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Flag-transitive, point-imprimitive symmetric 2-(v, k, λ) designs with $k > \lambda(\lambda - 3)/2$

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ABSTRACT

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive, point-imprimitive automorphism group G that leaves invariant a non-trivial partition Σ of $\mathcal{P}$. Praeger and Zhou [42] have shown that, there is a constant k_0 such that, for each $B \in \mathcal{B}$ and $\Delta \in \Sigma$, the size of $|B \cap \Delta|$ is either 0 or k_0. In the present paper we show that, if $k > \lambda(\lambda - 3)/2$ and $k_0 \geqslant 3$, $\mathcal{D}$ is isomorphic to one of the known flag-transitive, point-imprimitive symmetric 2-designs with parameters $(45, 12, 3)$ or $(96, 20, 4)$.

1. Introduction and main result

A 2-(v, k, λ) *design* $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. We say $\mathcal{D}$ is *non-trivial* if $2 < k < v$, and symmetric if $v = b$. All 2-(v, k, λ) designs in this paper are assumed to be non-trivial. An automorphism of $\mathcal{D}$ is a permutation of the point set which preserves the block set. The set of all automorphisms of $\mathcal{D}$ with the composition of permutations forms a group, denoted by $\mathrm{Aut}(\mathcal{D})$. For a subgroup G of $\mathrm{Aut}(\mathcal{D})$, G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$, and said to be *point-imprimitive* otherwise. In this setting, we also say that $\mathcal{D}$ is either *point-primitive* or *point-imprimitive*, respectively. A *flag* of $\mathcal{D}$ is a pair (x, B) where x is a point and B is a block containing x. If $G \leqslant \mathrm{Aut}(\mathcal{D})$ acts transitively on the set of flags of $\mathcal{D}$, then we say that G is *flag-transitive* and that $\mathcal{D}$ is a *flag-transitive design*.

Flag-transitive symmetric designs are widely studied. If $\lambda = 1$, that is, $\mathcal{D}$ is a projective plane of order n, Kantor [26] proved that either $\mathcal{D}$ is Desarguesian and $PSL_3(n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of order $(n^2 + n + 1)(n + 1)$, and $n^2 + n + 1$ is a prime. In both cases, the action of G is point-primitive. For $\lambda > 1$, flag-transitive point-imprimitive symmetric designs do exist. In 1945 Hussain [21] and, independently, in 1946 Nandi [39] discovered that there are exactly three symmetric 2-$(16, 6, 2)$-designs. In 2006, O'Reilly Regueiro [44] showed that, if $\lambda \leqslant 4$ then the parameters of $\mathcal{D}$ are $(16, 2, 2)$, $(45, 12, 3)$, $(15, 8, 4)$, $(96, 20, 4)$ and that exactly two of the three 2-designs discovered by Hussain and Nandi are flag-transitive and point-imprimitive. In 2006, Praeger and Zhou [42] proved that there is exactly one flag-transitive, point-imprimitive symmetric 2-$(15, 8, 3)$ design, in 2007 Praeger [40] showed that there is exactly one flag-transitive, point-imprimitive symmetric 2-$(45, 12, 3)$ design, and in 2009, Law, Praeger and Reichard [29] proved that there are exactly four flag-transitive point-imprimitive symmetric 2-$(96, 40, 4)$ designs. Apart from two possible numerical

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exceptions, the classification of the flag-transitive point-imprimitive symmetric 2-designs has been recently extended to $\lambda \leqslant 10$ by Mandić and Šubasić [34].

It is worth noting that one of the four 2-(96, 40, 4) designs is a special case of a beautiful, general construction of flag-transitive, point-imprimitive symmetric 2-designs due to Cameron and Praeger [8] based on a previous work of Sane [45]. It is an open problem whether the remaining three 2-designs arise or not from the Cameron-Praeger construction.

An upper bound on k, when $\mathcal{D}$ is flag-transitive and point-imprimitive, was given by O'Reilly Regueiro in [44] and subsequently refined by Praeger and Zhou in [42]. Among the other results, the authors determined the parameters of $\mathcal{D}$ as functions of λ when $k > \lambda(\lambda - 3)/2$. Recently, the flag-transitive 2-designs with $\lambda = 2$ have been investigated by Devillers, Liang, Praeger and Xia in [12], where, it is shown that, apart from the two known symmetric 2-(16, 6, 2) designs, G is primitive of affine or almost simple type.

The present paper is a contribution to the problem of classifying flag-transitive, point-imprimitive symmetric 2-(v, k, λ) designs. In particular, we classify those with $k > \lambda(\lambda - 3)/2$ and such that a block of the 2-design intersects a block of imprimitivity in at least 3 points. More precisely, our result is the following.

Theorem 1.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive point-imprimitive automorphism group G that leaves invariant a non-trivial partition Σ of $\mathcal{P}$. If $k > \lambda(\lambda - 3)/2$ and there is block of $\mathcal{D}$ intersecting an element of Σ in at least 3 points, then one of the following holds:*

(1) *$\mathcal{D}$ is isomorphic to the 2-(45, 12, 3) design as in [40, Construction 4.2].*
(2) *$\mathcal{D}$ is isomorphic to one of the four 2-(96, 20, 4) designs as in [29].*

The outline of the proof is as follows. The group G preserves a set of imprimitivity Σ on the point set of $\mathcal{D}$ consisting of d classes each of size c. By [42], each block B of $\mathcal{D}$ intersects any block of imprimitivity either in 0 or in a constant number k_0 of points. In Lemma 2.1 we show that the number of blocks intersecting a block of imprimitivity in the same k_0-set of points is constant and is independent on the choice of the block of $\mathcal{D}$ and of the element of Σ. We call such a number *the overlap number of* $\mathcal{D}$ and we denote it by θ.

If $k_0 \geqslant 3$, in Theorems 2.3 and 2.4 we show that the blocks of imprimitivity have the structure of flag-transitive 2-$(c, k_0, \lambda/\theta)$ designs, where (c, k_0) is either (λ^2, λ), or $(\lambda + 6, 3)$ with $\lambda \equiv 1, 3 \pmod 6$. Moreover, in Lemma 2.6 we prove that, such 2-designs are also point-primitive. Flag-transitive, point-primitive 2-$(\lambda^2, \lambda, \lambda/\theta)$ designs are classified in [35,37,38], whereas flag-transitive, point-primitive 2-$(\lambda + 6, 3, \lambda/\theta)$ designs are shown to be embedded in $\mathcal{D}$ only for $\lambda = 3$ by using [7,36]. Finally, we complete the proof of Theorem 1.1 by combining the previous information on the structure of the blocks of imprimitivity with the constraints on the action of G on $\mathcal{D}$ given in [28] and on the structure of G essentially provided in [3].

2. The overlap number of $\mathcal{D}$

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive point-imprimitive automorphism group G that leaves invariant a non-trivial partition Σ of $\mathcal{P}$ with d classes each of size c. Then there is a constant k_0 such that, for each $B \in \mathcal{B}$ and $\Delta_i \in \Sigma$, $i = 1, \ldots, d$, the size $|B \cap \Delta_i|$ is either 0 or k_0 by [42, Theorem 1.1]. If we pick two distinct points x, y in a block of imprimitivity, then there are exactly λ blocks of $\mathcal{D}$ incident with them. Thus $k_0 \geqslant 2$. Moreover, $v > k$ since $\mathcal{D}$ is non-trivial, and hence $k_0 < c$ by [42, (4) and (7)]. Therefore, $2 \leqslant k_0 < c$. If $k_0 = 2$, then the flag-transitivity of G on $\mathcal{D}$ implies the 2-transitivity of $G_{\Delta_i}^{\Delta_i}$ on Δ_i for each $i = 1, \ldots, d$.

Let $\mathcal{B}_i = \{B \in \mathcal{B} : B \cap \Delta_i \neq \varnothing\}$, where $i = 1, \ldots, d$. For any $B \in \mathcal{B}_i$ define

$$\mathcal{B}_i(B) = \left\{B' \in \mathcal{B}_i : B' \cap \Delta_i = B \cap \Delta_i\right\} \text{ and } \theta(i, B) = |\mathcal{B}_i(B)|.$$

Clearly, $1 \leqslant \theta(i, B) \leqslant \lambda$.

Lemma 2.1. *$\theta(i, B) = \theta(j, B')$ for each $i, j \in \{1, \ldots, d\}$ and for each $B \in \mathcal{B}_i$ and $B' \in \mathcal{B}_j$.*

Proof. Let $B \in \mathcal{B}_i$ and $B' \in \mathcal{B}_j$, where $i, j \in \{1, \ldots, d\}$, and let $x \in B \cap \Delta_i$ and $x' \in B' \cap \Delta_j$. Then there is $\gamma \in G$ such that $(x, B)^\gamma = (x', B')$ since G is flag-transitive. Hence, $(B \cap \Delta_i)^\gamma = B' \cap \Delta_j$.

Let $C \in \mathcal{B}_i(B)$. Then $C \cap \Delta_i = B \cap \Delta_i$, and hence

$$C^\gamma \cap \Delta_j = (C \cap \Delta_i)^\gamma = (B \cap \Delta_i)^\gamma = B' \cap \Delta_j.$$

Thus $C^\gamma \in \mathcal{B}_j(B')$, and hence $\theta(i, B) \leqslant \theta(j, B')$. Now, switching the role of B and B' in the previous argument, we get $\theta(j, B') \leqslant \theta(i, B)$. Thus $\theta(i, B) = \theta(j, B')$, which is the assertion. □

In view of the previous lemma, we may denote $\theta(i, B)$ simply by θ and call it *the overlap number of* $\mathcal{D}$.

Corollary 2.2. *Let $B \in \mathcal{B}_i$ and $x \in B \cap \Delta_i$, then $\theta = \left|G_{x,B\cap\Delta_i} : G_{x,B}\right|$.*

Proof. Let $B \in \mathcal{B}_i$ and $x \in B \cap \Delta_i$, then $G_{x,B} \leqslant G_{x,B\cap\Delta_i}$. Thus, $\left|G_{x,B\cap\Delta_i} : G_{x,B}\right| \leqslant \theta$.

Let $B' \in \mathcal{B}_i(B)$. Then there is $\varphi \in G_x$ such that $B^{\varphi} = B'$. Thus $(B \cap \Delta_i)^{\varphi} = B' \cap \Delta_i = B \cap \Delta_i$, and hence $\varphi \in G_{x,B\cap\Delta_i}$ and $G_{x,B}\varphi \subseteq G_{x,B\cap\Delta_i}$. Therefore $\left|G_{x,B\cap\Delta_i} : G_{x,B}\right| \geqslant \theta$, and hence $\left|G_{x,B\cap\Delta_i} : G_{x,B}\right| = \theta$. □

Theorem 2.3. *If $k_0 \geqslant 3$, then $\mathcal{D}_i = (\Delta_i, \mathcal{B}_i^*)$, where $\mathcal{B}_i^* = \{B \cap \Delta_i : B \in \mathcal{B}_i\}$, is a non-trivial 2-$(c, k_0, \lambda/\theta)$ design with $\theta \mid \lambda$ admitting $G_{\Delta_i}^{\Delta_i}$ as a flag-transitive automorphism group.*

Proof. Clearly, the number of points in $\mathcal{D}_i$ is c and each element of $\mathcal{B}_i^*$ contains k_0 points of Δ_i. Let $x_1, x_2 \in \Delta_i$ with $x_1 \neq x_2$, then there are precisely λ blocks of $\mathcal{D}$ incident with them, say $B_1, \ldots, B_\lambda$. For each B_j there are precisely θ blocks among $B_1, \ldots, B_\lambda$ whose intersection set with Δ_i is $B_j \cap \Delta_i$, hence there are exactly λ/θ distinct elements of $\mathcal{B}_i^*$ incident with $x_1 \neq x_2$. Thus, $\mathcal{D}_i$ is a 2-$(c, k_0, \lambda/\theta)$ design. Also, $\mathcal{D}_i$ is non-trivial since $k_0 < c$ by [42, (4) and (7)] and $k_0 \geqslant 3$ by our assumption. Finally, the flag-transitivity of G on $\mathcal{D}$ implies the flag-transitivity of $G_{\Delta_i}^{\Delta_i}$ on $\mathcal{D}_i$. □

The following theorem is an improvement of [42, Theorem 1.1] on the basis of Theorem 2.3.

Theorem 2.4. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a symmetric 2-design admitting a flag-transitive point-imprimitive automorphism group G that leaves invariant a non-trivial partition $\Sigma = \{\Delta_1, \ldots, \Delta_d\}$ of $\mathcal{P}$ such that $|\Delta_i| = c$ for each $i = 1, \ldots, d$. Then the following hold:*

I. *There is a constant k_0 such that, for each $B \in \mathcal{B}$ and $\Delta_i \in \Sigma$, the size $|B \cap \Delta_i|$ is either 0 or k_0.*
II. *There is a constant θ such that, for each $B \in \mathcal{B}$ and $\Delta_i \in \Sigma$ with $|B \cap \Delta_i| > 0$, the number of blocks of $\mathcal{D}$ whose intersection set with Δ_i coincides with $B \cap \Delta_i$ is θ.*
III. *If $k_0 = 2$, then $G_{\Delta_i}^{\Delta_i}$ acts 2-transitively on Δ_i for each $i = 1, \ldots, d$.*
IV. *If $k_0 \geqslant 3$, then $\mathcal{D}_i = \left(\Delta_i, (B \cap \Delta_i)^{G_{\Delta_i}^{\Delta_i}}\right)$ is a flag-transitive non-trivial 2-$(c, k_0, \lambda/\theta)$ design for each $i = 1, \ldots, d$.*

Moreover, if $k > \lambda(\lambda - 3)/2$ then one of the following holds:

V. *$k_0 = 2$ and one of the following holds:*
   1. *$\mathcal{D}$ is a symmetric 2-$(\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ design and $(c, d) = (\lambda + 2, \lambda^2)$.*
   2. *$\mathcal{D}$ is a symmetric 2-$\left(\left(\frac{\lambda+2}{2}\right)\left(\frac{\lambda^2-2\lambda+2}{2}\right), \frac{\lambda^2}{2}, \lambda\right)$ design, $(c, d) = \left(\frac{\lambda+2}{2}, \frac{\lambda^2-2\lambda+2}{2}\right)$, and either $\lambda \equiv 0 \pmod 4$, or $\lambda = 2u^2$ with u odd, $u \geqslant 3$ and $2(u^2 - 1)$ square.*

VI. *$k_0 \geqslant 3$ and one of the following holds:*
   1. *$\mathcal{D}$ is a symmetric 2-$(\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$ design, $d = \lambda + 2$, and $\mathcal{D}_i$ is a 2-$(\lambda^2, \lambda, \lambda/\theta)$ design with $\theta \mid \lambda$ for each $i = 1, \ldots, d$.*
   2. *$\mathcal{D}$ is a symmetric 2-$((\lambda + 6)\frac{\lambda^2+4\lambda-1}{4}, \lambda\frac{\lambda+5}{2}, \lambda)$ design with $\lambda \equiv 1, 3 \pmod 6$, $d = \frac{\lambda^2+4\lambda-1}{4}$, and $\mathcal{D}_i$ is a 2-$(\lambda + 6, 3, \lambda/\theta)$ design with $\theta \mid \lambda$ for each $i = 1, \ldots, d$.*

Proof. The assertion follows from [42, Theorem 1.1] and Theorem 2.3. □

From now on we assume that $k > \lambda(\lambda - 3)/2$ and $k_0 \geqslant 3$. Hence, we will focus on the symmetric 2-designs in (VI.1) and (VI.2) of Theorem 2.4, and we will refer to them as 2-*designs of type 1 and 2*, respectively.

Lemma 2.5. $\lambda \geqslant 3$.

Proof. If $\lambda = 2$, then $k_0 = 2$ by [42, Corollary 1.3 and Table 1], which is contrary to our assumption. Thus, $\lambda \geqslant 3$. □

Lemma 2.6. *If $k > \lambda(\lambda - 3)/2$ and $k_0 \geqslant 3$, then $G_{\Delta_i}^{\Delta_i}$ acts point-primitively on $\mathcal{D}_i$.*

Proof. The assertion follows from [10, 2.3.7.(c)] or [23, Theorem 4.8.(i)]. □

Lemma 2.7. *Let N be a minimal normal subgroup of G. Then one of the following holds:*

(1) *Σ is the N-orbit decomposition of the point set of $\mathcal{D}$;*
(2) *N acts point-transitively on $\mathcal{D}$;*

or, for $c = \lambda^2$ and $d = \lambda + 2$ the following additional possibility arises:

(3) *The N-orbit decomposition of the point set of $\mathcal{D}$ is a further G-invariant partition $\Sigma' = \left\{\Delta'_1, \ldots, \Delta'_{\lambda^2}\right\}$ such that the following hold:*
(a) $\left|\Delta'_j\right| = \lambda + 2$ *for each* $j = 1, \ldots, \lambda^2$;
(b) *For each* $B \in \mathcal{B}$ *and* $\Delta'_j \in \Sigma'$, *the size* $\left|B \cap \Delta'_j\right|$ *is either* 0 *or* 2;
(c) *For each* $\Delta_i \in \Sigma$ *and* $\Delta'_j \in \Sigma'$, $\left|\Delta_i \cap \Delta'_j\right| = 1$.
(d) $G_{\Delta'_j}^{\Delta'_j}$ *acts* 2*-transitively on* Δ'_j *for each* $j = 1, \ldots, \lambda^2$.

Proof. Let N be a minimal normal subgroup of G. Assume that $G_{\Delta_i}N$ acts point-transitively on $\mathcal{D}$. Then N acts transitively on Σ. If there is $j_0 \in \{1, \ldots, d\}$ such that $N_{\Delta_{j_0}}^{\Delta_{j_0}} = 1$, then $N_{\Delta_{j_0}} \leqslant G_{(\Delta_{j_0})}$. Hence, $N_{\Delta_i} \leqslant G_{(\Delta_i)}$ for each i since G acts transitively on Σ and $N \trianglelefteq G$. Thus, the point set of $\mathcal{D}$ is split into c' orbits under N each of length d', where $(c', d') = (d, c)$, since N acts transitively on Σ. Hence, $\Sigma' = \left\{\Delta'_1, \ldots, \Delta'_c\right\}$, where $\Delta'_j = x_j^N$ for each $j = 1, \ldots, c$, is a set of imprimitivity for G. Moreover, $N_{x_i} = N_{(\Delta_i)}$ for each $x_i \in \Delta_i$ and $i = 1, \ldots, c$. By Theorem 2.4, there is a constant k'_0 such that for each $B \in \mathcal{B}$ and $\Delta'_i \in \Sigma'$, the size $\left|B \cap \Delta'_i\right|$ is either 0 or k'_0.

If $k'_0 = 2$, then either $\left(c', d'\right) = \left(\lambda + 2, \lambda^2\right)$, or $\left(c', d'\right) = \left(\frac{\lambda+2}{2}, \frac{\lambda^2-2\lambda+2}{2}\right)$ and either $\lambda \equiv 0 \pmod 4$, or $\lambda = 2u^2$, where u is odd, $u \geqslant 3$, and $2(u^2-1)$ is a square by Theorem 2.4. On the other hand, we know that $(c', d') = (d, c)$ and either $(d, c) = (\lambda + 2, \lambda^2)$, or $\left(\frac{\lambda^2+4\lambda-1}{4}, \lambda + 6\right)$ and $\lambda \equiv 1, 3 \pmod 6$ again by Theorem 2.4 since $k_0 \geqslant 3$. By comparing the values of (c', d'), we see that the unique admissible value is $(c', d') = (d, c) = (\lambda + 2, \lambda^2)$, and we obtain (3a) and (3b).

Let $\Delta_i \in \Sigma$ and $\Delta'_j \in \Sigma'$. Since $N_{x_i} = N_{(\Delta_i)}$ for each $x_i \in \Delta_i$ and $i = 1, \ldots, \lambda + 2$, and Δ'_j is a N-orbit for each $j = 1, \ldots, \lambda^2$, it follows that $\left|\Delta_i \cap \Delta'_j\right| = 1$. Also, $G_{\Delta'_j}^{\Delta'_j}$ acts 2-transitively on $\Delta'_j \in \Sigma'$ since $k'_0 = 2$. Thus, we get (3c) and (3d).

If $k'_0 \geqslant 3$, then either $\left(c', d'\right) = \left(\lambda^2, \lambda + 2\right)$, or $\left(c', d'\right) = \left(\lambda + 6, \frac{\lambda^2+4\lambda-1}{4}\right)$ and $\lambda \equiv 1, 3 \pmod 6$ by Theorem 2.4. On the other hand, we know that $(c', d') = (d, c)$ and either $(d, c) = (\lambda + 2, \lambda^2)$, or $\left(\frac{\lambda^2+4\lambda-1}{4}, \lambda + 6\right)$ and $\lambda \equiv 1, 3 \pmod 6$. By comparing the values of (c', d') no admissible λ's arise since $\lambda \geq 3$ by Lemma 2.5.

Assume that $N_{\Delta_i}^{\Delta_i} \neq 1$ for each $i = 1, \ldots, d$. Hence, $N_{\Delta_i}^{\Delta_i}$ acts point-transitively on $\mathcal{D}_i$ for each $i = 1, \ldots, d$ since $G_{\Delta_i}^{\Delta_i}$ acts point-primitively on $\mathcal{D}_i$ by Lemma 2.6. Therefore, N acts point-transitively on $\mathcal{D}$, as N acts transitively on Σ, which is (2).

Assume that $G_{\Delta_i}N$ acts point-intransitively on $\mathcal{D}$. Hence, $G_{\Delta_i}N \neq G$. Then $\Delta_i \subseteq \Delta''_i$, where $\Delta''_i = x^{G_{\Delta_i}N} = \Delta_i^N$ and $x \in \Delta_i$. Also, $\Sigma'' = \left\{\left(\Delta''_i\right)^g : g \in G\right\}$ is a set of imprimitivity for G by [14, Theorem 1.5A]. If $B \in \mathcal{B}$ is such that $B \cap \Delta''_i \neq \varnothing$, then $k''_0 = \left|B \cap \Delta''_i\right| \geqslant |B \cap \Delta_i| = k_0 \geqslant 3$, and hence we may apply Theorem 2.4 referred to the set of imprimitivity Σ'', and we obtain that $\mathcal{D}''_i = (\Delta''_i, \left(B \cap \Delta''_i\right)^{G_{\Delta''_i}^{\Delta''_i}})$ is a flag-transitive non-trivial 2-$(c'', k''_0, \lambda/\theta'')$ design. Moreover, either $c'' = \lambda^2$ or $c'' = \lambda + 6$ since $k > \lambda(\lambda - 3)/2$. It is easily seen that $c'' = c$ since $c \mid c''$, being $\Delta''_i = \Delta_i^N$. Thus $\Delta_i = \Delta''_i$, and hence $N \trianglelefteq G_{\Delta_i}$ for each $i = 1, \ldots, d$. If there $i_0 \in \{1, \ldots, d\}$ such that N fixes a point in Δ_{i_0}, then N fixes each point of $\mathcal{D}$ since $N \trianglelefteq G$ and G acts point-transitively on $\mathcal{D}$, and we reach a contradiction. Thus, $N^{\Delta_i} \neq 1$ for each $i = 1, \ldots, d$, and hence N acts point-transitively on $\mathcal{D}_i$ since $N^{\Delta_i} \trianglelefteq G_{\Delta_i}^{\Delta_i}$ and $G_{\Delta_i}^{\Delta_i}$ acts point-primitively on $\mathcal{D}_i$ by Lemma 2.6. Therefore, Σ is the orbit decomposition of the point set of $\mathcal{D}$ under N, which is (1). □

Let $\Delta \in \Sigma$ and $x \in \Delta$. Since $G_{(\Sigma)} \trianglelefteq G_\Delta$ and $G_{(\Delta)} \trianglelefteq G_x$ it is immediate to verify that $(G^\Sigma)_\Delta = (G_\Delta)^\Sigma$ and $\left(G_\Delta^\Delta\right)_x = (G_x)^\Delta$. Hence, $(G^\Sigma)_\Delta$ and $\left(G_\Delta^\Delta\right)_x$ will simply be denoted by G_Δ^Σ and G_x^Δ, respectively.

3. The case where $\mathcal{D}$ is of type 1

In this section, we assume that $\mathcal{D}$ is of type 1. Hence, $\mathcal{D}$ is a symmetric 2-$(\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$ design with $d = \lambda + 2$. Moreover, $\mathcal{D}_i$ is a 2-$(\lambda^2, \lambda, \lambda/\theta)$ design with $\theta \mid \lambda$ admitting $G_{\Delta_i}^{\Delta_i}$ as a flag-transitive point-primitive automorphism group for each $i = 1, \ldots, d$. Our aim is to prove the following result.

Theorem 3.1. *If $\mathcal{D}$ is of type 1, then one of the following holds:*

(1) *$\mathcal{D}$ is isomorphic to the* 2-$(45, 12, 3)$ *design as in [40, Construction 4.2].*
(2) *$\mathcal{D}$ is isomorphic to one of the four* 2-$(96, 20, 4)$ *designs as in [29].*

Proposition 3.2. *G induces a 2-transitive group on Σ.*

Proof. It is clear that G acts transitively on Σ. Let B be any block of $\mathcal{D}$ and define $\Sigma(B)=\{\Delta_i \in \Sigma : \Delta_i \cap B \neq \varnothing\}$. Then $|\Delta_i \cap B|=\lambda$ for each $\Delta_i \in \Sigma(B)$. Further, $|\Sigma(B)|=\lambda+1$ and $\Sigma \setminus \Sigma(B)=\{\Delta_{i_0}\}$ for some $i_0 \in \{1,\dots,\lambda+2\}$ since $k=\lambda(\lambda+1)$ and $|\Sigma|=\lambda+2$. Since G_B acts transitively on B and preserves Σ, it follows that G_B acts transitively on $\Sigma(B)$. Thus G_B preserves Δ_{i_0}, and hence $G_B \leqslant G_{\Delta_{i_0}}$. Therefore $G_{\Delta_{i_0}}$ acts transitively on $\Sigma \setminus \{\Delta_{i_0}\}$, and hence G induces a 2-transitive group on Σ. □

Lemma 3.3. *If $G_{(\Delta_i)} \neq 1$, then either the primes dividing the order of $G_{(\Delta_i)}$ divide λ, or $\mathcal{D}_i$ is a translation plane.*

Proof. Assume that $G_{(\Delta_i)} \neq 1$. Let W be any Sylow w-subgroup of $G_{(\Delta_i)}$, where w is a prime not dividing λ. Clearly, W fixes the $\frac{\lambda^2}{\theta}(\lambda+1)$ blocks of $\mathcal{D}_i$. Let B be any block of $\mathcal{D}$ such that $B \cap \Delta_i$ is a block of $\mathcal{D}_i$. Then W preserves $B \cap \Delta_i$ and there are θ blocks of $\mathcal{D}$ whose intersection set with Δ_i is $B \cap \Delta_i$. Therefore W fixes at least one of these θ blocks, as $w \nmid \lambda$ and $\theta \mid \lambda$, and hence W fixes at least $\frac{\lambda^2}{\theta}(\lambda+1)$ blocks of $\mathcal{D}$. Then any non-trivial element of W fixes at least $\frac{\lambda^2}{\theta}(\lambda+1)$ points of $\mathcal{D}$ by [28, Theorem 3.1], and hence $\frac{\lambda^2}{\theta}(\lambda+1) \leqslant \frac{\lambda}{k-\sqrt{(k-\lambda)}} v$ by [28, Corollary 3.7]. Since $k=\lambda(\lambda+1)$ and $v=\lambda^2(\lambda+2)$, it follows that $\frac{\lambda^2}{\theta}(\lambda+1) \leqslant \lambda(\lambda+2)$. Thus $\theta=\lambda$, and hence $\mathcal{D}_i$ is a 2-$(\lambda^2,\lambda,1)$ design, that is, an affine plane. Then $\mathcal{D}_i$ is a translation plane by [51] since $G_{\Delta_i}^{\Delta_i}$ acts flag-transitively on $\mathcal{D}_i$. □

The following theorem classifies the flag-transitive 2-$(\lambda^2,\lambda,\lambda/\theta)$ designs $\mathcal{D}_i$.

