Article

On Multidimensional Congestion Games †

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Abstract: We introduce multidimensional congestion games, that is, congestion games whose set of players is partitioned into $d + 1$ clusters $C_0, C_1, \ldots, C_d$. Players in C_0 have full information about all the other participants in the game, while players in C_i, for any $1 \le i \le d$, have full information only about the members of $C_0 \cup C_i$ and are unaware of all the others. This model has at least two interesting applications: (*i*) it is a special case of graphical congestion games induced by an undirected social knowledge graph with independence number equal to d, and (*ii*) it represents scenarios in which players have a type and the level of competition they experience on a resource depends on their type and on the types of the other players using it. We focus on the case in which the cost function associated with each resource is affine and bound the price of anarchy and stability as a function of d with respect to two meaningful social cost functions and for both weighted and unweighted players. We also provide refined bounds for the special case of $d = 2$ in presence of unweighted players.

Keywords: congestion games; pure Nash equilibrium; potential games; price of anarchy; price of stability

1. Introduction

Congestion games [1–4] are, perhaps, the most famous class of non-cooperative games due to their capability to model several interesting competitive scenarios, while maintaining nice properties. In these games, there is a set of players sharing a set of resources. Each resource has an associate cost function which depends on the number of players using it (the so-called *congestion*). Players aim at choosing subsets of resources so as to minimize the sum of the resource costs.

Congestion games were introduced by Rosenthal in Reference [2]. He proved that each such a game admits a bounded potential function whose set of local minima coincides with the set of *pure Nash equilibria* [5] of the game, that is, strategy profiles in which no player can decrease her cost by unilaterally changing her strategic choice. This existence result makes congestion games particularly appealing especially in all those applications in which pure Nash equilibria are elected as the ideal solution concept.

In these contexts, the study of inefficiency due to selfish and non-cooperative behavior has affirmed as a fervent research direction. To this aim, the notions of *price of anarchy* [6] and *price of stability* [7] are widely adopted. The price of anarchy (resp. stability) compares the performance of the worst (resp. best) pure Nash equilibrium with that of an optimal cooperative solution.

Congestion games with unrestricted cost functions are general enough to model the Prisoner's Dilemma game, whose unique pure Nash equilibrium is known to perform arbitrarily bad with

respect to the solution in which all players cooperate. Hence, in order to deal with significative bounds on the prices of anarchy and stability, some kind of regularity needs to be imposed on the cost functions associated with the resources. To this aim, lot of research attention has been devoted to the case of polynomial cost functions [8–17], and more general latency functions verifying some mild assumptions [10,18,19]. Among these, the particular case of affine functions occupies a predominant role. In fact, as shown in References [20–22], they represent the only case, together with that (perhaps not particularly meaningful) of exponential cost functions, for which *weighted congestion games* [20], that is the generalization of congestion games in which each player has a weight and the congestion of a resource becomes the sum of the weights of its users, still admit a potential function.

1.1. Motivations

Traditional (weighted) congestion games are defined under a *full information* scenario—each player knows all the other participants in the game as well as their available strategies. These requirements, anyway, become too strong in many practical applications, where players may be unaware about even the mere existence of other potential competitors. This observation, together with the widespread of competitive applications in social networks, has drawn some attention on the model of *graphical (weighted) congestion games* [23–25].

A graphical (weighted) congestion game $(\mathcal{G}, G)$ is obtained by coupling a traditional (weighted) congestion game $\mathcal{G}$ with a *social knowledge graph* G expressing the *social context* in which the players operate. In G, each node is associated with a player of $\mathcal{G}$ and there exists a directed edge from node i to node j if and only if player i has full information about player j. A basic property of (weighted) congestion games is that the congestion of a resource r in a given strategy profile σ is the same for all players. The existence of a social context in graphical (weighted) congestion games, instead, makes the congestion of each resource player dependent. In these games, in fact, the congestion presumed by player i on resource r in the strategy profile σ is obtained by excluding from the set of players choosing r in σ those of whom player i has no knowledge. Clearly, if G is complete, then there is no difference between $(\mathcal{G}, G)$ and $\mathcal{G}$. In all the other cases, however, there may be a big difference between the cost that a player *presumes* to pay on a certain strategy profile and the real cost that she effectively *perceives* because of the presence of players she was unaware of during her decisional process (We observe that the model of graphical congestion games is sufficiently powerful to describe an alternative scenario in which players never perceive their real costs, which are perceived and evaluated by a central entity only. In such case, the central entity aims at evaluating the global and real impact on the performance of the game caused by the players' strategic behaviour).

Graphical congestion games have been introduced by Bilò et al. in Reference [24]. They focus on affine cost functions and provide a complete characterization of the cases in which existence of pure Nash equilibria can be guaranteed. In particular, they show that equilibria always exist if and only if G is either undirected or directed acyclic. Then, for all these cases, they give bounds on the price of anarchy and stability expressed as a function of the number of players in the game and of the maximum degree of G. These metrics are defined with respect to both the sum of the perceived costs and the sum of the presumed ones.

Fotakis et al. [25] argue that the maximum degree of G is not a proper measure of the level of social ignorance in a graphical congestion game and propose to bound the prices of anarchy and stability as a function of the independence number of G, denoted by $\delta(G)$. They focus on graphical weighted congestion games with affine cost functions and show that they still admit a potential function when G is undirected. Then, they prove that the price of anarchy is between $\delta(G)(\delta(G) + 1)$ and $\delta(G)(\delta(G) + 2 + \sqrt{\delta(G)^2 + 4\delta(G)})/2$ with respect to both the sum of the perceived costs and the sum of the presumed ones, and that the price of stability is between $\delta(G)$ and $2\delta(G)$ with respect to the sum of the perceived costs.

1.2. Our Contribution and Significance

The works of Bilò et al. [24] and Fotakis et al. [25] aim at characterizing the impact of social ignorance in the most general case, that is, without imposing any particular structure on the social knowledge graphs defining the graphical game. Nevertheless, real-world-based knowledge relationships usually obey some regularities and present recurrent patterns: for instance, people tend to cluster themselves into well-structured collaborative groups (cliques) due to family memberships, mutual friendships, interest sharing, business partnerships.

To this aim, we introduce and study *multidimensional (weighted) congestion games*, that is, (weighted) congestion games whose set of players are partitioned into $d+1$ clusters $C_0, C_1, \ldots, C_d$. Players in C_0 have full information about all the other participants in the game, while players in C_i, for any $1 \leq i \leq d$, have full information only about the members of $C_0 \cup C_i$ and are unaware of all the others. It is not difficult to see (and we provide a formal proof of this fact in the next section) that each multidimensional (weighted) congestion game is a graphical (weighted) congestion game whose social knowledge graph G is undirected and verifies $\delta(G) = d$. In addition, G possesses the following, well-structured, topology: it is the union of $d+1$ disjoint cliques (each corresponding to one of the $d+1$ clusters in the multidimensional (weighted) congestion game) with the additional property that there exists an edge from all the nodes belonging to one of these cliques (the one corresponding to cluster C_0) to all the nodes in all the other cliques.

We believe that the study of graphical games restricted to some prescribed social knowledge graphs like the ones we consider here, may be better suited to understand the impact of social ignorance in non-cooperative systems coming from practical and real-world applications. Moreover, the particular social knowledge relationships embedded in the definition of multidimensional (weighted) congestion games perfectly model the situation that generates when several independent games with full information are gathered together by some promoting parties so as to form a sort of "global super-game". The promoting parties become players with full information in the super-game, while each player in the composing sub-games maintains full information about all the other players in the same sub-game, acquires full information about all the promoting parties in the super game, but completely ignores the players in the other sub-games. Such a composing scheme resembles, in a sense, the general architecture of the Internet, viewed as a self-emerged network resulting from the aggregation of several autonomous systems (AS). Users in an AS have full information about anything happening within the AS, but, at the same time, they completely ignore the network global architecture and how it develops outside their own AS, except for the existence of high-level network routers. High-level network routers, instead, have full information about the entire network.

Furthermore, multidimensional (weighted) congestion games are also useful to model games in which players belong to different types and the level of competition that each player experiences on each selected resource depends on her type and on the types of the other players using the resource. Consider, for instance, a machine which is able to perform d different types of activities in parallel and a set of tasks requiring the use of this machine. Tasks are of two types: simple and complex. Simple tasks take the machine busy on one particular activity only, while complex tasks require the completion of all the d activities. Hence, complex tasks compete with all the other tasks, while simple ones compete only with the tasks requiring the same machine (thus, also with complex tasks). A similar example is represented by a facility location game where players want to locate their facilities so as to minimize the effect of the competition due to the presence of neighbor competitors. If we assume that the facilities can be either specialized shops selling particular products (such as perfumeries, clothes shops, shoe shops) or shopping centers selling all kinds of products, we have again that the shopping centers compete with all the other participants in the game, while specialized shops compete only with shops of the same type and with shopping centers.

In this paper, we focus on multidimensional (weighted) congestion games with affine cost functions. In such a setting, we bound the price of anarchy and the price of stability with respect to the two social cost functions, which are the sum of the perceived costs and the sum of the presumed

costs. In fact, when multidimensional (weighted) congestion games are viewed as graphical (weighted) congestion games with highly clustered knowledge relationships, the sum of the perceived costs is more appropriate to define the overall quality of a profile: players decide according to their knowledge, but then, when the solution is physically realized, their cost becomes influenced also by the players of which they were not aware. Hence, under this social cost function, the notions of price of anarchy and price of stability effectively measure the impact of social ignorance in the system. On the other hand, when multidimensional (weighted) congestion games are used to model players belonging to different types, the perceived cost of a player coincides with the presumed one since there is no real social ignorance, even if the fact that players can be of different types allows for a reinterpretation of the model as a special case of graphical (weighted) congestion games. Hence, in such a setting, the inefficiency due to selfish behavior has to be analyzed with respect to the sum of the presumed costs.

We determine general upper bounds for the price of anarchy and the price of stability as a function of d. For the sum of the presumed costs, we show that the price of anarchy and stability of weighted games are at most $\frac{(\sqrt{d+4}+\sqrt{d})(\sqrt{d}\sqrt{d+4}+d+4)}{4\sqrt{d+4}} \leq d+2$ and 2, respectively. Instead, for the sum of the perceived costs, the results of Reference [25] yield upper bounds of $d(d+2+\sqrt{d^2+4d})/2$ and $2d$ for the price of anarchy and the price of stability, respectively.

Then, we investigate the case of unweighted games with $d=2$ (i.e., bidimensional congestion games) in higher depth and provide refined bounds. In particular, we prove that price of anarchy is $119/33 \approx 3.606$ with respect to the sum of the presumed costs and it is $35/8 = 4.375$ with respect to the sum of the perceived ones, and that the price of stability is between $(1+\sqrt{5})/2 \approx 1.618$ and $1+2/\sqrt{7} \approx 1.756$ for the sum of the presumed costs as social cost function, and between $(5+\sqrt{17})/4 \approx 2.28$ and 2.92 for the sum of the perceived ones. These results are derived by exploiting the primal-dual method developed in Reference [11].

A preliminary version of this paper has been presented at SIROCCO 2012 [26].

1.3. Paper Organization

Next section contains all formal definitions and notation. In Section 3 we discuss the existence of pure Nash equilibria in multidimensional weighted congestion games. In Sections 4 and 5, we present our bounds for the price of anarchy and the price of stability, respectively. Finally, in the last section, we give some concluding remarks and discuss open problems.

2. Model and Definitions

For an integer $n \geq 2$, we denote $[n] := \{1, 2, \ldots, n\}$. In a *weighted congestion game* $\mathcal{G}$, we have n players and a set of resources R, where each resource $r \in R$ has an associated cost function ℓ_r. The set of strategies for each player $i \in [n]$, denoted as S_i, can be any subset of the powerset of R, that is, $S_i \subseteq 2^R$. Each player $i \in [n]$ is associated with a positive weight $w_i > 0$. Given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, the congestion of resource r in σ, denoted as $n_r(\sigma)$, is the total weight of players choosing r in σ, that is, $n_r(\sigma) = \sum_{i \in [n]: r \in \sigma_i} w_i$. The perceived cost paid by player i in σ is $c_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(n_r(\sigma))$. An *unweighted congestion game* (congestion game, for brevity) is a weighted congestion game in which all players have unitary weights. An *affine weighted congestion game* is a weighted congestion game such that, for any $r \in R$, it holds that $\ell_r(x) = \alpha_r x + \beta_r$, with $\alpha_r, \beta_r \geq 0$; the game is *linear* if $\beta_r = 0$ for any $r \in R$.

