g f)(z) := \frac{1}{z}\int_0^z f(\xi)g'(\xi)\, d\xi,\ z \in \mathbb{D}\setminus\{0\}, \quad (T_g f)(0) := f(0)g'(0), \tag{5.3}$$

for each $f \in H(\mathbb{D})$. Clearly, $T_g \in \mathcal{L}(H(\mathbb{D}))$. Suppose that $T_g f = 0$ for some $f \in H(\mathbb{D})$. Then (5.3) implies that $\int_0^z f(\xi)g'(\xi)\, d\xi = 0$, that is, $(V_g f)(z) = 0$ for all $z \in \mathbb{D}\setminus\{0\}$. Since also $(V_g f)(0) = 0$, it follows that $V_g f = 0$ in $H(\mathbb{D})$. By the injectivity of $V_g \in \mathcal{L}(H(\mathbb{D}))$ we can conclude that $f = 0$. Accordingly, $T_g \in \mathcal{L}(H(\mathbb{D}))$ is *injective* and so its optimal domain spaces $[T_g, H_v^\infty]$ and $[T_g, H_v^0]$ are *normed spaces* for every weight v such that $T_g(H_v^\infty) \subseteq H_v^\infty$ (resp. $T_g(H_v^0) \subseteq H_v^0$); see Proposition 2.4(ii) with $T = T_g$ and $X = H_v^\infty$ (resp. H_v^0).

Proposition 5.3. *Let $v\colon [0,1) \to (0,\infty)$ be a weight function and $g \in H(\mathbb{D})$.*

(i) *The operator $V_g\colon H_v^\infty \to H_v^\infty$ is continuous if and only if the operator $T_g\colon H_v^\infty \to H_v^\infty$ is continuous.*
(ii) *Let $V_g\colon H_v^\infty \to H_v^\infty$ be continuous. Then $[V_g, H_v^\infty] = [T_g, H_v^\infty]$ as linear spaces in $H(\mathbb{D})$ and with equivalent norms.*
(iii) *The operator $V_g\colon H_v^0 \to H_v^0$ is continuous if and only if the operator $T_g\colon H_v^0 \to H_v^0$ is continuous.*
(iv) *Let $V_g\colon H_v^0 \to H_v^0$ be continuous. Then $[V_g, H_v^0] = [T_g, H_v^0]$ as linear spaces in $H(\mathbb{D})$ and with equivalent norms.*

Proof. (i) Suppose that $T_g\colon H_v^\infty \to H_v^\infty$ is continuous. By Proposition 5.2(i) also $V_g = S \circ T_g\colon H_v^\infty \to \mathcal{E}_v^\infty \subseteq H_v^\infty$ is continuous.

Conversely, suppose that $V_g\colon H_v^\infty \to H_v^\infty$ is continuous. Clearly, $V_g(H_v^\infty) \subseteq \mathcal{E}_v^\infty$ and so Proposition 5.2(i) implies that $T_g = T \circ V_g\colon H_v^\infty \to H_v^\infty$ is continuous.

(ii) The assumption on V_g ensures that $T_g\colon H_v^\infty \to H_v^\infty$ is also continuous; see part (i).

Let $h \in [V_g, H_v^\infty]$. Then $h \in H(\mathbb{D})$ and $V_g h \in H_v^\infty$. Since $(V_g h)(0) = 0$, it follows that actually $V_g h \in \mathcal{E}_v^\infty$. So, $h \in H(\mathbb{D})$ and $T_g h = (T \circ V_g)h \in H_v^\infty$, after recalling that $T \in \mathcal{L}(\mathcal{E}_v^\infty, H_v^\infty)$ by Proposition 5.2(i). Moreover,

$$\|h\|_{[T_g,H_v^\infty]} := \|T_g h\|_{\infty,v} = \|(T\circ V_g)h\|_{\infty,v} \le \|T\|_{\mathcal{E}_v^\infty\to H_v^\infty}\|V_g h\|_{\infty,v} = \|T\|_{\mathcal{E}_v^\infty\to H_v^\infty}\|h\|_{[V_g,H_v^\infty]} < \infty,$$

that is, $h\in[T_g,H_v^\infty]$. Accordingly, $[V_g,H_v^\infty]\subseteq[T_g,H_v^\infty]$ with a continuous inclusion.