Theorem 3.4. *If $\mathcal{D}_i$ is a 2-$(\lambda^2,\lambda,\lambda/\theta)$ design admitting a flag-transitive automorphism group $G_{\Delta_i}^{\Delta_i}$, then one of the following holds*

(1) *$G_{\Delta_i}^{\Delta_i}$ is almost simple and one of the following holds:*
  (a) *$\mathcal{D}_i$ is isomorphic to the 2-$(6^2,6,2)$ design constructed in [35], $\theta=3$ and $PSL_2(8) \trianglelefteq G_{\Delta_i}^{\Delta_i} \leqslant P\Gamma L_2(8)$.*
  (b) *$\mathcal{D}_i$ is isomorphic to one of the three 2-$(6^2,6,6)$ designs constructed in [35], $\theta=1$ and $G_{\Delta_i}^{\Delta_i} \cong P\Gamma L_2(8)$.*
  (c) *$\mathcal{D}_i$ is isomorphic to the 2-$(12^2,12,3)$ design constructed in [37], $\theta=4$ and $G_{\Delta_i}^{\Delta_i} \cong PSL_3(3)$.*
  (d) *$\mathcal{D}_i$ is isomorphic to the 2-$(12^2,12,6)$ design constructed in [37], $\theta=2$ and $G_{\Delta_i}^{\Delta_i} \cong PSL_3(3):Z_2$.*

(2) *$G_{\Delta_i}^{\Delta_i}=T:G_0^{\Delta_i}$, $\lambda=p^m$, p prime, $m \geqslant 1$, and one of the following holds:*
  (a) *$\mathcal{D}_i$ is a translation plane of order p^m, $\theta=p^m$, and one of the following holds:*
    (i) *$\mathcal{D}_i \cong AG_2(p^m)$ and the possibilities $G_0^{\Delta_i}$ are given [15,32].*
    (ii) *$\mathcal{D}_i$ is the Lüneburg plane of order 2^m, $m \equiv 2 \pmod 4$, $m \geqslant 6$, and $Sz(2^{m/2}) \trianglelefteq G_0^{\Delta_i} \leqslant \left(Z_{2^{m/2}-1} \times Sz(2^{m/2})\right).Z_{m/2}$;*
    (iii) *$\mathcal{D}_i$ is the Hall plane of order 3^2 and the possibilities for $G_0^{\Delta_i}$ are given [16];*
    (iv) *$\mathcal{D}_i$ is the Hering plane of order 3^3 and $G_0^{\Delta_i} \cong SL_2(13)$.*
  (b) *$\mathcal{D}_i$ is a 2-(p^{2m},p^m,p^{m-t}) design, $\theta=p^t$, where $0 \leqslant t \leqslant m$, the blocks are subspaces of $AG_{2m}(p)$ and $G_0^{\Delta_i} \leqslant \Gamma L_1(p^{2m})$;*
  (c) *$\mathcal{D}_i$ is isomorphic to one of the following 2-design constructed in [38]:*
    (i) *$\mathcal{D}$ is a 2-$(p^{2m},p^m,p^{m/2})$ design, m is even, $\theta=p^{m/2}$ and either $SL_2(p^m) \trianglelefteq G_0^{\Delta_i} \leqslant (Z_{p^{m/2}-1} \circ SL_2(p^m)).Z_{(2,p^m-1)}.Z_m$, or $SL_2(5) \trianglelefteq G_0^{\Delta_i} \leqslant \left(\langle -1 \rangle .S_5^-\right):Z_2$ for $p=3$ and $m=2$.*
    (ii) *$\mathcal{D}_i$ is a 2-$(p^{3m},p^{3m/2},p^m)$ design, p odd and m even, $\theta=p^{m/2}$ and $SU_3(p^{m/2}) \trianglelefteq G_0^{\Delta_i} \leqslant (Z_{p^{m/2}-1} \times SU_3(p^{m/2})).Z_m$.*
    (iii) *$\mathcal{D}_i$ is a 2-(p^{4m},p^{2m},p^{2m}) design, $\theta=1$ and $Sp_4(p^m) \trianglelefteq G_0^{\Delta_i} \leqslant \Gamma Sp_4(p^m)$.*
    (iv) *$\mathcal{D}_i$ is one of the four 2-$(2^{4m},2^{2m},\lambda)$ designs with $\lambda=2^m,2^{2m-1},2^{2m},2^{2m}$ respectively, $m>1$ is odd, and $Sz(2^m) \trianglelefteq G_0 \leqslant \left(Z_{2^m-1} \times Sz(2^m)\right).Z_m$. Further, $\theta=2^m,2,1,1$ respectively.*
    (v) *$\mathcal{D}_i$ is a 2-$(2^{6m},2^{3m},2^{3m})$ design, $\theta=1$ and $G_2(2^m) \trianglelefteq G_0^{\Delta_i} \leqslant (Z_{2^m-1} \times G_2(2^m)):Z_m$.*
    (vi) *$\mathcal{D}_i$ is one of the two 2-$(2^6,2^3,2^2)$-designs, $\theta=1$ and $G_0^{\Delta_i}$ is either one of the groups $3^{1+2}:Q_8$, $3^{1+2}:Z_8$ or $3^{1+2}:SD_{16}$, or $3^{1+2}:Z_8 \leqslant G_0^{\Delta_i} \leqslant PSU_3(3)$.*
    (vii) *$\mathcal{D}_i$ is a 2-$(2^6,2^3,2^3)$-design, $\theta=1$ and $G_0^{\Delta_i}$ is one of the groups $3^{1+2}:Q_8$, $3^{1+2}:Z_8$, $3^{1+2}:SD_{16}$, $\left(3^{1+2}:Q_8\right):Z_3:Z_2$.*

See [35,37,38] for a proof.

Proposition 3.5. *$G_{\Delta_i}^{\Delta_i}$ is of affine type and $\lambda=p^m$.*

Proof. Assume that $G_{\Delta_i}^{\Delta_i}$ is almost simple. Then either $PSL_2(8) \trianglelefteq G_{\Delta_i}^{\Delta_i} \leqslant P\Gamma L_2(8)$ and $\lambda=6$, or $PSL_3(3) \trianglelefteq G_{\Delta_i}^{\Delta_i} \leqslant PSL_3(3):Z_2$ and $\lambda=12$ by Theorem 3.4.

Assume that the former occurs. Since G^{Σ} acts 2-transitively on Σ by Proposition 3.2, and $|\Sigma|=8$, one of the following holds by [25, Section 2, (A) and (B)]:

(1) $AGL_1(8) \trianglelefteq G^{\Sigma} \leqslant A\Gamma L_1(8)$;
(2) $G^{\Sigma} \cong E_8 : SL_3(2)$;
(3) $PSL_2(7) \trianglelefteq G^{\Sigma} \leqslant PGL_2(7)$;
(4) $A_8 \trianglelefteq G^{\Sigma} \leqslant S_8$.

Assume that (4) holds. Since $G_{(\Sigma)}G_{(\Delta_i)} \trianglelefteq G_{\Delta_i}$ and $A_7 \trianglelefteq G^{\Sigma}_{\Delta_i} \leqslant S_7$, either $G_{(\Delta_i)} \trianglelefteq G_{(\Sigma)}$ or $G_{(\Delta_i)}/(G_{(\Delta_i)} \cap G_{(\Sigma)})$ contains a subgroup isomorphic to A_7. The latter is ruled out by Lemma 3.3 since $\lambda = 6$, whereas the former implies that a quotient group of $G^{\Delta_i}_{\Delta_i}$ is isomorphic to A_7, which is impossible since $PSL_2(8) \trianglelefteq G^{\Delta_i}_{\Delta_i} \leqslant P\Gamma L_2(8)$. Thus, (4) is ruled out.

Assume that one of (1)–(3) occurs. Since $G_{(\Sigma)}G_{(\Delta_i)} \trianglelefteq G_{\Delta_i}$ and $PSL_2(8) \trianglelefteq G^{\Delta_i}_{\Delta_i} \leqslant P\Gamma L_2(8)$, either $G_{(\Sigma)} \trianglelefteq G_{(\Delta_i)}$ or $PSL_2(8) \trianglelefteq G_{(\Sigma)}/(G_{(\Sigma)} \cap G_{(\Delta_i)})$. The former implies $G^{\Delta_i}_{\Delta_i} \cong G^{\Sigma}_{\Delta_i}/G^{\Sigma}_{(\Delta_i)}$ and hence a quotient group of $G^{\Sigma}_{\Delta_i}$ has a subgroup isomorphic to $PSL_2(8)$, but this is clearly impossible. So $PSL_2(8) \trianglelefteq G_{(\Sigma)}/(G_{(\Sigma)} \cap G_{(\Delta_i)})$ and $A_7 \trianglelefteq G^{\Sigma}_{\Delta_i}$. Hence, if W is any Sylow 7-subgroup of G_{Δ_i}, $7^2 \mid |W|$. Then $7 \mid |W(\Delta_i)|$, since $PSL_2(8) \trianglelefteq G^{\Delta_i}_{\Delta_i} \leqslant P\Gamma L_2(8)$, but this contradicts Lemma 3.3.

Assume that $PSL_3(3) \trianglelefteq G^{\Delta_i}_{\Delta_i} \leqslant PSL_3(3) : Z_2$ and $\lambda = 12$. Since G^{Σ} acts 2-transitively on Σ with $|\Sigma| = 14$, one of the following holds by [25, Section 2, (A) and (B)]:

(1) $PSL_2(13) \trianglelefteq G^{\Sigma} \leqslant PGL_2(13)$;
(2) $A_{14} \trianglelefteq G^{\Sigma} \leqslant S_{14}$.

We may proceed as the $PSL_2(8)$-case to rule out (1) and (2), this time W is a Sylow 13-subgroup of G_{Δ_i}. □

Lemma 3.6. *The following hold:*

(1) $G_{(\Delta_i)} \trianglelefteq G_{(\Sigma)} \leqslant G_{\Delta_i}$ *for each* $i = 1, \dots, p^m + 2$.
(2) $G_{(\Delta_i)} \cap G_{(\Delta_j)} = 1$ *for each* $i, j = 1, \dots, p^m + 2$ *with* $i \neq j$.

Proof. Since G_{Δ_i} acts transitively on $\Sigma \setminus \{\Delta_i\}$ by Proposition 3.2, and since $G_{(\Delta_i)} \trianglelefteq G_{\Delta_i}$, it follows that $\Sigma \setminus \{\Delta_i\}$ is union of $G_{(\Delta_i)}$-orbits of the equal length z, where z is a divisor of $p^m + 1$ by Proposition 3.5. Assume that $z > 1$. Then $\mathcal{D}_i$ is a translation plane of order p^m by Lemma 3.3. Let U be a Sylow u-subgroup of $G_{(\Delta_i)}$, where u is a prime divisor of z. Arguing as in Lemma 3.3, with U in the role of W, we see that U fixes at least $p^m(p^m+1)$ blocks of $\mathcal{D}$ and each of these intersects Δ_i in p^m points, since $\mathcal{D}$ is a translation plane and $\theta = p^m$. Let B be any of such blocks. Then U preserves Δ_i and at least one the p^m elements of $\Sigma \setminus \{\Delta_i\}$ intersecting B, say Δ_j. Then $\left|\Delta_j^{G_{(\Delta_i)}}\right|$ is coprime to u, whereas $\left|\Delta_j^{G(\Delta_i)}\right| = z$ and $u \mid z$. Thus $G_{(\Delta_i)}$ preserves each element of Σ and hence $G_{(\Delta_i)} \leqslant G_{(\Sigma)}$. Actually, $G_{(\Delta_i)} \trianglelefteq G_{(\Sigma)}$ as $G_{(\Sigma)} \leqslant G_{\Delta_i}$.

Let $\gamma \in G_{(\Delta_i)} \cap G_{(\Delta_i)}$, with $i \neq j$, then γ fixes $2p^{2m}$ points of $\mathcal{D}$. If $\gamma \neq 1$ then $2p^{2m} \leqslant p^m(p^m+2)$ by [28, Corollary, 3.7]. So $\lambda = p^m \leqslant 2$, which is contrary to Lemma 2.5. Thus $\gamma = 1$ and hence $G_{(\Delta_i)} \cap G_{(\Delta_j)} = 1$ for $i \neq j$. □

Corollary 3.7. $G_{(\Sigma)} \neq 1$.

Proof. Suppose that $G_{(\Sigma)} = 1$. Then $G_{(\Delta)} = 1$ for each $\Delta \in \Sigma$ by Lemma 3.6(1), and hence $Soc(G_{\Delta})$ is elementary abelian of order p^{2m} by Proposition 3.5. Then $\Sigma \setminus \{\Delta\}$ is partitioned in $Soc(G_{\Delta})$-orbits of equal length p^h with $h > 0$ since G_{Δ} acts transitively on $\Sigma \setminus \{\Delta\}$, $Soc(G_{\Delta}) \trianglelefteq G_{\Delta}$ and $G_{(\Sigma)} = 1$. Then p divides $|\Sigma \setminus \{\Delta\}|$, thus contradicting $|\Sigma \setminus \{\Delta\}| = p^m + 1$. □

Proposition 3.8. *Let* V *be a minimal normal subgroup of* G *contained in* $G_{(\Sigma)}$.

(1) $V^{\Delta_i} \cong Soc(G^{\Delta_i}_{\Delta_i})$ *for each* $i = 1, \dots, p^m + 2$.
(2) V *is an elementary abelian* p*-group of order* p^{2m+t}, *where* $0 \leqslant t \leqslant 2m$.
(3) $C_G(V) \cap G_{(\Sigma)}$ *is an elementary abelian* p*-group order* p^{2m+y}, *where* $t \leqslant y \leqslant 2m$, *containing* V.

Proof. Let V be a minimal normal subgroup of G contained in $G_{(\Sigma)}$. Then V acts transitively on Δ_i for each $i = 1, \dots, p^m + 2$ by Lemma 2.7. Moreover, $V^{\Delta_i} \trianglelefteq G^{\Delta_i}_{\Delta_i}$. If R is a minimal normal subgroup of $G^{\Delta_i}_{\Delta_i}$ contained in V^{Δ_i}, then $Soc(G^{\Delta_i}_{\Delta_i}) = R \trianglelefteq V^{\Delta_i}$ by [14, Theorem 4.3B(i)] since $G^{\Delta_i}_{\Delta_i}$ acts primitively on Δ_i by Lemma 2.6 and since $Soc(G^{\Delta_i}_{\Delta_i})$ is an elementary abelian group of order p^{2m} by Proposition 3.5. Thus V^{Δ_i} contains a normal subgroup isomorphic to $Soc(G^{\Delta_i}_{\Delta_i})$, and hence V is an elementary abelian p-group, since V is a minimal normal subgroup of G. Therefore $V^{\Delta_i} \cong Soc(G^{\Delta_i}_{\Delta_i})$, which is (1).

It follows from (1) that $\left|V^{\Delta_i}\right| = p^{2m}$. Let $p^t = \left|V_{(\Delta_{i_0})}\right|$ with $t \geqslant 0$ for some i_0, then $\left|V_{(\Delta_i)}\right| = p^t$ for each $i = 1, \dots, p^m + 2$ since $V \trianglelefteq G$ and G acts transitively on Σ. Moreover, it follows from Lemma 3.6(2) that, $V_{(\Delta_s)} \cap V_{(\Delta_{s'})} = 1$ for each $s, s' = 1, \dots p^m + 2$ with $s \neq s'$. Thus, $V_{(\Delta_s)}$ is isomorphic to a subgroup of $V^{\Delta_{s'}}$. Therefore $p^t \leqslant p^{2m}$, and hence $0 \leqslant t \leqslant 2m$. So $|V| = p^{2m+t}$ with $0 \leqslant t \leqslant 2m$, and we obtain (2).

Set $C = C_G(V)$ and $K = G_{(\Sigma)}$ and recall that G is permutationally isomorphic to a subgroup of $G^{\Delta}_{\Delta} \wr G^{\Sigma}$ by [41, Theorem 5.5]. Since

$$C \cap K \leqslant \prod_{\Delta \in \Sigma} (C \cap K)^{\Delta} \leqslant \prod_{\Delta \in \Sigma} C_{K^{\Delta}}(V^{\Delta}) = \prod_{\Delta \in \Sigma} V^{\Delta},$$

it follows $C \cap K$ is an elementary abelian p-group. By repeating the final part of the argument in (2) with $C \cap K$ in the role of V, we see that the order of $C \cap K$ is p^{2m+y} with $t \leqslant y \leqslant 2m$ since $V \leqslant C$, and we obtain (3). □

Remark 3.9. The proof of (3) is a slight modification of an argument contained in the proof of [29, Lemma 3.2]: here we consider a minimal normal subgroup of G instead of $O_p(G_{(\Sigma)})$, the largest normal p-subgroup of $G_{(\Sigma)}$. In this way $G/C_G(V)$ is an irreducible subgroup of $GL(V)$, and this will play an important role in completing the proof of Theorem 3.1.

Corollary 3.10. *Let $\Delta \in \Sigma$ and $x \in \Delta$, then G^{Σ}_{Δ} is isomorphic to a quotient group of G^{Δ}_{x}.*

Proof. It follows from Lemma 3.6(1) and Proposition 3.8(1) that $G_{\Delta} = G_{(\Sigma)} G_x$ with $G_{(\Delta)} \trianglelefteq G_{(\Sigma),x}$. Thus

$$G^{\Sigma}_{\Delta} \cong G_x / G_{(\Sigma),x} \cong G^{\Delta}_{x} / G^{\Delta}_{(\Sigma),x} \tag{3.1}$$

which is the assertion. □

Lemma 3.11. *Let $u = p^m + 2$ with u prime and $u \geqslant 5$. If u divides $p^z - 1$ for some $0 < z \leqslant 4m$, then either $(p^m, u, z) = (3, 5, 4)$ or $(p^m, u, z) = (9, 11, 5)$.*

Proof. Clearly $z > m$. Set $z = m + w$, where $w > 0$. Then u divides $p^w(p^m + 2) - 2p^w - 1$ and hence $2p^w + 1$. Then $p^w \geqslant p^m/2 + 1/2$, and hence $w = m + y$ for some $y \geqslant 0$, as p is odd. Thus u divides $2p^y(p^m + 2) - (4p^y - 1)$, and hence $4p^y - 1$. Then either $p^m + 2 = 4p^y - 1$, or $2(p^m + 2) \leqslant 4p^y - 1$. The former yields $p^y = 3$, $p^m = 3^2$ and hence $(p^m, u, z) = (9, 11, 5)$, the latter implies $p^y \geqslant p^m/2 + 5/4$ and hence $y = m + x$ for some $0 \leqslant x \leqslant 2m$ since $z \leqslant 4m$. Therefore u divides $(4p^m + 8)p^x - (8p^x + 1)$, and hence $8p^x + 1$. Thus $a(p^m + 2) = 8p^x + 1$ for some $a \geqslant 1$. If $x < m$, then $(ap^{m-x} - 8)p^x + 2a - 1 = 0$. Therefore $ap^{m-x} < 8$, and hence either $a = 1$ and $p^{m-x} = 3, 5$ or 7, or $a = 2$ and $p^{m-x} = 3$ since p is odd. It is easy to verify that no solutions arise in these cases since u is a prime. Thus $x = m$, and hence $(8 - a)p^m = 2a - 1$ with $1 \leqslant a < 8$. Thus either $a = 3$ and $p^m = 3$, or $a = 7$ and $p^m = 13$. However, the latter cannot occur since $u = p^m + 2$ and u is a prime, hence $(p^m, u, z) = (3, 5, 4)$. □

Proposition 3.12. *One of the following holds:*

(1) $C_G(V) \leqslant G_{(\Sigma)}$.
(2) $C_G(V) = V \times U$, *where U is a minimal normal subgroup of G of order u^h, with u prime and $h \geqslant 1$, satisfying the following properties:*
   (a) $u^h = p^m + 2$;
   (b) *The U-orbit decomposition of the point set of $\mathcal{D}$ is a G-invariant partition $\Sigma' = \left\{\Delta'_1, \dots, \Delta'_{p^{2m}}\right\}$ such that $\left|\Delta'_j\right| = p^m + 2$ for each $j = 1, \dots, p^{2m}$ and the following hold:*
      (i) *For each $B \in \mathcal{B}$ and $\Delta'_j \in \Sigma'$, the size $\left|B \cap \Delta'_j\right|$ is either 0 or 2.*
      (ii) $\left|\Delta_i \cap \Delta'_j\right| = 1$ *for each $\Delta_i \in \Sigma$ and $\Delta'_j \in \Sigma'$;*
      (iii) $G^{\Delta'_j}_{\Delta'_j}$ *acts 2-transitively on Δ'_j for each $\Delta'_j \in \Sigma'$;*
      (iv) $U^{\Delta'_j} = Soc(G^{\Delta'_j}_{\Delta'_j})$ *for each $\Delta'_j \in \Sigma'$.*

Proof. Let $C = C_G(V)$ and assume that $C \nleq G_{(\Sigma)}$. Then $1 \neq C^{\Sigma} \trianglelefteq G^{\Sigma}$. Thus $Soc(G^{\Sigma}) \leqslant C^{\Sigma}$, as G^{Σ} acts 2-transitively on Σ. Therefore, C^{Σ} acts transitively on Σ.

Assume that $(C^{\Sigma})_{\Delta_1} \neq 1$, where $\Delta_1 \in \Sigma$. Then $V < C_{\Delta_1}$ as $V \leqslant C \cap G_{(\Sigma)}$ by Proposition 3.8(3). So $C_{\Delta_1} = C_x V$, and hence $C_x \leqslant G_{(\Delta_1)}$ since V acts transitively on Δ_1 and $C = C_G(V)$. Therefore, $C_x \leqslant G_{(\Sigma)}$ by Lemma 3.6(1). So $C_{\Delta_1} \leqslant G_{(\Sigma)}$, and hence $(C^{\Sigma})_{\Delta_1} = 1$, a contradiction.

Assume that $(C^{\Sigma})_{\Delta_1} = 1$. Thus C^{Σ} acts regularly on Σ, and hence $C^{\Sigma} = Soc(G^{\Sigma})$ since $Soc(G^{\Sigma}) \leqslant C^{\Sigma}$. Therefore $C/(C \cap G_{(\Sigma)}) \cong Soc(G^{\Sigma})$, where $\left|Soc(G^{\Sigma})\right| = p^m + 2$, with $p^m + 2 \not\equiv 0 \pmod 4$ since $p^m \geqslant 3$ by Lemma 2.5. Thus $Soc(G^{\Sigma})$ is an elementary abelian group of order u^h for some prime u and some integer $h \geqslant 1$ since G^{Σ} acts 2-transitively on Σ. Therefore $u^h = p^m + 2$ and hence $u \neq p$. Then $C = X : U$, where $X = C \cap G_{(\Sigma)}$ and U is an elementary abelian of order u^h, by [18, Theorem 6.2.1] since $C/(C \cap G_{(\Sigma)}) \cong Soc(G^{\Sigma})$ and since X is an elementary abelian p-group by Proposition 3.8(3). In particular $U \leqslant GL(X)$. Then $X = X_1 \oplus \cdots \oplus X_\ell$, with $\ell \geqslant 1$, where the X_s's, $s = 1, \ldots, \ell$, are irreducible U-invariant subspaces of X by [18, Theorem 3.3.1]. Moreover, for each $s = 1, \ldots, \ell$ there is a subgroup of U_s of U of index at most u, fixing X_s pointwise by [18, Theorem 3.2.3] since U is elementary abelian.

Assume that $V < X$. Then there is X_{s_0} containing an element, say x, such that $x \notin V$. If $h > 1$, let ψ be an element of U_{s_0}, $\psi \neq 1$. Then ψ centralizes V, $\langle x \rangle$ and hence $V \oplus \langle x \rangle$. Now $V \oplus \langle x \rangle$ acts on Δ_1, and since it contains p^{2m+1} elements, there is an element $\alpha \in V \oplus \langle x \rangle$, $\alpha \neq 1$, such that $\alpha \in G_{(\Delta_1)}$ and α and ψ commute. On the other hand U acts regularly on Σ since $U^\Sigma = C/(C \cap G_{(\Sigma)}) \cong Soc(G^\Sigma)$ with $C \cap G_{(\Sigma)}$ a p-group by Proposition 3.8(3). Hence, ψ maps Δ_1 onto Δ_e for some $e > 1$. So $\alpha \in G_{(\Delta_1)} \cap G_{(\Delta_e)}$ with $\alpha \neq 1$ since α and ψ commute, but this contradicts Lemma 3.6(2). Thus $h = 1$ and hence $U \cong Z_u$. If U fixes an element in $X \setminus V$, we reach a contradiction by using the previous argument. Thus U does not fix points in $X \setminus V$ and hence $u \mid p^{2m+y} - p^{2m+t}$ with $0 \leqslant t < y \leqslant 2m$ by Lemma 3.8(2),(3), since $V < X$. Then $u \mid p^{y-t} - 1$ with $0 < y - t \leqslant 2m$, which is impossible for Lemma 3.11. Thus $X = V$, and hence $C = V \times U$ with U is elementary abelian of order u^h and $u^h = p^m + 2$. Moreover $U \trianglelefteq G$ as $C \trianglelefteq G$.

Let U^* be a minimal normal subgroup of G contained in U. The set Σ' of all point-U^*-orbits is G-invariant partition of the point set of $\mathcal{D}$. If either $\Sigma' = \Sigma$ or U^* acts point-transitively on $\mathcal{D}$, then c or v is a power of u respectively. However both these cases lead to a contradiction since $c = p^{2m}$, $v = p^{2m}(p^m + 2)$ and $u \neq p$. Thus, $\Sigma' = \left\{\Delta'_1, \ldots, \Delta'_{p^{2m}}\right\}$, with $\left|\Delta'_j\right| = p^m + 2 = u^h$ for each $j = 1, \ldots, p^{2m}$, by Lemma 2.7. Hence $U^* = U$ and we obtain (2a). Moreover, for each $B \in \mathcal{B}$ and $\Delta'_j \in \Sigma'$, the size $\left|B \cap \Delta'_j\right|$ is either 0 or 2, $G_{\Delta'_j}^{\Delta'_j}$ acts 2-transitively on Δ'_j and $U^{\Delta'_j} = Soc(G_{\Delta'_j}^{\Delta'_j})$. Finally, $\left|\Delta_i \cap \Delta'_j\right| = 1$ for each $\Delta_i \in \Sigma$ and $\Delta'_j \in \Sigma'$. Thus (2b.i)–(2b.iv) follow. □

The Diophantine equation in Proposition 3.12(2.a) is a special case of the Pillai Equation (e.g. see [47]). It has at most one solution in positive integers (m, h) by [47, Theorem 6]. Moreover, $p > 3$ for $h > 1$, and $(p^m, u^h) = (5^2, 3^3)$ for $h > 1$ and m even by [47, Lemmas 2 and 4].

4. Reduction to the case $C_G(V) \leqslant G_{(\Sigma)}$

In this section, we show that only (1) of Proposition 3.12 occurs. Hence, assume that $C_G(V) = V \times U$, where U is a minimal normal elementary abelian u-subgroup of G of order u^h, with $u^h = p^m + 2$, satisfying properties (2b.i)–(2b.iv) of Proposition 3.12.