For any integer $d \geq 2$, a *d-dimensional weighted congestion game* $(\mathcal{G}, \mathcal{C})$ consists of a weighted congestion game $\mathcal{G}$ whose set of players is partitioned into $d+1$ clusters $C_0, C_1, \ldots, C_d$. For a player i, we denote by $f(i) \in \{0, \ldots, d\}$ the cluster she belongs to. We say that players in C_0 are *omniscient* and that players in C_i, for any $i \in [d]$, are *ignorant*. Given a strategy profile σ, we denote by $n_{r,j}(\sigma)$ the total weight of players belonging to C_j who are using resource r in σ. The presumed cost of a player i in σ is $\hat{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(n_{r,f(i)}(\sigma) + n_{r,0}(\sigma))$ if she is ignorant and $\hat{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(\sum_{j=0}^{d} n_{r,j}(\sigma)) =$

$\sum_{r \in \sigma_i} \ell_r(n_r(\sigma)) = c_i(\sigma)$ if she is omniscient. A *d-dimensional weighted affine congestion game* is a pair $(\mathcal{G}, \mathcal{C})$ such that $\mathcal{G}$ is an affine weighted congestion game.

Given a strategy profile σ and a strategy $s \in S_i$ for a player $i \in [n]$, we denote with (σ_{-i}, s) the strategy profile obtained from σ by replacing the strategy σ_i played by i in σ with s. A *pure Nash equilibrium* is a strategy profile σ such that, for any player $i \in [n]$ and for any strategy $s \in S_i$, it holds that $\hat{c}_i(\sigma_{-i}, s) \geq \hat{c}_i(\sigma)$.

Let Σ be the set of all possible strategy profiles which can be realized in $(\mathcal{G}, \mathcal{C})$. We denote with $\mathcal{NE}(\mathcal{G}, \mathcal{C}) \subseteq \Sigma$ the set of pure Nash equilibria of $(\mathcal{G}, \mathcal{C})$. Let SF $: \Sigma \to \mathbb{R}_{\geq 0}$ be a *social cost function* measuring the overall quality of each strategy profile in Σ. We denote with σ^* a *social optimum* of $(\mathcal{G}, \mathcal{C})$ with respect to SF, that is, a strategy profile minimizing the social cost function SF. We consider two social cost functions, namely, the (weighted) sum of the presumed costs of all players and the (weighted) sum of their perceived ones denoted, respectively, as Pres and Perc. Technically, they assume the following expressions:

$$
\begin{aligned}
\mathsf{Pres}(\sigma) &= \sum_{i \in [n]} w_i \hat{c}_i(\sigma) \\
&= \sum_{i \in C_0} w_i \sum_{r \in \sigma_i} (\alpha_r n_r(\sigma) + \beta_r) + \sum_{i \notin C_0} w_i \sum_{r \in \sigma_i} \left(\alpha_r \left(n_{r,f(i)}(\sigma) + n_{r,0}(\sigma) \right) + \beta_r \right) \\
&= \sum_{r \in R} \left(\alpha_r n_{r,0}(\sigma) \sum_{j=0}^{d} n_{r,j}(\sigma) + \beta_r n_{r,0}(\sigma) \right) \\
&\quad + \sum_{r \in R} \left(\alpha_r n_{r,0}(\sigma) \sum_{j=1}^{d} n_{r,j}(\sigma) + \alpha_r \sum_{j=1}^{d} n_{r,j}(\sigma)^2 + \beta_r \sum_{j=1}^{d} n_{r,j}(\sigma) \right) \\
&= \sum_{r \in R} \left(\alpha_r \left(\sum_{j=0}^{d} n_{r,j}(\sigma)^2 + 2 n_{r,0}(\sigma) \sum_{j=1}^{d} n_{r,j}(\sigma) \right) + \beta_r \sum_{j=0}^{d} n_{r,j}(\sigma) \right)
\end{aligned}
$$

and

$$
\mathsf{Perc}(\sigma) = \sum_{i \in [n]} w_i c_i(\sigma) = \sum_{i \in [n]} w_i \sum_{r \in \sigma_i} (\alpha_r n_r(\sigma) + \beta_r) = \sum_{r \in R} \left(\alpha_r n_r(\sigma)^2 + \beta_r n_r(\sigma) \right).
$$

For a fixed social cost function SF, the *price of anarchy* of $(\mathcal{G}, \mathcal{C})$, denoted by $PoA(\mathcal{G}, \mathcal{C})$, is the ratio between the social value of the *worst* pure Nash equilibrium of $(\mathcal{G}, \mathcal{C})$ and that of a social optimum, that is, $PoA(\mathcal{G}, \mathcal{C}) = \max_{\sigma \in \mathcal{NE}(\mathcal{G}, \mathcal{C})} \frac{\mathsf{SF}(\sigma)}{\mathsf{SF}(\sigma^*)}$. The *price of stability*, denoted by $PoS(\mathcal{G}, \mathcal{C})$, considers the best case, so that $PoS(\mathcal{G}, \mathcal{C}) = \min_{\sigma \in \mathcal{NE}(\mathcal{G}, \mathcal{C})} \frac{\mathsf{SF}(\sigma)}{\mathsf{SF}(\sigma^*)}$.

3. Existence of Pure Nash Equilibria

In this section, we prove that multidimensional unweighted (resp. weighted affine) congestion games are graphical unweighted (resp. weighted affine) congestion games defined by an underlying undirected social knowledge graph. This allows us to use a result in Reference [24] (resp. [25]) stating that these games are potential games, thus admitting pure Nash equilibria.

A *graphical weighted congestion game* $(\mathcal{G}, G)$ consists of a weighted congestion game $\mathcal{G}$ and a directed graph $G = (N, A)$ such that each node of N is associated with a player in $\mathcal{G}$ and there exists a directed edge from node i to node j if and only if player i has full information about player j. The congestion presumed by player i on resource r in the profile σ is $\tilde{n}_{r,i}(\sigma) = \sum_{p \in N : r \in \sigma_p, (i,p) \in A} w_p + w_i$ and the presumed cost paid by player i in σ is $\tilde{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(\tilde{n}_{r,i}(\sigma))$. A *graphical weighted affine congestion game* is a pair $(\mathcal{G}, G)$ such that $\mathcal{G}$ is an affine weighted congestion game. The *independence number* $\delta(G)$ of $(\mathcal{G}, G)$ is the cardinality of a maximum independent set of graph G.

A function $\Phi : \Sigma \to \mathbb{R}$ is a *weighted potential function* for a graphical weighted congestion game $(\mathcal{G}, G)$, if for any profile σ, any player $i \in [n]$ and any strategy $s \in S_i$, it holds that $\Phi(\sigma) - \Phi(\sigma_{-i}, s) = a_i(\tilde{c}_i(\sigma) - \tilde{c}_i(\sigma_{-i}, s))$ for some $a_i > 0$; if $a_i = 1$, Φ is an *exact potential function*. In Reference [24]

(resp. [25]), it is shown that each graphical unweighted (resp. weighted affine) congestion game $(\mathcal{G}, G)$ such that G is undirected admits an exact potential function (resp. weighted potential function).

The following result shows that d-dimensional weighted congestion games are a subclass of graphical weighted congestion games.

Proposition 1. *Each d-dimensional weighted congestion game is a graphical weighted congestion game whose social knowledge graph is undirected.*

Proof. Fix a d-dimensional weighted congestion game $(\mathcal{G}, \mathcal{C})$. We define a graph $G = (N, A)$ such that each node in N is associated with a player in $\mathcal{G}$ and there is an undirected edge $\{u, v\} \in A$ if and only if either $u, v \in C_i$ for some $0 \leq i \leq d$ or $u \in C_0$. We show that, for any strategy profile σ of $\mathcal{G}$ and for any $i \in [n]$, $\hat{c}_i(\sigma) = \tilde{c}_i(\sigma)$.

Consider first an omniscient player $i \in C_0$. In $(\mathcal{G}, \mathcal{C})$, it holds that

$$\hat{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(n_r(\sigma)) = \sum_{r \in \sigma_i} \ell_r \left(\sum_{p \in [n]: r \in \sigma_p} w_p \right),$$

while in $(\mathcal{G}, G)$, it holds that

$$\tilde{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(\tilde{n}_{r,i}(\sigma)) = \sum_{r \in \sigma_i} \ell_r \left(\sum_{p \in N: r \in \sigma_p, \{i,p\} \in A} w_p + w_i \right) = \sum_{r \in \sigma_i} \ell_r \left(\sum_{p \in [n]: r \in \sigma_p} w_p \right),$$

where the last equality follows from the fact that, by construction of G, it holds that $\{i, p\} \in A$, for any $p \in [n]$ with $p \neq i$.

Next, consider an ignorant player $i \in C_j$ for some $j \in [d]$. In $(\mathcal{G}, \mathcal{C})$, it holds that

$$\hat{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(n_{r,f(i)}(\sigma) + n_{r,0}(\sigma)) = \sum_{r \in \sigma_i} \ell_r \left(\sum_{p \in C_0 \cup C_j: r \in \sigma_p} w_p \right),$$

while in $(\mathcal{G}, G)$, it holds that

$$\tilde{c}_i(\sigma) = \sum_{r \in \sigma_i} \ell_r(\tilde{n}_{r,i}(\sigma)) = \sum_{r \in \sigma_i} \ell_r \left(\sum_{p \in N: r \in \sigma_p, \{i,p\} \in A} w_p + w_i \right) = \sum_{r \in \sigma_i} \ell_r \left(\sum_{p \in C_0 \cup C_j: r \in \sigma_p} w_p \right),$$

where the last equality follows from the fact that, by construction of G, for any $p \in [n]$ with $p \neq i$, it holds that $\{i, p\} \in A$ if and only if $p \in C_0 \cup C_j$. □

Each game admitting an exact or weighted potential function always admits pure Nash equilibria. Hence, by Proposition 1 and the existence of an exact (resp. weighted) potential function for graphical unweighted (resp. weighted affine) congestion games with undirected social knowledge graphs, we have that d-dimensional unweighted (resp. weighted affine) congestion games always admit pure Nash equilibria.

For weighted affine games, the potential function assume the following expression:

$$\Phi(\sigma) = \sum_{r \in R} \left(\alpha_r \left(\sum_{i \in [n]: r \in \sigma_i} w_i^2 + \sum_{\{i,p\} \in A: r \in \sigma_i \cap \sigma_p} w_i w_p \right) + \beta_r \sum_{i \in [n]: r \in \sigma_i} w_i \right)$$

$$= \frac{1}{2} \sum_{r \in R} \left(\alpha_r \left(\sum_{j=0}^{d} n_{r,j}(\sigma)^2 + \sum_{i \in [n]: r \in \sigma_i} w_i^2 + 2 n_{r,0}(\sigma) \sum_{j=1}^{d} n_{r,j}(\sigma) \right) + 2 \beta_r \sum_{j=0}^{d} n_{r,j}(\sigma) \right). \quad (1)$$

4. Bounds for the Price of Anarchy

In this section, we provide an upper bound for the price of anarchy of multidimensional weighted affine congestion games as a function of d.

Fix a pure Nash equilibrium σ and a social optimum σ^*, thus fixing the congestions $n_{r,i}(\sigma)$ and $n_{r,i}(\sigma^*)$ for each $i \in [n]$ and $r \in R$. The pure Nash equilibrium condition implies that no player lowers her presumed cost by deviating to the strategy she uses in σ^*. For any player $i \in C_0$, this yields

$$\sum_{r\in\sigma_i} (\alpha_r n_r(\sigma) + \beta_r) - \sum_{r\in\sigma_i^*} (\alpha_r(n_r(\sigma) + w_i) + \beta_r) \leq 0, \tag{2}$$

that is a necessary condition satisfied by any pure Nash equilibrium (The equilibrium condition yields the stronger inequality $\sum_{r\in\sigma_i\setminus\sigma_i^*} (\alpha_r n_r(\sigma) + \beta_r) - \sum_{r\in\sigma_i^*\setminus\sigma_i} (\alpha_r(n_r(\sigma) + w_i) + \beta_r) \leq 0$, so that inequality $\sum_{r\in\sigma_i} (\alpha_r n_r(\sigma) + \beta_r) - \sum_{r\in\sigma_i^*} (\alpha_r(n_r(\sigma) + w_i) + \beta_r) \leq 0$ is a relaxation of the equilibrium condition.). For weighted games, by using $w_i \leq n_{r,0}(\sigma^*)$ for any $r \in R$ and $i \in [n]$ such that $r \in \sigma_i$, by multiplying (2) for w_i and summing it for each $i \in C_0$, we get

$$\sum_{r\in R} \left(\alpha_r n_{r,0}(\sigma) \sum_{j=0}^{d} n_{r,j}(\sigma) + \beta_r n_{r,0}(\sigma) - \alpha_r n_{r,0}(\sigma^*) \left(n_{r,0}(\sigma^*) + \sum_{j=0}^{d} n_{r,j}(\sigma) \right) - \beta_r n_{r,0}(\sigma^*) \right) \leq 0, \tag{3}$$

that is a further necessary condition satisfied by any pure Nash equilibrium. For unweighted games, we simply fix $w_i = 1$ for any $i \in [n]$ and sum the inequality for each $i \in C_0$, thus getting

$$\sum_{r\in R} \left(\alpha_r n_{r,0}(\sigma) \sum_{j=0}^{d} n_{r,j}(\sigma) + \beta_r n_{r,0}(\sigma) - \alpha_r n_{r,0}(\sigma^*) \left(1 + \sum_{j=0}^{d} n_{r,j}(\sigma) \right) - \beta_r n_{r,0}(\sigma^*) \right) \leq 0. \tag{4}$$