Let $f\in[T_g,H_v^\infty]$. Then $f\in H(\mathbb{D})$ and $T_g f\in H_v^\infty$. By Proposition 5.2(i) it follows that $V_g f=(S\circ T_g)f\in H_v^\infty$. Accordingly, $f\in[V_g,H_v^\infty]$. Moreover,

$$\|f\|_{[V_g,H_v^\infty]} = \|V_g f\|_{\infty,v} = \|(S\circ T_g)f\|_{\infty,v} \le \|S\|_{H_v^\infty\to\mathcal{E}_v^\infty}\|T_g f\|_{\infty,v} = \|S\|_{H_v^\infty\to\mathcal{E}_v^\infty}\|f\|_{[T_g,H_v^\infty]} < \infty,$$

that is, $f\in[V_g,H_v^\infty]$. Accordingly, $[T_g,H_v^\infty]\subseteq[V_g,H_v^\infty]$ with a continuous inclusion.

It follows that $[T_g,H_v^\infty]=[V_g,H_v^\infty]$ as linear spaces of $H(\mathbb{D})$ and that the norms $\|\cdot\|_{[T_g,H_v^\infty]}$ and $\|\cdot\|_{[V_g,H_v^\infty]}$ are *equivalent*.

In view of Proposition 5.2(ii), the proofs of parts (iii) and (iv) are similar. $\square$

Let us consider a particular case. Define $g_0(z):=-\mathrm{Log}(1-z)$, for $z\in\mathbb{D}$. Then, $g_0'(z)=\frac{1}{1-z}$, for $z\in\mathbb{D}$, and hence, $g\in\mathcal{B}$. Moreover, for each $f\in H(\mathbb{D})$, the operator T_{g_0} is given by $(T_{g_0}f)(0)=f(0):=(Cf)(0)$ and

$$(T_{g_0})f(z)=\frac{1}{z}\int_0^z\frac{f(\xi)}{1-\xi}\,d\xi=:(Cf)(z),\ \ z\in\mathbb{D}\setminus\{0\}, \tag{5.4}$$

that is, $T_{g_0}=C$ is the classical *Cesàro operator*. We also point out that

$$(V_{g_0}f)(z)=\int_0^z\frac{f(\xi)}{1-\xi}\,d\xi,\quad z\in\mathbb{D}. \tag{5.5}$$

Aleman and Persson have made an extensive investigation of various properties of generalized Cesàro operators acting in a large class of Banach spaces of analytic functions on $\mathbb{D}$, [4,5,30]. In particular, their results apply to the classical Cesàro operator C given by (5.4) when it acts in the Korenblum growth spaces $A^{-\gamma}$ and $A_0^{-\gamma}$ for $\gamma>0$. Additional results which complement and extend their work can be found in [3].

For a detailed investigation of the optimal domain spaces $[C,H^p]$, for $1\le p<\infty$, we refer to [18]. We now turn our attention to the setting when the H^p-spaces are replaced by the Korenblum spaces. Note, whenever $g\in\mathcal{B}$ is non-constant, that both of the optimal domain spaces $[V_{g_0},A^{-\gamma}]$ and $[V_{g_0},A_0^{-\gamma}]$ are Banach spaces (cf. Proposition 3.3). In view of Proposition 5.3(ii), (iv), also $[C,A^{-\gamma}]=[T_{g_0},A^{-\gamma}]$ and $[C,A_0^{-\gamma}]=[T_{g_0},A_0^{-\gamma}]$ are Banach spaces (see also [3]). The following result is a consequence of Theorems 3.1 and 3.2 in [3].

Proposition 5.4. *Let $\gamma>0$ and $g_0(z)=-\mathrm{Log}(1-z)$ for $z\in\mathbb{D}$. In each of the following cases the optimal domain space is genuinely larger than the original domain space.*

(i) $A^{-\gamma} \subsetneqq [C, A^{-\gamma}]$ *and* $A^{-\gamma} \subsetneqq [V_{g_0}, A^{-\gamma}]$.
(ii) $A_0^{-\gamma} \subsetneqq [C, A_0^{-\gamma}]$ *and* $A_0^{-\gamma} \subsetneqq [V_{g_0}, A_0^{-\gamma}]$.