Lemma 4.1. *Let $\Delta \in \Sigma$ and $\Delta' \in \Sigma'$ and let x be their intersection point. Then the following hold:*

(1) *G acts faithfully on Δ;*
(2) *$G_x^{\Delta'}$ is isomorphic to a quotient group of G_x.*

Proof. Let $\Delta_i \in \Sigma$ and $\Delta'_j \in \Sigma'$ and let x_{ij} be their (unique) intersection point. Since $G_{(\Delta)} \leqslant G_{(\Sigma)}$ by Lemma 3.6(1), it follows that $G_{(\Delta)}$ preserves each Δ_i. On the other hand, $G_{(\Delta)}$ preserves each Δ'_j since these ones intersect Δ in a unique point and Σ' is a G-invariant partition of the point set of $\mathcal{D}$. Thus $G_{(\Delta)}$ fixes each x_{ij}, and hence $G_{(\Delta)}$ fixes $\mathcal{D}$ pointwise. Therefore $G_{(\Delta)} = 1$, which is (1). Finally, assertion (2) holds since $G_{(\Delta')} \trianglelefteq G_x$. □

Lemma 4.2. *The following hold:*

(1) *$\frac{p^m}{\theta}(p^m + 1)$ divides $|G_x|$;*
(2) *$\theta(p^m + 2)(p^m + 1)$ divides $\left|G_{\Delta'}^{\Delta'}\right|$.*

Proof. Let $\Delta \in \Sigma$ and $\Delta' \in \Sigma'$ and let x be their intersection point. The replication number $r = \frac{p^m}{\theta}(p^m + 1)$ of $\mathcal{D}_i$ divides $|G_x|$ since $\mathcal{D}_i$ is a flag-transitive 2-$(p^{2m}, p^m, \frac{p^m}{\theta})$ design, hence (1) holds.

Since $G_{\Delta'}^{\Delta'}$ acts 2-transitively on Δ' and $U^{\Delta'} = Soc(G_{\Delta'}^{\Delta'})$ with $U^{\Delta'}$ elementary abelian of order $p^m + 2$, by Proposition 3.12(2a), (2.b.iii) and (2.b.iv), it follows that $(p^m + 2)(p^m + 1)$ divides $\left|G_{\Delta'}^{\Delta'}\right|$.

Let B be any block of $\mathcal{D}$ such that $x \in B$. Then $B \cap \Delta' = \{x, y\}$ for some $y \neq x$ by Proposition 3.12(2.b.i). Let $\gamma \in G_{x,B\cap\Delta} \cap G_{(\Delta')}$. Then $B^\gamma \cap \Delta = B \cap \Delta$, and $y \in B^\gamma$ since $B \cap \Delta' = \{x, y\}$. Thus $(B \cap \Delta) \cup \{y\} \subseteq B^\gamma \cap B$ with $y \notin B \cap \Delta$, as $y \in \Delta'$, $y \neq x$, and $\Delta \cap \Delta' = \{x\}$. Therefore $\left|B^\gamma \cap B\right| \geqslant \lambda + 1$, and hence $B^\gamma = B$. So $\gamma \in G_{x,B}$, as $x^\gamma = x$. Thus $G_{x,B\cap\Delta} \cap G_{(\Delta')} \leqslant G_{x,B}$, and hence θ divides $\left|\left(G_{x,B\cap\Delta}\right)_{\Delta'}^{\Delta'}\right|$ since $\left|G_{x,B\cap\Delta} : G_{x,B}\right| = \theta$ by Corollary 2.2. So, θ divides $\left|G_{\Delta'}^{\Delta'}\right|$. Hence, $\theta(p^m + 2)(p^m + 1) \mid \left|G_{\Delta'}^{\Delta'}\right|$, which is (2), since $\theta \mid p^m$ and $(p^m + 2)(p^m + 1)$ divides $\left|G_{\Delta'}^{\Delta'}\right|$. □

Theorem 4.3. $C_G(V) \leqslant G_{(\Sigma)}$.

Proof. Let $\Delta_i \in \Sigma$ and $\Delta' \in \Sigma'$ and let x_i be their intersection point. Recall that $\mathcal{D}_i$ is isomorphic to one of the 2-designs listed in Theorem 3.4. Since $p^m + 2 = u^h$ with u prime by Proposition 3.12(2.a) and $p^m \geqslant 3$ by Lemma 2.5, it follows that $p > 2$. Thus, cases (2.a.ii) and (2.c.iv)–(2.c.vii) of Theorem 3.4 are ruled out.

Assume that $h > 1$. If $\mathcal{D}_i$ is as in case (2.c.i) of Theorem 3.4, then $(p^m, u^h) = (5^2, 3^3)$ or $(11^2, 5^3)$ by [47, Lemma 4] since m is even. Further, either $SL_3(u) \trianglelefteq G_\Delta^\Sigma \leqslant \Gamma L_3(u)$, or $G_\Delta^\Sigma \leqslant \Gamma L_1(u^3)$ by [25] since G^Σ is an affine group acting 2-transitively on Σ by Propositions 3.2 and 3.12. However, no cases occur by Corollary 3.10. Case (2.c.iii) of Theorem 3.4 is ruled out similarly.

Assume that case (2.c.ii) of Theorem 3.4 occurs. Then $p^{3m/2} + 2 = u^h$ with p odd and m even, and hence u is odd and $u \neq 5, 7, 11, 23, 29, 59$ for $h = 2$ and $u^h \neq 3^4, 3^6$. Then either G_Δ^Σ contains one of the groups $SL_h(u)$ or $Sp_h(u)$ as a normal subgroup, or $G_\Delta^\Sigma \leqslant \Gamma L_1(u^3)$ by [25] since G^Σ is an affine group acting 2-transitively on Σ. Again, we reach a contradiction by Corollary 3.10 since $p^{3m/2} + 1$ divides the order of G_Δ^Σ.

Assume that $h = 1$. Hence, $G_{\Delta'}^{\Delta'} \cong AGL_1(u)$ by Proposition 3.12(2.b.iii). Then $\theta = 1$ by Lemma 4.2(2), and hence (2.a.iii)–(2.a.iv) and in (2.c.v) of Theorem 3.4 are ruled out. Therefore, bearing in mind Lemma 4.1(1), one of the following holds:

(i). $\mathcal{D}_i$ is a 2-(p^{2m}, p^m, p^{m-s}) design, $p > 2$ and $p^m \neq 3^2, 3^3$, $\theta = p^s$ with $0 \leqslant s \leqslant m$, and the blocks are subspaces of $AG_{2m}(p)$ and $G_{x_i} \leqslant \Gamma L_1(p^{2m})$.
(ii). $\mathcal{D}_i$ is a 2-(p^{4m}, p^{2m}, p^{2m}) design, $\theta = 1$ and $Sp_4(p^m) \trianglelefteq G_{x_i} \leqslant \Gamma Sp_4(p^m)$

Assume that (i) occurs. Then p^{m-s} divides the order of G_{x_i} by Lemma 4.2(1). On the other hand, p^s divides the order of $G_{x_i}^{\Delta'}$, and hence $\left|G_{x_i}\right|$ by Lemmas 4.1(2) and 4.2(2). Therefore p^a divides the order of G_{x_i}, and hence $\left|\Gamma L_1(p^{2m})\right|$, where $a = \max\{s, m-s\} \geqslant m/2$. So $p^{m/2} \leqslant m$, which is a contradiction since p is odd.

Assume that (ii) occurs. Then $G_\Delta^\Sigma \cong GL_1(u)$, with $u = p^m + 2$ is isomorphic to a quotient group of G_{x_i} with $Sp_4(p^m) \trianglelefteq G_{x_i} \leqslant \Gamma Sp_4(p^m)$. Hence $p^m + 2 \mid (p^m - 1)m$, which is not the case since p is odd. Thus (2) of Proposition 3.12 is ruled out, and hence the assertion follows. □

5. Proof of Theorem 3.1

In this section, we complete the proof of Theorem 3.1. In the sequel, $C_G(V)$ and $G/C_G(V)$ will simply be denoted by C and H, respectively.

Proposition 5.1. *H is an irreducible subgroup of $GL_{2m+t}(p)$ with $0 \leqslant t \leqslant 2m$.*

Proof. Since V is a minimal normal elementary abelian subgroup of G of order p^{2m+t} with $0 \leqslant t \leqslant 2m$ by Proposition 3.8(2), the assertion follows. □

For each divisor n of $2m + t$ the group $\Gamma L_n(p^{(2m+t)/n})$ has a natural irreducible action on V. By Proposition 5.1 we may choose n to be minimal such that $H \leqslant \Gamma L_n(p^{(2m+t)/n})$ in this action and write $q = p^{(2m+t)/n}$.

Let a, e be integers. A divisor w of $a^e - 1$ that is coprime to each $a^i - 1$ for $1 \leqslant i < e$ is said to be a *primitive divisor*, and we call the largest primitive divisor $\Phi_e^*(a)$ of $a^e - 1$ the *primitive part* of $a^e - 1$. One should note that $\Phi_e^*(a)$ is strongly related to cyclotomy in that it is equal to the quotient of the cyclotomic number $\Phi_e(a)$ and $(e, \Phi_e(a))$ when $e > 2$. Also, $\Phi_e^*(a) > 1$ for $a \geqslant 2$, $e > 2$ and $(a, e) \neq (2, 6)$ by Zsigmondy's Theorem (for instance, see [43, P1.7]).

Since G^Σ is a 2-transitive group by Proposition 3.2, either G^Σ is of affine type or an almost simple group. We analyze the two cases separately.

5.1. G^Σ is of affine type

In this subsection, we assume that G^Σ is of affine type. Hence, $Soc(G^\Sigma)$ is an elementary abelian u-group for some prime u. Let u^h be the order of $Soc(G^\Sigma)$, then $u^h = p^m + 2$ since $|\Sigma| = p^m + 2$. In the sequel, U will denote the pre-image of $Soc(G^\Sigma)$ in G.

Lemma 5.2. *The following hold:*

(1) *A quotient group of H has a 2-transitive permutation representation of degree $p^m + 2$.*
(2) $(p^m + 1)\left(p^m + 2\right) \mid |H|$. *In particular, $\Phi_{2m}^*(p) \mid |H|$.*
(3) *Either $\Phi_{2m}^*(p) > 1$, or $(p^m, u^h) = (3, 5)$ or $(7, 9)$.*

Proof. Since $C \leqslant G_{(\Sigma)}$ by Theorem 4.3 and G^Σ acts 2-transitively on Σ by Proposition 3.2, a quotient group of H is isomorphic to G^Σ. Thus, (1) and (2) follow.

Suppose that $\Phi^*_{2m}(p)=1$, then either $m=1$ and q is a Mersenne prime, or $(p,m)=(2,3)$ by [43, P1.7]. The latter is ruled out since $p^m+2=10$ in this case, the former yields $u^h=p^m+2=2^y+1$, for some prime $y\geqslant 0$. Then either $(p^m,u^h,y)=(7,9,3)$, or $h=1$ by [43, B1.1]. If $h=1$ then u is a Fermat prime, hence $y=2$ since y is a prime, and we obtain $(u^h,y)=(5,2)$. Therefore, either $(p^m,u^h)=(3,5)$ or $(p^m,u^h)=(7,9)$. □

From now on, we assume that $\Phi^*_{2m}(p)>1$. The cases $(p^m,u^h)=(3,5),(7,9)$ are tackled at the end of this subsection.

Lemma 5.3. *$n>1$.*

Proof. Suppose that $n=1$. Then $H\leqslant \Gamma L_1(q)$, and hence $(p^m+2)\Phi^*_{2m}(p)\mid (p^{2m+t}-1)\cdot(2m+t)$. Then $2m\mid 2m+t$ by [27, Proposition 5.2.15(i)], and hence either $t=0$ or $t=2m$ since $0\leqslant t\leqslant 2m$. If $t=0$, then $p^m+2\mid (p^{2m}-1)\cdot 2m$, and hence $p^m+2\mid 3m$, which is impossible since $p^m>3$ by our assumption. Thus $t=2m$, and hence $(p^m+2)\mid(p^{4m}-1)\cdot 4m$. So $p^m+2\mid 15m$, and hence $p^m=3$, but this contradicts our assumption. □

Proposition 5.4. *Let $H^*=H\cap GL_n(q)$, then $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,(2m+t)/n)}\mid |H^*|$.*

Proof. By Lemma 5.2(2), one has $\frac{\Phi^*_{2m}(p)(p^m+2)}{(\Phi^*_{2m}(p)(p^m+2),(2m+t)/n)}\mid |H^*|$. Assume that $(\Phi^*_{2m}(p),(2m+t)/n))>1$. Then there is a primitive prime divisor w of $p^{2m}-1$ dividing $(2m+t)/n$. So $w\mid 2m+t$. On the other hand, $w=2ma+1$ for some $a\geqslant 1$ by [27, Proposition 5.2.15(ii)]. Hence $(2ma+1)s=2m+t$ for some $s\geqslant 1$ and hence $t=s=a=1$. So $w=n=2m+1$ and $q=p$, whereas $w\mid(2m+1)/n$, a contradiction. Thus $(\Phi^*_{2m}(p),(2m+t)/n))=1$, and the assertion follows since $(\Phi^*_{2m}(p),p^m+2)=1$. □

Proposition 5.5. *$n=4m$, $q=p$ and one of the following holds:*

(1) *$H\leqslant GL_{2m}(p)\wr Z_2$ and H preserves a sum decomposition $V=V_1\oplus V_2$ with $\dim V_1=\dim V_2=2m$.*
(2) *$H\leqslant GL_2(p)\circ GL_{2m}(p)$, $m>1$, and H preserves a tensor decomposition $V=V_1\otimes V_2$ with $\dim V_1=2$ and $\dim V_2=2m$.*

Proof. Assume that $Y\trianglelefteq H$, where Y is isomorphic to one of the groups $SL_n(q)$, $Sp_n(q)$ for n even, $SU_n(q^{1/2})$ for q square, or $\Omega^{\varepsilon}_n(q)$ with $\varepsilon=\pm$ for n even, and $\varepsilon=\circ$ for nq odd by [3, Theorem 3.1]. Then $H\leqslant(Z\circ Y).Out(\bar{Y})$, where $Z=Z(GL_n(q))$ and $\bar{Y}=Y/(Y\cap Z)$, and hence $|H/Y|\mid (q-1)\left|Out(\bar{Y})\right|$.

Let X be the pre-image of Y in G. Then $X\trianglelefteq G$, and hence $X^\Sigma\trianglelefteq G^\Sigma$. Either $X^\Sigma=1$ or $U^\Sigma\trianglelefteq X^\Sigma$ since G^Σ acts 2-transitively on Σ and since $U^\Sigma=Soc(G^\Sigma)$. The latter implies that U^Σ is isomorphic to a normal subgroup of $X/(X\cap G_{(\Sigma)})$, with $C\leqslant X\cap G_{(\Sigma)}$ by Theorem 4.3. So $X/(X\cap G_{(\Sigma)})$ is a quotient group of the classical group Y and contains a normal elementary abelian group of order u^h. Then $C\leqslant X\cap G_{(\Sigma)}\leqslant W$, where W is the pre-image of $Z(Y)$ in G, and hence the normal elementary abelian subgroup of order u^h of $X/(X\cap G_{(\Sigma)})$ is contained in $W/(X\cap G_{(\Sigma)})$. Thus, u^h divides $|Z(Y)|$ and hence $q-1$, which is impossible since $u^h=q^m+2\geqslant 5$ by Lemma 5.2. Therefore $X^\Sigma=1$, and hence $X\leqslant G_{(\Sigma)}$. Thus, $Y\leqslant G_{(\Sigma)}/C$ by Theorem 4.3. On the other hand, we have $G^\Sigma\cong H/(G_{(\Sigma)}/C)$, where $H/(G_{(\Sigma)}/C)$ is a quotient group of H/Y. Thus $\left|G^\Sigma\right|$ divides the order of H/Y, and hence $\left|G^\Sigma\right|\mid (q-1)\left|Out(\bar{Y})\right|$.

Note that, $\left|Out(\bar{Y})\right|\mid 4\cdot n\cdot\mu(2m+t)/n$, where $\mu=2$ or 3 according to whether X is isomorphic or not to $\Omega^+_8(q)$, respectively. Then

$$\Phi^*_{2m}(p)(p^m+2)\mid \mu\left(p^{\frac{2m+t}{n}}-1\right)(2m+t) \tag{5.1}$$

since $\Phi^*_{2m}(p)(p^m+2)$ divides the order G^Σ by Proposition 3.2 and $\Phi^*_{2m}(p)$ and p are odd. Let w be a primitive prime divisor of p^m+1. If $w\mid p^{\frac{2m+t}{n}}-1$, then $2m\mid\frac{2m+t}{n}$ by [27, Proposition 5.2.15(i)], and hence $n=2$ and $t=2m$ since $t\leqslant 2m$ and $n>1$ by Lemma 5.3. Thus $\mu=2$ and $p^m+2\mid m\left(p^m-1\right)$, and hence $p^m+2\mid 3m$ since p is odd, which is impossible by $p^m\geqslant 3$. Thus, $w\mid\mu\,(2m+t)$.

If $\mu=2$, then $w\mid 2m+t$. If $\mu=3$ then $n=8$, and hence $m\geqslant 2$ since $n\mid 2m+t$ with $t\leqslant 2m$. Moreover, $w\equiv 1 \pmod{2m}$ by [27, Proposition 5.2.15(ii)], and hence $w\neq 3$. Thus, $w\mid 2m+t$ also in this case. Again from [27, Proposition 5.2.15(ii)], it results that $w=2m+1$ and $t=1$. Moreover, $n=2m+1$, and hence $q=p$ since $n\mid 2m+1$ with $2m+1$ prime and $n>1$ by Lemma 5.3. Then $\Phi^*_{2m}(p)(p^m+2)\mid(p-1)\mu n$ by (5.1) with n dividing $\Phi^*_{2m}(p)$, hence $p^m+2\mid 3\mu$, which is impossible since p is odd and $p^m\neq 7$ by our assumption. Thus $Y\ntrianglelefteq H$, and hence H lies either in a member of $\mathcal{C}_i(\Gamma)$ for some i such that $1\leqslant i\leqslant 7$, or is a member of $\mathcal{S}(\Gamma)$, where Γ denotes $\Gamma L_n(q)$, by the Aschbacher's Theorem (see [27]).

Assume that H lies in $\mathcal{S}(\Gamma)$. Then $S\trianglelefteq H/(H\cap Z)\leqslant Aut(S)$, where S is a non-abelian simple group and Z is the center of $GL_n(q)$. Then the pre-image N of S in H is absolutely irreducible and is not a classical group over a subfield of $GF(q)$ in its natural representation. Let M and Q be the pre-images in G of N and of $Z(N)$, respectively. Since $M,U\trianglelefteq G$, we may use the above argument with M, Q, N and $Z(N)$ in the role of X, W, Y and Z, respectively, to obtain that either a quotient of $Z(N)$ contains a normal elementary subgroup of order u^h, or $M\trianglelefteq G_{(\Sigma)}$ and $\left|G^\Sigma\right|\mid(q-1)\left|Out(S)\right|$. The former implies that there is a subgroup of the Schur multiplier of S containing a normal elementary subgroup of order u^h. Then $h=1$ by [27, Theorem 5.1.4]. Let ψ be an element of $Z(N)$ of order u. Then ψ does not fix non-zero vectors of V, and so $u \mid p^{2m+t}-1$, with $0<2m+t\leqslant 4m$. Thus, $p^m=9$ by Lemma 3.11 since $p^m>3$ by our assumption, and hence $u=11$ divides $|Z(N)|$ and $n\leqslant 4m=8$. Also, H is an irreducible subgroup of $\Gamma L_n(3^{(4+t)/n})$ of order divisible by 55 by Lemma 5.2(2). Then either $t=1$, $n=5$, $q=3$ and $H=N$ with N is isomorphic to $PSL_2(11)$ or M_{11}, or $t=2$, $n=6$, $q=3$ and $H=N\cong Z_2.M_{12}$ by [6, Tables 8.2, 8.4, 8.9, 8.19. 8.25, 8.36 and 8.43]. However, 11 does not divide the order $Z(N)$ in any of these groups, and hence they are ruled out. Thus $M\trianglelefteq G_{(\Sigma)}$. Moreover, $1\neq M^{\Delta_i}\trianglelefteq G_{\Delta_i}^{\Delta_i}$ for each $\Delta_i\in\Sigma$ since $C\leqslant M$, $M\cap G_{(\Delta_i)}\leqslant Q$ for each $\Delta_i\in\Sigma$ and $M/Q\cong N/Z(N)\cong S$ with S non-abelian simple. Hence, $M^{\Delta_i}=T:(M^{\Delta_i})_0$ and a quotient group of M^{Δ_i} is non-abelian simple.

Since M^{Δ_i} is non-solvable and $p^m+2=u^h$ with u prime and $p^m>3$, one of the following holds by Theorem 3.4:

(1) $SL_2(p^m)\trianglelefteq(M^{\Delta_i})_0\leqslant\Gamma L_2(p^m)$;
(2) $SU_3(p^{m/3})\trianglelefteq(M^{\Delta_i})_0\leqslant(Z_{p^{m/3}-1}\times SU_3(p^{m/3})).Z_{2m/3}$, $m\equiv 0 \pmod 3$;
(3) $Sp_4(p^{m/2})\trianglelefteq(M^{\Delta_i})_0\leqslant\Gamma Sp_4(p^{m/2})$, m even;
(4) $(M^{\Delta_i})_0\leqslant(Q_8\circ D_8).S_5$ and $p^m=9$.

Note that, in the previous list some automorphism groups of non-isomorphic 2-designs listed in Theorem 3.4 are brought together. Indeed, the group in (2.c.i) of Theorem 3.4 is a subgroup of $\Gamma L_2(p^m)$ as well as that is (2.c.i) for $p^m=9$ is a subgroup of the full translation complement of the Hall plane of order 9, which is $(Q_8\circ D_8).S_5$ by [33, Theorem II.8.3]. Either $S\cong PSL_2(p^m)$, or $S\cong PSU_3(p^{m/3})$ or $S\cong PSp_4(p^{m/2})$, or $S\cong PSL_2(5)$ for $p^m=9$ since $M^{\Delta_i}\cong M/(M\cap G_{(\Delta_i)})$, $M\cap G_{(\Delta_i)}\leqslant Q$ and $M/Q\cong N/Z(N)\cong S$ with S non-abelian simple. However, both cases are ruled out since they violate $\left|G^{\Sigma}\right| \mid (q-1)\,|Out(S)|$, being $\left|G^{\Sigma}\right|$ divisible by p^m+2 with $p^m>3$. Thus, H lies in a member of $\mathcal{C}_i(\Gamma)$ for some i such that $1\leqslant i\leqslant 7$.

The group H does not lie in a member of $\mathcal{C}_1(\Gamma)$ since H is irreducible subgroup of Γ by Proposition 5.1 and subsequent remark, and does not lie in a member of $\mathcal{C}_3(\Gamma)$ by the definition of q. Also, H does not lie in a member of $\mathcal{C}_5(\Gamma)$. Indeed, if not so, then $n<4m$ since $q=p$ for $n=4m$. Then H^* lies in a member of $\mathcal{C}_5(GL_n(q))$, but this is impossible by [3, Theorem 3.1] since $\Phi^*_{2m}(p)\mid |H^*|$ by Proposition 5.4. Assume that H lies in a member of $\mathcal{C}_2(\Gamma)$. Then H stabilizes a decomposition of $V=V_1\oplus\cdots\oplus V_{n_0}$ with $n_0>1$ and $\dim V_1=\cdots=\dim V_{n_0}=m_0\geqslant 1$. Thus $n=m_0n_0$, and hence $H^*\leqslant GL_{m_0}(q)\wr S_{n_0}$.

If $n<4m$, then $m_0=1$, $n_0=n$, $q=p$, and either $p^m=3^2$ and $n\leqslant 7$, or $p^m=3^3,5^3$, and $n\leqslant 11$ by [3, Theorem 3.1] since $\Phi^*_{2m}(p)\mid|H^*|$ and p is odd. Then $u=11,29$ or 127, respectively. Since u does not divide the order of the corresponding $GL_1(q)\wr S_n$, then u does not divide the order of H^*, whereas u must divide it by Proposition 5.4. Indeed, in each of these cases $(u,(2m+t)/n)=1$ since u a prime such that $u>4m\geqslant(2m+t)/n$. Hence, these cases are ruled out.

If $n=4m$, then $q=p$ and hence $H=H^*\leqslant GL_{m_0}(p)\wr S_{n_0}$. Let w be a prime divisor of $p^{2m}-1$. Then w divides either the order of $GL_{m_0}(p)$ or that of S_{n_0}. The former yields $2m\leqslant m_0$ by [27, Proposition 5.2.15(i)]. Therefore $(2m)\,n_0\leqslant m_0n_0=n=4m$, and hence $n_0=2$ and $m_0=2m$ since $n_0>1$.

The case where w divides the order of S_{n_0} yields $w=2m+1$ since $w\equiv 1 \pmod{2m}$ by [27, Proposition 5.2.15(ii)] since $n_0\leqslant 4m$. Actually, $n_0=4m$ and $m_0=1$ since $n_0\mid 4m$ and $n_0\geqslant w=2m+1>2$. Note that, $n_0=2$ and $m_0=2m$ is clearly not compatible with $n_0=4m$ and $m_0=1$. Hence, in the latter case, $\Phi^*_{2m}(p)=(2m+1)^s$ for some $s\geqslant 1$ is a divisor of $n_0!$. Then $s<\frac{n_0-1}{2m}<4m/2m=2$ by [14, Exercise 2.6.8], and hence $\Phi^*_{2m}(p)=2m+1$. Thus $p^m=3^2,3^3,5^3$ by [3, Lemma 6.1.(i)], and hence $p^m+2=11,29,127$, respectively. Then p^m+2 does not divide the order of the corresponding H since $H\leqslant GL_1(p)\wr S_{4m}$, and hence these cases are ruled out by Lemma 5.2(2). Therefore $n_0=2$ and $m_0=2m$, hence $H\leqslant GL_{2m}(p)\wr S_2$ preserves a decomposition $V=V_1\oplus V_2$, which is (1).

Assume that H lies in a member of $\mathcal{C}_4(\Gamma)$. Then H preserves a tensor decomposition $V=V_1\otimes V_2$, with $\dim V_i=n_i$ and $1\leqslant n_1<n_2$. Therefore, $H^*\leqslant GL_{n_1}(q)\circ GL_{n_2}(q)$. No cases arise for $n<4m$ by [3, Theorem 3.1] since $\Phi^*_{2m}(p)\mid|H^*|$. Thus $n=4m$ and $q=p$, hence $H=H^*\leqslant GL_2(p)\circ GL_{2m}(p)$, and we obtain (2).

Assume that H lies in a member of $\mathcal{C}_6(\Gamma)$. Then H lies in the normalizer in $\Gamma L_n(q)$ of an absolutely irreducible symplectic type s-group R, with $s\neq p$. Hence $n=s^y$ for some $y\geqslant 1$ by [27, Definition (c) at p. 150.]

If $n<4m$, then $(q,n)=(3,4)$ or $(3,8)$ by [3, Theorem 3.1] since $\Phi^*_{2m}(p)\mid|H^*|$ and $u^h=q^{n/2}+2$ with q odd. Thus, $h=1$. Moreover, $n=2m+t$ since $q=p$, and therefore $(m,t)=(2,2)$ for $(q,n)=(3,4)$, and $(m,t)=(3,2),(4,0)$ for $(q,n)=(3,8)$ since $n<4m$. A similar argument to that of the $\mathcal{S}$-case yields $u\mid 3^{2m+t-f}-1$ with $0<2m+t-f\leqslant 4m$, where 3^f is the number of fixed points of an element of order u of H. Either $m=1$ and $2+t-f=4$, or $m=2$ and $4+t-f=5$ by Lemma 3.11, whereas $m=2$ or $3,4$, respectively. Thus, H does not lie in a member of $\mathcal{C}_6(\Gamma)$ for $n<4m$.