For any player $i \in C_j$, with $j \in [d]$, the equilibrium condition yields

$$\sum_{r\in\sigma_i} (\alpha_r (n_{r,j}(\sigma) + n_{r,0}(\sigma)) + \beta_r) - \sum_{r\in\sigma_i^*} (\alpha_r (n_{r,j}(\sigma) + n_{r,0}(\sigma) + w_i) + \beta_r) \leq 0.$$

For weighted games, again, by using $w_i \leq n_{r,0}(\sigma^*)$ for any $r \in R$ and $i \in [n]$ such that $r \in \sigma_i$, by multiplying this inequality for w_i, and by summing it for each $i \in C_j$, we get

$$\sum_{r\in R} \left(\alpha_r n_{r,j}(\sigma) \left(n_{r,j}(\sigma) + n_{r,0}(\sigma) \right) + \beta_r n_{r,j}(\sigma) - \alpha_r n_{r,j}(\sigma^*) \left(n_{r,j}(\sigma) + n_{r,0}(\sigma) + n_{r,j}(\sigma^*) \right) - \beta_r n_{r,j}(\sigma^*) \right) \leq 0.$$

By further summing for each $j \in [d]$, we obtain

$$\sum_{r\in R} \left(\sum_{j=1}^{d} (n_{r,j}(\sigma) (\alpha_r(n_{r,j}(\sigma) + n_{r,0}(\sigma) + \beta_r)) - \sum_{j=1}^{d} (n_{r,j}(\sigma^*) (\alpha_r(n_{r,j}(\sigma) + n_{r,0}(\sigma) + n_{r,j}(\sigma^*)) + \beta_r)) \right) \leq 0. \tag{5}$$

For unweighted games, by setting $w_i = 1$, and by summing the equilibrium constraint for any $i \in [n]$ and $j \in [d]$, we analogously get

$$\sum_{r\in R} \left(\sum_{j=1}^{d} (n_{r,j}(\sigma) (\alpha_r(n_{r,j}(\sigma) + n_{r,0}(\sigma) + \beta_r)) - \sum_{j=1}^{d} (n_{r,j}(\sigma^*) (\alpha_r(n_{r,j}(\sigma) + n_{r,0}(\sigma) + 1) + \beta_r)) \right) \leq 0. \tag{6}$$

In the sequel, for the sake of conciseness, we adopt $k_{r,j}$ and $l_{r,j}$ as short-hands for $n_{r,j}(\sigma)$ and $n_{r,j}(\sigma^*)$, respectively.

Theorem 1 provides an upper bound for the price of anarchy of multidimensional weighted affine congestion games with respect to social cost function Pres.

Theorem 1. *For each d-dimensional weighted affine congestion game* $(\mathcal{G}, \mathcal{C})$,

$$PoA(\mathcal{G}, \mathcal{C}) \leq \frac{(\sqrt{d+4} + \sqrt{d})(\sqrt{d} \cdot \sqrt{d+4} + d + 4)}{4\sqrt{d+4}} \leq d + 2$$

under the social cost function Pres.

Proof. Let σ and σ^* be a worst-case equilibrium and a social optimum of $(\mathcal{G}, \mathcal{C})$, respectively. Let $k_r = (k_{r,0}, \ldots, k_{r,d})$, $l_r = (l_{r,0}, \ldots, l_{r,d})$, and let $P = (p_{i,j})_{i,j \in [d] \cup \{0\}}$ be the $(d+1) \times (d+1)$ binary matrix such that: (i) $p_{i,j} = 1$ if either $i = j$, or $i = 0$, or $j = 0$; (ii) $p_{i,j} = 0$ otherwise. By summing inequalities (3) and (5), we get the following compact inequality involving the product between vectors, matrices, and scalars:

$$\sum_{r \in R} \left(\alpha_r (k_r \cdot P \cdot k_r^T) + \beta_r \sum_{j=0}^{d} k_{r,j} - \alpha_r (l_r \cdot P \cdot k_r^T + l_r \cdot l_r^T) - \beta_r \sum_{j=0}^{d} l_{r,j} \right) \leq 0 \tag{7}$$

Let $Q = (q_{i,j})_{i,j \in [d] \cup \{0\}}$ be the $(d+1) \times (d+1)$ matrix such that: (i) $q_{i,j} = \sqrt{d}$ if $i = j$; (ii) $q_{i,j} = 1$ if either $i = 0$, or $j = 0$, with $(i, j) \neq (0, 0)$; (iii) $q_{i,j} = 0$ otherwise. As $0 \leq p_{i,j} \leq q_{i,j}$ for any i, j we have that

$$l_r \cdot P \cdot k_r^T \leq l_r \cdot Q \cdot k_r^T. \tag{8}$$

We have that matrix Q is a symmetric positive-semidefinite matrix (see Lemma A1 in the Appendix A for the proof of this fact), thus, the following inequality holds for any $u > 0$:

$$0 \leq \left(\sqrt{u} \cdot k_r - \frac{1}{2\sqrt{u}} \cdot l_r \right) \cdot Q \cdot \left(\sqrt{u} \cdot k_r - \frac{1}{2\sqrt{u}} \cdot l_r \right)^T = u \cdot k_r \cdot Q \cdot k_r^T + \frac{1}{4u} \cdot l_r \cdot Q \cdot l_r^T - l_r \cdot Q \cdot k_r^T. \tag{9}$$

Finally, as $0 \leq q_{i,j} \leq \sqrt{d} \cdot p_{i,j}$ for any i, j, we have that

$$x \cdot Q \cdot x^T \leq \sqrt{d} \cdot x \cdot P \cdot x^T \tag{10}$$

for any vector $x = (x_0, \ldots, x_d)$ of non-negative real numbers. By exploiting (7), (9), and (10), for any fixed $u > 0$ we get

$$\text{Pres}(\sigma) = \sum_{r \in R} \left(\alpha_r (k_r \cdot P \cdot k_r^T) + \beta_r \sum_{j=0}^{d} k_{r,j} \right)$$

$$\leq \sum_{r \in R} \left(\alpha_r (l_r \cdot P \cdot k_r^t + l_r \cdot l_r^T) + \beta_r \sum_{j=0}^{d} l_{r,j} \right) \tag{11}$$

$$\leq \sum_{r \in R} \left(\alpha_r (l_r \cdot P \cdot k_r^T + l_r \cdot P \cdot l_r^T) + \beta_r \sum_{j=0}^{d} l_{r,j} \right)$$

$$\leq \sum_{r \in R} \left(\alpha_r \left(u \cdot k_r \cdot Q \cdot k_r^T + \frac{1}{4u} \cdot l_r \cdot Q \cdot l_r^T + l_r \cdot P \cdot l_r^T \right) + \beta_r \sum_{j=0}^{d} l_{r,j} \right) \tag{12}$$

$$\leq \sum_{r \in R} \left(\alpha_r \left(\sqrt{d} \cdot u \cdot k_r \cdot P \cdot k_r^T + \left(\frac{\sqrt{d}}{4u} + 1 \right) \cdot l_r \cdot P \cdot l_r^T \right) + \beta_r \sum_{j=0}^{d} l_{r,j} \right) \tag{13}$$

$$\leq \sqrt{d} \cdot u \cdot \sum_{r \in R} \left(\alpha_r (k_r \cdot P \cdot k_r^T) + \beta_r \sum_{j=0}^{d} k_{r,j} \right) + \left(\frac{\sqrt{d}}{4u} + 1 \right) \cdot \sum_{r \in R} \left(\alpha_r (l_r \cdot P \cdot l_r^T) + \beta_r \sum_{j=0}^{d} l_{r,j} \right)$$

$$= \sqrt{d} \cdot u \cdot \text{Pres}(\sigma) + \left(\frac{\sqrt{d}}{4u} + 1 \right) \cdot \text{Pres}(\sigma^*), \tag{14}$$

where (11), (12), and (13), come from (7), (9), and (10), respectively (Inequalities (11)–(14) can be stated within the smoothness framework of Roughgarden [19], and show that multidimensional weighted affine congestion games are (λ, μ)-smooth with $\lambda = \frac{\sqrt{d}}{4u} + 1$ and $\mu = \sqrt{d} \cdot u$ for any $u > 0$.). Finally, by manipulating (14), we get

$$PoA(\mathcal{G}, \mathcal{C}) = \frac{\text{Pres}(\sigma)}{\text{Pres}(\sigma^*)} \leq \inf_{u>0} \frac{\frac{\sqrt{d}}{4u} + 1}{1 - \sqrt{d} \cdot u} = \frac{(\sqrt{d+4} + \sqrt{d})(\sqrt{d} \cdot \sqrt{d+4} + d + 4)}{4\sqrt{d+4}}, \tag{15}$$

thus showing the claim. A simpler upper bound of $d + 2$ can be obtained by setting $u = \frac{1}{2\sqrt{d}}$ in (15). □

Relatively to the social cost function Perc, the following upper bound is derived as a corollary of a result in Reference [25].

Corollary 1. *For each d-dimensional affine congestion game* $(\mathcal{G}, \mathcal{C})$, $PoA(\mathcal{G}, \mathcal{C}) \leq d(d + 2 + \sqrt{d^2 + 4d})/2$ *under the social cost function* Perc.

Proof. Theorem 2 of Reference [25] states that $\delta(G)(\delta(G) + 2 + \sqrt{\delta(G)^2 + 4\delta(G)})/2$ is an upper bound for the price of anarchy of any graphical congestion having independence number $\delta(G)$. As the graphical congestion game equivalent to $(\mathcal{G}, \mathcal{C})$ has independence number equal to d, the claim follows. □

5. Bounds for the Price of Stability

In order to bound the price of stability with respect to the social cost function Pres, we consider a pure Nash equilibrium that minimizes the potential function Φ defined in (1), which leads to the following upper bound.

Theorem 2. *For each d-dimensional weighted affine congestion game* $(\mathcal{G}, \mathcal{C})$, $PoS(\mathcal{G}, \mathcal{C}) \leq 2$ *under the social cost function* Pres.

Proof. Let σ and σ^* be a pure Nash equilibrium minimizing the potential function Φ defined in (1), and let σ^* be a social optimum. We have that

$$
\begin{aligned}
\text{Pres}(\sigma) &= \sum_{r \in R} \left(\alpha_r \left(\sum_{j=0}^{d} k_{r,j}^2 + 2k_{r,0} \sum_{j=1}^{d} k_{r,j} \right) + \beta_r \sum_{j=0}^{d} k_{r,j} \right) \\
&\leq \sum_{r \in R} \left(\alpha_r \left(\sum_{j=0}^{d} k_{r,j}^2 + \sum_{i \in [n]: r \in \sigma_i} w_i^2 + 2k_{r,0} \sum_{j=1}^{d} k_{r,j} \right) + 2\beta_r \sum_{j=0}^{d} k_{r,j} \right) \\
&= 2 \cdot \Phi(\sigma) \tag{16} \\
&\leq 2 \cdot \Phi(\sigma^*) \tag{17} \\
&= \sum_{r \in R} \left(\alpha_r \left(\sum_{j=0}^{d} l_{r,j}^2 + \sum_{i \in [n]: r \in \sigma_i^*} w_i^2 + 2l_{r,0} \sum_{j=1}^{d} l_{r,j} \right) + 2\beta_r \sum_{j=0}^{d} l_{r,j} \right) \tag{18} \\
&\leq \sum_{r \in R} \left(\alpha_r \left(2 \sum_{j=0}^{d} l_{r,j}^2 + 2l_{r,0} \sum_{j=1}^{d} l_{r,j} \right) + 2\beta_r \sum_{j=0}^{d} l_{r,j} \right) \\
&\leq 2 \sum_{r \in R} \left(\alpha_r \left(\sum_{j=0}^{d} l_{r,j}^2 + 2l_{r,0} \sum_{j=1}^{d} l_{r,j} \right) + \beta_r \sum_{j=0}^{d} l_{r,j} \right) \\
&= 2 \cdot \text{Pres}(\sigma^*), \tag{19}
\end{aligned}
$$

where (17) holds since σ minimizes Φ, and (16) and (18) hold by exploiting (1). By (19), we get $PoS(\mathcal{G},\mathcal{C}) \leq \frac{\text{Pres}(\sigma)}{\text{Pres}(\sigma^*)} \leq 2$, and the claim follows. $\square$

Relatively to the social cost function Perc, the following upper bound is derived as a corollary of a result in Reference [25].

Corollary 2. *For each d-dimensional affine congestion game* $(\mathcal{G},\mathcal{C})$, $PoS(\mathcal{G},\mathcal{C}) \leq 2d$ *under the social cost function* Perc.

Proof. Theorem 6 of Reference [25] states that $2\delta(G)$ is an upper bound for the price of stability of any graphical congestion game having independence number $\delta(G)$. As the graphical congestion game equivalent to $(\mathcal{G},\mathcal{C})$ has independence number equal to d, the claim follows. $\square$

6. Bounds for Bidimensional Unweighted Games

In this section, we investigate in more detail the case of unweighted affine games with $d = 2$, that is, bidimensional affine congestion games, and provide refined bounds for the price of anarchy and the price of stability under both social cost functions. The technique we adopt is the primal-dual framework introduced in Reference [11].