Proof. It is already known that $A^{-\gamma} \neq [C, A^{-\gamma}]$ and $A_0^{-\gamma} \neq [C, A_0^{-\gamma}]$; see [3, Theorems 3.1 and 3.2]. So, by Proposition 5.3(ii), (iv) we can conclude that also $A^{-\gamma} \neq [V_{g_0}, A^{-\gamma}]$ and $A_0^{-\gamma} \neq [V_{g_0}, A_0^{-\gamma}]$. For the stated inclusions we refer to Proposition 2.4(iii). $\square$

The following result, [3, Proposition 3.4], shows that the largest Korenblum space $A^{-\beta}$ that is contained in $[C, A^{-\gamma}]$ is $A^{-\gamma}$. The same is true for $A_0^{-\beta}$ and $[C, A_0^{-\gamma}]$. This property is particular for $C = T_{g_0}$ and is not valid for T_g for general functions $g \in \mathcal{B}$; see Proposition 5.3(ii), (iv) and Example 5.13(iii).

Proposition 5.5. *Let $\gamma > 0$. For each $\beta > \gamma$, the space $A^{-\beta} \not\subseteq [C, A^{-\gamma}]$ and the space $A_0^{-\beta} \not\subseteq [C, A_0^{-\gamma}]$.*

The next result, [3, Proposition 3.6], shows that the Banach spaces of analytic functions $A^{-\gamma}$ and $[C, A_0^{-\gamma}]$ on $\mathbb{D}$ are two *non-comparable, proper* linear subspaces of the optimal domain space $[C, A^{-\gamma}]$. In particular, C maps both of these spaces into $A^{-\gamma}$.

Proposition 5.6. *Let $\gamma > 0$. Both $A^{-\gamma}$ and $[C, A_0^{-\gamma}]$ are proper subspaces of $[C, A^{-\gamma}]$, with $[C, A_0^{-\gamma}]$ being closed. Moreover, $A^{-\gamma}$ and $[C, A_0^{-\gamma}]$ are non-comparable. That is,*

$$A^{-\gamma} \not\subseteq [C, A_0^{-\gamma}] \quad \textit{and} \quad [C, A_0^{-\gamma}] \not\subseteq A^{-\gamma}.$$

Observe that the closedness of $[C, A_0^{-\gamma}]$ in $[C, A^{-\gamma}]$ is a special case of Proposition 2.4(v) for $T = C$ and with $X = A^{-\gamma}$ and $Y = A_0^{-\gamma}$.

Proposition 5.7. *Let $\gamma > 0$ and $g_0(z) = -\mathrm{Log}(1 - z)$ for $z \in \mathbb{D}$.*

(i) *Both of the Cesàro operators $C\colon A^{-\gamma} \to A^{-\gamma}$ and $C\colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range.*
(ii) *Both of the operators $V_{g_0}\colon A^{-\gamma} \to A^{-\gamma}$ and $V_{g_0}\colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range.*

Proof. (i) The Cesàro operator $C \in \mathcal{L}(H(\mathbb{D}))$ is an isomorphism and satisfies

$$C^{-1}(z^n) = (n + 1)(1 - z)z^n, \quad n \in \mathbb{N}_0. \tag{5.6}$$

Indeed, for fixed $n \in \mathbb{N}_0$, the function $f_n(z) := (n + 1)(1 - z)z^n$, for $z \in \mathbb{D}$, satisfies

$$(Cf_n)(z) = \frac{1}{z} \int_0^z \frac{(n + 1)(1 - \xi)\xi^n}{1 - \xi}\, d\xi = z^n, \quad n \in \mathbb{N}_0,$$

from which (5.6) follows. It is clear from (5.6) that the range $C(A_0^{-\gamma}) \subseteq A_0^{-\gamma}$ contains the polynomials and hence, it is dense in $A_0^{-\gamma}$. Suppose that $C\colon A_0^{-\gamma} \to A_0^{-\gamma}$ has closed range, that is, $C(A_0^{-\gamma})$ is a closed subspace of $A_0^{-\gamma}$. It follows that $C(A_0^{-\gamma}) = A_0^{-\gamma}$ and hence, $C\colon A_0^{-\gamma} \to A_0^{-\gamma}$ is a surjective isomorphism. This is a contradiction because 0 belongs to the spectrum $\sigma(C; A_0^{-\gamma})$; see [30, Theorems 4.1, 5.1 and Corollaries 2.1, 5.1], [4, Theorem 4.1]. So, $C \in \mathcal{L}(A_0^{-\gamma})$ fails to have closed range.