If $n=4m$, then $q=p$ and hence $s=2$ and $y\geqslant 2$ as $n=s^y$. Therefore, $m=2^{y-2}$. Let w be a primitive prime divisor of $p^{2m}-1$. Then $w=2^{y-1}j+1=2mj+1$ for some $j\geqslant 1$ by [27, Proposition 5.2.15(ii)]. On the other hand, w divides the order of H, where $H/(H\cap Z)$ is a subgroup of one of the groups given in [27, Table 4.6.A] for $s=2$, and Z is the center of $GL_{4m}(p)$. It follows that w divides the order of $Sp_{2y}(2)$, and hence it divides either 2^i-1 or 2^i+1 for some $1\leqslant i\leqslant y$. Then either $w=2^{y-1}+1=2m+1$ or $w=2^y+1=2(2m)+1$, respectively, and hence $\Phi^*_{2m}(p)$ is either $2m+1$ or $2(2m)+1$, or $(2m+1)\,(2(2m)+1)$. Then $p^m=3^2,3^3,5^3,3^9$ or 17^3 by [3, Lemma 6.1] since $q=p$ and p is odd. Actually, $p^m\neq 3^9,17^3$ since they do not fulfill $u^h=p^m+2$, whereas $h=1$ in the remaining cases. Then $u\mid p^{4m-f}-1$, and arguing as above we obtain $p^m = 3^2$, $u = 11$ and $f = 3$ and $n = 8$. Since $q = p = 3$, it follows that $Z(R) \cong Z_2$, $R \cong D_8 \circ Q_8$, and hence $n = 2$ by [27, Definition (c) and Table 4.6.B at p. 150]. However, it is ruled out since it contradicts $n = 8$.

Assume that H lies in a member of $\mathcal{C}_7(\Gamma)$. Then H stabilizes a decomposition of $V = V_1 \otimes \cdots \otimes V_{n_0}$ with $\dim V_1 = \cdots = \dim V_{n_0} = m_0$. Hence, $n = m_0^{n_0}$ with $m_0 \geqslant 1$ and $n_0 \geqslant 2$, and $H^* \leqslant (GL_{m_0}(p) \circ \cdots \circ GL_{m_0}(p)).S_{n_0}$. If $m_0 < 3$, then H lies in a member of $\mathcal{C}_8(\Gamma)$ (see remark before Proposition 4.7.3 in [27]), which is not the case. Hence, $m_0 \geqslant 3$. No cases arise for $n < 4m$ by [3, Theorem 3.1] since $\Phi^*_{2m}(p) \mid |H^*|$ by Proposition 5.4. Thus $n = 4m$ and $q = p$, and hence $H = H^*$. A prime divisor w of $p^{2m} - 1$ divides either the order of $GL_{m_0}(p)$ or that of S_{n_0}. Since $w \equiv 1 \pmod{2m}$ by [27, Proposition 5.2.15(ii)], the latter implies $n_0 \geqslant 2m + 1$, and hence $3^{2m+1} \leqslant m_0^{2m+1} \leqslant 4m$, a contradiction. Thus, w divides the order of $GL_{m_0}(p)$. Then $2m \leqslant n_0$ by [27, Proposition 5.2.15(i)], and hence $(2m)^{n_0} \leqslant m_0^{n_0} = n = 4m$. So $n_0 = m_0 = 2$, and we reach a contradiction as $m_0 \geqslant 3$. This completes the proof. □

Lemma 5.6. *The following hold in case (1) of Proposition 5.5:*

(1) $H_{V_j}^{V_j}$ *acts irreducibly on* V_j *for* $j = 1, 2$;
(2) $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,3)} \mid \left|H_{V_j}^{V_j}\right|$ *for* $j = 1, 2$;
(3) $H_{V_j}^{V_j}$ *contains a normal subgroup* Q_j *of order divisible by* $\frac{p^m+2}{(p^m+2,3)}$ *and such that* $\Phi^*_{2m}(p) \mid \left|H_{V_j}^{V_j} : Q_j\right|$ *for* $j = 1, 2$.

Proof. The length of $V_{(\Delta_i)}^G$ is $p^m + 2$ since G acts transitively on Σ and $V \trianglelefteq G$. Then the length of $V_{(\Delta_i)}^H$ is $p^m + 2$ since $C = C_G(V)$. Let w be a primitive prime divisor of $p^{2m} - 1$, and since $G_{\Delta_i}^{\Delta_i}$ acts flag-transitively on $\mathcal{D}_i$, let W_i be a Sylow w-subgroup of G preserving Δ_i. Then W_i normalizes $V_{(\Delta_i)}$. Moreover, W_i acts faithfully on V inducing a Sylow w-subgroup of H since C is a p-group by Proposition 3.8(3) and Theorem 4.3.

Assume that $(W_i)_{(V_1)} \neq 1$. Then $(W_i)_{(V_1)}$ acts irreducibly on V_2. If it is not so, there is $\zeta \in (W_i)_{(V_1)}$, $\zeta \neq 1$, fixing V_2 pointwise since $\dim V_2 = 2m$ and ζ is a w-element with w a primitive prime divisor of $p^{2m} - 1$. Then $\zeta \in C$, and hence ζ is a p-element since C is a p-group, whereas ζ is a non-trivial w-element with $w \neq p$. Clearly, $(W_i)_{(V_1)}$ preserves $\langle x_1 \rangle \oplus V_2$ for each $x_1 \in V_1$, $x_1 \neq 0$. Since $\dim \langle x_1 \rangle \oplus V_2 = 2m + 1$ and $\dim V_{(\Delta_i)} = 2m$, it follows that $V_{(\Delta_i)} \cap (\langle x_1 \rangle \oplus V_2) \neq 0$. Let $\mu \in GF(p)$ and $x_2 \in V_2$ such that $\mu x_1 + x_2 \in V_{(\Delta_i)} \cap (\langle x_1 \rangle \oplus V_2)$. If $x_2 \neq 0$, there is a non-trivial $\alpha \in (W_i)_{(V_1)}$ such that $x_2^\alpha \neq x_2$. Then $\mu x_1 + x_2^\alpha \in V_{(\Delta_i)} \cap (\langle x_1 \rangle \oplus V_2)$ since $(W_i)_{(V_1)}$, and hence α, preserves $V_{(\Delta_i)} \cap (\langle x_1 \rangle \oplus V_2)$. Hence, $x_2^\alpha - x_2$ is a non-zero element of $V_{(\Delta_i)} \cap V_2$ since $x_2^\alpha - x_2 = (\mu x_1 + x_2^\alpha) - (\mu x_1 + x_2)$ and $x_2^\alpha \neq x_2$. Thus $V_{(\Delta_i)} = V_2$ since $(W_i)_{(V_1)}$ acts irreducibly on V_2 and preserves $V_{(\Delta_i)}$. So, $p^m + 2 = 2$ since the length of $V_{(\Delta_i)}^H$ is $p^m + 2$ and H switches V_1 and V_2, and we reach a contradiction. Thus $x_2 = 0$, $\mu \neq 0$ and $\mu x_1 \in V_{(\Delta_i)} \cap V_1$. Then $V_{(\Delta_i)} = V_1$, and hence $\left|V_{(\Delta_i)}^H\right| = p^m + 2 = 2$, which is a contradiction. Thus, W_i acts faithfully and irreducibly on V_1. Similarly, we prove that W_i acts faithfully and irreducibly on V_2. Thus, $H_{V_j}^{V_j}$ acts irreducibly on V_j, which is (1), and $\Phi^*_{2m}(p) \mid \left|H_{V_j}^{V_j}\right|$.

Recall that U is the pre-image of $Soc(G^\Sigma)$. Let S be a Sylow u-subgroup of U, where $u^h = p^m + 2$. Then S acts faithfully on V inducing a Sylow u-subgroup of H since C is a p-group. Assume that $S_{(V_1)} \neq 1$. Since $(p^m + 2, p^{2m} - 1) \mid 3$, there is a subgroup of Y of $S_{(V_1)}$ such that $\left|S_{(V_1)} : Y\right| \leqslant (3, u)$ fixing an non-zero element z_2 of V_2. Then Y fixes $V_1 \oplus \langle z_2 \rangle$ pointwise. Since $V_{(\Delta_i)} \cap (V_1 \oplus \langle z_2 \rangle) \neq 0$ for each $i = 1, \ldots, p^m$, there is an element of $v_i \in V_{(\Delta_i)}$, $v_i \neq 0$, fixed by Y for each $i = 1, \ldots, p^m$.

Assume that $Y \nleq G_{(\Sigma)}$. Then there are $\eta \in Y$ and $i_0 \in \{1, \ldots, p^m + 2\}$ such that $[v_{i_0}, \eta] = 1$ and $\Delta_{i_0}^\eta \neq \Delta_{i_0}$. Then $v_{i_0} \in V_{\left(\Delta_{i_0}\right)} \cap V_{\left(\Delta_{i_0}^\eta\right)}$ with $v_i \neq 0$, whereas $V_{\left(\Delta_{i_0}\right)} \cap V_{\left(\Delta_{i_0}^\eta\right)} = 0$ by Lemma 3.6(2) since $\Delta_{i_0}^\eta \neq \Delta_{i_0}$. Thus $Y \leqslant G_{(\Sigma)}$, and hence $\left|S_{(V_1)} : S_{(V_1)} \cap G_{(\Sigma)}\right| \leqslant (3, u)$. Similarly, we have $\left|S_{(V_2)} : S_{(V_2)} \cap G_{(\Sigma)}\right| \leqslant (3, u)$. Then

$$\left|S^{V_j}\right| = \frac{|S|}{\left|S_{(V_j)}\right|} \geqslant \frac{|S|}{(3, u)\left|S_{(V_2)} \cap G_{(\Sigma)}\right|} \geqslant \frac{|S|}{(3, u)\left|S \cap G_{(\Sigma)}\right|} = \frac{\left|U^\Sigma\right|}{(3, u)} = \frac{p^m + 2}{(p^m + 2, 3)},$$

since $U = G_{(\Sigma)}S$. Then $\frac{p^m+2}{(p^m+2,3)} \mid \left|S^{V_j}\right|$ since $u^h = p^m + 2$ and S^{V_j} is a u-group, and hence $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,3)} \mid \left|H_{V_j}^{V_j}\right|$ since we have already proven that $\Phi^*_{2m}(p) \mid \left|H_{V_j}^{V_j}\right|$. Therefore, we get (2). Moreover $S^{V_j} \leqslant Q_j \trianglelefteq H^{V_j}$, where $Q_j = (U/C)^{V_j}$, since $C \trianglelefteq G_{(\Sigma)} \trianglelefteq U$ by Theorem 4.3. Hence, the order of Q_j is divisible by $\frac{p^m+2}{(p^m+2,3)}$. Also, since G^Σ acts 2-transitively on Σ, it follows that $\Phi^*_{2m}(p)$ divides $\left|G^\Sigma : U^\Sigma\right|$ and hence $|G : U|$. Then $\Phi^*_{2m}(p)$ divides $|H : U/C|$, since $C \trianglelefteq U$ and C is a p-group, and hence $\left|H_{V_j}^{V_j} : Q_j\right|$ for each $j = 1, 2$, which is (3). □

Lemma 5.7. *The following hold in case (2) of Proposition 5.5:*

(1) H^{V_2} *acts irreducibly on* V_2;
(2) $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,3)} \mid \left|H^{V_2}\right|$;

(3) *H^{V_2} contains a normal subgroup Q_2 of order divisible by $\frac{p^m+2}{(p^m+2,3)}$ and such that $\Phi^*_{2m}(p) \mid \left|H^{V_2} : Q_2\right|$.*

Proof. Set $M = GL_2(p) \circ GL_{2m}(p)$. Let w be primitive prime divisor of $p^{2m} - 1$ and let W be a Sylow w-subgroup of H. Then $W^{V_1} \leqslant H^{V_1}$, and hence $W^{V_1} = 1$ since $m > 1$ by Proposition 5.5(2). Therefore, $W^{V_2} \cong W$ since $V = V_1 \otimes V_2$ and $W \cap Z(M) = 1$, being $Z(M) \leqslant Z_{p-1}$. Thus H^{V_2} acts irreducibly on V_2, which is (1), and $\Phi^*_{2m}(p) \mid \left|H^{V_2}\right|$.

Let S be a Sylow u-subgroup of U, where $U \cong Soc(G^{\Sigma})$. Then S acts faithfully on V inducing a Sylow u-subgroup of H with $u^h = p^m + 2$ and p odd since C is a p-group by Proposition 3.8(3) and Theorem 4.3. Note that, S^{V_1} is a subgroup of $GL_2(p)$ of order at most 9 since $(p^m + 2, (p-1)^2) \mid 9$. Hence, $S = S_{(V_1)}$ or $\left|S : S_{(V_1)}\right| \leqslant 9$ according to whether 3 does not divide or does divide the order of $Z(M)$. Thus, $\left|S^{V_2}\right| \geqslant \frac{|S|}{(p^m+2,3)}$ since $S_{(V_1)} \cap S_{(V_2)} \leqslant Z(M) \leqslant Z_{p-1}$. Then $\frac{p^m+2}{(p^m+2,3)} \mid \left|S^{V_2}\right|$ since $u^h = p^m + 2$ and S^{V_2} is a u-group, hence $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,3)} \mid \left|H^{V_2}\right|$, which is (2).

Since $C \trianglelefteq G_{(\Sigma)} \trianglelefteq U$ by Theorem 4.3, let $Q_2 = (U/C)^{V_2}$. Then the order of Q_2 is divisible by $\frac{p^m+2}{(p^m+2,3)}$ since $S^{V_2} \leqslant Q_2 \trianglelefteq H^{V_2}$. Now, a similar argument to that used in the proof of Lemma 5.6(3) can be applied here to obtain that $\Phi^*_{2m}(p)$ divides $\left|H^{V_2} : Q_2\right|$. Thus, we get (3). □

Remark 5.8. In view of Lemmas 5.6 and 5.7, there is a quotient group X of a subgroup of H in (1) and in (2) of Proposition 5.5 with the following properties:

(1) X is an irreducible subgroup of $GL(V_2)$ of order divisible by $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,3)}$;
(2) X contains a normal subgroup Q of order divisible by $\frac{p^m+2}{(p^m+2,3)}$ and such that $\Phi^*_{2m}(p) \mid |X : Q|$;
(3) $p^m + 2 = u^h$, where u is a prime and $h \geqslant 1$.

We are going to show that a quotient group of H with such constraints does not exist. We derive from this fact that $\Phi^*_{2m}(p) = 1$, hence $(p^m, u^h) = (3,5), (7,9)$ by Lemma 5.2(3).

For each divisor ℓ of $2m$ the group $\Gamma L_\ell(p^{2m/\ell})$ has a natural irreducible action on V_2. By Proposition 5.1 we may choose ℓ to be minimal such that $X \leqslant \Gamma L_\ell(p^{2m/\ell})$ in this action and write $a = p^{2m/\ell}$.

Lemma 5.9. *The following hold*

(1) $\ell > 1$;
(2) $Q \not\leqslant Z(GL_\ell(a))$.

Proof. Suppose that $\ell = 1$. Then $X \leqslant \Gamma L_1(a)$, and hence $\Phi^*_{2m}(p)\frac{p^m+2}{(p^m+2,3)} \mid (p^{2m} - 1)\cdot(2m)$. Then we obtain (1) by proceeding as in Lemma 5.3 (for $t = 0$) with $\frac{p^m+2}{(p^m+2,3)}$ in the role of $p^m + 2$ and bearing in mind that $p^m \neq 7$.

Suppose the contrary. Then $\frac{p^m+2}{(p^m+2,3)} \mid p^{2m/\ell} - 1$, and hence $p^m + 2 \mid 9$. So, $(p^m, u^h) = (7,9)$ since $p^m \geqslant 3$ by Lemma 2.5, but this case is ruled out since it contradicts the assumption $\Phi^*_{2m}(p) > 1$. Thus, we obtain (2). □

Let $X^* = X \cap GL_\ell(a)$. Then $\Phi^*_{2m}(p)$ divides the order of X^* since $(\Phi^*_{2m}(p), \ell) = 1$ by [27, Proposition 5.2.15.(ii)], being $\ell \mid 2m$. Then one of the following holds by [3, Theorem 3.1] and by the minimality of ℓ:

(i). X^* contains a normal subgroup Y isomorphic either to group $SL_\ell(a)$ with $\ell \geqslant 2$, or to $Sp_\ell(a)$, or to $\Omega^-_\ell(a)$ with ℓ even and $\ell \geqslant 2$, or to $SU_\ell(a^{1/2}) \trianglelefteq X$ with a square, ℓ odd and $\ell \geqslant 3$.
(ii). $X^* \leqslant (D_8 \circ Q_8).S_5$ and $(\ell, a) = (4,3)$.
(iii). X^* is *nearly simple*, that is, $S \trianglelefteq X^*/(X^* \cap Z) \leqslant Aut(S)$, where Z is the center of $GL_\ell(a)$ and S is a non-abelian simple group. Moreover, if Y is the pre-image of S in X^*, then Y is absolutely irreducible on V and Y is not a classical group defined over a subfield of $GF(a)$ in its natural representation.

Theorem 5.10. $(p^m, u^h) = (3,5), (7,9)$.

Proof. Assume that (i) occurs. Then $X \leqslant (Z \circ X^*).Out(\bar{X^*})$ with $\bar{X^*} = X^*/(X^* \cap Z)$, where $Z = Z(GL_\ell(a))$. It follows that $X^* \trianglelefteq Q$ by Lemma 5.9(2) since $Q \trianglelefteq X$. Then $\Phi^*_{2m}(p) \mid (a-1)\,|Out(X^*)|$, since $\Phi^*_{2m}(p) \mid |X : Q|$ by Remark 5.8(3), and hence $\Phi^*_{2m}(p) \mid (p^{2m/\ell} - 1)2m$. However, this impossible by [27, Proposition 5.2.15] since $\ell > 1$ by Lemma 5.9(1).

Case (ii) is ruled out since $a^{\ell/2} + 2 = 11$ does not divide the order of X

Assume that case (iii) occurs. Suppose that $S \cong A_s$, $s \geqslant 5$ and that V_2 is the fully deleted permutation module for A_s. Then $a = p$, $n = 2m$, $A_s \trianglelefteq X^* \leqslant S_s \times Z$, where Z is the center of $GL_{2m}(p)$, and either $s = 2m + 1$ or $s = 2m + 2$ according

Table 1
Admissible S and corresponding ϱ.

S	ℓ	a	ϱ
$PSL_2(7)$	6	3	29
	6	5	127
	3	9	29
	3	5^2	127
$PSL_2(13)$	6	3	29
$PSL_3(2^2)$	6	3	29
$PSU_3(3)$	6	5	127

to whether p does not divide or does divide s, respectively, by [27, Lemma 5.3.4]. Moreover, $p^m = 3^2, 3^3, 5^3$ by [3, Theorem 3.1] since p is odd, and hence $p^m + 2 = 11, 29, 127$ respectively. However, such values of $p^m + 2$ do not divide $|X|$, and hence they are ruled out (see Remark 5.8).

Assume that $S \cong A_s$, $s \geqslant 5$ and V_2 is not the fully deleted permutation module for A_s. Then $(\ell, a) = (4, 7)$, $Z_2.A_7 \trianglelefteq H \leqslant Z_2.S_7 \times Z_3$ and $V_2 = V_4(7)$ by [3, Theorem 3.1] since p is odd. However, it is ruled out since $7^2 + 2$ does not divide the order of X.

Assume that S is sporadic. Then $S \cong J_2$ and $(\ell, a) = (6, 5)$ by [3, Theorem 3.1], since p is odd. However, $a^{\ell/2} + 2 = 127$ does not divide the order of X, and hence this case is excluded.

Assume that S is a Lie type simple group in characteristic p'. Then S is given in [3, Theorem 3.1] and recorded in Table 1. Since $\frac{a^{\ell/2}+2}{(a^{\ell/2}+2,3)} \mid |X|$ and since $S \trianglelefteq X^*/(X^* \cap Z) \leqslant Aut(S)$, it follows that

$$\varrho = \frac{a^{\ell/2} + 2}{\left(a^{\ell/2} + 2, 3(a-1)\,|Aut(S)|\right)}$$

must divide the order of S. However, the order of S is divisible by the corresponding ϱ in none of the cases listed in Table 1. Hence, all the groups in Table 1 are excluded.

Assume that S is a Lie type simple group in characteristic p. Then $p = 2$ by [3, Theorem 3.1], whereas p must be odd, and hence no cases arise. Thus $\Phi^*_{2m}(p) = 1$, and hence $(p^m, u^h) = (3, 5), (7, 9)$ by Lemma 5.2(3). □

Theorem 5.11. *If G^Σ is of affine type, then $\mathcal{D}$ is isomorphic to the* 2-(45, 12, 3) *design as in [40, Construction 4.2].*

Proof. It follows from Theorem 5.10 that $(p^m, u^h) = (3, 5)$ or $(7, 9)$, and in the former case the assertion follows by [40, Corollary 4.2]. Hence, in order to complete the proof we need to rule out $(p^m, u^h) = (7, 9)$. We are going to prove this in a series of steps.

(1). **Let $\Delta_i \in \Sigma$ and $x_i \in \Delta_i$. Then $G^\Sigma \cong AGL_1(9)$, $G_{x_i}^{\Delta_i} \cong Z_{3^j} \times Z_{16}$ and $G_{(\Sigma),x_i}^{\Delta_i} \cong Z_{3^j} \times Z_2$ where $j = 0, 1$.**

$G_{\Delta_i}^{\Sigma} \cong Z_8, Q_8, SD_{16}, SL_2(3), GL_2(3)$ by [25, Section 2, (B)] since G^Σ is an affine group acting 2-transitively on Σ by Proposition 3.2. On the other hand, since $\mathcal{D}_i \cong AG_2(7)$ by Theorem 3.4 and $m = 1$, it follows from [15, Theorem 1' and Table II] that either $SL_2(7) \trianglelefteq G_{x_i}^{\Delta_i} \leqslant GL_2(7)$, or $G_{x_i}^{\Delta_i} \leqslant \Gamma L_1(7^2)$, or $G_{x_i}^{\Delta_i} \cong SL_2(3).Z_2$. Moreover, if $G_{x_i}^{\Delta_i} \leqslant \Gamma L_1(7^2)$, it is not difficult to see that $G_{x_i}^{\Delta_i} \cong Z_{3^j} \times Z_{16}, Z_{3^j} \times Q_{16}, Z_{3^j} \times SD_{32}$, with $j = 0, 1$. Using (3.1) in Corollary 3.10 with $G_{\Delta_i}^{\Sigma} \cong Z_8, Q_8, SD_{16}, SL_2(3), GL_2(3)$, we see that the unique possibilities are $G_{\Delta_i}^{\Sigma} \cong Z_8$ and either $G_{x_i}^{\Delta_i} \cong Z_{16}$ and $G_{(\Sigma),x_i}^{\Delta_i} \cong Z_2$, or $G_{x_i}^{\Delta_i} \cong Z_{48}$ and $G(\Sigma)_{x_i}^{\Delta_i} \cong Z_8$. Thus $G^\Sigma \cong AGL_1(9)$, $G_{x_i}^{\Delta_i} \cong Z_{3^j} \times Z_{16}$ and $G_{(\Sigma),x_i}^{\Delta_i} \cong Z_{3^j} \times Z_2$ with $j = 0, 1$.

(2). $G_{(\Delta_i)} \leqslant E_{7^2} : (Z_{3^j} \times Z_2)$ **for each** $i = 1, \dots, 9$.

Since $G_{(\Delta_i)} \trianglelefteq G_{(\Sigma)}$ and $G_{(\Delta_i)} \cap G_{(\Delta_s)} = 1$ for each $s \in \{1, \dots, 9\}$, $s \neq i$, by Lemma 3.6(2), it follows that $G_{(\Delta_i)}$ is isomorphic to a normal subgroup of $G_{(\Sigma)}^{\Delta_s}$. On the other hand, $G_{(\Sigma)}^{\Delta_s} \cong E_{7^2} : Z_{3^j} \times Z_2$ with $j = 0, 1$ since $G_{(\Sigma)} = G_{(\Sigma),x_s} V$ by Proposition 3.8(1)(3) and $G_{(\Sigma),x_s}^{\Delta_s} \cong Z_{3^j} \times Z_2$ with $j = 0, 1$ by (1). Thus, $G_{(\Delta_i)} \leqslant E_{7^2} : Z_{3^j} \times Z_2$.

(3). G **is solvable.**

It follows from (2) that $G_{(\Delta_i)}$ is solvable. Thus $G_{(\Sigma),x_i}$ is solvable since $G_{(\Sigma),x_i}^{\Delta_i} \cong Z_{3^j} \times Z_2$ by (1), and hence $G_{(\Sigma)}$ is solvable since $G_{(\Sigma)} = G_{(\Sigma),x_i} V$. Therefore, G is solvable since $G^\Sigma \cong AGL_1(9)$.

(4). H **is a solvable irreducible subgroup of** $GL_4(7)$.

Recall that H is an irreducible subgroup of $GL_{2+t}(7)$ with $t \leqslant 2$ by Proposition 5.1, and let R be the pre-image of $H \cap SL_{2+t}(7)$ in G. Then $R \lhd G$. Therefore $R^{\Sigma} \lhd G^{\Sigma}$, and hence either $R^{\Sigma} = 1$, or $E_9 \unlhd R^{\Sigma}$ since $G^{\Sigma} \cong AGL_1(9)$. The former implies $R \leqslant G_{(\Sigma)}$ and hence $9 \mid |G/R|$. Then 9 divides the index of $SL_{2+t}(7)$ in $GL_{2+t}(7)$ since $G/R \cong H/(H \cap SL_{2+t}(7))$, which is a contradiction. Thus $E_9 \unlhd R^{\Sigma}$, and hence a quotient group of $H \cap SL_{2+t}(7)$ contains a normal subgroup isomorphic to E_9 since C is a 7-group by Proposition 3.8(2) and Theorem 4.3.

Let M be the pre-image of $H \cap Z(GL_{2+t}(7))$ in G. Clearly, $M \lhd G$. Therefore $M^{\Sigma} \lhd G^{\Sigma}$, and hence either $M^{\Sigma} = 1$, or $9 \mid \left|M^{\Sigma}\right|$ since $G^{\Sigma} \cong AGL_1(9)$. The latter implies $9 \mid |M/C|$ since C is a 7-group, whereas $M/C = H \cap Z(GL_{2+t}(V)) \leqslant Z_6$. Thus $M^{\Sigma} = 1$, and hence $M \leqslant G_{(\Sigma)}$.

Set $A = (H \cap SL_{2+t}(7))/(H \cap Z(SL_{2+t}(7)))$, then A is isomorphic to a solvable subgroup of $PSL_{2+t}(7)$. Moreover, a quotient group of A contains E_9 as a normal subgroup since $A \cong R/M$, $E_9 \unlhd R^{\Sigma}$ and $M \leqslant G_{(\Sigma)}$. Thus, $t \neq 0$.

Assume that $t = 1$. Since $Z_{16} \leqslant G_{x_i}^{\Delta_i}$, then H contains 2-elements of order at least 16, and hence A contains elements of order at least 8. Hence, A is isomorphic to a solvable subgroup of $PSL_3(7)$ of order divisible by 72. Then $A \cong E_9 : Q_8$ by [9], whereas A contains elements of order at least 8. Thus $t \neq 1$, and hence the claim follows from (3) and from Proposition 5.1.