6.1. Price of Anarchy

We first consider the price of anarchy. Let $(\mathcal{G},\mathcal{C})$ be an arbitrary d-dimensional unweighted congestion game, and let σ and σ^* be a worst-case equilibrium and a social optimum of $(\mathcal{G},\mathcal{C})$, respectively. For $\mathsf{SF} = \mathsf{Pres}$, we get the following primal linear program $\mathsf{LP}(\mathsf{Pres},\sigma,\sigma^*)$ in variables $(\alpha_r,\beta_r)_{r\in R}$, whose optimal solution provides an upper bound to $PoA(\mathcal{G},\mathcal{C})$:

$$\textit{max} \quad \sum_{r\in R}\left(\alpha_r\left(\sum_{j=0}^{d}k_{r,j}^2 + 2k_{r,0}\sum_{j=1}^{d}k_{r,j}\right) + \beta_r\sum_{j=0}^{d}k_{r,j}\right)$$

$$\textit{s.t.}$$

$$\sum_{r\in R}\left(\alpha_r k_{r,0}\sum_{j=0}^{d}k_{r,j} + \beta_r k_{r,0} - \alpha_r l_{r,0}\left(1 + \sum_{j=0}^{d}k_{r,j}\right) - \beta_r l_{r,0}\right) \leq 0$$

$$\sum_{r\in R}\left(\sum_{j=1}^{d}\left(k_{r,j}\left(\alpha_r(k_{r,j}+k_{r,0}+\beta_r)\right)\right) - \sum_{j=1}^{d}\left(l_{r,j}\left(\alpha_r(k_{r,j}+k_{r,0}+1)+\beta_r\right)\right)\right) \leq 0$$

$$\sum_{r\in R}\left(\alpha_r\left(\sum_{j=0}^{d}l_{r,j}^2 + 2l_{r,0}\sum_{j=1}^{d}l_{r,j}\right) + \beta_r\sum_{j=0}^{d}l_{r,j}\right) = 1$$

$$\alpha_r,\beta_r \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall r \in R.$$

The optimal solution of the above linear program is an upper bound to the price of anarchy as the objective function is equal to $\mathsf{Pres}(\sigma)$, the first two constraints are the pure Nash equilibrium conditions derived in (4) and (6), respectively (which are necessary conditions satisfied by any equilibrium), and the last normalization constraint imposes without loss of generality that $\mathsf{Pres}(\sigma^*) = 1$ (When applying the primal dual method, we observe that, once σ and σ^* are fixed, the coefficients $(\alpha_r)_{r\in R}$ and $(\beta_r)_{r\in R}$ are chosen in such a way that the value $\mathsf{Pres}(\sigma) = \mathsf{Pres}(\sigma)/\mathsf{Pres}(\sigma^*)$ is maximized, thus getting an upper bound on the price of anarchy. We also observe that $(\alpha_r)_{r\in R}$ and $(\beta_r)_{r\in R}$ are the unique variables in the considered LP formulation, and the other quantities (e.g., the congestions) are considered as fixed parameters (w.r.t. the LP formulation). See Reference [11] for further details on the primal-dual method and how to apply it to measure the performance of congestion games under different quality metrics.).

By associating the three dual variables x, y and γ, with the three constraints of $\mathsf{LP}(\mathsf{Pres}, \sigma, \sigma^*)$, the dual formulation $\mathsf{DLP}(\mathsf{Pres}, \sigma, \sigma^*)$ becomes

$$
\begin{aligned}
\min \quad & \gamma \\
\text{s.t.} \quad &
\end{aligned}
$$

$$
x\left(k_{r,0}\sum_{j=0}^{d}k_{r,j} - l_{r,0} - l_{r,0}\sum_{j=0}^{d}k_{r,j}\right) + y\sum_{j=1}^{d}\left(k_{r,j}(k_{r,j}+k_{r,0}) - l_{r,j}(k_{r,j}+k_{r,0}+1)\right)
$$
$$
+\gamma\left(\sum_{j=0}^{d}l_{r,j}^2 + 2l_{r,0}\sum_{j=1}^{d}l_{r,j}\right) \geq \sum_{j=0}^{d}k_{r,j}^2 + 2k_{r,0}\sum_{j=1}^{d}k_{r,j} \qquad\qquad \forall r \in R
$$

$$
x(k_{r,0}-l_{r,0}) + y\sum_{j=1}^{d}(k_{r,j}-l_{r,j}) + \gamma\sum_{j=0}^{d}l_{r,j} \geq \sum_{j=0}^{d}k_{r,j} \qquad\qquad \forall r \in R
$$

$$
x, y \geq 0.
$$

By the Weak Duality Theorem, each feasible solution for $\mathsf{DLP}(\mathsf{Pres}, \sigma, \sigma^*)$ provides an upper bound to the optimal solution of $\mathsf{LP}(\mathsf{Pres}, \sigma, \sigma^*)$, that is on the price of anarchy achievable by the particular choice of σ and σ^*. Anyway, if the provided dual solution is independent of this choice, we obtain an upper bound on the price of anarchy for any possible game.

For the case of the social cost function Perc, we only need to replace the objective function and the third constraint in $\mathsf{LP}(\mathsf{Pres}, \sigma, \sigma^*)$, respectively, with

$$
\sum_{r \in R}\left(\alpha_r\left(\sum_{j=0}^{d}k_{r,j}\right)^2 + \beta_r\sum_{j=0}^{d}k_{r,j}\right) \quad \text{and} \quad \sum_{r \in R}\left(\alpha_r\left(\sum_{j=0}^{d}l_{r,j}\right)^2 + \beta_r\sum_{j=0}^{d}l_{r,j}\right) = 1.
$$

This results in the following dual program $\mathsf{DLP}(\mathsf{Perc}, \sigma, \sigma^*)$:

$$
\begin{aligned}
\min \quad & \gamma \\
\text{s.t.} \quad &
\end{aligned}
$$

$$
x\left(k_{r,0}\sum_{j=0}^{d}k_{r,j} - l_{r,0} - l_{r,0}\sum_{j=0}^{d}k_{r,j}\right) + y\sum_{j=1}^{d}\left(k_{r,j}(k_{r,j}+k_{r,0}) - l_{r,j}(k_{r,j}+k_{r,0}+1)\right)
$$
$$
+\gamma\left(\sum_{j=0}^{d}l_{r,j}\right)^2 \geq \left(\sum_{j=0}^{d}k_{r,j}\right)^2 \qquad\qquad \forall r \in R
$$

$$
x(k_{r,0}-l_{r,0}) + y\sum_{j=1}^{d}(k_{r,j}-l_{r,j}) + \gamma\sum_{j=0}^{d}l_{r,j} \geq \sum_{j=0}^{d}k_{r,j} \qquad\qquad \forall r \in R
$$

$$
x, y \geq 0.
$$

Note that the second constraint is the same in both $\mathsf{DLP}(\mathsf{Pres}, \sigma, \sigma^*)$ and $\mathsf{DLP}(\mathsf{Perc}, \sigma, \sigma^*)$. For the sake of conciseness, in the sequel, we shall drop the subscript r from the notation; moreover, when fixed a dual solution, we shall denote the first and second constraint of a given dual program as $g_1(k, l) \geq 0$ and $g_2(k, l) \geq 0$, respectively, where we set $k = (k_0, \ldots, k_d)$ and $l = (l_0, \ldots, l_d)$.

When $d = 2$, we exploit an equivalent, but nicer, representation of the dual inequalities. With this aim, we set $k_r := n_r(\sigma)$ and $l_r := n_r(\sigma^*)$ and replace $k_{r,0}$ and $l_{r,0}$ with $k_r - k_{r,1} - k_{r,2}$ and $l_r - l_{r,1} - l_{r,2}$, respectively. By substituting and rearranging, $\mathsf{DLP}(\mathsf{Pres}, \sigma, \sigma^*)$ becomes

$$
\begin{aligned}
min \quad & \gamma \\
s.t. \quad & \\
& x\left((k_r - k_{r,1} - k_{r,2})k_r - (l_r - l_{r,1} - l_{r,2})(k_r + 1)\right) \\
& \quad +y\left(k_{r,1}(k_r - k_{r,2}) - l_{r,1}(k_r - k_{r,2} + 1) + k_{r,2}(k_r - k_{r,1}) - l_{r,2}(k_r - k_{r,1} + 1)\right) \\
& \quad +\gamma\left(l_r^2 - 2l_{r,1}l_{r,2}\right) \geq k_r^2 - 2k_{r,1}k_{r,2} && \forall r \in R \\
& x(k_r - k_{r,1} - k_{r,2} - l_r + l_{r,1} + l_{r,2}) + y(k_{r,1} + k_{r,2} - l_{r,1} - l_{r,2}) + \gamma l_r \geq k_r && \forall r \in R \\
& x, y \geq 0.
\end{aligned}
$$

Similarly, the dual program DLP(Perc, σ, σ^*) can be rewritten as:

$$
\begin{aligned}
min \quad & \gamma \\
s.t. \quad & \\
& x\left((k_r - k_{r,1} - k_{r,2})k_r - (l_r - l_{r,1} - l_{r,2})(k_r + 1)\right) \\
& \quad +y\left(k_{r,1}(k_r - k_{r,2}) - l_{r,1}(k_r - k_{r,2} + 1) + k_{r,2}(k_r - k_{r,1}) - l_{r,2}(k_r - k_{r,1} + 1)\right) + \gamma l_r^2 \geq k_r^2 && \forall r \in R \\
& x(k_r - k_{r,1} - k_{r,2} - l_r + l_{r,1} + l_{r,2}) + y(k_{r,1} + k_{r,2} - l_{r,1} - l_{r,2}) + \gamma l_r \geq k_r && \forall r \in R \\
& x, y \geq 0.
\end{aligned}
$$

In the following theorem we provide upper bounds for the price of anarchy of bidimensional affine congestion games under social cost functions Pres and Perc.

Theorem 3. *For each bidimensional affine congestion game* $(\mathcal{G}, \mathcal{C})$, $PoA(\mathcal{G}, \mathcal{C}) \leq \frac{119}{33}$ *under the social cost function* Pres *and* $PoA(\mathcal{G}, \mathcal{C}) \leq \frac{35}{8}$ *under the social cost function* Perc.

We now show the existence of two matching lower bounding instances (the proof is deferred to the Appendix B).

Theorem 4. *There exist two bidimensional linear congestion games* $(\mathcal{G}, \mathcal{C})$ *and* $(\mathcal{G}', \mathcal{C}')$ *such that* $PoA(\mathcal{G}, \mathcal{C}) \geq \frac{119}{33}$ *under the social cost function* Pres *and* $PoA(\mathcal{G}', \mathcal{C}') \geq \frac{35}{8}$ *under the social cost function* Perc *Appendix B.*

6.2. Price of Stability

In order to bound the price of stability, we can use the same primal formulations exploited for the determination of the price of anarchy with the additional constraint $\Phi(\sigma) \leq \Phi(\sigma^*)$, which, by Equation (1), becomes

$$
\sum_{r \in R}\left(\alpha_r\left(\sum_{j=0}^{d}\left(k_{r,j}^2 + k_{r,j} - l_{r,j}^2 - l_{r,j}\right) + 2k_{r,0}\sum_{j=1}^{d}k_{r,j} - 2l_{r,0}\sum_{j=1}^{d}l_{r,j}\right) + 2\beta_r\sum_{j=0}^{d}(k_{r,j} - l_{r,j})\right) \leq 0.
$$

Hence, the dual program for the social cost function Pres becomes the following one.

$$\min \quad \gamma$$
$$s.t.$$

$$x \left(k_{r,0} \sum_{j=0}^{d} k_{r,j} - l_{r,0} - l_{r,0} \sum_{j=0}^{d} k_{r,j} \right) + y \sum_{j=1}^{d} \left(k_{r,j}(k_{r,j} + k_{r,0}) - l_{r,j}(k_{r,j} + k_{r,0} + 1) \right)$$

$$+ z \left(\sum_{j=0}^{d} \left(k_{r,j}^2 + k_{r,j} - l_{r,j}^2 - l_{r,j} \right) + 2k_{r,0} \sum_{j=1}^{d} k_{r,j} - 2l_{r,0} \sum_{j=1}^{d} l_{r,j} \right)$$

$$+ \gamma \left(\sum_{j=0}^{d} l_{r,j}^2 + 2l_{r,0} \sum_{j=1}^{d} l_{r,j} \right) \geq \sum_{j=0}^{d} k_{r,j}^2 + 2k_{r,0} \sum_{j=1}^{d} k_{r,j} \qquad \forall r \in R$$

$$x(k_{r,0} - l_{r,0}) + y \sum_{j=1}^{d}(k_{r,j} - l_{r,j}) + 2z \sum_{j=0}^{d}(k_{r,j} - l_{r,j}) + \gamma \sum_{j=0}^{d} l_{r,j} \geq \sum_{j=0}^{d} k_{r,j} \qquad \forall r \in R$$

$$x, y, z \geq 0.$$

Again, for the social cost function Perc, we obtain mutatis mutandis the following dual program.