Suppose now that $C\colon A^{-\gamma} \to A^{-\gamma}$ has closed range. By the open mapping theorem it follows that there exists $D > 0$ such that $\|Cf\|_{-\gamma} \geq D\|f\|_{-\gamma}$ for all $f \in A^{-\gamma}$. Since $A_0^{-\gamma} \subseteq A^{-\gamma}$, this implies that $\|Cf\|_{-\gamma} \geq D\|f\|_{-\gamma}$ for all $f \in A_0^{-\gamma}$ and hence, that $C\colon A_0^{-\gamma} \to A_0^{-\gamma}$ has closed range. A contradiction.

(ii) First observe, by Proposition 5.3(ii), (iv), that $[C, A^{-\gamma}] = [V_{g_0}, A^{-\gamma}]$ and $[C, A_0^{-\gamma}] = [V_{g_0}, A_0^{-\gamma}]$ as linear spaces and topologically. Moreover, as noted above, both $[C, A^{-\gamma}]$ and $[C, A_0^{-\gamma}]$ are Banach spaces. By part (i) combined with Proposition 2.9 (or with Corollary 2.10) it follows necessarily that $A^{-\gamma}$ ($A_0^{-\gamma}$, resp.) is *not* a closed subspace of $[C, A^{-\gamma}]$ (resp. of $[C, A_0^{-\gamma}]$). Accordingly, $A^{-\gamma}$ ($A_0^{-\gamma}$, resp.) is not a closed subspace of $[V_{g_0}, A^{-\gamma}]$ (resp. of $[V_{g_0}, A_0^{-\gamma}]$) either. Since $[V_{g_0}, A^{-\gamma}]$ and $[V_{g_0}, A_0^{-\gamma}]$ are Banach spaces, again by Proposition 2.9 we can conclude that both of the operators $V_{g_0}\colon A^{-\gamma} \to A^{-\gamma}$ and $V_{g_0}\colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range. □

The following fact is known. We include a proof for the sake of completeness.

Lemma 5.8. *Let $(X, \|\cdot\|_X)$ be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$ be injective and compact. Then T does not have closed range in X.*

Proof. Since $T\colon X \to X$ is injective, necessarily $T(X)$ is an infinite dimensional subspace of X. Suppose that $T \in \mathcal{L}(X)$ does have closed range. Then $T(X)$ is a closed subspace of X and hence, by the open mapping theorem, the operator $T\colon X \to T(X)$ is an isomorphism. But, $T \in \mathcal{L}(X)$ is compact. So, $T(B_X) \subseteq T(X)$ is a relatively compact 0-neighborhood of $T(X)$, which implies that $T(X)$ is finite dimensional. A contradiction. □

The function $g_0(z) = -\mathrm{Log}(1 - z)$, for $z \in \mathbb{D}$, belongs to $\mathcal{B} \setminus \mathcal{B}_0$. The following result has similarities with Proposition 5.4.

Proposition 5.9. *Let $g \in \mathcal{B}_0$ and $\gamma > 0$.*

(i) *Both of the operators $V_g\colon A^{-\gamma} \to A^{-\gamma}$ and $V_g\colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range.*
(ii) *Each of the containments $A^{-\gamma} \subseteq [V_g, A^{-\gamma}]$ and $A_0^{-\gamma} \subseteq [V_g, A_0^{-\gamma}]$ is proper. That is, the optimal domain space is genuinely larger than the original domain space.*
(iii) *Both of the operators $T_g\colon A^{-\gamma} \to A^{-\gamma}$ and $T_g\colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range.*
(iv) *Each of the containments $A^{-\gamma} \subseteq [T_g, A^{-\gamma}]$ and $A_0^{-\gamma} \subseteq [T_g, A_0^{-\gamma}]$ is proper. That is, the optimal domain space is genuinely larger than the original domain space.*

Proof. (i) By Proposition 3.2 both of the operators $V_g\colon A^{-\gamma}\to A^{-\gamma}$ and $V_g\colon A_0^{-\gamma}\to A_0^{-\gamma}$ are compact. On the other hand, the operators $V_g\colon A^{-\gamma}\to A^{-\gamma}$ and $V_g\colon A_0^{-\gamma}\to A_0^{-\gamma}$ are also injective. So, the result follows from Lemma 5.8.