(5). $G = C : (Q : J)$, **where** $|Q| = 3^{2+j}$**, with** $0 \leqslant j \leqslant 1$**, and** $J \cong Z_{16}$**.**

Let S be a Sylow 7-subgroup of G containing C. Since $G^{\Sigma} \cong AGL_1(9)$, it follows that $S \leqslant G_{(\Sigma)}$. Since $G_{\Delta_i}^{\Delta_i} \cong E_{7^2} : (Z_{3^j} \times Z_{16})$ and $G_{(\Delta_i)} \leqslant E_{7^2} : (Z_{3^j} \times Z_2)$ by (1) and (2), it follows that $|S| \leqslant 7^4$. On the other hand, $7^4 \leqslant |C| \leqslant |S|$ by (4) and Proposition 3.8(3). Hence, $S = C \unlhd G$. Moreover, $|H| = 2^{4+e} \cdot 3^{2+f}$, where $e \leqslant 1$ and $j \leqslant f \leqslant 2$, again by (1) and (2). Then $G = C : K$, where K is a group of order $2^{4+e} \cdot 3^{2+f}$, by [18, Theorem 6.2.1(i)].

Since $K^{\Sigma} \cong AGL_1(9)$, it results $\left|K_{(\Sigma)}\right| = 2^{1+e} \cdot 3^f$ with $e \leqslant 1$ and $j \leqslant f \leqslant 2$. Let S be a Sylow w-subgroup of $K_{(\Sigma)}$ with $w \in \{2,3\}$. If either $e = 1$ or $f = 2$, assume that w is 2 or 3, respectively. Then $Z_w \leqslant S_{(\Delta_i)}$ for each $i \in \{1,\dots,9\}$ since $|S| = w^2$ and $G_{(\Sigma),x_i}^{\Delta_i} \cong Z_{3^j} \times Z_2$, with $j = 0,1$, for each $i \in \{1,\dots,9\}$ by (1). Then $S_{(\Delta_i)} \cap S_{(\Delta_{i'})} \neq 1$ for some $i, i' \in \{1,\dots,9\}$, with $i \neq i'$ since the number of cyclic subgroup of $Q_{(\Sigma)}$ is $w+1$, which is at most 4, and $|\Sigma| = 9$. However, this case is excluded since it contradicts Lemma 3.6(2). Thus $e = 0$, $f \leqslant 1$, and hence $|K(\Sigma)| = 2 \cdot 3^f$. From this fact and $K^{\Sigma} \cong AGL_1(9)$, it results that the pre-image P in K of $Z_3 \times Z_3$ is $Q : Z_2$, where Q a Sylow 3-subgroup of K. Moreover, the Frattini's argument implies $K = N_K(Q)P$, and hence $K = Q : J$ with J is a Sylow 2-subgroup of K. Finally, $J \cong Z_{16}$ since J is of order 16 and $G_{x_i}^{\Delta_i} \cong Z_{3^j} \times Z_{16}$ by (1).

(6). Q **is abelian.**

Suppose the contrary. Then $j = 1$, and hence Q is extraspecial. If there is an element ϕ in Q of order 9, then $Fix(\phi^2) \neq 1$ since $V = V_4(7)$. Then K preserves $Fix(\phi^2)$ since $\langle \phi^2 \rangle = Z(Q)$, whereas K acts irreducibly on V by (4) since $K = G/C = H$. Thus, Q is of exponent 3. Now, $Z(Q) \leqslant K_{(\Sigma)}$ since $K^{\Sigma} \cong AGL_1(9)$, therefore $Z(Q)$ preserves each Δ_i in Σ. Hence, $Z(Q)$ normalizes $V_{(\Delta_i)}$. Thus, $Z(Q)$ is a reducible subgroup of $GL_4(7)$. Then Q is reducible by [18, Theorem 3.4.1]. Then either $V = X_1 \oplus X_2 \oplus X_3 \oplus X_4$, where X_s, $s = 1,2,3,4$, is a Q-invariant 1-dimensional subspace of V and $K \leqslant GL_1(7) \wr S_4$, or $V = Y_1 \oplus Y_2$, where Y_1, Y_2 are Q-invariant 2-dimensional subspaces of V and $K \leqslant GL_2(7) \wr Z_2$. In each case, there is a Q-invariant subspace of V fixed pointwise by a non-trivial normal subgroup of Q since the order of Q is 3^3. Also, such a group contains $Z(Q)$. So, $Fix(Z(Q))$ is a H-invariant subspace of V of dimension at least 1 since $Z(Q) \unlhd K$, and we reach a contradiction since K acts irreducibly on V. Thus, Q is abelian.

(7). **The final contradiction.**

Q acts reducibly on V by [18, Theorem 3.2.3] since $Z_3 \times Z_3 \leqslant Q$, hence K acts transitively on a Q-invariant decomposition of V in subspaces of equal dimension by [18, Theorem 3.4.1] since K acts irreducibly on V. Therefore, either $K \leqslant GL_1(7) \wr S_4$, or $K \leqslant GL_2(7) \wr Z_2$. However, the former does not contain cyclic subgroups of order 16. Hence, $K \leqslant GL_2(7) \wr Z_2$ and $Q = \langle \alpha \rangle \times \langle \beta \rangle \times \langle \gamma \rangle$, where $o(\alpha) = 3^j$ and $o(\beta) = o(\gamma) = 3$. Also $\alpha \in Z(K)$, whereas $N_K(\langle \delta \rangle) \cong Z_3 : Z_4$ for each $\delta \in Q \setminus \langle \alpha \rangle$.

Let $V = V_1 \oplus V_2$ be the decomposition preserved by K. Clearly, $J_{V_1} \cong Z_8$ acts faithfully on V_1 since $J \cong Z_{16}$ switches V_1 and V_2. Also, Q preserves V_1 and $Q_{(V_1)} \neq 1$. If $Q_{(V_1)}$ is of order 9, then $Q_{(V_1)} \cap Q_{(V_2)} \neq 1$ since $Q \unlhd K$, the order of Q is 3^{2+f} with $f \leqslant 1$ and K switches V_1 and V_2. However, this is impossible since it contradicts Lemma 3.6(2). Thus $Q_{(V_1)}$ is of order 3, and hence $(Z_3 \times Z_3) : Z_8 \leqslant K_{V_1}^{V_1} \leqslant GL_2(7)$, which is also impossible. So, this case is ruled out and the proof is completed. □

5.2. G^{Σ} *is an almost simple group*

In this subsection, we assume that $Soc(G^{\Sigma})$ is a non-abelian simple group.

Proposition 5.12. *One of the following holds:*

I. $p^m=4$ *and* $\mathcal{D}$ *is isomorphic to one of the four* 2-(96, 20, 4) *designs constructed in* [29].
II. $p^m=9$ *and* $G^{\Sigma}\cong PSL_2(11), M_{11}$.
III. $p^m=16$ *and* $G^{\Sigma}\cong PSL_2(17)$.

Proof. Suppose that G^{Σ} is almost simple and let $S=Soc(G^{\Sigma}_{\Sigma})$. Since G^{Σ} acts 2-transitively on Σ and $|\Sigma|=p^m+2$, one of the following holds v:

(1) $S\cong A_{p^m+2}$ and $p^m\geqslant 3$;
(2) $S\cong PSL_h(u)$ with $h\geqslant 2$, $(h,u)\neq(2,2),(2,3)$ and $\frac{u^h-1}{u-1}=p^m+2$;
(3) $S\cong PSU_3(u)$ with $u\neq 2$ and $u^3+1=p^m+2$;
(4) $S\cong Sz(2^{2t+1})$ with $t\geqslant 1$, and $2^{4t+2}+1=p^m+2$;
(5) $S\cong {}^2G_2(3^{2t+1})'$ with $t\geqslant 1$ and $3^{6t+3}+1=p^m+2$;
(6) $S\cong Sp_{2n}(2)$ with $n\geqslant 3$ and $2^{2n-1}\pm 2^{n-1}=p^m+2$;
(7) $S\cong PSL_2(11), M_{11}$ and $p^m=9$;
(8) $S\cong A_7$ and $p^m=13$.

Assume that (1) occurs. Let $\Delta_i\in\Sigma$ and $x_i\in\Delta_i$, then $A_{p^m+1}\trianglelefteq G^{\Sigma}_{\Delta_i}\leqslant S_{p^m+1}$, and hence a quotient group of $G_{x_i}^{\Delta_i}$ contains A_{p^m+1} by Corollary 3.10. Thus, one of the following holds by Theorem 3.4:

(i) $G_{x_i}^{\Delta_i}\leqslant \Gamma L_2(p^m)$;
(ii) $G_{x_i}^{\Delta_i}\leqslant (Q_8\circ D_8).S_5$ and $p^m=9$;
(iii) $G_{x_i}^{\Delta_i}\cong SL_2(13)$ and $p^m=27$;
(iv) $G_{x_i}^{\Delta_i}\leqslant (Z_{p^{m/3}-1}\times SU_3(p^{m/3})).Z_{2m/3}$, with $m\equiv 0 \pmod 3$.
(v) $Sp_4(p^m)\trianglelefteq G_{x_i}^{\Delta_i}\leqslant \Gamma Sp_4(p^m)$, with m even.
(vi) $Sz(2^{m/2})\trianglelefteq G_{x_i}^{\Delta_i}\leqslant \left(Z_{2^{m/2}-1}\times Sz(2^{m/2})\right).Z_{m/2}$, with $m\equiv 2 \pmod 4$;
(vii) $G_2(2^{m/3})'\trianglelefteq G_{x_i}^{\Delta_i}\leqslant \left(Z_{2^{m/3}-1}\times G_2(2^{m/3})\right).Z_{m/3}$, with $m\equiv 0 \pmod 3$.

Cases (i)–(vii) bring together some of the automorphism groups of the 2-designs listed in Theorem 3.4. For instance, $\Gamma L_2(p^m)$ contains $G_{x_i}^{\Delta_i}$ when this one is as in (2.a.i), (2.a.iii) or in (2.c.i) of Theorem 3.4, $(Q_8\circ D_8).S_5$ contains $G_{x_i}^{\Delta_i}$ when this one is as in (2.a.iii) or in (2.c.i) for $p^m=9$ and the group in case (vii) contains the groups in (2.c.v) or (2.c.vi) for $p^m=8$ (the non solvable case) of Theorem 3.4.

It is easy to check that only groups in (i) for $p^m=2,3,4,5$ and in (vi) admit a quotient group containing A_{p^m+1} as a normal subgroup. Actually, $p^m\neq 2$ by Lemma 2.5, and $p^m\neq 5$ since A_6 occurs only in (i) for $p^m=9$ and in (vi) for $p^m=4$. If $p^m=3$, then G is solvable by [40], and this case is ruled out. Thus $p^m=4$, and hence the assertion (I) follows from [29].

Assume that (2) occurs. Thus $[u^{h-1}]:SL_{h-1}(u)\trianglelefteq G^{\Sigma}_{\Delta_i}$ (e.g. see [27, Proposition 4.1.17(II)]), and hence a quotient group of $G_{x_i}^{\Delta_i}$ contains $[u^{f(h-1)}]:SL_{h-1}(u^f)$ as a normal subgroup by Corollary 3.10. This is clearly impossible for $h\geqslant 3$, hence $h=2$. Then $u^f=p^m+1$. By [43, B1.1], either $(p^m,u^f)=(2^3,3^2)$, or $f=1$, $p=2$ and u is a Fermat prime, or $m=1$, $u=2$ and p is a Mersenne prime. In each case, $G^{\Sigma}_{\Delta_i}$ contains a normal Frobenius group of order $(p^m+1)\frac{p^m}{(p^m,2)}$ with kernel of order p^m+1 and complement of order $\frac{p^m}{(p^m,2)}$. Since a quotient group of $G_{x_i}^{\Delta_i}$ is isomorphic to $G^{\Sigma}_{\Delta_i}$ by Corollary 3.10, it follows from Theorem 3.4 that either $\mathcal{D}_i\cong AG_2(p^m)$, where p^m is either 8, or 2^{2^e} with $e\geqslant 1$, or a Fermat prime, or $\mathcal{D}_i$ is as in (2.b), or $\mathcal{D}_i$ is as in (2.c.i) of Theorem 3.4.

Assume that $\mathcal{D}_i\cong AG_2(p^m)$, where p^m is either 8, or 2^{2^e} with $e\geqslant 1$, or a Fermat prime. Then either $G_{x_i}^{\Delta_i}\leqslant \Gamma L_1(p^{2m})$, or $SL_2(p^m)\trianglelefteq G_{x_i}^{\Delta_i}$, or $p^m=5$ and $SL_2(3)\trianglelefteq G_{x_i}^{\Delta_i}$ by [15,32].

If $SL_2(p^m)\trianglelefteq G_{x_i}^{\Delta_i}$, then $(p^m+1)\frac{p^m}{(p^m,2)}$ divides $\left|\Gamma L_2(p^m):SL_2(p^m)\right|$, and hence $(p^m-1)m$, since a quotient group of $G_{x_i}^{\Delta_i}$ contains a normal Frobenius group of order $(p^m+1)\frac{p^m}{(p^m,2)}$. This is clearly is impossible. Case $p^m=5$ and $SL_2(3)\trianglelefteq G_{x_i}^{\Delta_i}$ is ruled out similarly. Thus $G_{x_i}^{\Delta_i}\leqslant \Gamma L_1(p^{2m})$, and hence $(p^m+1)\frac{p^m}{(p^m,2)}$ divides the order of $\Gamma L_1(p^{2m})$. Then $\frac{p^m}{(p^m,2)}\mid 2m$, and hence $p^m=4,16$ since $p^m\geqslant 3$ by Lemma 2.5.

Case (2.c.i) of Theorem 3.4 is ruled out by the previous argument since $SL_2(p^m)\trianglelefteq G_{x_i}^{\Delta_i}\leqslant (Z_{q^{m/2}-1}\circ SL_2(p^m)).Z_m$, and we obtain $p^m=4,16$ for $\mathcal{D}_i$ as in (2.b) of Theorem 3.4 since $G_{x_i}^{\Delta_i}\leqslant \Gamma L_1(p^{2m})$ in this case. Hence, $p^m=4,16$ in each case. If $p^m=4$, then $A_5\trianglelefteq G^{\Sigma}\leqslant S_5$, and hence the assertion (I) follows from [29] in this case.

Assume that $p^m=16$. Then $PSL_2(17)\trianglelefteq G^{\Sigma}\leqslant PGL_2(17)$. If $G^{\Sigma}\cong PGL_2(17)$, then a quotient group of $G_{x_i}^{\Delta_i}$ is isomorphic to Frobenius group of order $17\cdot 16$. However, this is impossible since $G_{x_i}^{\Delta_i}\leqslant \Gamma L_1(2^8)$. Thus $G^{\Sigma}\cong PSL_2(17)$, which is (III).

Cases (3), (4) and (5) yield an equation of type $u^i = p^m + 1$, with $i = 3, 2, 3$ respectively. Only (4) is admissible with $m = 1$ and $u = 2$ by [43, B1.1(2)], however it is ruled out since $u \neq 2$ in (4). Case (6) cannot occur since $2^{2n-1} \pm 2^{n-1} = p^m + 2$ with $n \geqslant 3$ has no solutions. In (7), $G^{\Sigma} \cong PSL_2(11), M_{11}$ by [9], and hence (II) holds.

Finally, assume that (8) occurs. Then $G^{\Sigma} \cong A_7$ by [9]. Moreover, $\mathcal{D}_i \cong AG_2(13)$ and $G_{x_i}^{\Delta_i} \leqslant GL_2(13)$ by Theorem 3.4. On the other hand, a quotient group of $G_{x_i}^{\Delta_i}$ contains $PSL_2(7)$ by Corollary 3.10 since $PSL_2(7) \trianglelefteq G_{\Delta_i}^{\Sigma}$, being $|\Sigma| = 15$, and we reach a contradiction. This completes the proof. □

Theorem 5.13. *If G^{Σ} is almost simple, then $p^m = 4$ and $\mathcal{D}$ is isomorphic to one of the four 2-(96, 20, 4) designs constructed in [29].*

Proof. Assume that Case (II) of Proposition 5.12 occurs. Then H is an irreducible subgroup of $GL_{8+t}(2)$ with $t \leqslant 8$ by Lemma 5.1. Moreover, a quotient group of H is isomorphic either to $PSL_2(11)$ or to M_{11}, since $C \leqslant G_{(\Sigma)}$ by Theorem 4.3. From [6, Tables 8.18–8.19, 8.25–8.26, 8.35–8.36, 8.44–8.45], it follows that either $H = G^{\Sigma}$ and $t = 1, 2$, or $H \cong SL_2(11) < Z_2.M_{12}$ for $t = 2$. However, the action of H on $V_{4+t}(3)$ is irreducible only for $t = 1$ by [54]. Therefore $G_{(\Sigma)} = C$ and hence $G_{(\Sigma),x}^{\Delta} = 1$ for $\Delta \in \Sigma$ and $x \in \Delta$. It follows that $G_{\Delta}^{\Sigma} \cong G_x^{\Delta}$ by (3.1) of Corollary 3.10. Thus, $G_{(\Sigma),x}^{\Delta}$ is isomorphic either to A_5 or to M_{10} according to whether H is isomorphic to $PSL_2(11)$ or M_{11}, respectively, by [9]. On the other hand, G_x^{Δ} is contained in one of the groups $\Gamma L_2(9)$, $GSp_4(3)$, or $(D_8 \circ Q_8).S_5$ according to whether either one of (2.a.i) or (2.c.i), or (2.c.iii), or one of (2.a.iii) or (2.c.vi) occurs, respectively, by Theorem 3.4 since $p^m = 9$ and G_x^{Δ} is non-solvable. However, none of these groups contains A_5 or M_{10} (see [6, Tables 8.12–8.13]), and hence Case (II) is ruled out.

Assume that Case (III) occurs. Then H is an irreducible subgroup of $GL_{8+t}(2)$, with $t \leqslant 8$, by Lemma 5.1. Moreover, a quotient group of H is isomorphic to $PSL_2(17)$ since $C \leqslant G_{(\Sigma)}$ by Theorem 4.3. If $t < 8$, then $t = 0$ and $H \cong PSL_2(17)$ by [3, Theorem 3.1] since $\Phi_8^*(2) = 17$ divides the order of H. Thus $|G| = 2^{12} \cdot 3^2 \cdot 17$, and hence $|G_x| = 2^3 \cdot 17$ since $v = 2^9 \cdot 3^2$. Then this case is excluded since $k = 2^4 \cdot 17$ does not divide the order of G_x. Therefore $t = 8$, and hence $C = V$ by Proposition 3.8(3). It is easy to verify that H is not a geometric subgroup of $GL_{16}(2)$ by using [27, Section 4]. Thus, H is a nearly simple subgroup of $GL_{16}(2)$, and hence $H \cong PSL_2(17)$ by [54].

Let Q be a Sylow 17-subgroup of G. Simple computations with the aid of GAP [17] show that Q preserves a decomposition of $V = V_1 \oplus V_2$, where V_1 and V_2 are the unique Q-invariant proper subspaces of V. Moreover, V_1^H and V_2^H are two distinct orbits each of length 18, and $\left|V_2 \cap V_1^{\eta}\right| = 2^3$ for each $\eta \in H \setminus H_{V_1}$ and $\left|V_1 \cap V_2^{\sigma}\right| = 2^3$ for each $\sigma \in H \setminus H_{V_2}$.

The group Q fixes a point x of $\mathcal{D}$ since $v = 2^{17} \cdot 3^2$. Thus Q preserves the unique element Δ of Σ containing x, and hence normalizes $V_{(\Delta)}$, being $V_x = V_{(\Delta)}$. Then V_x is either V_1 or V_2 since V_1 and V_2 are the unique Q-invariant proper subspaces of V. Dualizing, Q preserves a block B of $\mathcal{D}$, and hence also V_B is either V_1 or V_2. Actually, (V_x, V_B) is either (V_1, V_2) or (V_2, V_1) since $\left|G_B : G_{B,x}\right| = 2^4 \cdot 17$ and $G_x/V_x \cong G_B/V_B \cong F_{136}$. In particular, x and B are the unique point and block of $\mathcal{D}$ fixed by Q respectively. Moreover, $|B \cap \Delta| = 0$ and $|B \cap \Delta| = 2^4$ for each $\Delta' \in \Sigma \setminus \{\Delta\}$ since Q acts regularly on $\Sigma \setminus \{\Delta\}$.

Assume that $(V_x, V_B) = (V_1, V_2)$. Then $V_1^H = \{V_{(\Delta')} : \Delta' \in \Sigma\}$, and hence $\left|V_B \cap V_{(\Delta')}\right| = 2^3$ for each $\Delta' \in \Sigma \setminus \{\Delta\}$. On the contrary, $V_B = V_{B \cap \Delta'}$, and hence $\left|V_B \cap V_{(\Delta')}\right| = 2^4$ for each $\Delta' \in \Sigma \setminus \{\Delta\}$ since V_B preserves each element of $\Sigma \setminus \{\Delta\}$, $|V_B| = 2^8$ and $\left|B \cap \Delta'\right| = 2^4$. So, we obtain a contradiction, and hence this case is excluded. The case $(V_x, V_B) = (V_2, V_1)$ is ruled out similarly, and the proof is thus completed. □

6. The case where $\mathcal{D}$ is of type 2

In this section, we assume that $\mathcal{D}$ is of type 2. Hence, $\mathcal{D}$ is a symmetric 2-$((\lambda+6)\frac{\lambda^2+4\lambda-1}{4}, \lambda\frac{\lambda+5}{2}, \lambda)$ design with $\lambda \equiv 1, 3 \pmod 6$ admitting a flag-transitive automorphism group G preserving a partition Σ of the point set of $\mathcal{D}$ in $\frac{\lambda^2+4\lambda-1}{4}$ classes each of size $\lambda + 6$. Then $\mathcal{D}_i$ is a 2-$(\lambda+6, 3, \lambda/\theta)$ design with $\theta \mid \lambda$ admitting $G_{\Delta_i}^{\Delta_i}$ as a flag-transitive point-primitive automorphism group for each $i = 1, \ldots, \frac{\lambda^2+4\lambda-1}{4}$ by Theorem 2.4. Our aim is to prove the following result, thus completing the proof of Theorem 1.1.

Theorem 6.1. *If $\mathcal{D}$ is of type 2, then $\mathcal{D}$ is isomorphic to the 2-(45, 12, 3) design as in [40, Construction 4.2].*

6.1. Hypothesis

If $\lambda = 3$, the assertion follows from [40, Corollary 1.2]. Therefore, in the sequel, we assume that $\lambda > 3$.

Lemma 6.2. *Let $\gamma \in G$, $\gamma \neq 1$, then $\left|\mathrm{Fix}(\gamma)\right| < \frac{3(\lambda^2+4\lambda-1)}{4}$.*

Proof. Let $\gamma \in G$, $\gamma \neq 1$, then

$$\left|\mathrm{Fix}(\gamma)\right| \leqslant \frac{\lambda}{\lambda(\lambda+5)/2 - \sqrt{\lambda(\lambda+5)/2 - \lambda}} \cdot (\lambda+6)\frac{\lambda^2+4\lambda-1}{4} < \frac{3(\lambda^2+4\lambda-1)}{4}$$

by [28, Corollary 3.7] since $c = \lambda + 6$, $d = \frac{\lambda^2+4\lambda-1}{4}$ and $\lambda > 3$. $\square$

Lemma 6.3. *Let x be any point of $\mathcal{D}$, then $G_{(\Sigma),x}$ is a $\{2,3\}$-group. Moreover, $\left|G_{(\Sigma)} : G_{(\Sigma),x}\right| \mid \lambda + 6$.*

Proof. Let x be any point of $\mathcal{D}$ and let φ be any w-element of $G_{(\Sigma),x}$ with w a prime and $w \neq 2, 3$. Then φ fixes at least a block B of $\mathcal{D}$ by [28, Theorem 3.1]. Then φ fixes B pointwise since B intersects each element of Σ in either 0 or 3 points. Therefore, φ fixes at least k blocks of $\mathcal{D}$ again by [28, Theorem 3.1]. Let B' and B'' be further blocks fixed by φ. We may repeat the previous argument with B' and B'' in the role of B, thus obtaining that φ fixes B' and B'' pointwise, respectively. Then φ fixes at least $3(k-\lambda)$ points of $\mathcal{D}$, and hence $|\mathrm{Fix}(\varphi)| \geqslant 3\,(k-\lambda)$. Then

$$3\left(\lambda\frac{\lambda+5}{2} - \lambda\right) \leqslant |\mathrm{Fix}(\varphi)| < \frac{3(\lambda^2+4\lambda-1)}{4}$$

by Lemma 6.2, and we reach a contradiction. Thus, $G_{(\Sigma),x}$ is a $\{2,3\}$-group.

Assume that $G_{(\Sigma)} \neq G_{(\Sigma),x}$ and let Δ be the element of Σ containing x. Then $G_{(\Sigma)} \neq G_{(\Delta)}$ since $G_{(\Delta)} \leqslant G_x$. Thus $G(\Sigma)^{\Delta} \neq 1$, and hence $Soc(G)^{\Delta}_{\Delta} \leqslant G(\Sigma)^{\Delta}$ by Lemma 2.6 and [14, Theorem 4.3B(i)]. Therefore, $G_{(\Sigma)}$ acts transitively on Δ, and hence $\left|G_{(\Sigma)} : G_{(\Sigma),x}\right| = \lambda + 6$. $\square$

Lemma 6.4. *Let $\Delta \in \Sigma$, then each prime divisor of $\left|G_{(\Delta)}\right|$ divides $\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$.*

Proof. Let $\Delta \in \Sigma$ and γ be a w-element of $G_{(\Delta)}$ with w a prime such that $w \nmid \lambda$. Let $\mathcal{B}(\Delta) = \{B \cap \Delta \neq \varnothing : B \in \mathcal{B}\}$. Then $(\Delta, \mathcal{B}(\Delta))$ is a non-trivial 2-$(\lambda+6, 3, \lambda/\theta)$ design by Theorem 2.4(VI.2). In particular, if $B \cap \Delta$ is any fixed element of $\mathcal{B}(\Delta)$, θ is a constant number denoting the number of blocks of $\mathcal{D}$ whose intersection set with Δ coincides with $B \cap \Delta$. Now, γ preserves each of the $\frac{(\lambda+6)(\lambda+5)}{6}\frac{\lambda}{\theta}$ elements of $\mathcal{B}(\Delta)$ since γ fixes Δ pointwise, and hence γ fixes at least $\mu\frac{(\lambda+6)(\lambda+5)}{6}\frac{\lambda}{\theta}$ blocks of $\mathcal{D}$ with $\mu \equiv \theta \pmod{w}$. Here, $\mu \geqslant 1$ since $\theta \mid \lambda$ but $w \nmid \lambda$. Therefore,

$$\mu\frac{\lambda}{\theta}\frac{(\lambda+6)(\lambda+5)}{6} \leqslant |\mathrm{Fix}(\varphi)| < \frac{3(\lambda^2+4\lambda-1)}{4}$$

by [28, Theorem 3.1] and Lemma 6.2. Thus $\mu\frac{\lambda}{\theta} \leqslant 4$, and hence $\frac{\lambda}{\theta}$ is either 1 or 3 since λ is odd. In the former case $\lambda = \theta$, $\mu \leqslant 4$ and $\mu \equiv \lambda \pmod{w}$, whereas $\theta = \frac{\lambda}{3}$, $\mu = 1$ and $\lambda \equiv 3 \pmod{w}$ in the latter. Therefore, in both cases, the assertion holds. $\square$

Throughout the remainder of the paper we will make use of the following fact:

$$\frac{G^{\Sigma}_{\Delta}}{G^{\Sigma}_{(\Delta)}} \cong \frac{G_{\Delta}}{G_{(\Delta)}G_{(\Sigma)}} \cong \frac{G^{\Delta}_{\Delta}}{G^{\Delta}_{(\Sigma)}}. \tag{6.1}$$

6.2. The 2-design $\mathcal{D}^{\Sigma}$

In this section, we show that Σ can be endowed with the structure of a 2-design admitting G^{Σ} as a flag-transitive point-primitive automorphism group by using [7]. Moreover, G^{Σ} is either almost simple or affine by [55].