$$\min \quad \gamma$$
$$s.t.$$

$$x \left(k_{r,0} \sum_{j=0}^{d} k_{r,j} - l_{r,0} - l_{r,0} \sum_{j=0}^{d} k_{r,j} \right) + y \sum_{j=1}^{d} \left(k_{r,j}(k_{r,j} + k_{r,0}) - l_{r,j}(k_{r,j} + k_{r,0} + 1) \right)$$

$$+ z \left(\sum_{j=0}^{d} \left(k_{r,j}^2 + k_{r,j} - l_{r,j}^2 - l_{r,j} \right) + 2k_{r,0} \sum_{j=1}^{d} k_{r,j} - 2l_{r,0} \sum_{j=1}^{d} l_{r,j} \right)$$

$$+ \gamma \left(\sum_{j=0}^{d} l_{r,j} \right)^2 \geq \left(\sum_{j=0}^{d} k_{r,j} \right)^2 \qquad \forall r \in R$$

$$x(k_{r,0} - l_{r,0}) + y \sum_{j=1}^{d}(k_{r,j} - l_{r,j}) + 2z \sum_{j=0}^{d}(k_{r,j} - l_{r,j}) + \gamma \sum_{j=0}^{d} l_{r,j} \geq \sum_{j=0}^{d} k_{r,j} \qquad \forall r \in R$$

$$x, y, z \geq 0.$$

Again, by setting $k_r := n_r(\sigma)$ and $l_r := n_r(\sigma^*)$ and replacing $k_{r,0}$ and $l_{r,0}$ with $k_r - k_{r,1} - k_{r,2}$ and $l_r - l_{r,1} - l_{r,2}$, respectively, DLP$(\text{Pres}, \sigma, \sigma^*)$ becomes

$$\min \quad \gamma$$
$$s.t.$$

$$x \left((k_r - k_{r,1} - k_{r,2})k_r - (l_r - l_{r,1} - l_{r,2})(k_r + 1) \right)$$
$$+ y \left(k_{r,1}(k_r - k_{r,2}) - l_{r,1}(k_r - k_{r,2} + 1) + k_{r,2}(k_r - k_{r,1}) - l_{r,2}(k_r - k_{r,1} + 1) \right)$$
$$+ z \left(k_r^2 - 2k_{r,1}k_{r,2} - l_r^2 + 2l_{r,1}l_{r,2} + k_r - l_r \right) + \gamma \left(l_r^2 - 2l_{r,1}l_{r,2} \right) \geq k_r^2 - 2k_{r,1}k_{r,2} \qquad \forall r \in R$$
$$x(k_r - k_{r,1} - k_{r,2} - l_r + l_{r,1} + l_{r,2}) + y(k_{r,1} + k_{r,2} - l_{r,1} - l_{r,2}) + 2z(k_r - l_r) + \gamma l_r \geq k_r \qquad \forall r \in R$$
$$x, y, z \geq 0.$$

Similarly, the dual program DLP$(\text{Perc}, \sigma, \sigma^*)$ can be rewritten as:

$$\min \quad \gamma$$
$$s.t.$$

$$x \left((k_r - k_{r,1} - k_{r,2})k_r - (l_r - l_{r,1} - l_{r,2})(k_r + 1) \right)$$
$$+ y \left(k_{r,1}(k_r - k_{r,2}) - l_{r,1}(k_r - k_{r,2} + 1) + k_{r,2}(k_r - k_{r,1}) - l_{r,2}(k_r - k_{r,1} + 1) \right)$$
$$+ z \left(k_r^2 - 2k_{r,1}k_{r,2} - l_r^2 + 2l_{r,1}l_{r,2} + k_r - l_r \right) + \gamma l_r^2 \geq k_r^2 \qquad \forall r \in R$$
$$x(k_r - k_{r,1} - k_{r,2} - l_r + l_{r,1} + l_{r,2}) + y(k_{r,1} + k_{r,2} - l_{r,1} - l_{r,2}) + 2z(k_r - l_r) + \gamma l_r \geq k_r \qquad \forall r \in R$$
$$x, y, z \geq 0.$$

Theorem 5. *For each bidimensional affine congestion game* $(\mathcal{G}, \mathcal{C})$, $PoS(\mathcal{G}, \mathcal{C}) \leq 1 + \frac{2}{\sqrt{7}}$ *under the social cost function* Pres *and* $PoS(\mathcal{G}, \mathcal{C}) \leq 2.92$ *under the social cost function* Perc.

Proof. For the social cost function Pres, set $\gamma = 1 + \frac{2}{\sqrt{7}}$, $x = y = \frac{1}{\sqrt{7}}$ and $z = \frac{1}{2} + \frac{1}{2\sqrt{7}}$. The second dual constraint is always satisfied, as $\min\{x, y\} \geq 1$ and $\max\{x, y\} + 2z \leq \gamma$. Thus, we shall focus again on the first constraint $g_1(\boldsymbol{k}, \boldsymbol{l}) \geq 0$. For any $r \in R$, $g_1(\boldsymbol{k}, \boldsymbol{l})$ becomes

$$k^2(3 - \sqrt{7}) - k(2l - 1 - \sqrt{7}) + 2k_1 k_2(\sqrt{7} - 3) + 2(k_1 l_2 + k_2 l_1) + (l^2 - l)(3 + \sqrt{7}) - 2l_1 l_2(3 + \sqrt{7}) \geq 0.$$

The claim follows by applying Lemma A9 reported in the Appendix A.

For the social cost function Perc, set $\gamma = 2.92$, $x = 0.68$, $y = 1.3$ and $z = 0.81$. Again, the second dual constraint is always satisfied, as $\min\{x, y\} \geq 1$ and $\max\{x, y\} + 2z \leq \gamma$. Thus, we shall focus again on the first constraint $g_1(\boldsymbol{k}, \boldsymbol{l}) \geq 0$. For any $r \in R$, $g_1(\boldsymbol{k}, \boldsymbol{l})$ become $49k^2 + k(62k_1 + 62k_2 - 68l - 62l_1 - 62l_2 + 81) + 130k_1 l_2 + 130k_2 l_1 - 422k_1 k_2 + 211l^2 - 149l + 2(81l_1 l_2 - 31l_1 - 31l_2) \geq 0$. The claim follows by applying Lemma A13 reported in the Appendix A. $\square$

For these cases, unfortunately, we are not able to devise matching lower bounds. The following result is obtained by suitably extending the lower bounding instance given in Reference [17] for the price of stability of congestion games (the proof is deferred to the Appendix).

Theorem 6. *For any $\epsilon > 0$, there exist two bidimensional linear congestion games* $(\mathcal{G}, \mathcal{C})$ *and* $(\mathcal{G}', \mathcal{C}')$ *such that* $PoS(\mathcal{G}, \mathcal{C}) \geq \frac{1 + \sqrt{5}}{2} - \epsilon$ *under the social cost function* Pres *and* $PoS(\mathcal{G}', \mathcal{C}') \geq \frac{5 + \sqrt{17}}{4} - \epsilon$ *under the social cost function* Perc.

7. Conclusions and Open Problems

We have introduced d-dimensional (weighted) congestion games: a generalization of (weighted) congestion games able to model various interesting scenarios of applications. They can also be reinterpreted as a particular subclass of that of graphical (weighted) congestion games defined by an undirected social knowledge graph whose independence number is equal to d. We have provided bounds for the price of anarchy and the price of stability of these games as a function of d under the two fundamental social cost functions sum of the players' perceived costs and sum of the players' presumed costs. We have also considered in deeper detail the case of $d = 2$ in presence of unweighted players only.

Closing the gap between upper and lower bounds is an intriguing and challenging open problem. In particular, we conjecture that the upper bound of $O(d)$ for the price of anarchy of d-dimensional weighted congestion games is asymptotically tight (with respect to d), even for unweighted games.

Along the line of research of improving the performance of congestion games via some feasible strategies or coordination (e.g., taxes [27,28] or Stackelberg strategies [29,30]), another interesting research direction is partitioning the players into $d + 1$ clusters similarly as in d-dimensional games, to improve as much as possible the price of anarchy or the price of stability.

A further research direction is that of combining the model of multidimensional congestion games with other variants of congestion games (e.g., risk-averse congestion games [31–34] and congestion games with link failures [35–37]).

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Appendix A. Technical Lemmas

In this section we gather all technical lemmas needed to prove our main theorems.

Lemma A1. *For any $d \geq 0$, let $Q = (q_{i,j})_{i,j \in [d] \cup \{0\}}$ be the $(d+1) \times (d+1)$ matrix such that: (i) $q_{i,j} = \sqrt{d}$ if $i = j$; (ii) $q_{i,j} = 1$ if either $i = 0$, or $j = 0$, with $(i,j) \neq (0,0)$; (iii) $q_{i,j} = 0$ otherwise. We have that Q is a positive-semidefinite matrix.*

Proof. To show the claim, we resort to the Sylvester's criterion, stating that a symmetric matrix M is positive-semidefinite if and only if the determinant of each principal minor of M (i.e., each upper upper left h-by-h corner of M) is non-negative. Let $A_{h,x} = (a_{h,x,i,j})_{i,j \in [h]}$ be a $h \times h$ matrix such that: (i) $a_{h,x,i,j} = x$ if $i = j$; (ii) $a_{h,x,i,j} = 1$ if $(i,j) \neq (1,1)$, and, either $i = 1$, or $j = 1$; (iii) $a_{h,x,i,j} = 0$ otherwise. We have that each principal minor of matrix Q is of type $A_{h,\sqrt{d}}$ for some $h \in [d+1]$. Thus, it is sufficient showing that the determinant of matrix $A_{h,\sqrt{d}}$, denoted as $Det(A_{h,\sqrt{d}})$, is non-negative for any $h \in [d+1]$.

We first show by induction on integers $h \geq 1$ that $Det(A_{h,x}) = x^h - (h-1) \cdot x^{h-2}$ for any fixed $x \in \mathbb{R}$. If $h = 0$ we trivially get $Det(A_{h,x}) = x = x^h - (h-1) \cdot x^{h-2}$. Now, we assume that $Det(A_{h,x}) = x^h - (h-1)x^{h-2}$ holds for some $h \geq 1$, and we show that $Det(A_{h+1,x}) = x^{h+1} - h \cdot x^{h-1}$. We get $Det(A_{h+1,x}) = x \cdot Det(A_{h,x}) - x^{h-1} = x(x^h - (h-1)x^{h-2}) - x^{h-1} = x^{h+1} - h \cdot x^{h-1}$, where the first equality comes from the Laplace expansion for computing the determinant, and the second equality comes from the inductive hypothesis.

By using the fact that $Det(A_{h,x}) = x^h - (h-1) \cdot x^{h-2}$ holds for any $x \in \mathbb{R}$ and any integer $h \geq 1$, we have that $Det(A_{h,\sqrt{d}}) = (\sqrt{d})^h - (h-1)(\sqrt{d})^{h-2} \geq 0$ for any $h \in [d+1]$, where the last inequality holds since quantity $x^h - (h-1)x^{h-1}$ is always non-negative for any $x \geq \sqrt{h-1}$ if $h \leq d+1$. Thus each principal minor of Q has a non-negative determinant, and the claim follows. $\square$

Lemma A2. *Let $\theta : \mathbb{Z}_{\geq 0}^6 \to \mathbb{Q}$ be the function such that $\theta(a,b,c,d,e,f) = 18a^2 - a(b+c+51d-e-f) + 50bf + 50ce - 34bc + 119d^2 - 51d + e + f - 238ef$. For any $(a,b,c,d,e,f) \in \mathbb{Z}_{\geq 0}^6$ such that $a \geq b+c$ and $d \geq e+f$, it holds that $\theta(a,b,c,d,e,f) \geq 0$.*

Proof. At a first glance, in order to use standard arguments from calculus, we allow the 6-tuples (a,b,c,d,e,f) to take values in the set of non-negative real numbers.

We first show that, in such an extended scenario, θ attains its minimum for 6-tuples (a,b,c,d,e,f) such that $b = c$ and $e = f$. Consider to this aim the 6-tuple $(a,b,b+h,d,e,e+k)$, where $h,k \in \mathbb{R}$. By definition of θ, we get $\theta(a,b+\frac{h}{2},b+\frac{h}{2},d,e+\frac{k}{2},e+\frac{k}{2}) = \theta(a,b,b+h,d,e,e+k) - \frac{17h^2 - 50hk + 119k^2}{2} \leq \theta(a,b,b+h,d,e,e+k) - \frac{(4h-10k)^2}{2} \leq \theta(a,b,b+h,d,e,e+k)$.