(ii) Since $[V_g, A^{-\gamma}]$ and $[V_g, A_0^{-\gamma}]$ are Banach spaces, the result follows from part (i) combined with Proposition 2.9 (or with Corollary 2.10).

(iii) Let T be the operator given by (5.2). Since $T_g = T\circ V_g$ with $T\in\mathcal{L}(\mathcal{E}_v^\infty; A^{-\gamma})\cap\mathcal{L}(\mathcal{E}_v^0, A_0^{-\gamma})$ (cf. Proposition 5.2 with $v(z):=(1-|z|)^\gamma$, for $z\in\mathbb{D}$, in which case $H_v^\infty = A^{-\gamma}$ and $H_v^0 = A_0^{-\gamma}$) and both of the operators $V_g\colon A^{-\gamma}\to A^{-\gamma}$ and $V_g\colon A_0^{-\gamma}\to A_0^{-\gamma}$ are compact, also the operators $T_g\colon A^{-\gamma}\to A^{-\gamma}$ and $T_g\colon A_0^{-\gamma}\to A_0^{-\gamma}$ are compact. Moreover, both of the operators $T_g\colon A^{-\gamma}\to A^{-\gamma}$ and $T_g\colon A_0^{-\gamma}\to A_0^{-\gamma}$ are also injective. So, the result follows again from Lemma 5.8.

(iv) Follows by arguing as in part (ii) and using part (iii). $\square$

We conclude this section by giving some further classes of functions $g\in\mathcal{B}$ for which the operators V_g and T_g fail to have closed range. In order to do this, we need some preliminary results.

Lemma 5.10. *Let $\gamma>0$ and $g\in\mathcal{B}$ be non-constant.*

(i) *Suppose that $V_g(A^{-\gamma}) = \{f\in A^{-\gamma}:\ f(0)=0\}$. Then $A^{-\gamma} = [V_g, A^{-\gamma}]$.*
(ii) *Suppose that $V_g(A_0^{-\gamma}) = \{f\in A_0^{-\gamma}:\ f(0)=0\}$. Then $A_0^{-\gamma} = [V_g, A_0^{-\gamma}]$.*

Proof. (i) We know that $A^{-\gamma}\subseteq [V_g, A^{-\gamma}]$.

Let $f\in[V_g, A^{-\gamma}]$. Then $f\in H(\mathbb{D})$ and $V_g f\in A^{-\gamma}$. Since $(V_g f)(0)=0$ (see (3.1)), it follows that $V_g f\in\{u\in A^{-\gamma}:\ u(0)=0\}$ and so, by assumption, there exists $h\in A^{-\gamma}$ such that $V_g h = V_g f$. The injectivity of V_g implies that $h=f$. Accordingly, $f\in A^{-\gamma}$.

(ii) Follows by arguing as in part (i). $\square$

Lemma 5.11. *Let $g\in\mathcal{B}$ satisfy $|g'|>0$ on $\mathbb{D}$ and $\frac{1}{g'}\in H^\infty$. Suppose, for $\gamma>0$, that $V_g(A_0^{-\gamma}) = \{f\in A_0^{-\gamma}:\ f(0)=0\}$. Then $V_g(A^{-\gamma}) = \{f\in A^{-\gamma}:\ f(0)=0\}$.*

Proof. Let $f\in A^{-\gamma}$ satisfy $f(0)=0$. Select a sequence $(q_n)_{n\in\mathbb{N}}$ of polynomials satisfying $\|q_n\|_{-\gamma}\le\|f\|_{-\gamma}$ for all $n\in\mathbb{N}$, and $q_n\to f$ in $H(\mathbb{D})$ as $n\to\infty$, [10]. In particular, $q_n(0)\to f(0)=0$ as $n\to\infty$, and hence, $p_n := q_n - q_n(0)\to f$ in $H_0(\mathbb{D})\subseteq H(\mathbb{D})$ as $n\to\infty$. Therefore, $\|p_n\|_{-\gamma}\le\|q_n\|_{-\gamma}+|q_n(0)|\le M$ for all $n\in\mathbb{N}$ and some $M>0$. Recall that $V_g\colon H(\mathbb{D})\to H_0(\mathbb{D})$ is bijective and continuous. Moreover, $V_g^{-1}\colon H_0(\mathbb{D})\to H(\mathbb{D})$ is given by $h\mapsto\frac{h'}{g'}$ for $h\in H_0(\mathbb{D})$. Since $H_0(\mathbb{D})$ is a closed subspace of $H(\mathbb{D})$, it is a Fréchet space and so, by the open mapping theorem, V_g^{-1} is continuous. Hence, $V_g^{-1}p_n\to V_g^{-1}f$ in $H(\mathbb{D})$ as $n\to\infty$. On the other hand, for each $n\in\mathbb{N}$ we have that $p_n(0)=0$ and $p_n\in A_0^{-\gamma}$. So, by the assumption and the injectivity of V_g it follows that $V_g^{-1}p_n\in A_0^{-\gamma}$ for all $n\in\mathbb{N}$.