For each block B of $\mathcal{D}$ define $B' = \{\Delta : \Delta \cap B \neq \varnothing\}$ and $\mathcal{B}' = \{B' : B \in \mathcal{B}\}$. Now, define the following equivalence relation on $\mathcal{B}' \times \mathcal{B}'$:

$$\mathcal{R} = \{(B'_1, B'_2) \in \mathcal{B}' \times \mathcal{B}' : B'_1 = B'_2\}.$$

Denote by B^{Σ} the equivalence class determined by the block B of $\mathcal{D}$, and by $\mathcal{B}^{\Sigma}$ the quotient set $\{B^{\Sigma} : B \in \mathcal{B}\}$. Finally, we define

$$\mathcal{I} = \{\left(\Delta, B^{\Sigma}\right) \in \Sigma \times \mathcal{B}^{\Sigma} : \Delta \in B' \text{ for each } B' \in B^{\Sigma}\}$$

Now, consider the incidence structure $\mathcal{D}^{\Sigma} = (\Sigma, \mathcal{B}^{\Sigma}, \mathcal{I})$. Then the following hold:

Proposition 6.5. *$\mathcal{D}^{\Sigma} = (\Sigma, \mathcal{B}^{\Sigma}, \mathcal{I})$ is a 2-$\left(\frac{\lambda^2+4\lambda-1}{4}, \frac{\lambda(\lambda+5)}{6}, \frac{(\lambda+6)^2\lambda}{9\eta}\right)$ design with $\eta = \left|B^{\Sigma}\right|$ admitting G^{Σ} as a flag-transitive automorphism group. In particular, $\lambda \equiv 3 \pmod{6}$.*

Proof. By [7, Proposition 2.3] and since $\lambda \equiv 1, 3 \pmod{6}$, either the assertion follows, and hence $\lambda \equiv 3 \pmod{6}$, or $\mathcal{D}^{\Sigma}$ is a symmetric 1-design with $\frac{\lambda^2+4\lambda-1}{4}$ points and $\frac{\lambda^2+4\lambda-1}{4} - 1$ points on any block. In the latter case, one has $\frac{\lambda^2+4\lambda-1}{4} = \frac{\lambda(\lambda+5)}{6}$, and hence $\lambda = 3$, which is contrary to our assumptions. $\square$

Remark 6.6. In the sequel, we will denote the replication number of $\mathcal{D}^{\Sigma}$ and $\frac{(\lambda+6)^2\lambda}{9\eta}$ by r^{Σ} and λ^{Σ}, respectively. Further, we set $R=\frac{r^{\Sigma}}{(r^{\Sigma},\lambda^{\Sigma})}$.

Lemma 6.7. *Let B be any block of $\mathcal{D}$, then $G_B \leqslant G_{B'}$ and $\eta = |G_{B'}:G_B|$.*

Proof. Let B be any block of $\mathcal{D}$, then $G_B \leqslant G_{B'}$ since $B'=\{\Delta : \Delta \cap B \neq \varnothing\}$. Thus, $\eta \geqslant |G_{B'}:G_B|$. Now, let C be any other block of $\mathcal{D}$ such that $C'=B'$. Then $B^g = C$ for some $g\in G$ since G acts blocks-transitively on $\mathcal{D}$. Therefore $(B')^g = C' = B'$, and hence $g \in G_{B'}$. Thus $\eta \leqslant |G_{B'}:G_B|$, and hence $\eta = \left|B^{\Sigma}\right| = |G_{B'}:G_B|$. □

Proposition 6.8. *Either $\eta = 1$ or $\eta = \lambda+6$.*

Proof. Assume that $\eta > 1$. Hence, $G_B \neq G_{B'}$ by Lemma 6.7. Further, $G_{B'} \neq G$ since $\eta \mid \frac{(\lambda+6)^2\lambda}{9}$ by Proposition 6.5 and $|G:G_B| = (\lambda+6)\frac{\lambda^2+4\lambda-1}{4}$. Then $\Psi = \Theta^G$, where $\Theta = B^{G_{B'}}$, is a G-invariant point-partition for $\mathcal{D}'=(\mathcal{B},\mathcal{P})$, the dual of $\mathcal{D}$, by [14, Theorem 1.5A]. Hence, $\mathcal{D}'$ is a symmetric 2-design with the same parameters of $\mathcal{D}$ admitting G as a flag-transitive automorphism group preserving the point-partition Ψ with d' classes each of size $c'=\eta$. Then η is as in Theorem 2.4. If $\eta = \lambda+2$, then $9(\lambda+2) \mid (\lambda+6)^2\lambda$ and hence $9(\lambda+2) \mid 32$, a contradiction. Similarly, one has $\eta \neq \frac{\lambda+2}{2}$. If $\eta=\lambda^2$, then $9\lambda^2 \mid (\lambda+6)^2\lambda$ and hence $9\lambda \mid (\lambda+6)^2$. Thus $\lambda \mid 6$, and hence $\lambda = 3$ since λ is odd. However, this contradicts the assumption $\lambda > 3$. Therefore $\eta = \lambda+6$, which is the assertion. □

Corollary 6.9. *If $\eta=1$, then $G_{(\Sigma)}=1$.*

Proof. If $\eta = 1$ then $G_{(\Sigma)} \leqslant G_{B'} = G_B$ for any block B of $\mathcal{D}$, and hence $G_{(\Sigma)}=1$ by [28, Theorem 3.1]. □

Lemma 6.10. *Let $\Delta \in \Sigma$, then the following hold:*

(1) $R=\frac{\lambda+5}{2}$ *and* $R > |\Sigma|^{1/2}$;
(2) R *divides the length of any non-trivial point-*G^{Σ}_{Δ}*-orbit on* $\mathcal{D}^{\Sigma}$.

Proof. Since $r^{\Sigma} = \frac{(\lambda+5)}{2}\frac{(\lambda+6)\lambda}{3\eta}$ and $\lambda^{\Sigma} = \frac{(\lambda+6)^2\lambda}{9\eta}$, and $\eta = 1$ or $\lambda+6$ by Lemma 6.8, it follows that $(r^{\Sigma},\lambda^{\Sigma}) = \frac{(\lambda+6)\lambda}{3\eta}$. Thus $R=\frac{\lambda+5}{2}$, and hence (1) follows.

Let $\Delta\in\Sigma$ and let $\mathcal{B}^{\Sigma}_{\Delta}$ be the set of blocks of $\mathcal{D}^{\Sigma}$ incident with Δ. Now, let $\Delta_0 \in \Sigma\setminus\{\Delta\}$, then $\left(\mathcal{B}^{\Sigma}_{\Delta}, \Delta_0^{G^{\Sigma}_{\Delta}}\right)$ with the incidence relation inherited from that of $\mathcal{D}^{\Sigma}$ is a 1-design by [10, 1.2.6] since G^{Σ} acts flag-transitively on $\mathcal{D}^{\Sigma}$, and hence $r^{\Sigma} \mid \left|\Delta_0^{G^{\Sigma}_{\Delta}}\right|\lambda^{\Sigma}$. Thus $R \mid \left|\Delta_0^{G^{\Sigma}_{\Delta}}\right|$, which is (2). □

Theorem 6.11. *G^{Σ} acts primitively on Σ. Moreover, $Soc(G^{\Sigma})$ is either non-abelian simple or an elementary abelian w-group for some odd prime w.*

Proof. Since $\frac{\lambda^2+4\lambda-1}{4}-1 = \left(\frac{\lambda+5}{2}\right)\left(\frac{\lambda-1}{2}\right)$ and $\frac{\lambda(\lambda+5)}{6}-1 = \left(\frac{\lambda+6}{3}\right)\left(\frac{\lambda-1}{2}\right)$, it follows that

$$\left(\frac{\lambda^2+4\lambda-1}{4}-1, \frac{\lambda(\lambda+5)}{6}-1\right)^2 = \left(\frac{\lambda-1}{2}\right)^2 \leqslant \frac{\lambda^2+4\lambda-1}{4}-1,$$

and hence G^{Σ} acts primitively on Σ by [55, Lemma 2.7]. Further, since $|\Sigma|$ is odd, $Soc(G^{\Sigma})$ is either non-abelian simple or an elementary abelian w-group for some odd prime w, or G^{Σ} is a rank 3 group and there is an integer $x>1$ such that $|\Sigma| = x^2$ by [55, Theorem 1.1]. In the latter case, $\frac{(\lambda+2)^2-5}{4} = x^2$ since $|\Sigma| = \frac{\lambda^2+4\lambda-1}{4} = \frac{(\lambda+2)^2-5}{4}$. Hence $(\lambda+2-2x)(\lambda+2+2x)=5$, which does not have admissible solutions for $x>1$. □

Corollary 6.12. $\lambda \neq 9, 21, 75, 723$.

Proof. Assume that $\lambda = 9$, 21 or 75. Then $d = 29$, 131 or 1481, respectively, and in all these cases d is a prime. By Theorem 6.11 and [19, Theorem 1], either $A_d \trianglelefteq G^{\Sigma} \leqslant S_d$ and $A_{d-1} \trianglelefteq G^{\Sigma}_{\Delta} \leqslant S_{d-1}$, or $G^{\Sigma} \leqslant AGL_1(d)$. In the latter case, $\frac{\lambda(\lambda+5)}{6}$ divides $\left|G^{\Sigma}\right|$, and hence $|AGL_1(d)|$, since G^{Σ} acts flag-transitively on $\mathcal{D}^{\Sigma}$. However, this is impossible since $\frac{\lambda(\lambda+5)}{6}$ is 21, 91, or 10^3 according as $d = 29$, 31, and 1481, respectively. Thus, $A_d \trianglelefteq G^{\Sigma} \leqslant S_d$ and $A_{d-1} \trianglelefteq G^{\Sigma}_{\Delta} \leqslant S_{d-1}$. Further, either $G^{\Sigma}_{(\Delta)} = 1$ or $G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)} \leqslant Z_2$.

Note that, $\frac{\lambda+5}{2}$ divides the order G_{Δ}^{Δ} by Theorem 2.3, and $\frac{\lambda+5}{2}$ is $7, 13$ or 40 according as $d = 29$, 31, and 1481, respectively. On the other hand, $\left|G_{(\Sigma)}^{\Delta}\right|$ divides $\left|G_{(\Sigma)}\right|$ and hence divides $(\lambda+6)\left|G_{(\Sigma),x}\right|$ with $G_{(\Sigma),x}$ a $\{2,3\}$-group by Lemma 6.3. Therefore 7, 13 or 5 divides the order of $G_{\Delta}^{\Delta}/G_{(\Sigma)}^{\Delta}$ according as $d = 29$, 31 or 1481, respectively. Thus 7, 13 or 5 divides the order of $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$, respectively, by (6.1), and this forces $G_{(\Delta)}^{\Sigma} = 1$. Therefore, $A_{d-1} \trianglelefteq G_{\Delta}^{\Delta}/G_{(\Sigma)}^{\Delta}$ again by (6.1) since $A_{d-1} \trianglelefteq G_{\Delta}^{\Sigma}$. Then $\lambda + 6 = |\Delta| \geqslant d-1$ since the minimal non-trivial transitive permutation representation degree of A_{d-1} is $d-1$, which is not the case.

Assume that $\lambda = 723$. Then $d = 5 \cdot 41 \cdot 641$. Note that, $\frac{\lambda(\lambda+5)}{6} = 2^2 \cdot 7 \cdot 13 \cdot 241$ divides the order of G^{Σ} since G^{Σ} acts flag-transitively on $\mathcal{D}^{\Sigma}$ by Proposition 6.5. Let U be a Sylow 241-subgroup of G. Then $U \leqslant G_{\Delta}$ for some $\Delta \in \Sigma$ since $(d, 241) = 1$, and hence $Fix_{\Sigma}(U)$, the set of the elements of Σ fixed by U, is of size $60 + 241t$ for some $t \geqslant 0$ since $d \equiv 60 \pmod{241}$. Actually, $t = 0$ by Lemma 6.2.

Let $\Delta' \in Fix_{\Sigma}(U) \setminus \{\Delta\}$, then $G_{\Delta} = N_{G_{\Delta}}(U)G_{(\Delta)}$ and $G_{\Delta,\Delta'} = N_{G_{\Delta,\Delta'}}(U)G_{(\Delta\cup\Delta')}$ by the Frattini argument since $G_{(\Delta)} \trianglelefteq G_{\Delta}$ and $G_{(\Delta\cup\Delta')} \trianglelefteq G_{\Delta,\Delta'}$. Hence,

$$\left|G_{\Delta} : G_{\Delta,\Delta'}\right| = \frac{\left|N_{G_{\Delta}}(U) : N_{G_{\Delta,\Delta'}}(U)\right| \cdot \left|G_{(\Delta)} : G_{(\Delta\cup\Delta')}\right|}{\left|N_{G_{(\Delta)}}(U) : N_{G_{(\Delta\cup\Delta')}}(U)\right|}.$$

Now, $\frac{\lambda+5}{2} = 2^2 \cdot 7 \cdot 13$ divides $\left|G_{\Delta}^{\Sigma} : G_{\Delta,\Delta'}^{\Sigma}\right|$ by Lemma 6.10(2), and hence $7 \cdot 13$ divides $\left|G_{\Delta} : G_{\Delta,\Delta'}\right|$ by Lemma 6.3. On the other hand, 13 is coprime to $\left|G_{(\Delta)}\right|$ by Lemma 6.4. Therefore, $13 \mid \left|N_{G_{\Delta}}(U) : N_{G_{\Delta,\Delta'}}(U)\right|$, and hence $13 \mid \left|(\Delta')^{N_{G_{\Delta}}(U)}\right|$ with $(\Delta')^{N_{G_{\Delta}}(U)} \subseteq Fix_{\Sigma}(U) \setminus \{\Delta\}$. So, $13 \mid |Fix_{\Sigma}(U)| - 1$ Since $Fix_{\Sigma}(U) \setminus \{\Delta\}$ is a disjoint union of $N_{G_{\Delta}}(U)$-orbits, but this is impossible since $|Fix_{\Sigma}(U)| = 60$. □

6.3. 2-designs $\mathcal{D}_i$

The aim of this section is to prove the following result by analyzing the structure of the 2-designs $\mathcal{D}_i$ defined in Theorem 2.4(VI.2):

Theorem 6.13. *One of the following holds:*

(1) $\lambda = \frac{z^h-1}{z-1} - 6$, *where z a power of 2, $z-1 \mid h-6$, h is even for $z \equiv 2 \pmod 3$ and divisible by 3 for $z \equiv 1 \pmod 3$. In particular, $h > 3$ and $(h, z) \neq (4, 2)$. Moreover, the following facts hold:*
  (a) $\frac{\lambda+5}{2} = \frac{z}{2}\frac{z^{h-1}-1}{z-1}$ *is divisible by a primitive prime divisor of $z^{h-1}-1$;*
  (b) $G_{(\Sigma)} = 1$;
  (c) G_{Δ}^{Δ} *is divisible by a primitive prime divisor of $z^{h-1}-1$;*
  (d) *$Soc(G)$ is non-abelian simple and $PSL_h(z) \trianglelefteq Soc(G)_{\Delta}^{\Delta} \leqslant P\Gamma L_h(z)$.*

(2) $\lambda = 3^h - 6$, *$h > 4$ and $h \neq 6$, G_{Δ}^{Δ} is of affine type and the following facts hold:*
  (a) $\frac{\lambda+5}{2} = \frac{3^h-1}{2}$ *admits a primitive prime divisor;*
  (b) *$G_{(\Sigma)}$ is a $\{2,3\}$-group;*
  (c) *$G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is divisible by a primitive prime divisor of 3^h-1;*
  (d) *Either $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is solvable and $G_{\Delta}^{\Delta} \leqslant A\Gamma L_1(3^h)$, or a quotient group of $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is an almost simple group with socle isomorphic either to $PSL_{h/e}(3^e)$ with $h/e \geqslant 2$ or to $PSp_{h/e}(3^e)$ with h/e even and $h/e \geqslant 2$. Moreover, in both cases, $(h/e, e) \neq (2,1), (2,2), (3,2), (6,1)$.*

Theorem 6.13 relies on the following theorem, which is an immediate consequence of the classification of the flag-transitive 2-(v,k,λ) designs with $v \equiv 6 \pmod{\lambda}$ and $\lambda \equiv 1, 3 \pmod 6$ admitting a flag-transitive automorphism group provided in [36].

Theorem 6.14. *Let $\mathcal{D}_i$ be a 2-$(\lambda+6, 3, \lambda/\theta)$-design with $\lambda \equiv 3 \pmod 6$ and $\lambda > 3$ admitting a flag-transitive automorphism group $G_{\Delta_i}^{\Delta_i}$. Then one of the following holds:*

(1) *$\mathcal{D}_i$ is a 2-$(3^h, 3, \lambda/\theta)$ design and $G_{\Delta_i}^{\Delta_i}$ is a primitive $3/2$-transitive rank 3 subgroup of $A\Gamma L_1(3^h)$.*
(2) *$\mathcal{D}_i$ is a 2-$\left(\frac{z^h-1}{z-1}, 3, z-1\right)$ design as in [36, Example 1.3(1)] with z even, $z-1 \mid h-6$, h even for $z \equiv 2 \pmod 3$ and h divisible by 3 for $z \equiv 1 \pmod 3$. Moreover, one of the following holds:*
  (a) $PSL_h(z) \trianglelefteq G_{\Delta_i}^{\Delta_i} \leqslant P\Gamma L_h(z)$;
  (b) $G_{\Delta_i}^{\Delta_i} \cong A_7$ *and $(h, q) = (4, 2)$.*

(3) *$\mathcal{D}_i \cong AG_h(3)$, $h \geqslant 2$, and $G_{\Delta_i}^{\Delta_i}$ is a 2-transitive group of affine type.*

Proof. It is an immediate consequence of [36] since $\lambda \equiv 3 \pmod 6$ by Proposition 6.5. $\square$

Proof of Theorem 6.13. Let $\Delta \in \Sigma$ and assume that G_Δ^Δ is non-abelian simple. Then $\lambda + 5 = z\frac{z^{h-1}-1}{z-1}$ and $PSL_h(z) \trianglelefteq G_\Delta^\Delta \leqslant P\Gamma L_h(z)$ with z even, $z-1 \mid h-6$, h even for $z \equiv 2 \pmod 3$ or h divisible by 3 for $z \equiv 1 \pmod 3$ by Theorem 6.14. Moreover, $(h,z) \neq (3,4),(4,2)$ since $\lambda \neq 9, 21$ by Lemma 6.12. If $h=2$ then $\lambda+5=z$, and hence $z>2$, but this is in contrast to $z-1 \mid h-6$ and z even. Thus $h \geqslant 3$. Now, if $h=3$, then $z=4$ since $z-1 \mid h-6$, a contradiction. Therefore, $h-1 \geqslant 3$. Further, if $z=2$ then h is even, and hence $h-1 \neq 6$. Thus, $\frac{\lambda+5}{2}$ admits a primitive prime divisor by [27, Theorem 5.2.14]. This proves (1.a).

Note that, either $G_{(\Sigma)} \leqslant G_{(\Delta)}$, or $Soc(G_\Delta^\Delta) \leqslant G_{(\Sigma)}^\Delta$ by Lemma 2.6 since $G_{(\Sigma)} \trianglelefteq G_\Delta$. In the latter case, one has $Soc(G_\Delta^\Delta)_x \leqslant G_{(\Sigma),x}^\Delta$, where $x \in \Delta$. Therefore $Soc(G_\Delta^\Delta)_x$ is solvable since $G_{(\Sigma),x}$, and hence $G_{(\Sigma),x}^\Delta$, is a $\{2,3\}$-group by Lemma 6.3. Then $h=2$ by [27, Proposition 4.1.17(II)], whereas $h>3$. Thus $G_{(\Sigma)} \leqslant G_{(\Delta)}$, and hence $G_{(\Sigma)}=1$ since $G_{(\Sigma)} \trianglelefteq G$ and G acts transitively on Σ. This proves (1.b). Now, (1.a) and (1.b) together with (6.1) and $PSL_h(z) \trianglelefteq G_\Delta^\Delta \leqslant P\Gamma L_h(z)$ imply (1.c).

Let $L = Soc(G)$. Then L acts point-transitively on $\mathcal{D}$ by Lemma 2.7 since $G_{(\Sigma)}=1$. Further, L is non-abelian simple by Theorem 6.11 since $|\Delta| = \lambda+6$ is divisible by 3, whereas $|\Sigma| = \frac{\lambda^2+4\lambda-1}{4}$ is coprime to 3. Moreover, $PSL_h(z) \trianglelefteq L_\Delta^\Delta \leqslant P\Gamma L_h(z)$ by Lemma 2.6 with (h,z) fulfilling the above listed properties, and so we obtain (1.d).

Assume that G_Δ^Δ is of affine type. Hence, $\lambda = 3^h - 6$ with $h>4$ and $h \neq 6$ by Theorem 6.14 and Lemma 6.12 since $\lambda > 3$. Then (2.a) and (2.b) follow from [27, Theorem 5.2.14] and Lemma 6.3, respectively. Again by Theorem 6.14, either G_Δ^Δ is a primitive 3/2-transitive subgroup of $A\Gamma L_1(3^h)$ of rank 3 and the first part of (2.d) holds by (6.1), or G_Δ^Δ acts 2-transitively on Δ by Theorem 6.14. In both cases, G_Δ^Δ is divisible by a primitive prime divisor of 3^h-1, hence (2.c) follows from (2.b) and (6.1). Further, if G_Δ^Δ acts 2-transitively on Δ, one of the following holds by [25, (B)] since $h>4$ and $h \neq 6$:

(i) $G_x^\Delta \leqslant \Gamma L_1(3^h)$;
(ii) $SL_{h/e}(3^e) \trianglelefteq G_x^\Delta \leqslant \Gamma L_{h/e}(3^e)$ and $h/e \geqslant 2$;
(iii) $Sp_{h/e}(3^e) \trianglelefteq G_x^\Delta \leqslant \Gamma Sp_{h/e}(3^e)$ and h/e is even, $h/e \geqslant 2$;

Now, (i) implies the first part of (2.d) again by (6.1). In (ii) and (iii), $G_\Delta^\Delta/G_{(\Sigma)}^\Delta$ is non-solvable since $G_{(\Sigma)}^\Delta$ is $\{2,3\}$-group by (2.b), and hence $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma$ is non-solvable by (6.1). Now, a quotient group of $G_\Delta^\Delta/G_{(\Sigma)}^\Delta$ is an almost simple group with socle isomorphic either to $PSL_{h/e}(3^e)$ with $h/e \geqslant 2$, or to $PSp_{h/e}(3^e)$ with h/e even and $h/e \geqslant 2$ by (ii) and (iii), respectively, again by (6.1). Moreover, in both cases $(h/e, e) \neq (2,1),(2,2),(3,2),(6,1)$ since $h>4$ and $h \neq 6$. Therefore the second part of (2.d) follows, and hence the proof is completed. $\square$

Remark 6.15. Cases (1) and (2) of Theorem 6.13 have no common values of λ. Indeed, if it is not so there would be integers $h_1, h_2 > 3$ such that $\frac{z^{h_1}-1}{z-1} = 3^{h_2}$ with z a power of 2, which is contrary to [43, A8.5(2)].

6.4. The case where G^Σ is of affine type

In this section, we assume that the socle of G^Σ is an elementary abelian w-group for some odd prime w acting regularly on Σ. Set $T = Soc(G^\Sigma)$, then Σ can be identified with a t-dimensional $GF(w)$-vector space V in a way that T is the translation group of V and $G^\Sigma = T : G_\Delta^\Sigma \leqslant AGL(V)$ since G^Σ acts point-primitively on $\mathcal{D}^\Sigma$. Hence, $|\Sigma| = |T| = w^t$. Further, $\lambda = 3^h - 6$ and G_Δ^Δ is of affine type by Theorem 6.13. Hence,

$$\frac{3^{2h} - 8 \cdot 3^h + 11}{4} = |\Sigma| = w^t. \tag{6.2}$$

Lemma 6.16. *The following hold:*

(1) *t is odd, $w \equiv 2 \pmod 3$ and $w \geqslant 17$.*
(2) *$(w, \lambda - j) = 1$ for each $j = 0, \dots, 4$.*
(3) *$|G_\Delta|_w = \left|G_\Delta^\Delta\right|_w$.*

Proof. Rewriting (6.2), we obtain $\left(3^h-4\right)^2 - 4w^t = 5$. If t is even, then

$$\left(3^h - 4 - 2w^{t/2}\right)\left(3^h - 4 + 2w^{t/2}\right) = 5,$$

and hence $3^h - 4 - 2w^{t/2} = 1$ and $3^h - 4 + 2w^{t/2} = 5$. Therefore, $4w^t = 4$, whereas w is a prime. Thus, t is odd.

Clearly, $w > 3$. Further, both $3^{2h} - 8 \cdot 3^h + 11 = 4w^t$ and $3^{2h} - 8 \cdot 3^h + 11 \equiv 2 \pmod 3$ imply $w^t \equiv 2 \pmod 3$ and hence $w \equiv 2 \pmod 3$. Moreover, if $w = 5$ then $\left(3^h - 4\right)^2 - 4 \cdot 5^t = 5$. Therefore $25 \mid \left(3^h - 4\right)^2$, and hence $t = 1$ and $h = 2$, whereas $\lambda > 3$. Thus, $w \geqslant 11$ since $w \equiv 2 \pmod 3$. If $w = 11$, then $3^{2h} - 8 \cdot 3^h \equiv 0 \pmod{11}$, and hence $3^h \equiv 8 \pmod{11}$. However, this is impossible since the residues of 3^h modulo 11 are $1, 3, 4, 5$, or 9. Thus $w > 11$, and hence $w \geqslant 17$ since $w \equiv 2 \pmod 3$. This proves (1).

Assume that $\lambda \equiv j \pmod w$ for some $j = 0, 1, 2, 3$ or 4. Then $j^2 + 4j - 1 \equiv 0 \pmod w$ since $\frac{\lambda^2 + 4\lambda - 1}{4} = w^t$. Then $(w, j) = (11, 2)$, $(5, 3)$ or $(31, 4)$ since w is odd, but these are ruled out by (1). Thus, we obtain (2).

Finally, both (2) and Corollary 6.4 imply $\left|G_{(\Delta)}\right|_w = 1$. Thus, $|G_\Delta|_w = \left|G_\Delta^\Delta\right|_w$. □

For each divisor m of t the group $\Gamma L_m(w^{t/m})$ has a natural irreducible action on $U \cong V_m(w^{t/m})$. Now, G_Δ^Σ acts irreducibly on $V = V_t(w)$ since G^Σ primitively on V by Lemma 6.11, so choose m to be minimal such that $G_0 \leqslant \Gamma L_m(w^{t/m})$ and write $s = w^{t/m}$. Thus, $G_0 \leqslant \Gamma L_m(s)$ where $s^m = w^t$. Note that, m is odd, s is coprime to 3 and $s \geqslant 17$ by Lemma 6.16(1).

Lemma 6.17. $m > 1$.