Hence, we do not lose in generality by restricting to 6-tuples of non-negative real values (a,b,b,d,e,e) such that $a \geq 2b$ and $d \geq 2e$. In this case θ becomes $18a^2 - a(2b+51d-2e) - 34b^2 + 100be + 119d^2 - 51d - 238e^2 + 2e$. Consider the two partial derivatives $\frac{\delta\theta}{\delta b} = 100e - 2a - 68b$ and $\frac{\delta\theta}{\delta e} = 2(a+50b+1-238e)$. Since they are linear and decreasing in b and e, respectively, it follows that θ is minimized at one of the following four cases: $b = 0 \wedge e = 0$, $b = 0 \wedge e = \frac{d}{2}$, $b = \frac{a}{2} \wedge e = 0$ and $b = \frac{a}{2} \wedge e = \frac{d}{2}$.

In the first case, θ becomes $18a^2 - 51ad + 119d^2 - 51d$. Since $\frac{\delta\theta}{\delta a} = 36a - 51d$, θ is minimized at $a = \frac{17d}{12}$. By substituting, θ becomes $\frac{1}{8}(663d^2 - 408d)$ which is always non-negative for any $d \in \mathbb{Z}$.

In the second case, θ becomes $36a^2 - 100ad + 119d^2 - 100d$. Since $\frac{\delta\theta}{\delta a} = 36a - 50d$, θ is minimized at $a = \frac{25d}{18}$. By substituting, θ becomes $\frac{1}{9}(223d^2 - 450d)$ which is always non-negative for any $d \in \mathbb{Z} \setminus \{1,2\}$.

In the third case, θ becomes $\frac{17}{2}(a^2 - 6ad + 14d^2 - 6d)$. Since $\frac{\delta\theta}{\delta a} = 17(a-3d)$, θ is minimized at $a = 3d$. By substituting, θ becomes $\frac{17}{2}(5d^2 - 6d)$ which is always non-negative for any $d \in \mathbb{Z} \setminus \{1\}$.

In the fourth case, θ becomes $\frac{1}{2}(17a^2 - 50ad + 119d^2 - 100d)$. Since $\frac{\delta\theta}{\delta a} = 17a - 25d$, θ is minimized at $a = \frac{25d}{17}$. By substituting, θ becomes $\frac{1}{17}(699d^2 - 850d)$ which is always non-negative for any $d \in \mathbb{Z} \setminus \{1\}$.

Hence, in order to complete the proof, we are left to settle the following cases: $(a, 0, 0, 1, 0, 0)$, $(a, 0, 0, 2, 1, 1)$, $(a, 0, 0, 1, 1, 0)$, $(a, 0, 0, 1, 0, 1)$, $(a, \frac{a}{2}, \frac{a}{2}, 1, 1, 0)$, $(a, \frac{a}{2}, \frac{a}{2}, 1, 0, 1)$ and $(a, \frac{a}{2}, \frac{a}{2}, 1, 0, 0)$.

In the case $(a, 0, 0, 1, 0, 0)$, θ becomes $18a^2 - 51a + 68$ which is always non-negative for any $a \in \mathbb{R}$. In the case $(a, 0, 0, 2, 1, 1)$, θ becomes $18a^2 - 100a + 138$ which is always non-negative for any $a \in \mathbb{Z}$. In the cases $(a, 0, 0, 1, 1, 0)$ and $(a, 0, 0, 1, 0, 1)$, θ becomes $18a^2 - 50a + 69$ which is always non-negative for any $a \in \mathbb{R}$. In the cases $(a, \frac{a}{2}, \frac{a}{2}, 1, 1, 0)$ and $(a, \frac{a}{2}, \frac{a}{2}, 1, 0, 1)$, θ becomes $\frac{17a^2 - 50a + 138}{2}$ which is always non-negative for any $a \in \mathbb{R}$. Finally, in the case $(a, \frac{a}{2}, \frac{a}{2}, 1, 0, 0)$, θ becomes $\frac{17}{2}(a^2 - 6a + 8)$ which is always non-negative for any $a \in \mathbb{Z} \setminus \{3\}$. Hence, we are only left to consider the case $(3, b, c, 1, 0, 0)$ for which θ becomes $77 - 34bc - 3(b + c)$. Since $b + c \leq 3$, it holds that $77 - 34bc - 3(b + c) \geq 68 - 34bc$ which is always non-negative since $bc \leq 2$ for any $b, c \in \mathbb{Z}_{\geq 0}$ such that $b + c \leq 3$. $\square$

Lemma A3. *Let $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ be the function such that $\lambda(a, d) = a^2 - 3ad + 5d^2 - 3d$. For any $(a, d) \in \mathbb{Z}_{\geq 0}^2$ such that $d \neq 1$, it holds that $\lambda(a, d) \geq 0$.*

Proof. Since $\frac{\delta\lambda}{\delta a} = 2a - 3d$, λ is minimized at $a = \frac{3}{2}d$. By substituting, we get $11d^2 - 12d$ which is non-negative for any $d \in \mathbb{Z} \setminus \{1\}$. $\square$

Lemma A4. *Let $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ be the function such that $\lambda(a, d) = a^2 - 6ad + 14d^2 - 6d$. For any $(a, d) \in \mathbb{Z}_{\geq 0}^2$ such that $d \neq 1$, it holds that $\lambda(a, d) \geq 0$.*

Proof. Since $\frac{\delta\lambda}{\delta a} = 2a - 6d$, λ is minimized at $a = 3d$. By substituting, we get $5d^2 - 6d$ which is non-negative for any $d \in \mathbb{Z} \setminus \{1\}$. $\square$

Lemma A5. *Let $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ be the function such that $\lambda(a, d) = 20a^2 - 84ad + 259d^2 - 168d$. For any $(a, d) \in \mathbb{Z}_{\geq 0}^2$, it holds that $\lambda(a, d) \geq 0$.*

Proof. Since $\frac{\delta\lambda}{\delta a} = 40a - 84d$, λ is minimized at $a = \frac{21}{10}d$. By substituting, we get $61d^2 - 60d$ which is non-negative for any $d \in \mathbb{Z}$. $\square$

Lemma A6. *Let $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ be the function such that $\lambda(a, d) = 13a^2 - 21ad + 35d^2 - 21d$. For any $(a, d) \in \mathbb{Z}_{\geq 0}^2$, it holds that $\lambda(a, d) \geq 0$.*

Proof. Since $\frac{\delta\lambda}{\delta a} = 26a - 21d$, λ is minimized at $a = \frac{21}{26}d$. By substituting, we get $197d^2 - 156d$ which is non-negative for any $d \in \mathbb{Z}$. $\square$

Lemma A7. *Let $\theta : \mathbb{Z}_{\geq 0}^6 \to \mathbb{Q}$ be the function such that $\theta(a, b, c, d, e, f) = 7a^2 + 3a(2b + 2c - 5d - 2e - 2f) + 21bf + 21ce - 42bc + 35d^2 - 15d - 6e - 6f$. For any $(a, b, c, d, e, f) \in \mathbb{Z}_{\geq 0}^6$ such that $a \geq b + c$ and $d \geq e + f$, it holds that $\theta(a, b, c, d, e, f) \geq 0$.*

Proof. At a first glance, in order to use standard arguments from calculus, we allow the 6-tuples (a, b, c, d, e, f) to take values in the set of non-negative real numbers. Since $\frac{\delta\theta}{\delta c} = 3(2a - 14b + 7e)$ and $\frac{\delta\theta}{\delta f} = 3(7b - 2a - 2)$, θ is minimized at one of the following four cases: $c = 0 \wedge f = 0$, $c = 0 \wedge f = d - e$, $c = a - b \wedge f = 0$ and $c = a - b \wedge f = d - e$.

In the first case, we get $\theta = 7a^2 + 3a(2b - 5d - 2e) + 35d^2 - 15d - 6e$. Since $\frac{\delta\theta}{\delta b} = 6a$, θ is minimized at $b = 0$ which yields $\theta = 7a^2 - 3a(5d + 2e) + 35d^2 - 15d - 6e$. Since $\frac{\delta\theta}{\delta e} = -6(a + 1)$, θ is minimized at $e = d$ which yields $\theta = 7(a^2 - 3ad + 5d^2 - 3d)$. The claim then follows for any $d \neq 1$ by applying Lemma A3. For the leftover tuples of the form $(a, 0, 0, 1, 1, 0)$, we get $\theta = 7(a^2 - 3a + 2)$ which is always non-negative for any $a \in \mathbb{Z}$.

In the second case, we get $\theta = 7a^2 + 3a(2b - 7d) + 7(3b(d - e) + 5d^2 - 3d)$. Since $\frac{\delta\theta}{\delta b} = 3(2a + 7(d - e))$, θ is minimized at $b = 0$, which yields $\theta = 7(a^2 - 3ad + 5d^2 - 3d)$. The claim then follows for any $d \neq 1$ by applying Lemma A3. For the leftover tuples of the form $(a, 0, 0, 1, e, 1 - e)$, we get $\theta = 7(a^2 - 3a + 2)$ which is always non-negative for any $a \in \mathbb{Z}$.

In the third case, we get $\theta = 13a^2 - 3a(14b + 5(d - e)) + 42b^2 - 21be + 35d^2 - 15d - 6e$. Since $\frac{\delta\theta}{\delta e} = 3(5a - 7b - 2)$, θ is minimized at either $e = 0$ or $e = d$. For $e = d$, we get $\theta = 13a^2 - 42ab + 42b^2 - 21bd + 35d^2 - 21d$. Since $\frac{\delta\theta}{\delta b} = -21(2a - 4b + d)$, θ is minimized at $b = \frac{2a+d}{4}$. This yields $\theta = 20a^2 - 84ad + 259d^2 - 168d$ and the claim then follows by applying Lemma A5. For $e = 0$, we get $\theta = 13a^2 - 3a(14b + 5d) + 42b^2 + 35d^2 - 15d$. Since $\frac{\delta\theta}{\delta b} = 42(2b - a)$, θ is minimized at $b = \frac{a}{2}$ which yields $\theta = 5(a^2 - 6ad + 14d^2 - 6d)$ and the claim then follows for any $d \neq 1$ by applying Lemma A4. For the leftover tuples of the form $(a, \frac{a}{2}, \frac{a}{2}, 1, 0, 0)$, we get $\theta = \frac{5}{2}(a^2 - 6a + 8)$ which is always non-negative for any $a \in \mathbb{Z} \setminus \{3\}$. Hence, we are still left to prove what happens for the tuples of the form $(3, b, 3 - b, 1, 0, 0)$. In this case, we get $\theta = 42b^2 - 126b + 92$ which is always non-negative for any $b \in \mathbb{Z}$.

In the fourth case, we get $\theta = 13a^2 - 21a(2b + d - e) + 7(6b^2 + 3b(d - 2e) + 5d^2 - 3d)$. Since $\frac{\delta\theta}{\delta e} = 21(a - 2b)$, θ is minimized at either $e = 0$ or $e = d$. For $e = 0$, we get $\theta = 13a^2 - 21a(2b + d) + 7(6b^2 + 3bd + 5d^2 - 3d)$. Since $\frac{\delta\theta}{\delta b} = -21(2a - 4b - d)$, θ is minimized at either $b = 0$ or $b = \frac{2a-d}{4}$. The first case yields $\theta = 13a^2 - 21ad + 35d^2 - 21d$ and the claim then follows by applying Lemma A6, while the second one yields $\theta = \frac{20a^2 - 84ad + 259d^2 - 168d}{8}$ and the claim then follows by applying Lemma A5. For $e = d$, we get $\theta = 13a^2 - 41ab + 7(6b^2 - 3bd + 5d^2 - 3d)$. Since $\frac{\delta\theta}{\delta b} = -21(2a - 4b + d)$, θ is minimized at $b = \frac{2a+d}{4}$ which yields $\theta = \frac{20a^2 - 84ad + 259d^2 - 168d}{8}$ and the claim then follows by applying Lemma A5. □

Lemma A8. *Let* $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ *be the function such that* $\lambda(a, d) = \frac{3 - \sqrt{7}}{2}a^2 + (1 + \sqrt{7} - 2d)a + (3 + \sqrt{7})(\frac{d^2}{2} - d)$. *For any* $(a, d) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 1), (1, 1), (1, 2)\}$, *it holds that* $\lambda(a, d) \geq 0$.

Proof. Since $\frac{\delta\lambda}{\delta a} = (3 + \sqrt{7})a - 2d + 1 + \sqrt{7}$, λ is minimized at either $a = 0$ or $a = \frac{2d - 1 - \sqrt{7}}{3 + \sqrt{7}}$.

In the first case, λ becomes $\frac{3 - \sqrt{7}}{2}d(d - 2)$ which is always non-negative for any $d \in \mathbb{Z}_{\geq 0} \setminus \{1\}$.