Now, observe that the operator $V_g \colon A_0^{-\gamma} \to \{f \in A_0^{-\gamma} : f(0) = 0\}$ is continuous and bijective. By the open mapping theorem the inverse operator $V_g^{-1} \colon \{f \in A_0^{-\gamma} : f(0) = 0\} \to A_0^{-\gamma}$ exists and is continuous. So, there exists $D > 0$ such that $\|V_g^{-1} p_n\|_{-\gamma} \leq D$ for all $n \in \mathbb{N}$. Accordingly, for each $n \in \mathbb{N}$, we have

$$(1 - |z|)^\gamma |(V_g^{-1} p_n)(z)| \leq D, \quad z \in \mathbb{D}.$$

Since $V_g^{-1} p_n \to V_g^{-1} f$ in $H(\mathbb{D})$ as $n \to \infty$, it follows that

$$(1 - |z|)^\gamma |(V_g^{-1} f)(z)| \leq D, \quad z \in \mathbb{D}.$$

This implies that $V_g^{-1} f \in A^{-\gamma}$. So, $f = V_g(V_g^{-1} f) \in V_g(A^{-\gamma})$ and we can conclude that $\{f \in A^{-\gamma} : f(0) = 0\} \subseteq V_g(A^{-\gamma})$.

The reverse inclusion $V_g(A^{-\gamma}) \subseteq \{f \in A^{-\gamma} : f(0) = 0\}$ is clear. $\square$

Proposition 5.12. *Let $\gamma > 0$ and $g \in \mathcal{B}$ satisfy $|g'| > 0$ on $\mathbb{D}$ and $\frac{1}{g'} \in H^\infty$. Suppose that there exists $w \in \mathbb{C}$ with $|w| = 1$ such that $\lim_{r \to 1^-} |g'(rw)|(1 - r) = 0$. Then both of the operators $V_g \colon A^{-\gamma} \to A^{-\gamma}$ and $V_g \colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range.*

Proof. Observe that the set $\mathcal{P}_0$ of all polynomials on $\mathbb{D}$ vanishing at 0 is contained in $V_g(A_0^{-\gamma})$. Indeed, for a fixed $n \in \mathbb{N}$, the function $f(z) := \frac{nz^{n-1}}{g'(z)}$, for $z \in \mathbb{D}$, clearly belongs to $A_0^{-\gamma}$ because $\frac{1}{g'} \in H^\infty$ and $nz^{n-1} \in A_0^{-\gamma}$. Moreover,

$$(V_g f)(z) = \int_0^z f(\xi) g'(\xi)\, d\xi = \int_0^z n\xi^{n-1}\, d\xi = z^n, \quad z \in \mathbb{D},$$

and hence, $z^n \in V_g(A_0^{-\gamma})$. Since $\{z^n : n \in \mathbb{N}\} \subseteq V_g(A_0^{-\gamma})$, it follows that $\mathcal{P}_0$ is contained in $V_g(A_0^{-\gamma})$.

We point out that $\mathcal{P}_0$ is also dense in the closed subspace $\{f \in A_0^{-\gamma} : f(0) = 0\}$ of $A_0^{-\gamma}$. To see this fix $h \in \{f \in A_0^{-\gamma} : f(0) = 0\}$. Since $\mathcal{P}$ is dense in $A_0^{-\gamma}$, there exists a sequence $(q_n)_{n \in \mathbb{N}}$ of polynomials such that $q_n \to h$ in $A_0^{-\gamma}$ as $n \to \infty$. In particular, also $q_n(0) \to h(0) = 0$ as $n \to \infty$. Therefore, the polynomials $p_n := q_n - q_n(0) \in \mathcal{P}_0$, for all $n \in \mathbb{N}$, and satisfy $p_n \to h$ in $A_0^{-\gamma}$ as $n \to \infty$.