Proof. Assume that $m = 1$, then $G_\Delta^\Sigma \leqslant \Gamma L_1(w^t)$. Further $\lambda/3$ divides $\left|G_\Delta^\Sigma\right|$, and hence $\left|\Gamma L_1(w^t)\right|$, since G^Σ acts flag-transitively on $\mathcal{D}^\Sigma$ and λ is odd. So $3^{h-1} - 2 \mid (w^t - 1)t$, and hence $3^{h-1} - 2 \mid 5t$ since

$$w^t - 1 = |\Sigma| - 1 = \frac{(3^h - 1)(3^h - 7)}{4}.$$

Therefore

$$3^{2h} - 8 \cdot 3^h + 11 = 4w^t \geqslant 3^{\frac{3^{h-1} - 2}{5}}$$

by (6.2), which does not have admissible solutions since $h > 4$ by Theorem 6.13(2). This completes the proof. □

Lemma 6.18. *G_Δ^Σ does not contain any of the classical groups $SL_m(s)$, $Sp_m(s)$, $SU_m(s^{1/2})$, or $\Omega_m^\varepsilon(s)$ as a normal subgroup.*

Proof. Let X be any of the classical groups $SL_m(s)$, $Sp_m(s)$, $SU_m(s^{1/2})$ or $\Omega_m^\varepsilon(s)$, and assume that $X \trianglelefteq G_\Delta^\Sigma$. Actually, X is neither $Sp_m(s)$ nor $\Omega_m^\pm(s)$ since m is odd. In the remaining cases, we have $X \trianglelefteq G_\Delta^\Sigma \leqslant N_{\Gamma L_m(s)}(X)$, and hence there are no quotient groups of G_Δ^Σ containing $PSL_{h/\ell}(3^e)$ or $PSp_{h/e}(3^e)$ as a normal subgroup since $h/e \geqslant 2$, $(h/e, e) \neq (2,1), (2,2), (3,2), (6,1)$ and s is coprime to 3. Thus $G_\Delta^\Delta \leqslant A\Gamma L_1(3^h)$ by Theorem 6.13(2.d) since we saw that G_Δ^Δ is of affine type. Then $s^{m(m-1)/2}$, $s^{m(m-1)/4}$ or $s^{(m-1)^2/4}$ divides $\left|G_\Delta^\Delta\right|$, and hence $\left|A\Gamma L_1(3^h)\right|$, by Lemma 6.16(3), according as X is $SL_m(s)$, $SU_m(s^{1/2})$ or $\Omega_m(s)$, respectively. Thus, $s^{(m-1)^2/4} \mid (3^h - 1)h$ in each case. Actually, $s^{(m-1)^2/4} \mid h$ by (6.2). Therefore,

$$3^{s^{(m-1)^2/4}} \leqslant 3^h = \lambda + 6 < |\Sigma| = s^m \tag{6.3}$$

since $\lambda > 5$. However, (6.3) does not have admissible solutions since m is odd, $m > 1$ and $s \geqslant 17$. □

6.5. Aschbacher's theorem

Recall that $G_\Delta^\Sigma \leqslant \Gamma$, where $\Gamma = N_{\Gamma L_m(s)}(X)$ and X is any of the classical groups $SL_m(s)$, $Sp_m(s)$, $SU_m(s^{1/2})$ or $\Omega^\varepsilon(s)$. The case where $X \leqslant G_\Delta^\Sigma$ is excluded in Lemma 6.18, hence, in the sequel, we assume that G_Δ^Σ does not contain X. Now, according to [2], one of the following holds:

(I) G_Δ^Σ is geometric, that is, it lies in a maximal member of one the geometric classes $\mathcal{C}_i$ of Γ, $i = 1, \ldots, 8$;
(II) $(G_\Delta^\Sigma)^{(\infty)}$ is a quasisimple group, and its action on $V_m(s)$ is absolutely irreducible and not realizable over any proper subfield of $GF(s)$.

Description of each class $\mathcal{C}_i$, $i = 1, \ldots, 8$, can be found in [27, Chapter 4].

Theorem 6.19. *G_Δ^Σ is not geometric.*

Proof. The group G_Δ^Σ does not lie in a maximal member of type $\mathcal{C}_1$ since G_Δ^Σ acts irreducibly on $V_m(s)$. Moreover, by the definition of s, G_Δ^Σ does not lie in a maximal member of type $\mathcal{C}_3$. Further, G_Δ^Σ does not lie in a maximal member of type $\mathcal{C}_8$ by Lemma 6.18.

Assume that G_Δ^Σ lies in a maximal $\mathcal{C}_2$-subgroup of Γ. Then G_0 preserves a sum decomposition of $V = V_a(s) \oplus \cdots \oplus V_a(s)$ with $m/a \geqslant 2$, and hence $\bigcup_{i=1}^{m/a} V_a^*(s)$ is a union of G_0-orbits. Then $R \mid \frac{m}{a}(s^a - 1)$ by Lemma 6.10(2) since the size of

$\bigcup_{i=1}^{m/a} V_a^*(s)$ is $\frac{m}{a}(s^a-1)$. Then $s^{m/2} < \frac{m}{a}(s^a-1)$ since $R > s^{m/2}$ by Lemma 6.10(1), but no admissible solutions arise since $s \geqslant 17$, m is odd and $m > 1$ by Lemmas 6.16(1) and 6.17.

Assume that G_Δ^Σ lies in a maximal member of type $\mathcal{C}_4$ or $\mathcal{C}_7$ of Γ. Then either $V = V_t(w) = V_{a_1}(w) \otimes V_{a_2}(w)$, $t = a_1 a_2$ with $2 \leqslant a_1 \leqslant a_2$, and $G_\Delta^\Sigma \leqslant N_{GL_t(w)}(GL_{a_1}(w) \circ GL_{a_2}(w))$ in its natural action on V, or $V = V_t(w) = V_a(w) \otimes \cdots \otimes V_a(w)$, $t = a^\mu$, and $G_0 \leqslant N_{GL_t(w)}(GL_a(w) \circ \cdots GL_a(w))$, respectively. Assume that the latter occurs with $\mu \geqslant 3$. The non-zero vectors of the form $x_1 \otimes \cdots \otimes x_\mu$ form a union of G_Δ^Σ-orbits, of size $\frac{(w^a-1)^\mu}{(w-1)^{\mu-1}}$, and this number is therefore divisible by R. Then $w^{a^\mu/2} < \frac{(w^a-1)^\mu}{(w-1)^{\mu-1}}$ since $R > w^{a^\mu/2}$, and hence $a = 2$ and $\mu = 3$. So t is even, but this contradicts Lemma 6.16(1). Thus, $V = V_t(w) = V_{a_1}(w) \otimes V_{a_2}(w)$, $t = a_1 a_2$ with $2 \leqslant a_1 \leqslant a_2$ and $G_\Delta^\Sigma \leqslant N_{GL_t(w)}(GL_{a_1}(w) \circ GL_{a_2}(w))$. As in the previous proof, R divides the number of non-zero vectors of the form $x_1 \otimes x_2$, which is $\frac{2(w^{a_1}-1)(w^{a_2}-1)}{(w-1)}$. This must be greater than $w^{a_1 a_2/2}$. Thus, $a_2 \leqslant 3$, and hence $a_1 = a_2 = 3$ since t is odd and $2 \leqslant a_1 \leqslant a_2$. Then R divides $((w^3-1)^2/(w-1), w^9-1)$, hence divides $3(w^3-1)$, which is less than $w^{9/2}$. This contradicts Lemma 6.10(1).

Assume that G_Δ^Σ lies in a maximal member of type $\mathcal{C}_5$ of Γ. Then $G_0 \leqslant N_\Gamma(GL_w(w_0))$ with $w = w_0^\ell$ and $\ell > 1$; but this normalizer lies in a subgroup of $GL_\ell(w_0) \circ GL_m(w_0)$ of $GL_{\ell m}(s_0) \leqslant GL_t(w)$, and hence G_0 lies in a maximal member of type $\mathcal{C}_4$ or $\mathcal{C}_7$ of $GL_t(w)$, which is not the case by the above argument.

Finally, assume that G_Δ^Σ lies in a maximal $\mathcal{C}_6$-subgroup of Γ. Hence, G_Δ^Σ lies in the normalizer in $GL_m(s)$ of a symplectic type σ-group with $\sigma \neq s$. As shown in [2, Section 11], we may assume that G_0 contains the σ-group, otherwise lies in some other families $\mathcal{C}_i$, which is not the case since these have been previously ruled out. Then $G_\Delta^\Sigma \leqslant Z_{s-1} \circ \sigma^{1+2y} \cdot Sp_{2a}(\sigma) \cdot t/m$, $m = \sigma^y$ and $\sigma \mid s-1$ since t is odd. Now, we may use the argument of [32, Lemmas 3.7–3.8] with s, σ and R in the role of q, s and r, respectively, to obtain $t = 2$ or 4, and hence a contradiction since t is odd by Lemma 6.16(1). □

Theorem 6.20. *If $\lambda > 3$, then G^Σ is not of affine type.*

Proof. Assume that $(G_\Delta^\Sigma)^{(\infty)}$ is quasi-simple. Let $Z = G_\Delta^\Sigma \cap GL_m(s)$ and denote the socle of $(G_\Delta^\Sigma)^{(\infty)} Z/Z$ by S. Then S is non-abelian simple and $S \trianglelefteq G_\Delta^\Sigma/Z \leqslant Aut(S)$. Then either $S \leqslant G_{(\Delta)}^\Sigma Z/Z$, or $G_{(\Delta)}^\Sigma \leqslant Z$ since $G_{(\Delta)}^\Sigma Z/Z \trianglelefteq G_\Delta^\Sigma/Z$.

Assume that $G_{(\Delta)}^\Sigma \leqslant Z$. Then G_Δ^Σ/Z is isomorphic to a quotient group of $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma$, and hence $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma$ is non-solvable. Then a quotient group of $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma$ is an almost simple group with socle isomorphic either to $PSL_{h/e}(3^e)$ with $h/e \geqslant 2$ and $(h/e, e) \neq (2,1), (2,2)$, or to $PSp_{h/e}(3^e)$ with h/e even, $h/e \geqslant 2$ and $(h/e, e) \neq (2,1), (2,2)$ by Theorem 6.13(2.d), forcing S to be isomorphic to one of these groups since S is non-abelian simple and $S \trianglelefteq G_\Delta^\Sigma/Z \leqslant Aut(S)$. Thus, $(G_\Delta^\Sigma)^{(\infty)}$ is isomorphic either to $PSL_{h/e}(3^e)$ with $h/e \geqslant 2$ and $(h/e, e) \neq (2,1), (2,2)$, or to $PSp_{h/e}(3^e)$ with h/e even, $h/e \geqslant 2$ and $(h/e, e) \neq (2,1), (2,2)$, or to a covering group of any of them since $(G_\Delta^\Sigma)^{(\infty)}$ is quasi-simple (see [27, p. 173]). Then either $m \geqslant 3^{h-e} - 1 \geqslant 3^{h/2} - 1$ or $m \geqslant \frac{3^{h/2}-1}{2}$ by [27, Theorem 5.3.9] for $h/e > 2$ since s is coprime to 3. On the other hand, $s^{m/2} < \frac{\lambda+5}{2} = \frac{3^h-1}{2}$ by Lemma 6.10(1). Then

$$17^{\frac{3^{h/2}-1}{4}} < \frac{\lambda+5}{2} < \frac{3^h-1}{2}$$

since $s \geqslant 17$, and no admissible solutions arise since $h > 4$ by Theorem 6.13(2). Thus $h/e = 2$, and hence $S \cong PSL_2(3^e) \cong PSp_2(3^e)$ with $e \geqslant 2$. Actually, $e \neq 2$ by Lemma 6.12. Then $m \geqslant \frac{3^e-1}{2}$ by [27, Theorem 5.3.9], and hence $17^{\frac{3^e-1}{4}} < \frac{\lambda+5}{2} < \frac{3^{2e}-1}{2}$, a contradiction.

Assume that $S \leqslant G_{(\Delta)}^\Sigma Z/Z$. Then $\left(G_\Delta^\Sigma/Z\right)/\left(G_{(\Delta)}^\Sigma Z/Z\right)$ is isomorphic to a quotient group $Out(S)$, and hence $\left(G_\Delta^\Sigma/Z\right)/\left(G_{(\Delta)}^\Sigma Z/Z\right)$ is solvable. Therefore $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma Z$ is solvable since $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma Z \cong \left(G_\Delta^\Sigma/Z\right)/\left(G_{(\Delta)}^\Sigma Z/Z\right)$, and hence the group $\left(G_\Delta^\Sigma/G_{(\Delta)}^\Sigma\right)/\left(G_{(\Delta)}^\Sigma Z/G_{(\Delta)}^\Sigma\right)$ is solvable since $\left(G_\Delta^\Sigma/G_{(\Delta)}^\Sigma\right)/\left(G_{(\Delta)}^\Sigma Z/G_{(\Delta)}^\Sigma\right) \cong G_\Delta^\Sigma/G_{(\Delta)}^\Sigma Z$. Therefore, $G_\Delta^\Sigma/G_{(\Delta)}^\Sigma$ is solvable since $G_{(\Delta)}^\Sigma Z/G_{(\Delta)}^\Sigma \cong Z/(Z \cap G_{(\Delta)}^\Sigma)$ is solvable. Then $G_\Delta^\Delta \leqslant A\Gamma L_1(3^h)$ by Theorem 6.13(2.d) since G_Δ^Δ is of affine type.

We know that $\frac{\lambda(\lambda+5)(\lambda+6)}{6} \mid \left|G_\Delta^\Sigma\right|$ since G^Σ acts flag-transitively on $\mathcal{D}^\Sigma$ and $r^\Sigma = \frac{(\lambda+5)}{2}\frac{(\lambda+6)\lambda}{3\eta}$ with $\eta = 1$ or $\lambda+6$ by Proposition 6.8. Then $\frac{\lambda(\lambda+5)(\lambda+6)}{6\eta} \mid (s-1)\,|Aut(S)|$. Now, $s-1 \mid s^m - 1$ and hence $s-1 \mid \frac{1}{4}(\lambda+5)(\lambda-1)$ since $s^m = |\Sigma|$ and $|\Sigma| - 1 = \frac{1}{4}(\lambda+5)(\lambda-1)$. Thus, $\frac{3^h}{\eta}(3^{h-1}-2) \mid |Aut(S)|$, with $\eta = 1$ or 3^h since $\lambda + 6 = 3^h$.

Assume that S is sporadic. Then $h \leqslant 113$ by [27, Table 5.1.C] since $3^{h-1} - 2 \mid |Aut(S)|$. On the other hand, $h > 4$ by Theorem 6.13(2). Now, easy computations show that (6.2) is fulfilled only for $h = 12, 16, 25, 28, 48$ or 79, and in each of these cases $w^t = s^m$ is a prime. So, $m = 1$, but this contradicts Lemma 6.17.

Assume that S is alternating of degree ℓ. If $\eta = 1$ then $3^h \mid |Aut(S)|$, and hence $\ell > 2h$ by [14, Exercise 2.6.8]. Thus $\ell \geqslant 9$ since $h > 4$, and hence $t \geqslant m \geqslant \ell - 2 > 2h - 2$ by [27, Proposition 5.3.7(i)] since $s^m = w^t$ with $m \mid t$. On the other hand $w > 9$ by Lemma 6.16(1), and this implies $|\Sigma| \geqslant 3^{2t}$. Thus $h \geqslant t + 1$ by (6.2), and hence $t > 2h - 2 \geqslant 2t$, a contradiction.

Assume that $\eta=\lambda+6$. Then $\mathcal{D}^{\Sigma}$ is symmetric, and hence $\mathcal{D}^{\Sigma}$ is the development of $\left(\frac{\lambda^2+4\lambda-1}{4}, \frac{\lambda(\lambda+5)}{6}, \frac{(\lambda+6)\lambda}{9}\right)$-difference set by [4, Theorem VI.1.6]. If $-1 \in G_{\Delta}^{\Sigma}$ then $\mathcal{D}^{\Sigma}$ is reversible, and hence $w=2$ by [4, Theorem VI.14.39(a)], a contradiction. Thus, $-1 \notin G_{\Delta}^{\Sigma}$. We have seen that there are no admissible solutions for $h \leqslant 113$, this forces $\ell > 16$. Therefore $(G_{\Delta}^{\Sigma})^{(\infty)} \cong A_{\ell}$ by [27, Theorem 5.1.4(i)], and $\ell > 16$.

If V is the fully deleted permutation module for A_{ℓ}, then $s=w$ and G_{Δ}^{Σ} has one orbit on V of length either $(w-1)\ell$ or $(w-1)\ell(\ell-1)/2$ according to whether $w \nmid \ell$ or $w \mid \ell$, respectively. Then either $(w-1)\ell > w^{(\ell-1)/2}$ or $(w-1)\ell(\ell-1)/2 > w^{(\ell-1)/2}$, respectively, by Lemma 6.10, but both cases give an immediate contradiction. Thus V is not the fully deleted permutation module for A_{ℓ}, and hence $m \geqslant \ell(\ell-5)/4$ by [22, Theorem 7]. Then $(s-1)\,(\ell!)_{w'} > s^{\ell(\ell-5)/8}$ by Lemma 6.10, which leads to $\ell \leqslant 16$ and hence to a contradiction.

Assume that S is a simple Lie group in characteristic w. Let U be any Sylow w-subgroup of G_{Δ}^{Σ}. Then U is isomorphic to a Sylow w-subgroup of G_{Δ} since $G_{(\Sigma)}$ is a $\{2,3\}$-group by Theorem 6.13(2.b) and $w>3$. Then U is isomorphic to a Sylow w-subgroup of G_{Δ}^{Δ} since $|G_{\Delta}|_w = \left|G_{\Delta}^{\Delta}\right|_w$ by Lemma 6.16(3). Hence, U is isomorphic to a w-subgroup of $\Gamma L_1(3^h)$ since $G_{\Delta}^{\Delta} \leqslant A\Gamma L_1(3^h)$ and $w \neq 3$. Now, $(w, 3^h-1)=1$ since $3^h-1 \mid w^t-1$ by (6.2). Therefore $w \mid h$, and hence U is cyclic. Then $S \cong PSL_2(w)$ by [5, Theorem 1 and its proof]. Therefore $(3^{h-1}-2)(3^h-1)$ divides $6(w-1)\,|PGL_2(w)|$, and hence $t=3$ and $3^h-1 \mid 6(w-1)$ since $(3^h-7)(3^h-1)=4(w^t-1)$ with t is odd by Lemma 6.16(1) and $t>1$ by Lemma 6.17. At this point, we reach a contradiction since $w>17$.

Assume that S is a simple Lie group in characteristic w'. We know that $m > R_{w'}(S)$, the smallest degree of a faithful projective representation of S over a field of w'-characteristic. Lower bounds for $R_{w'}(S)$ are given in [27, Theorem 5.3.9]. Also we have R divides $\left|G_{\Delta}^{\Sigma}\right|$ and hence $(s-1)\,|Aut(S)|_{w'}$. Thus

$$(s-1)\,|Aut(S)|_{w'} > R > s^{m/2} \geqslant s^{R_{w'}(S)/2},$$

and hence S is one of the numerical groups listed in [27, Lemma 7.1]. Thus $h \leqslant 55$ since $3^{h-1}-2 \mid |Aut(S)|$, and we have seen that no admissible cases occur for such values of h. □

6.6. The case where G^{Σ} is almost simple

In this section, we assume that the socle of G^{Σ} is a non-abelian simple group acting primitively on Σ. We will denote by L the pre-image of $Soc(G^{\Sigma})$ in G.

Lemma 6.21. *The following facts hold:*

(1) $|\Sigma|$ *is coprime to* 6*;*
(2) $\left|L^{\Sigma}\right| \leqslant 3\left|L_{\Delta}^{\Sigma}\right|^2\left|Out(L^{\Sigma})\right|$*;*
(3) $\left|G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}\right| \mid \left|Out(L^{\Sigma})\right|\left|L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma}\cap G_{(\Delta)}^{\Sigma})\right|$.

Proof. Assertion (1) is immediate since $\lambda \equiv 3 \pmod 6$ by Lemma 6.5.

Since $\left|G_{\Delta}^{\Sigma}\right|$ is divisible by $r^{\Sigma}=\frac{\lambda(\lambda+6)}{3\eta}\frac{\lambda+5}{2}$ with $\eta=1$ or $\lambda+6$, it follows that $3\left|G_{\Delta}^{\Sigma}\right| > |\Sigma| = \left|G^{\Sigma}:G_{\Delta}^{\Sigma}\right|$, and hence $\left|G^{\Sigma}\right| \leqslant 3\left|G_{\Delta}^{\Sigma}\right|^2$.

Since G_{Δ}^{Σ} acts primitively on Σ and $L^{\Sigma} \trianglelefteq G^{\Sigma}$, it follows that $\left|G^{\Sigma}:G_{\Delta}^{\Sigma}\right| = \left|L^{\Sigma}:L_{\Delta}^{\Sigma}\right|$. Then $\left|G_{\Delta}^{\Sigma}:L_{\Delta}^{\Sigma}\right| = \left|G^{\Sigma}:L^{\Sigma}\right|$, and hence $\left|G_{\Delta}^{\Sigma}:L_{\Delta}^{\Sigma}\right| \mid \left|Out(L^{\Sigma})\right|$. Thus, $\left|L^{\Sigma}\right| \leqslant 3\left|L_{\Delta}^{\Sigma}\right|^2\left|Out(L^{\Sigma})\right|$, which is (2).

Clearly, $\left|G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma} : L_{\Delta}^{\Sigma}G_{(\Delta)}^{\Sigma}/G_{(\Delta)}^{\Sigma}\right| = \left|G_{\Delta}^{\Sigma} : L_{\Delta}^{\Sigma}G_{(\Delta)}^{\Sigma}\right|$. Thus, it divides $\left|G_{\Delta}^{\Sigma}:L_{\Delta}^{\Sigma}\right|$ and hence $\left|Out(L^{\Sigma})\right|$. Then $\left|G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}\right| \mid \left|Out(L^{\Sigma})\right|\left|L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma}\cap G_{(\Delta)}^{\Sigma})\right|$ since $L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma}\cap G_{(\Delta)}^{\Sigma}) \cong L_{\Delta}^{\Sigma}G_{(\Delta)}^{\Sigma}/G_{(\Delta)}^{\Sigma}$, which is (3). □

Remark 6.22. We have seen that $\frac{\lambda+5}{2}$ is divisible by a primitive prime divisor of $z^{h-1}-1$ or 3^h-1 according as case (1) or (2) of Theorem 6.13 occurs, respectively. Throughout the remainder of the paper we denote such primitive prime divisor by ϑ for both cases, it will be clear from the context to which case we refer to. Moreover, we simply say that ϑ is a *primitive prime divisor of* $\frac{\lambda+5}{2}$.

Lemma 6.23. $\vartheta \geqslant 2h+1$.

Proof. In case (1) of Theorem 6.13, $h>3$, h is even and $h \neq 4$ when $z=2$ and h is divisible by 3 when $z=4$. Now, $h-1 \mid \vartheta-1$ by [27, Proposition 5.2.15(ii)]. Then $\vartheta \geqslant 3h-2$ by [20, Theorem 3.9(b)(c)]. In particular, $\vartheta \geqslant 2h+1$ since $h>3$.

In case (2) of Theorem 6.13, $h>4$ and $h \neq 6$. Further, $h \mid \vartheta-1$ by [27, Proposition 5.2.15(ii)]. Then $\vartheta \geqslant 2h+1$ by [20, Theorem 3.9(b)]. □

Lemma 6.24. ϑ *does not divide* $\left|Out(L^{\Sigma})\right|$.

Proof. Assume that $\vartheta \mid \left|Out(L^{\Sigma})\right|$. Then either $\vartheta \mid f$, or $L^{\Sigma} \cong PSL_n(q)$ and $\vartheta \mid (n, q-1)$, or $PSU_n(q)$ and $\vartheta \mid (n, q+1)$ by [27, Table 5.1.B].

Assume that the former occurs. Note that, $f \geqslant \vartheta \geqslant 2h+1$ by Lemma 6.23. Then $q = p^f \geqslant p^{2h+1} \geqslant 3^h p$, and hence $|\Sigma| < \frac{(\lambda+2)^2}{4} \leqslant \frac{(q/p+2)^2}{4}$. Therefore, $P(L^{\Sigma}) \leqslant |\Sigma| < \frac{(q/p+2)^2}{4}$, where $P(L^{\Sigma})$ denotes the minimal degree of the non-trivial transitive permutation representations of L^{Σ}. Then $L^{\Sigma} \cong PSL_2(q)$ with $q > 11$ by [27, Theorem 5.2.2] and [48–50] since $f \geqslant 2h+1$ and $h > 3$ in cases (1) and (2) of Theorem 6.13. Now, if G^{Σ}_{Δ} is the stabilizer of a point of $PG_1(q)$, then $|\Sigma| = q+1$, and hence $\frac{\lambda-1}{2} \cdot \frac{\lambda+5}{2} = |\Sigma| - 1 = q$, which is not the case since $\left(\frac{\lambda-1}{2}, \frac{\lambda+5}{2}\right) = 1$ being $\lambda \equiv 0 \pmod 3$. Then q is odd and L^{Σ}_{Δ} is dihedral, A_4, S_4, A_5 or $PGL_2(q^{1/2})$ by [30]. For each of these groups, it is easy to check $|\Sigma| = \left|L^{\Sigma} : L^{\Sigma}_{\Delta}\right|$ is greater than $\frac{(q/p+2)^2}{4}$ since q is odd and $q > 11$. Thus, $\vartheta \nmid f$.

Assume that $L^{\Sigma} \cong PSL_n(q)$ and $\vartheta \mid (n, q-1)$. Then $n \geqslant 2h+1$ and $q > 2h+1$ by Lemma 6.23. Moreover, $h > 3$ by Theorem 6.13. Then

$$q^{n-1} < \frac{q^n - 1}{q-1} \leqslant |\Sigma| < \lambda^2 \tag{6.4}$$

by [27, Theorem 5.2.2]. In case (1) of Theorem 6.13, $L = L^{\Sigma} \cong PSL_n(q)$, $PSL_h(z) \trianglelefteq L^{\Delta}_{\Delta}$ with $h > 3$ and $(h, z) \neq (4, 2)$ and $\lambda < z^h$. Then [30] together [27, Propositions 4.1.17(II), 4.1.22(II), 4.2.9(II) and 4.5.3(II)] force L_{Δ} to be parabolic and $q = z$. However, this is impossible since $\lambda < z^h$, $n \geqslant 2h+1$ and (6.4) imply $q^{2h} \leqslant q^{n-1} < z^{2h}$ and hence $q < z$.

In case (2) of Theorem 6.13, one has $\lambda < 3^h$. Hence, (6.4) implies $7^{2h} < q^{2h} < q^{n-1} < 3^{2h}$ since $n \geqslant 2h+1$ and $q > 2h+1$ with $h > 3$, which is clearly impossible.

Finally, the case $L^{\Sigma} \cong PSU_n(q)$ and $\vartheta \mid (n, q+1)$ is ruled out similarly. Indeed, $n \geqslant 2h+1 > 7$ and $q > 2h-1 > 5$ since $h > 3$, and hence $P(L^{\Sigma}) > q^n$ by [27, Theorem 5.2.2]. □

Proposition 6.25. *The following facts hold:*

(1) *ϑ divides $\left|L^{\Sigma}_{\Delta}/(L^{\Sigma}_{\Delta} \cap G^{\Sigma}_{(\Delta)})\right|$;*
(2) *Each L^{Σ}_{Δ}-orbit on $\Sigma \setminus \{\Delta\}$ is divisible by ϑ.*

Proof. It follows from Theorem 6.13(1.c),(2.c) that ϑ divides $\left|G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)}\right|$. Then (1) immediately follows from Lemma 6.21(3) since ϑ does not divide $\left|Out(L^{\Sigma})\right|$ by Lemma 6.24.