In the second case, λ becomes $\frac{1}{2}(3(\sqrt{7} - 1)d^2 + 2d(\sqrt{7} - 7) + \sqrt{7} - 5)$ which is always non-negative for any $d \in \mathbb{Z}_{\geq 0} \setminus \{1, 2\}$. For the leftover case $d = 2$, λ becomes $\frac{3 - \sqrt{7}}{2}a^2 + (\sqrt{7} - 3)a$ which is always non-negative for any $a \in \mathbb{Z} \setminus \{1\}$. For the other case $d = 1$, λ becomes $\frac{3 - \sqrt{7}}{2}a^2 + (\sqrt{7} - 1)a - \frac{3 + \sqrt{7}}{2}$ which is always non-negative for any $a \in \mathbb{Z} \setminus \{0, 1\}$. □

Lemma A9. *Let* $\theta : \mathbb{Z}_{\geq 0}^6 \to \mathbb{Q}$ *be the function such that* $\theta(a, b, c, d, e, f) = a^2(3 - \sqrt{7}) - a(2d - 1 - \sqrt{7}) + 2bc(\sqrt{7} - 3) + 2(bf + ce) + (d^2 - d)(3 + \sqrt{7}) - 2(3 + \sqrt{7})ef$. *For any* $(a, b, c, d, e, f) \in \mathbb{Z}_{\geq 0}^6$ *such that* $a \geq b + c$ *and* $d \geq e + f$, *it holds that* $\theta(a, b, c, d, e, f) \geq 0$.

Proof. Note first, that for 6-tuples of the form $(0, b, c, 1, e, f)$, it holds that $\theta = 0$, since $a = 0 \Rightarrow b = c = 0$ and $d = 1 \Rightarrow ef = 0$, for 6-tuples of the form $(1, b, c, 1, e, f)$, it holds that $\theta = 2(1 + bf + ce) > 0$, since $a = d = 1 \Rightarrow bc = ef = 0$, and for 6-tuples of the form $(1, b, c, 2, e, f)$, it holds that $\theta = 2bf + 2ce - 2(3 + \sqrt{7})ef + 2(3 + \sqrt{7}) \geq 0$, since $d = 2 \Rightarrow ef \leq 1$. Hence, in the sequel of the proof, we avoid to consider the cases $a = 0 \wedge d = 1$, $a = d = 1$ and $a = 1 \wedge d = 2$.

At a first glance, in order to use standard arguments from calculus, we allow the 6-tuples (a, b, c, d, e, f) to take values in the set of non-negative real numbers. Since it holds that $\frac{\delta\theta}{\delta c} = 2(b(\sqrt{7} - 3) + e)$ and $\frac{\delta\theta}{\delta f} = 2(b - (\sqrt{7} + 3)e)$, θ is minimized at one of the following four cases: $c = 0 \wedge f = 0$, $c = 0 \wedge f = d - e$, $c = a - b \wedge f = 0$ and $c = a - b \wedge f = d - e$.

In the first case, we get $\theta = (3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(d^2 - d)$. The claim follows by applying Lemma A8, since $\theta \geq \lambda$.

In the second case, we get $\theta = (3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(d^2 - d) + 2(d - e)(b - (3 + \sqrt{7})e)$. Since $\frac{\delta\theta}{\delta b} = 2(d - e)$, θ is minimized at $b = 0$, which yields $\theta = (3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(d^2 - d) - 2(d - e)(3 + \sqrt{7})e$. Since $\frac{\delta\theta}{\delta e} = 4(3 + \sqrt{7})e - 2d(3 + \sqrt{7})$, θ is minimized for $e = \frac{d}{2}$. In this case, θ becomes $(3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(\frac{d^2}{2} - d)$. The claim follows by applying Lemma A8, since $\theta \geq \lambda$.

In the third case, we get $\theta = (3 - \sqrt{7})a^2 + (\sqrt{7} + 1 + 2e - 2d)a + (3 + \sqrt{7})(d^2 - d) + 2b^2(3 - \sqrt{7}) + 2b((\sqrt{7} - 3)a - e)$. Since $\frac{\delta\theta}{\delta e} = 2(a - b)$, θ is minimized for $e = 0$, which yields $\theta = (3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(d^2 - d) + 2b^2(3 - \sqrt{7}) + 2ab(\sqrt{7} - 3)$. Since $\frac{\delta\theta}{\delta b} = 4(3 - \sqrt{7})b - 2a(3 - \sqrt{7})$, θ is minimized for $b = \frac{a}{2}$. In this case, θ becomes $\frac{3 - \sqrt{7}}{2}a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(d^2 - d)$. The claim follows by applying Lemma A8, since $\theta \geq \lambda$.

In the fourth case, we get $\theta = (3 - \sqrt{7})a^2 + (\sqrt{7} + 1 + 2e - 2d)a + (3 + \sqrt{7})(d^2 - d) + 2b^2(3 - \sqrt{7}) + 2(\sqrt{7} - 3)ab + 2bd - 4be + 2(3 + \sqrt{7})e^2$. Since $\frac{\delta\theta}{\delta b} = 4(3 - \sqrt{7})b + 2(\sqrt{7} - 3)a + 2d - 4e$, θ is minimized at either $b = 0$ or $b = \frac{(3 - \sqrt{7})a + 2e - d}{2(3 - \sqrt{7})}$. For $b = 0$, θ becomes $(3 - \sqrt{7})a^2 + (\sqrt{7} + 1 + 2e - 2d)a + (3 + \sqrt{7})(d^2 - d) + 2(3 + \sqrt{7})e^2$. Since $\frac{\delta\theta}{\delta e} = 2a - 2(3 + \sqrt{7})d + 4(3 + \sqrt{7})e$, θ is minimized at either $e = 0$ or $e = \frac{(3 + \sqrt{7})d - a}{2(3 + \sqrt{7})}$. In these two cases, θ becomes, respectively, $(3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - 2d)a + (3 + \sqrt{7})(d^2 - d)$ and $\frac{3}{4}(3 - \sqrt{7})a^2 + (\sqrt{7} + 1 - d)a + (3 + \sqrt{7})(\frac{d^2}{2} - d)$ which are always non-negative because of Lemma A8 and the fact that $\theta \geq \lambda$. For $b = \frac{(3 - \sqrt{7})a + 2e - d}{2(3 - \sqrt{7})}$, θ becomes $\frac{3 - \sqrt{7}}{2}a^2 + (\sqrt{7} + 1 - d)a + (3 + \sqrt{7})(\frac{3d^2}{4} - d) - (3 + \sqrt{7})de + (3 + \sqrt{7})e^2$. Since $\frac{\delta\theta}{\delta e} = (3 + \sqrt{7})(2e - d)$, θ is minimized at either $e = 0$ or $e = \frac{d}{2}$. In these two cases, θ becomes, respectively, $\frac{3 - \sqrt{7}}{2}a^2 + (\sqrt{7} + 1 - d)a + (3 + \sqrt{7})(\frac{3d^2}{4} - d)$ and $\frac{3 - \sqrt{7}}{2}a^2 + (\sqrt{7} + 1 - d)a + (3 + \sqrt{7})(\frac{d^2}{2} - d)$ which are always non-negative because of Lemma A8 and the fact that $\theta \geq \lambda$. □

Lemma A10. *Let* $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ *be the function such that* $\lambda(a, d) = 49a^2 + a(81 - 130d) + 211d^2 - 211d$. *For any* $(a, d) \in \mathbb{Z}_{\geq 0}^2$, *it holds that* $\lambda(a, d) \geq 0$.

Proof. Since $\frac{\delta\lambda}{\delta a} = 98a - 130d + 81$, λ is minimized at either $a = 0$ or $a = \frac{130d - 81}{98}$.

In the first case, λ becomes $d(d - 1)$ which is always non-negative for any $d \in \mathbb{Z}$.

In the second case, λ becomes $d(3057d - 2537) - \frac{6561}{8}$ which is always non-negative for any $d \in \mathbb{Z} \setminus \{0, 1\}$. For the leftover case $d = 0$, λ becomes $49a^2 + 81a$, which is non-negative for any $a \in \mathbb{R}$. For the other case of $d = 1$, λ becomes $a(a - 1)$ which is non-negative for any $a \in \mathbb{Z}$. □

Lemma A11. *Let* $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ *be the function such that* $\lambda(a, d) = 11a^2 + a(81 - 68d) + 422d^2 - 298d$. *For any* $(a, d) \in \mathbb{Z}_{\geq 0}^2$, *it holds that* $\lambda(a, d) \geq 0$.

Proof. Since $\frac{\delta\lambda}{\delta a} = 222a - 2(68d - 81)$, λ is minimized at either $a = 0$ or $a = \frac{68d - 81}{11}$.

In the first case, λ becomes $d(211d - 149)$ which is always non-negative for any $d \in \mathbb{Z}$.

In the second case, λ becomes $d(9d + 3869) - \frac{6561}{2}$ which is always non-negative for any $d \in \mathbb{Z}_{\geq 0} \setminus \{0\}$. For the leftover case of $d = 0$, λ becomes $11a^2 + 162a$ which is non-negative for any $a \in \mathbb{R}$. □

Lemma A12. *Let* $\lambda : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Q}$ *be the function such that* $\lambda(a, d) = 2321a^2 + 422a(81 - 65d) + 84817d^2 - 89042d$. *For any* $(a, d) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 1)\}$, *it holds that* $\lambda(a, d) \geq 0$.

Proof. Since $\frac{\delta\lambda}{\delta a} = 4642a - 422(65d - 81)$, λ is minimized at either $a = 0$ or $a = \frac{65d - 81}{11}$.

In the first case, λ becomes $84817d^2 - 89042d$ which is always non-negative for any $d \in \mathbb{Z} \setminus \{1\}$.

In the second case, λ becomes $d(5189d + 155296) - \frac{1384371}{8}$ which is always non-negative for any $d \in \mathbb{Z}_{\geq 0} \setminus \{0, 1\}$. For the leftover case $d = 0$, λ becomes $11a^2 + 162a$, which is non-negative for

any $a \in \mathbb{R}$. For the other case of $d = 1$, λ becomes $a(11a + 32) - \frac{4225}{211}$ which is non-negative for any $a \in \mathbb{Z}_{\geq 0} \setminus \{0\}$. □

Lemma A13. *Let $\theta : \mathbb{Z}_{\geq 0}^6 \to \mathbb{Q}$ be the function such that $\theta(a, b, c, d, e, f) = 49a^2 + a(62b + 62c - 68d - 62e - 62f + 81) + 130bf + 130ce - 422bc + 211d^2 - 149d + 162ef - 62e - 62f$. For any $(a, b, c, d, e, f) \in \mathbb{Z}_{\geq 0}^6$ such that $a \geq b + c$ and $d \geq e + f$, it holds that $\theta(a, b, c, d, e, f) \geq 0$.*

Proof. At a first glance, in order to use standard arguments from calculus, we allow the 6-tuples (a, b, c, d, e, f) to take values in the set of non-negative real numbers. Since it holds that $\frac{\delta\theta}{\delta c} = 62a - 42b + 130e$ and $\frac{\delta\theta}{\delta f} = -2(31a - 65b - 81e + 31)$, θ is minimized at one of the following four cases: $c = 0 \wedge f = 0$, $c = 0 \wedge f = d - e$, $c = a - b \wedge f = 0$ and $c = a - b \wedge f = d - e$.

In the first case, we get $\theta = 49a^2 + a(62b - 68d - 62e + 81) + 211d^2 - 149d - 62e$. Since $\frac{\delta\theta}{\delta e} = -62(a + 1)$, θ is minimized at $e = d$, which yields $\theta = 49a^2 + a(62b - 130d + 81) + 211d^2 - 211d$. Since $\frac{\delta\theta}{\delta b} = 62a$, θ is minimized for $b = 0$. In this case, θ becomes $49a^2 + a(81 - 130d) + 211d^2 - 211d$. The claim follows by applying Lemma A10.

In the second case, we get $\theta = 49a^2 + a(62b - 130d + 81) + 130b(d - e) + 211d^2 + d(162e - 211) - 162e^2$. Since $\frac{\delta\theta}{\delta b} = 62a + 130(d - e)$, θ is minimized at $b = 0$, which yields $\theta = 49a^2 + a(81 - 130d) + 211d^2 + d(162e - 211) - 162e^2$. Since $\frac{\delta\theta}{\delta e} = 162d - 324e$, θ is minimized at either $e = 0$ and $e = d$. In both cases θ becomes $49a^2 + a(81 - 130d) + 211d^2 - 211d$ and the claim follows by applying Lemma A10.

In the third case, we get $\theta = 111a^2 - a(422b + 68d - 68e - 81) + 422b^2 - 130be + 211d^2 - 149d - 62e$. Since $\frac{\delta\theta}{\delta b} = -422a + 844b - 130e$, θ is minimized at $b = \frac{211a + 65e}{422}$, which yields $\theta = 2321a^2 + 422a(3e + 81 - 68d) + 89042d^2 - 62878d - 4225e^2 - 26164e$. Since $\frac{\delta\theta}{\delta e} = 1266a - 8450e - 26164$, θ is minimized at either $e = 0$ or $e = d$. For $e =$, θ becomes $11a^2 + 2a(81 - 68d) + 422d^2 - 298d$ and the claim follows by applying Lemma A11. For $e = d$, θ becomes $2321a^2 + 422a(81 - 65d) + 84817d^2 - 89042d$ and the claim follows for any 6-tuple (a, b, c, d, e, f) such that $(a, d) \neq (0, 1)$ by applying Lemma A12. Hence, we are left to consider the 6-tuples of the form $(0, 0, 0, 1, e, 0)$. In this case θ becomes $62(1 - e)$ which is always non-negative since $e \in \{0, 1\}$.