Since $\mathcal{P}_0 \subseteq V_g(A_0^{-\gamma}) \subseteq \{f \in A_0^{-\gamma} : f(0) = 0\}$ and $\mathcal{P}_0$ is dense in $\{f \in A_0^{-\gamma} : f(0) = 0\}$, it follows that also $V_g(A_0^{-\gamma})$ is dense in $\{f \in A_0^{-\gamma} : f(0) = 0\}$. Suppose that $V_g \colon A_0^{-\gamma} \to A_0^{-\gamma}$ has closed range, in which case $V_g(A_0^{-\gamma}) = \{f \in A_0^{-\gamma} : f(0) = 0\}$. By Lemma 5.11 it follows that $V_g(A^{-\gamma}) = \{f \in A^{-\gamma} : f(0) = 0\}$. In view of Lemma 5.10(i) this implies that $A^{-\gamma} = [V_g, A^{-\gamma}]$ and hence, the operator $V_g \colon A^{-\gamma} \to A^{-\gamma}$ has closed range (cf. Proposition 2.9 or Corollary 2.10(ii)). But, $A^{-\gamma} = [V_g, A^{-\gamma}]$ is in contradiction with Example 3.7. So, we can conclude that both of the operators $V_g \colon A^{-\gamma} \to A^{-\gamma}$ and $V_g \colon A_0^{-\gamma} \to A_0^{-\gamma}$ fail to have closed range. $\square$

We conclude with some relevant examples concerning optimal domain spaces of V_g acting in Korenblum spaces.

Example 5.13. (i) If g is non-constant and $g' \in H^\infty$ then, for each $\gamma > 0$, we have $A^{-(\gamma+1)} \subseteq [V_g, A^{-\gamma}]$. Indeed, if $f \in A^{-(\gamma+1)}$, then also $fg' \in A^{-(\gamma+1)}$ because $g' \in H^\infty$. Moreover, $g' \in H^\infty$ implies that $g \in \mathcal{B}$. So, by Proposition 3.5(i), it follows that $f \in [V_g, A^{-\gamma}]$. In particular, $[V_g, A^{-\gamma}]$ is genuinely larger than $A^{-\gamma}$.

(ii) Let $g \in \mathcal{B}$ satisfy $|g'| > 0$ on $\mathbb{D}$ and $\frac{1}{g'} \in H^\infty$. Then, for each $\gamma > 0$, we have $[V_g, A^{-\gamma}] \subseteq A^{-(\gamma+1)}$. Indeed, if $f \in [V_g, A^{-\gamma}]$, then $f \in H(\mathbb{D})$ and $fg' \in A^{-(\gamma+1)}$ (cf. Proposition 3.5(i)). Since $\frac{1}{g'} \in H^\infty$, it follows that $f = \frac{1}{g'}(fg') \in A^{-(\gamma+1)}$.

(iii) Let $g(z);= z^n$ for all $z \in \mathbb{D}$ and some $n \in \mathbb{N}$. Then, for each $\gamma > 0$, we have that $A^{-(\gamma+1)} \subseteq [V_g, A^{-\gamma}]$ by part (i), as $g' \in H^\infty$.

If $g(z) := z$, for $z \in \mathbb{D}$, then $g'(z) = 1$ for all $z \in \mathbb{D}$ and so both $g', \frac{1}{g'} \in H^\infty$. By parts (i) and (ii) it follows that $[V_g, A^{-\gamma}] = A^{-(\gamma+1)}$.

Let $g(z) := z^2$, for $z \in \mathbb{D}$, in which case $g'(z) = 2z$ for all $z \in \mathbb{D}$. Given $f \in [V_g, A^{-\gamma}]$, Proposition 3.5(i) implies that $f \in H(\mathbb{D})$ and $(fg')(z) = 2zf(z) \in A^{-(\gamma+1)}$. Since $(zf(z))(0) = 0$, it follows that $zf(z) \in \mathcal{E}_v^\infty$, where $v(r) := (1-r)^{\gamma+1}$ for $r \in [0,1)$ and $\mathcal{E}_v^\infty = \{h \in A^{-(\gamma+1)} : \ h(0) = 0\}$. Hence, by Proposition 5.2(i) we see that $f = T(zf(z)) \in A^{-(\gamma+1)}$. So, $[V_g, A^{-\gamma}] \subseteq A^{-(\gamma+1)}$. Accordingly, $[V_g, A^{-\gamma}] = A^{-(\gamma+1)}$. Since $\frac{1}{g'} \notin H^\infty$, this condition is sufficient (cf. part (ii)) but, not necessary.