Let $\Delta' \in \Sigma \setminus \{\Delta\}$. Then $\left|(\Delta')^{L^{\Sigma}_{\Delta}}\right|$ divides $\left|(\Delta')^{G^{\Sigma}_{\Delta}}\right|$ since $L^{\Sigma}_{\Delta} \trianglelefteq G^{\Sigma}_{\Delta}$, and the ratio is a divisor of $\left|G^{\Sigma}_{\Delta} : L^{\Sigma}_{\Delta}\right|$. We have seen in the proof Lemma 6.21(2) that $\left|G^{\Sigma}_{\Delta} : L^{\Sigma}_{\Delta}\right|$ divides $\left|Out(L^{\Sigma})\right|$. Thus, $\left|(\Delta')^{G^{\Sigma}_{\Delta}}\right|$ divides $\left|(\Delta')^{L^{\Sigma}_{\Delta}}\right| \left|Out(L^{\Sigma})\right|$. This fact implies (2) since ϑ divides $\left|(\Delta')^{G^{\Sigma}_{\Delta}}\right|$ by Lemma 6.10(2) and ϑ does not divide $\left|Out(L^{\Sigma})\right|$ by Lemma 6.24. □

Lemma 6.26. *L^{Σ} in neither sporadic nor alternating.*

Proof. Assume that L^{Σ} is sporadic and let $\Delta \in \Sigma$. If L^{Σ}_{Δ} is non-maximal in L^{Σ}, then $G^{\Sigma} = L^{\Sigma}.Z_2$ and both G^{Σ} and G^{Σ}_{Δ} are listed in [52, Table 1] (see also [53]). However, none of them fulfills $(|\Sigma|, 6) = 1$, and hence they are ruled out by Lemma 6.21(1). Thus, $|\Sigma| = \left|L^{\Sigma} : L^{\Sigma}_{\Delta}\right|$ with L^{Σ}_{Δ} maximal in Σ. Now, it is easy to check in [9,53,13] that there are no sporadic groups with a primitive permutation representation degree of the form $\frac{(3^h-4)^2-5}{4}$, or with a quotient group of a maximal subgroup containing $PSL_h(z)$ with z power of 2, $h > 3$ and $(h, z) \neq (4, 2), (5, 2)$ as a normal subgroup. Thus, L^{Σ} is not sporadic since it contradicts Theorem 6.13.

Assume that L^{Σ} is alternating. Then either $A_x \times A_{d-x} \trianglelefteq G^{\Sigma}_{\Delta} \leqslant S_x \times S_{d-x}$ with $1 \leqslant x < d/2$, or $A_y \wr A_{d/y} \trianglelefteq G^{\Sigma}_{\Delta} \leqslant S_y \wr S_{d/y}$ with $y, d/y > 1$ by [30] since $(|\Sigma|, 6) = 1$. In both cases, no quotient group of G^{Σ}_{Δ} contain $PSL_h(z)$ with z a power of 2, $h > 3$ and $(h, z) \neq (4, 2)$ as a normal subgroup. Then $\lambda = 3^h - 6$, $h > 4$, and a quotient group of G^{Σ}_{Δ} is either solvable and divisible by a primitive prime divisor of $3^h - 1$, or contains a normal subgroup isomorphic to $PSL_{h/e}(3^e)$ with $h/e \geqslant 2$ and $(h/e, e) \neq (2, 2)$ or $PSp_{h/e}(3^e)$ with h/e even, $h/e \geqslant 2$ and $(h/e, e) \neq (2, 2)$ by Theorem 6.13. Clearly, both cases cannot occur. □

Lemma 6.27. *L^{Σ} is not isomorphic to $PSL_2(q)$.*

Proof. Assume that $L^{\Sigma} \cong PSL_2(q)$. The same argument used in Lemma 6.24 shows that q is odd and L^{Σ}_{Δ} is dihedral, A_4, S_4, A_5 or $PGL_2(q^{1/2})$. In each of these cases p divides $\left|L^{\Sigma} : L^{\Sigma}_{\Delta}\right|$, and hence p divides $|\Sigma|$. Then $p > 3$ by Lemma 6.21(1). Actually, L^{Σ}_{Δ} cannot be any of the groups A_4, S_4 and A_5 by Lemma 6.23 and Proposition 6.25(1).

Assume that $L^{\Sigma}_{\Delta} \cong PGL_2(q^{1/2})$. If $PSL_2(q^{1/2}) \trianglelefteq L^{\Sigma}_{\Delta} \cap G^{\Sigma}_{(\Delta)}$, then $L^{\Sigma}_{\Delta}/(L^{\Sigma}_{\Delta} \cap G^{\Sigma}_{(\Delta)}) \leqslant Z_2$ is not divisible by ϑ, thus contradicting Proposition 6.25(1). Therefore, $L^{\Sigma}_{\Delta}/(L^{\Sigma}_{\Delta} \cap G^{\Sigma}_{(\Delta)}) \cong PGL_2(q^{1/2})$, and hence $PGL_2(q^{1/2}) \trianglelefteq G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)}$ since $L^{\Sigma}_{\Delta}/(L^{\Sigma}_{\Delta} \cap G^{\Sigma}_{(\Delta)}) \cong L^{\Sigma}_{\Delta} G^{\Sigma}_{(\Delta)}/G^{\Sigma}_{(\Delta)} \trianglelefteq G^{\Sigma}_{\Delta}/G^{\Sigma}_{(\Delta)}$. However, this is impossible by Theorem 6.13 since $p > 3$ and $h > 3$.

Assume that L^{Σ}_{Δ} is dihedral. Then

$$\frac{q(q^2-1)}{2} \leqslant 3 \cdot (q \pm 1)^2 \cdot 2f,$$

by Lemma 6.21(2). Thus $q(q^2-1) \leqslant 12f(q \pm 1)^2$, and hence $q \leqslant 24f$. Thus either $q = 25$, or $q = p \leqslant 23$ since q is odd and $q > 3$. However, none of these cases yields a $|\Sigma|$ of the form $\frac{\lambda^2+4\lambda-1}{4}$ with $\lambda = 3^h - 6$ and $h > 4$. $\square$

Lemma 6.28. *L^{Σ}_{Δ} does not lie in a maximal $\mathcal{C}_1$-subgroup of L^{Σ}.*

Proof. Assume that L^{Σ}_{Δ} is contained in maximal parabolic subgroup M of L^{Σ}. Suppose that L^{Σ} is not isomorphic to one of the following groups:

(1) $PSL_n(q)$;
(2) $P\Omega^{+}_n(q)$ with q even and $n/2$ odd, and $M \cong P_{n/2}, P_{n/2-1}$;
(3) $E_6(q)$.

Then q is even by [30]. Moreover, there is a unique M-orbit $\mathcal{O}$ of size a power of 2 by [46, Lemma 2.6 and its proof]. Clearly, $\Delta \notin \mathcal{O}$ and $\mathcal{O}$ is a union of some non-trivial L^{Σ}_{Δ}-orbits. The ϑ divides $|\mathcal{O}|$ by Proposition 6.25(2), a contradiction. Thus, L^{Σ} is isomorphic to any of the groups $PSL_n(q)$, $P\Omega^{+}_n(q)$ with q even and $n/2$ odd, or $E_6(q)$. Now, we may apply the same argument as in [46, (1.a) in Section 5, (1) in Section 7 and Section 8] for linear spaces with our $|\Sigma|$ and R and in the role of v and r, respectively, by Lemma 6.10, and we see that only the following cases are admissible:

(a) $L^{\Sigma} \cong PSL_n(q)$ and one of the following holds:
  (i) $M = P_1$ and $|\Sigma| = \frac{q^n-1}{q-1}$. Moreover, G^{Σ} is a point-2-transitive automorphism group of $\mathcal{D}^{\Sigma}$;
  (ii) $M = P_2$, n is odd, $|\Sigma| = \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)}$ and $R = \frac{q(q^{n-2}-1)}{t(q-1)}\left(q+1, \frac{n-3}{2}\right)$ with $t \leqslant 2q$. Moreover, G^{Σ} is a primitive point-rank 3 automorphism group of $\mathcal{D}^{\Sigma}$.
(b) $L^{\Sigma} \cong P\Omega^{+}_{10}(q)$, $|\Sigma| = (q^4+1)(q^3+1)(q^2+1)(q+1)$ and $R = \frac{q(q^5-1)}{q-1}$.

Assume that case (a.i) holds. If $\eta = \lambda + 6$ then $\mathcal{D}^{\Sigma}$ is symmetric, and hence $\mathcal{D}^{\Sigma} \cong PG_{n-1}(q)$ and $\mathcal{B}^{\Sigma}$ is the set of hyperplanes of $PG_{n-1}(q)$ by [24]. Thus $k^{\Sigma} = \frac{q^{n-1}-1}{q-1}$, and hence $\frac{q^n-1}{q-1} < \frac{3}{2}\frac{q^{n-1}-1}{q-1}$ since $|\Sigma| < \frac{3}{2}k^{\Sigma}$, being $|\Sigma| = \frac{(\lambda^2+4\lambda-1)}{4}$ and $k^{\Sigma} = \frac{\lambda(\lambda+5)}{6}$. Therefore $(2q-3)q^{n-1} - 5 < 0$, which does not have admissible solutions since $(q,n) \neq (2,2)$, being L^{Σ} non-abelian simple. Thus $\eta = 1$, and hence $G^{\Sigma} = G$ by Corollary 6.9.

Assume that case (1) of Theorem 6.13 occurs. Then $z = q$ and $h = n-1$ by [27, Proposition 4.1.17(II)]. Therefore ϑ divides $|\Sigma| - 1 = \frac{\lambda+5}{2}\frac{\lambda-1}{2} = q\frac{q^{n-1}-1}{q-1}$, and hence $2(n-1)+1 \leqslant n-1$ by Lemma 6.23 and [27, Proposition 5.2.15(i)], a contradiction.

Assume that case (2) of Theorem 6.13 occurs. Then G^{Δ}_{Δ} contains $(Z_3)^h$, $h > 4$, as a normal subgroup. On the other hand, G_{Δ} contains a normal subgroup H isomorphic to $\left[q^{n-1}\right] : SL_{n-1}(q)$ and G_{Δ}/H is isomorphic to a quotient group of $Z_{q-1}.Z_{(n,q-1)}.Z_f$ by [27, Proposition 4.1.17(II)]. It follows that $h \leqslant 3$, whereas $h > 4$.

Assume that case (a.ii) holds. If $\eta = \lambda + 6$, then $\mathcal{D}^{\Sigma}$ is symmetric. However, this is impossible by [11] since $L^{\Sigma} \cong PSL_n(q)$ with n is odd. Thus $\eta = 1$, and hence $G^{\Sigma} = G$ by Corollary 6.9.

Assume that case (1) of Theorem 6.13 occurs. Then $z = q$ and $h = n-2$ by [27, Proposition 4.1.17(II)] since $h > 3$. Therefore ϑ divides

$$|\Sigma| - 1 = \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)} - 1 = \frac{q\left(q^{n-2}-1\right)}{(q+1)(q-1)^2}\left((q-1)(q+1)^2 + q^3(q^{n-3}-1)\right),$$

and hence $2(n-2)+1 \leqslant 2$ by Lemma 6.23 and [27, Proposition 5.2.15(i)]. So $n = 2$, which is not the case since n is odd.

Assume that case (2) of Theorem 6.13 occurs. As above, G^{Δ}_{Δ} contains $(Z_3)^h$, $h > 4$, as a normal subgroup. On the other hand, G_{Δ} contains a normal subgroup K isomorphic to $\left[q^{2(n-2)}\right] : (SL_2(q) \circ SL_{n-2}(q))$, and G_{Δ}/K is isomorphic to a quotient group of $(Z_{q-1} \times Z_{q-1}).Z_{(n,q-1)}.Z_f$ by [27, Proposition 4.1.17(II)], and this leads to a contradiction since $h > 4$.

In case (b), since $R = \frac{\lambda+5}{2} = \frac{q(q^5-1)}{q-1}$ and $|\Sigma| = \frac{\lambda^2+4\lambda-1}{4}$, one has

$$|\Sigma| = R^2 - 3R + 1 = \left(\frac{q(q^5-1)}{q-1} - 5\right)^2 - 3\left(\frac{q(q^5-1)}{q-1} - 5\right) + 1,$$

which compared to $|\Sigma|=(q^4+1)(q^3+1)(q^2+1)(q+1)$ leads to $q^9+2q^8+2q^7+3q^6-q^5-2q^4-3q^3-3q^2-4q=0$, which does not have admissible solutions.

Finally, assume that L^Σ is one of the groups $PSp_n(q)$, $PSU_n(q)$ or $P\Omega_n^\varepsilon(q)$, and that L^Σ_Δ is a non-degenerate subspace of $PG_n(q)$. Again, Lemma 6.10 allows us to argue as in [46, Sections 3–6] with $|\Sigma|$ and R and in the role of v and r, respectively, and we see that the unique admissible case is $L^\Sigma\cong PSU_n(q)$, L^Σ_Δ is the stabilizer of a non-isotropic point of $PG_{n-1}(q^2)$, $|\Sigma|=\frac{q^{n-1}(q^n-(-1)^n)}{q+1}$ and $R\mid(q+1)(q^{n-1}-(-1)^{n-1})$. Actually, n is odd since both $|\Sigma|$ and q are odd. Further, $R\mid|\Sigma|-1$ and hence $R\mid\left((q+1)(q^{n-1}-1),|\Sigma|-1\right)$. Then $R\mid\frac{q^{n-1}-1}{q+1}$ since $|\Sigma|-1=\frac{q^{n-1}-1}{q+1}(q^n+q+1)$, but this contradicts $R^2>|\Sigma|$. □

Lemma 6.29. *L^Σ_Δ does not lie in a maximal $\mathcal{C}_i$-subgroup of L^Σ with $i=2,5$.*

Proof. Assume that $L^\Sigma_\Delta\in\mathcal{C}_2(L^\Sigma)\cup\mathcal{C}_5(L^\Sigma)$ and let M be a maximal group of L^Σ containing L^Σ_Δ. Note that, p is odd by [30]. Further, $p\mid|\Sigma|$ by Lemma 6.28 and [46, Lemma 2.3], and hence $p>3$ by Lemma 6.21(1).

Assume that $L^\Sigma_\Delta\in\mathcal{C}_5(L^\Sigma)$. Then L^Σ_Δ normalizes a classical group over $GF(q^{1/m})$ with m. Also, $\left|L^\Sigma\right|\leqslant 3|M|^2\left|Out(L^\Sigma)\right|\leqslant|M|^3$ by Lemma 6.21(2), and hence one of the following holds by [1, Propositions 4.7, 4.17, 4.27 and 4.23]:

(1) $L^\Sigma\cong PSL_n(q)$ and M is a $\mathcal{C}_5$-subgroup of type $GL_n(q^{1/3})$;
(2) $L^\Sigma\cong PSU_n(q)$ and M is a $\mathcal{C}_5$-subgroup either of type $GU_n(q^{1/3})$.

In case (1), we have

$$q^{n^2-2}\leqslant\left|L^\Sigma\right|\leqslant 3|M|^2\left|Out(L^\Sigma)\right|\leqslant\frac{6(q-1,n)f(q-1)^2\left|PGL_n(q^{1/3})\right|^2}{(q-1,n)^2\left(q^{1/3}-1,\frac{q-1}{(q-1,n)}\right)^2}\tag{6.5}$$

by [27, Proposition 4.5.3(II)] and by [1, Corollary 4.3(i)], hence $q^{n^2-2}<6fq^{\frac{2n^2}{3}+2}$ again by [1, Corollary 4.3(i)]. Then $p^{f\frac{n^2-12}{3}}\leqslant 6f$, and hence $n=3$ since $n>2$ by Lemma 6.27, f is divisible by 3, $f\geqslant 1$ and $p>3$. Now, it is easy to verify that (6.5) is not fulfilled for $n=3$.

In case (2), we may assume that $n\geqslant 3$ since $PSU_2(q)\cong PSL_2(q)$ cannot occur by Lemma 6.27. Then

$$q^{n^2-3}\leqslant\left|L^\Sigma\right|\leqslant 3|M|^2\left|Out(L^\Sigma)\right|\leqslant\frac{6(q+1,n)f(q+1)^2\left|PGU_n(q^{1/3})\right|^2}{(q+1,n)^2\left(q^{1/3}+1,\frac{q+1}{(q+1,n)}\right)^2}\tag{6.6}$$

by [27, Proposition 4.5.3(II)] and [1, Corollary 4.3(ii)], and hence $q^{n^2-3}\leqslant 6fq^{\frac{2(n^2-1)}{3}+3}$ again by [1, Corollary 4.3(ii)]. Then $q^{\frac{n^2-16}{3}}\leqslant 6f$, and hence $n=3$ or 4 since $n\geqslant 3$ and f is divisible by 3, $f>1$ and $p>3$. Actually, (6.6) is not fulfilled for $n=4$, hence $n=3$. In this case, $\left|L^\Sigma\right|\leqslant 3|M|^2\left|Out(L^\Sigma)\right|$ becomes

$$q^3(q^3+1)(q^2-1)\leqslant 6f(q+1,3)^4q^2(q+1)^2(q^{1/3}-1)^2$$

and hence $q(q^2-q+1)(q^{2/3}+q^{1/2}+1)\leqslant 486f$, which has no admissible solutions since f is divisible by 3, $f>1$ and $p>3$.

Assume that $L^\Sigma_\Delta\in\mathcal{C}_2(L^\Sigma)$. Again by [1, Propositions 4.7, 4.17, 4.27 and 4.23], and bearing in mind that $p>3$, and the L^Σ_Δ-invariant decomposition of consists of non-degenerate isometric subspaces when L^Σ is not $PSL_n(q)$ by [30], one of the following holds:

(1) $L^\Sigma\cong PSL_n(q)$ and M is a $\mathcal{C}_2$-subgroup of type $GL_{n/t}(q)\wr S_t$, where $t=2$, or $t=3$ and either $q=5$ and n odd, or $(n,q)=(3,11)$;
(2) $L^\Sigma\cong PSU_n(q)$ and M is a $\mathcal{C}_2$ subgroup of type $GU_{n/t}(q)\wr S_t$, where $t=2$, or $t=3$ and either $q=5$ and $(n,6)=3$, or $q=7$ and n odd, or $q=13$ and $(n,14)=1$, or $t=n=4$ and $q=5$;
(3) $L^\Sigma\cong PSp_n(q)$ and M is a $\mathcal{C}_2$ subgroup of type $Sp_{n/t}(q)\wr S_t$, where either $t\leqslant 3$, or $(t,n)=(4,8)$;
(4) $L^\Sigma\cong P\Omega_n^\varepsilon(q)$ and M is a $\mathcal{C}_2$ subgroup of type $O^{\varepsilon'}_{n/t}(q)\wr S_t$, where $t=2$, or $t=n=7$ and $q=5$.

Note that, $L^\Sigma\cong P\Omega_4^+(5)$ is ruled out by Lemma 6.21(1), hence $L^\Sigma_\Delta=M$ in the remaining cases of (1)–(4) by [27, Tables 3.5.H–3.5.I].

Suppose that $L^\Sigma_\Delta/(L^\Sigma_\Delta\cap G^\Sigma_{(\Delta)})$ is solvable. Then only (1) or (2) holds with $\vartheta\mid(q\mp 1,n/2)$, respectively, by [27, Propositions 4.2.9(II)–4.2.11(II)] since ϑ divides $\left|L^\Sigma_\Delta/(L^\Sigma_\Delta\cap G^\Sigma_{(\Delta)})\right|$ by Proposition 6.25(1). Now, $\vartheta\mid(q\mp 1,n/2)$ implies $\vartheta\mid\left|Out(L^\Sigma)\right|$, which is contrary to Lemma 6.24.

Suppose that $L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma} \cap G_{(\Delta)}^{\Sigma})$ is non-solvable. Then $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is non-solvable by since $L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma} \cap G_{(\Delta)}^{\Sigma}) \cong L_{\Delta}^{\Sigma}G_{(\Delta)}^{\Sigma}/G_{(\Delta)}^{\Sigma} \trianglelefteq G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$. Then a quotient group of $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is almost simple with socle isomorphic either to $PSL_h(z)$ with $h > 3$ and z even or to $PSL_{h/e}(3^e)$ with $h/e \geqslant 2$, or to $PSp_{h/e}(3^e)$ with h/e even and $h/e \geqslant 2$ by Theorem 6.13(1.d),(2.d). Further, $(h/e, e) \neq (2,1), (2,2), (3,2), (6,1)$ in both latter cases. However, the above three possibilities for $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ are excluded by [27, Propositions 4.2.9(II)–4.2.11(II)] since $p > 3$. □

Lemma 6.30. *L^{Σ} is not a simple exceptional group of Lie type.*

Proof. Assume that $L^{\Sigma} = L(q)$ is a simple exceptional group of Lie type. Let M be a maximal group of L^{Σ} containing L_{Δ}^{Σ}. By [30], either $M = N_{L^{\Sigma}}(L(q^{1/s}))$ with s an odd prime, or M is one of the groups listed in [30, Table 1] since L_{Δ}^{Σ} is not contained in a maximal parabolic subgroup of L^{Σ} by Lemma 6.28. In the former case, $\left(L^{\Sigma}, M\right)$ is either $(E_6(q), N_{E_6(q)}(E_6(q^{1/3})))$ or $({}^2E_6(q), N_{{}^2E_6(q)}({}^2E_6(q^{1/3})))$ by [1, Theorem 5] since $\left|L^{\Sigma}\right| \leqslant 3\,|M|^2 \left|Out(L^{\Sigma})\right| \leqslant |M|^3$ by Lemma 6.21(2). However, in none of the cases $L^{\Sigma} \leqslant 6(3, q \pm 1)s\,|M|^2$ is fulfilled. Thus, M is one of the groups listed in [30, Table 1]. Again, Lemma 6.21(2) implies $L^{\Sigma} \leqslant |M|^3$, and hence we may preliminary filter the groups in [30, Table 1] with respect [1, Theorem 5]. The admissible groups are then filtered respect to $L^{\Sigma} \leqslant 3\,|M|^2 \left|Out(L^{\Sigma})\right|$, and we obtain the following admissible cases:

(1) $L^{\Sigma} \cong E_7(q)$ and $M \cong 2 \cdot \left(PSL_2(q) \times P\Omega_{12}^{+}(q)\right) \cdot 2$;
(2) $L^{\Sigma} \cong {}^2E_6(q)$ and $M \cong (4, q+1).\left(P\Omega_{10}^{-}(q) \times \frac{q+1}{(3,q+1)(4,q+1)}\right).(4, q+1)$;
(3) $L^{\Sigma} \cong {}^3D_4(q)$ and $M \cong G_2(q)$;
(4) $L^{\Sigma} \cong {}^2G_2(q)$, $q = 3^m$, m odd and $m > 1$, and $M \cong Z_2 \times PSL_2(q)$;
(5) $L^{\Sigma} \cong G_2(q)$ and $M \cong SL_3(q) : Z_2$ or $SU_3(q) : Z_2$ according as $q \equiv \pm 1 \pmod 4$
(6) $L^{\Sigma} \cong G_2(3)$ and $M \cong 2^3.SL_3(2)$;
(7) $L^{\Sigma} \cong F_2(q)$ and $M \cong 2.\Omega_9(q)$;
(8) $L^{\Sigma} \cong F_2(q)$ and $M \cong 2^2.P\Omega_8^{+}(q).S_3$;

Note that, case $L^{\Sigma} \cong E_6(q)$ and $(4, q-1).\left(P\Omega_{10}^{+}(q) \times \frac{q-1}{(3,q-1)(4,q-1)}\right).(4, q-1) \trianglelefteq G_{\Delta}^{\Sigma}$ with G_{Δ}^{Σ} containing a graph automorphism of order 2 is excluded from the previous list. Indeed, in this case, $L_{\Delta}^{\Sigma} = (4, q-1).\left(P\Omega_{10}^{+}(q) \times \frac{q-1}{(3,q-1)(4,q-1)}\right).(4, q-1)$ lies in a maximal parabolic subgroup of $E_6(q)$ of type D_5, and this is impossible by Lemma 6.28.

Case (2) is ruled out by Lemma 6.21(1). Indeed,

$$|\Sigma| = \left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right| = q^{16}\frac{q^9+1}{q+1}(q^8+q^4+1)$$

which is divisible by 3. Now, consider cases (1) and (3)–(8). Then $L_{\Delta}^{\Sigma} = M$ by [31]. Note that, L_{Δ}^{Σ} does not contain a quotient group as in Theorem 6.13(1). Thus $\lambda = 3^h - 6$ with $h > 4$ and $h \neq 6$ Theorem 6.13(2).

Let ϑ a primitive prime divisor of $\frac{\lambda+5}{2}$, then ϑ divides $\left|L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma} \cap G_{(\Delta)}^{\Sigma})\right|$ by Proposition 6.25(1). Thus $L_{\Delta}^{\Sigma}/L_{(\Delta)}^{\Sigma}$ is non-solvable in (1) and (3)–(8). Then $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is non-solvable, and hence a quotient subgroup of $G_{\Delta}^{\Sigma}/G_{(\Delta)}^{\Sigma}$ is almost simple with socle isomorphic to one of the groups $PSL_{h/e}(3^e)$ or $PSp_{h/e}(3^e)$ with $h/e \geqslant 2$ and $(h/e, e) \neq (2,1), (2,2)$ as a normal subgroup by Theorem 6.13(2.d) since q is odd. Then a quotient subgroup of $L_{\Delta}^{\Sigma}/(L_{\Delta}^{\Sigma} \cap G_{(\Delta)}^{\Sigma})$ contains one of the groups $PSL_{h/e}(3^e)$ or $PSp_{h/e}(3^e)$ with $h/e \geqslant 2$ and $(h/e, e) \neq (2,1), (2,2)$ as a normal subgroup since $G_{\Delta}^{\Sigma}/L_{\Delta}^{\Sigma}$ is solvable (see the proof of Lemma 6.21(3)). Then only (1) and (4) are admissible, and both with $q = 3^m$. Therefore $q \mid \left|L^{\Sigma} : L_{\Delta}^{\Sigma}\right|$, and hence q divides $|\Sigma|$, which is not the case by Lemma 6.21(1). □

Theorem 6.31. *If $\lambda > 3$, then G^{Σ} is not almost simple.*

Proof. Assume that G^{Σ} is almost simple. Then G^{Σ} is one of the groups listed in [30]. However, only the following cases are admissible as consequence of Lemmas 6.26–6.30:

(1) L^{Σ} is $P\Omega_7(q)$ or $P\Omega_8^{+}(q)$, q is prime and $q \equiv \pm 3 \pmod 8$, L_{Δ}^{Σ} is $\Omega_7(2)$ or $\Omega_8^{+}(2)$, respectively;
(2) $L^{\Sigma} \cong P\Omega_8^{+}(q)$, q is prime and $q \equiv \pm 3 \pmod 8$, G^{Σ} contains a triality automorphism of L^{Σ} and L_{Δ}^{Σ} is $2^3 \cdot 2^6 \cdot PSL_3(2)$;
(3) $L^{\Sigma} \cong PSU_3(5)$ and $L_{\Delta}^{\Sigma} \cong M_{10}$.

Assume that L^{Σ} is $P\Omega_7(q)$, q prime and $q \equiv \pm 3 \pmod 8$, and L_{Δ}^{Σ} is $\Omega_7(2)$. Then Lemma 6.21(2) implies

$$\frac{1}{4}q^{21} < |P\Omega_7(q)| \leqslant 3\,|\Omega_7(2)|^2 \cdot 2 = 2^{19} \cdot 3^9 \cdot 5^2 \cdot 7^2$$

by [1, Corollary 4.3(iv)] since q is a prime. Hence $q = 3$ since $q \equiv \pm 3 \pmod 8$. However, $|\Sigma| = |P\Omega_7(3) : \Omega_7(2)| = 3^5 \cdot 13$ is excluded since it is divisible by 3. The remaining cases in (1) and in (2) are ruled out similarly.

Finally, assume that $L^{\Sigma} \cong PSU_3(5)$ and $L^{\Sigma}_{\Delta} \cong M_{10}$. Then $|\Sigma| = \frac{\lambda^2+4\lambda-1}{4} = 5^2 \cdot 7$, a contradiction. □

Proof of Theorem 6.1. Proposition 6.5 and Theorems 6.11, 6.20 and 6.31 force $\lambda = 3$. At this point, the assertion follows from [40, Corollary 1.2]. □

Proof of Theorem 1.1. The result is an immediate consequence of Theorems 3.1 and 6.1. □

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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