In the fourth case, we get $\theta = 111a^2 - a(422b + 130d - 130e - 81) + 422b^2 + 130b(d - 2e) + 211d^2 + d(162e - 211) - 162e^2$. Since $\frac{\delta\theta}{\delta b} = -422a + 844b + 130(d - 2e)$, θ is minimized at either $b = 0$ or $b = \frac{211a - 65(d - 2e)}{422}$. For $b = 0$, θ becomes $111a^2 - a(130d - 130e - 81) + 211d^2 + d(162e - 211) - 162e^2$. Since $\frac{\delta\theta}{\delta e} = 130a + 162d - 324e$, θ is minimized at either $e = 0$ or $e = d$. In these two cases, θ becomes, respectively, $111a^2 + a(81 - 130d) + 211d^2 - 211d$ and $111a^2 + 81a + 211d^2 - 211d$ which are always non-negative because of Lemma A10 and the fact the $\theta \geq \lambda$. For $b = \frac{211a - 65(d - 2e)}{422}$, θ becomes $2321a^2 + 422a(81 - 65d) + 84817d^2 + 2d(42632e - 44521) - 85264e^2$. Since $\frac{\delta\theta}{\delta e} = 85264d - 170528e$, θ is minimized at either $e = 0$ or $e = d$. In both cases, θ becomes $2321a^2 + 422a(81 - 65d) + 84817d^2 - 89042d$ and the claim follows for any 6-tuple (a, b, c, d, e, f) such that $(a, d) \neq (0, 1)$ by applying Lemma A12. Hence, we are left to consider the 6-tuples of the form $(0, 0, 0, 1, e, 1 - e)$. In this case, θ becomes $162(1 - e)$ which is always non-negative since $e \in \{0, 1\}$. □

Appendix B. Missing Proofs

Theorem A1 (Claim of Theorem 4). *There exist two bidimensional linear congestion games $(\mathcal{G}, \mathcal{C})$ and $(\mathcal{G}', \mathcal{C}')$ such that $PoA(\mathcal{G}, \mathcal{C}) \geq \frac{119}{33}$ under the social cost function* Pres *and $PoA(\mathcal{G}', \mathcal{C}') \geq \frac{35}{8}$ under the social cost function* Perc.

Proof. For the social cost function Pres, consider the game $(\mathcal{G}, \mathcal{C})$ depicted in Figure A1a). First, we show that σ is a pure Nash equilibrium for $(\mathcal{G}, \mathcal{C})$, that is, no player can lower her perceived cost by switching to her optimal strategy. Player 1 is paying $27 \cdot 2 + 46 = 100$; by switching to σ_1^*, she pays $7 \cdot 4 + 18 \cdot 4 = 100$. Player 2 is paying $27 \cdot 2 + 42 + 56 = 152$; by switching to σ_2^*, she pays $17 \cdot 4 + 21 \cdot 4 = 152$. Player 3 is paying $27 \cdot 2 + 42 = 96$; by switching to σ_3^*, she pays $7 \cdot 4 + 17 \cdot 4 = 96$. Player 4 is paying $27 \cdot 2 + 46 + 56 = 156$; by switching to σ_4^*, she pays $18 \cdot 4 + 21 \cdot 4 = 156$.

Player 5 is paying $7 \cdot 3 + 17 \cdot 3 + 21 \cdot 3 = 135$; by switching to σ_5^*, she pays $27 \cdot 5 = 135$. Player 6 is paying $7 \cdot 3 + 18 \cdot 3 + 21 \cdot 3 = 138$; by switching to σ_6^*, she pays $46 \cdot 3 = 138$. Player 7 is paying $7 \cdot 3 + 18 \cdot 3 + 17 \cdot 3 = 126$; by switching to σ_7^*, she pays $42 \cdot 3 = 126$. Player 8 is paying $18 \cdot 3 + 17 \cdot 3 + 21 \cdot 3 = 168$; by switching to σ_8^*, she pays $56 \cdot 3 = 168$.

The price of anarchy of $(\mathcal{G}, \mathcal{C})$ is then lower bounded by the ratio

$$\frac{100 + 152 + 96 + 156 + 135 + 138 + 126 + 168}{25 + 38 + 24 + 39 + 27 + 46 + 42 + 56} = \frac{1071}{297} = \frac{119}{33}.$$

For the social cost function Perc, consider the game $(\mathcal{G}', \mathcal{C}')$ depicted in Figure A1b). First, we show that σ is a pure Nash equilibrium for $(\mathcal{G}', \mathcal{C}')$, that is, no player can lower her perceived cost by switching to her optimal strategy. Player 1 is paying $1418 + 958 + 189 \cdot 2 = 2754$; by switching to σ_1^*, she pays $918 \cdot 3 = 2754$. Player 2 is paying $616 + 221 + 189 \cdot 2 = 1215$; by switching to σ_2^*, she pays $405 \cdot 3 = 1215$. Player 3 is paying $1418 + 616 + 189 \cdot 2 = 2412$; by switching to σ_3^*, she pays $804 \cdot 3 = 2412$. Player 4 is paying $958 + 221 + 189 \cdot 2 = 1557$; by switching to σ_4^*, she pays $519 \cdot 3 = 1557$. Player 5 is paying $(918 + 405 + 804) \cdot 2 = 4254$; by switching to σ_5^*, she pays $1418 \cdot 3 = 4254$. Player 6 is paying $(918 + 519) \cdot 2 = 2874$; by switching to σ_6^*, she pays $958 \cdot 3 = 2874$. Player 7 is paying $(405 + 519) \cdot 2 = 1848$; by switching to σ_7^*, she pays $616 \cdot 3 = 1848$. Player 8 is paying $804 \cdot 2 = 1608$; by switching to σ_8^*, she pays $221 \cdot 3 + 189 \cdot 5 = 1608$.

By noting that the perceived cost of the first four players is exactly twice their presumed one, the price of anarchy of $(\mathcal{G}', \mathcal{C}')$ is then lower bounded by the ratio

$$\frac{2 \cdot (2754 + 1215 + 2412 + 1557) + 4254 + 2874 + 1848 + 1608}{1418 + 958 + 616 + 221 + 189 + 918 + 405 + 804 + 519} = \frac{26460}{6048} = \frac{35}{8}.$$

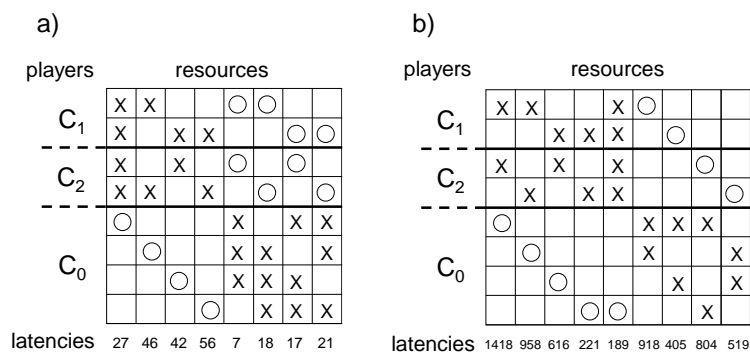$\square$

Figure A1. The games depicted in figures (**a,b**) represent the lower bound instances w.r.t. the social cost functions Pres and Perc, respectively. Each column in the matrix represents a resource of cost function $\ell(x) = \alpha x$ whose coefficient α is reported at the bottom of the column. Each row i in the matrix models the strategy set of player i as follows: circles represent resources belonging to σ_i, while crosses represent resources belonging to σ_i^*.

Theorem A2 (Claim of Theorem 6). *For any $\epsilon > 0$, there exist two bidimensional linear congestion games $(\mathcal{G}, \mathcal{C})$ and $(\mathcal{G}', \mathcal{C}')$ such that $PoS(\mathcal{G}, \mathcal{C}) \geq \frac{1+\sqrt{5}}{2} - \epsilon$ under the social cost function Pres and $PoS(\mathcal{G}', \mathcal{C}') \geq \frac{5+\sqrt{17}}{4} - \epsilon$ under the social cost function Perc.*

Proof. Let $(\mathcal{G}, \mathcal{C})$ be a bidimensional linear congestion game such that $|C_0| = n_0$ and $|C_1| = |C_2| = n_1$. Each player $i \in C_1 \cup C_2$ has two strategies σ_i and σ_i^*, while all players in C_0 have the same strategy $\bar{s}$.

There are three types of resources:

- n_1 resources r_i, $i \in [n_1]$, each with cost function $\ell_{r_i}(x) = \frac{n_1 + 2n_0 + 1 + \gamma}{2} x$, where γ is an arbitrarily small positive value. Resource r_i belongs only to σ_i^*;
- $n_1(n_1 - 1)$ resources $r'_{i,j}$, $i, j \in [n_1]$ with $i \neq j$, each with cost function $\ell_{r'_{ij}}(x) = \frac{1}{2}x$. Resource r'_{ij} belongs only to σ_i and to σ_j^*;
- one resource r'' with cost function $\ell_{r''}(x) = 1$. Resource r'' belongs to σ_i for each $i \in [n_1]$ and to $\bar{s}$.

Let σ (resp. σ^*) be the strategy profile in which each player $i \notin C_0$ plays strategy σ_i (resp. σ_i^*). The cost of each player $i \in C_j$, with $j \in \{1, 2\}$, for adopting strategy σ_i when there are exactly h players in C_j adopting the strategy played in σ (and thus there are $n_1 - h$ players in C_j adopting the strategy played in σ^*) is $cost_\sigma(h) = \frac{2n_1 - h - 1}{2} + n_0 + h$. Similarly, the cost of each player $i \in C_j$ for adopting strategy σ_i^* when there are exactly h players in C_j adopting the strategy played in σ is $cost_{\sigma^*}(h) = \frac{n_1 + 2n_0 + 1 + \gamma}{2} + \frac{n_1 + h - 1}{2}$. Since for any $h \in [n_1]$, it holds that $cost_{\sigma^*}(h - 1) > cost_\sigma(h)$, it follows that σ is the only pure Nash equilibrium for $(\mathcal{G}, \mathcal{C})$.

The price of stability of $(\mathcal{G}, \mathcal{C})$ is then lower bounded by the following ratio

$$\frac{n_1(n_1 - 1) + 2n_1(n_1 + n_0) + n_0(2n_1 + n_0)}{n_1(n_1 + 2n_0 + 1 + \gamma) + n_1(n_1 - 1) + n_0^2},$$

which, for n_0 going to infinity and $n_1 = \frac{1 + \sqrt{5}}{2} n_0$, tends to $\frac{1 + \sqrt{5}}{2}$.

Let $(\mathcal{G}', \mathcal{C}')$ be a bidimensional linear congestion game such that $C_0 = \emptyset$, $|C_1| = n_1$ and $|C_2| = n_2$. Each player $i \in C_1$ has two strategies σ_i and σ_i^*, while all players in C_2 have the same strategy $\bar{s}$.

There are three types of resources:

- n_1 resources r_i, $i \in [n_1]$, each with cost function $\ell_{r_i}(x) = \frac{n_1 + 1 + \gamma}{2} x$, where γ is an arbitrarily small positive value. Resource r_i belongs only to σ_i^*;
- $n_1(n_1 - 1)$ resources $r'_{i,j}$, $i, j \in [n_1]$ with $i \neq j$, each with cost function $\ell_{r'_{ij}}(x) = \frac{1}{2}x$. Resource r'_{ij} belongs only to σ_i and to σ_j^*;
- one resource r'' with cost function $\ell_{r''}(x) = 1$. Resource r'' belongs to σ_i for each $i \in [n_1]$ and to $\bar{s}$.

Let σ (resp. σ^*) be the strategy profile in which each player $i \notin C_0$ plays strategy σ_i (resp. σ_i^*). The cost of each player $i \in C_1$ for adopting strategy σ_i when there are exactly h players in C_1 adopting the strategy played in σ (and thus there are $n_1 - h$ players in C_1 adopting the strategy played in σ^*) is $cost_\sigma(h) = \frac{2n_1 - h - 1}{2} + h$. Similarly, the cost of each player $i \in C_1$ for adopting strategy σ_i^* when there are exactly h players in C_1 adopting the strategy played in σ is $cost_{\sigma^*}(h) = \frac{n_1 + 1 + \gamma}{2} + \frac{n_1 + h - 1}{2}$. Since for any $h \in [n_1]$, it holds that $cost_{\sigma^*}(h - 1) > cost_\sigma(h)$, it follows that σ is the only pure Nash equilibrium for $(\mathcal{G}', \mathcal{C}')$.

The price of stability of $(\mathcal{G}', \mathcal{C}')$ is then lower bounded by the following ratio

$$\frac{\frac{1}{2}n_1(n_1 - 1) + (n_1 + n_2)^2}{\frac{1}{2}n_1(n_1 + 1 + \gamma) + \frac{1}{2}n_1(n_1 - 1) + n_2^2},$$

which, for n_2 going to infinity and $n_1 = \frac{1 + \sqrt{17}}{4} n_2$, tends to $\frac{1 + \sqrt{17}}{4}$. $\quad \square$

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