Finally, if $g(z) = z^n$ for all $z \in \mathbb{D}$ (hence, $g'(z) = nz^{n-1}$ for all $z \in \mathbb{D}$) then, for any given $f \in [V_g, A^{-\gamma}]$, Proposition 3.5(i) implies that $f \in H(\mathbb{D})$ and $(fg')(z) = nf(z)z^{n-1} \in A^{-(\gamma+1)}$. To conclude that $f \in A^{-(\gamma+1)}$ it suffices to repeat $(n-1)$ times the argument above for the case $n = 2$, that is, to apply $(n-1)$ times Proposition 5.2(i). Accordingly, $[V_g, A^{-\gamma}] = A^{-(\gamma+1)}$ for all functions $g(z) := z^n$ with $n \in \mathbb{N}$.

(iv) Let $g \in \mathcal{B}$. For $0 < \beta \leq \gamma$, we have seen that $A^{-\beta} \subseteq A^{-\gamma} \subseteq [V_g, A^{-\gamma}]$.

Now, let $0 < \gamma < \beta$. Suppose $A^{-\beta} \subseteq [V_g, A^{-\gamma}]$. Then, for any $f \in A^{-\beta}$, it follows (cf. Proposition 3.5(i)) that $f \in H(\mathbb{D})$ and $fg' \in A^{-(\gamma+1)}$. This implies that the multiplication operator $M_{g'} : H(\mathbb{D}) \to H(\mathbb{D})$ satisfies $M_{g'}(A^{-\beta}) \subseteq A^{-(\gamma+1)}$. Since $M_{g'} \in \mathcal{L}(H(\mathbb{D}))$, it follows that the linear operator $M_{g'} : A^{-\beta} \to A^{-(\gamma+1)}$ is defined and has closed graph. Hence, $M_{g'} \in \mathcal{L}(A^{-\beta}, A^{-(\gamma+1)})$. By [13, Proposition 3.1] this implies that

$$\sup_{z \in \mathbb{D}} \frac{(1-|z|)^{\gamma+1}|g'(z)|}{(1-|z|)^\beta} < \infty. \tag{5.7}$$

If $\beta > (\gamma+1)$, then (5.7) yields that $\lim_{|z|\to 1^-} |g'(z)| = 0$. Accordingly, $g'(z) = 0$ for all $z \in \mathbb{D}$ and hence, g is a constant function. If $\gamma < \beta \leq (\gamma+1)$, then (5.7) yields that $g' \in A^{-(\gamma+1-\beta)}$. In particular, for $\beta = \gamma+1$ we can conclude that $g' \in H^\infty$.

(v) Let $g \in \mathcal{B}$ and $\gamma < \beta < (\gamma+1)$. Then $A^{-\beta} \subseteq [V_g, A^{-\gamma}]$ if and only if $g' \in A^{-(\gamma+1-\beta)}$. Indeed, if $A^{-\beta} \subseteq [V_g, A^{-\gamma}]$, then part (iv) implies that $g' \in A^{-(\gamma+1-\beta)}$. Conversely, suppose that $g' \in H^\infty$. Then, for each $f \in A^{-\beta}$, we have that

$$\sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+1}|(fg')(z)| =$$

$$\sup_{z\in\mathbb{D}}(1-|z|)^{\beta}|f(z)|(1-|z|)^{\gamma+1-\beta}|g'(z)| \leq \|f\|_{-\beta}\|g'\|_{-(\gamma+1-\beta)} < \infty.$$

Accordingly, $fg' \in A^{-(\gamma+1)}$. By Proposition 3.5(i) we can conclude that $f \in [V_g, A^{-\gamma}]$.

(vi) Observe, for $0 < \beta < (\gamma+1)$, that the condition $g' \in A^{-(\gamma+1-\beta)}$ implies that $g \in \mathcal{B}_0$.

Declaration of competing interest

There is no competing interest.

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Data availability

No data was used for the research described in the article.

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