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# On the Geometry of the Null Tangent Bundle of a Pseudo-Riemannian Manifold

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**Abstract:** When we consider a non-definite pseudo-Riemannian manifold, we obtain lightlike tangent vectors that constitute the null tangent bundle, whose fibers are lightlike cones in the corresponding tangent spaces. In this paper, we define and study a class of "$g$-natural" metrics on the tangent bundle of a pseudo-Riemannian manifold and we investigate the geometry of the null tangent bundle as a lightlike hypersurface equipped with an induced $g$-natural metric.

**Keywords:** pseudo-Riemannian metric; null tangent bundle; lightlike manifold; Ricci type tensor; extrinsic scalar curvature

**MSC:** 53B30, 53C50

**Citation:** Abbassi, M.T.K.; Boulagouaz, K.; Calvaruso, G. On the Geometry of the Null Tangent Bundle of a Pseudo-Riemannian Manifold. *Axioms* **2023**, *12*, 903. https://doi.org/10.3390/axioms12100903

Academic Editors: Juan De Dios Pérez, Yanlin Li and Zhigang Wang

Received: 24 August 2023
Revised: 18 September 2023
Accepted: 19 September 2023
Published: 22 September 2023



## 1. Introduction

In both special and general relativity, the main tool to study the causal structure of spacetime is the lightlike cone, since it encodes all the information about the behavior of lightlike geodesics. It is also well-known that the *null tangent bundle*, i.e., the set of all lightlike tangent vectors on a Lorentzian manifold determines its metric up to a conformal transformation ([1]) and, as a result, the null tangent bundle over the underlying manifold specifies its conformal geometry, which constitutes the framework of Conformal Cyclic Cosmology ([2]). To the best of our knowledge, although the study of the geometry of null tangent bundles seems natural for a better understanding of Conformal Cyclic Cosmology, surprisingly, we cannot find works on the subject in the literature. One of the reasons is probably a lack of research works on the geometry of tangent bundles of *non-definite* pseudo-Riemannian manifolds.

Actually, when the base manifold is Riemannian, the study of relationships between the geometric properties of the Riemannian base manifold $(M, g)$ and those of its *tangent bundle* $TM$, have been widely studied in the literature and led to several interesting results. Several well-known metrics on $TM$ fall within the wide family of *$g$-natural metrics*, which are built in some "natural" way from the Riemannian metric $g$ over $M$ ([3,4]). Some examples of investigation of the geometry of such metrics and their interplay with the properties of the base manifold may be found in [5–9] and references therein.

As it is well known, if $(M, g)$ is a pseudo-Riemannian manifold, then it admits some tangent vectors which do not have a Riemannian counterpart, namely, *null* (or *lightlike*) *vectors*. It is then natural to consider the *null tangent bundle* $T_0M$ of $(M, g)$, i.e., the set of all its null tangent vectors. To investigate the geometry of $T_0M$, it is interesting to equip $TM$ with pseudo-Riemannian metrics and to consider $T_0M$ as a lightlike hypersurface. Generally speaking, lightlike hypersurfaces play a very important role in mathematical physics, with particular regard to their relevance and many applications in relativity.

For this reason, in the last thirty years, the study of lightlike hypersurfaces has attracted the attention of a growing number of researchers, and the literature on the topic is very large. We may refer to the works [10,11] and references therein for some excellent introductions to the topic and its applications.

Like in the case of Riemannian base manifolds, we shall define in this paper a family of metrics on the tangent bundles of *pseudo-Riemannian manifolds* that we call also *g-natural metrics* and, equipping the null tangent bundles with the induced metrics, we start addressing the issue of the relationship between the geometry of a pseudo-Riemannian manifold and the one of its null tangent bundle.

The results of this paper will allow further research in this direction, starting from the case where the tangent bundle is spacetime. The first steps we accomplish here are a thorough investigation of $g$-natural metrics $G$ on the tangent bundle $TM$ of a pseudo-Riemannian manifold $(M, g)$, and an accurate description of the geometric features (connection and curvature) of $T_0M$ equipped with the metrics induced on this hypersurface of $(TM, G)$. From this starting point, the research of geometric features of the null tangent bundle can be developed in several different directions, like for example, harmonic maps defined on lightlike submanifolds [12–14], and CR-lightlike submanifolds [15–17].

This paper is organized in the following way. In Section 2, we report some basic information concerning the geometry of lightlike hypersurfaces and the tangent bundles. We then give, in Section 3, the general description of pseudo-Riemannian $g$-natural metrics on the tangent bundle $TM$ of a pseudo-Riemannian manifold $(M, g)$ and their possible signatures. In Section 4, we focus on the geometry of null tangent bundle $T_0M$. We first investigate the differentiable structure of $T_0M$. Then, equipping it with a $g$-natural metric, we construct a corresponding screen distribution, calculate its associated induced connection, and discuss some geometric properties related to the curvature, Ricci type tensor, and extrinsic curvature of $T_0M$. In particular, among other results, we completely characterize the case where the Ricci type tensor is symmetric for a base manifold of constant sectional curvature. To make the core of the paper compact and readable, we stated the details of calculations of the signatures of pseudo-Riemannian $g$-natural metrics on the tangent bundles, the very long expressions of their curvatures on the null tangent bundles, and the corresponding calculations at the end of this paper in Appendices A and B.

## 2. Preliminaries

### *2.1. Lightlike Hypersurfaces*

Let $(\overline{M}, \overline{g})$ denote a non-definite pseudo-Riemannian manifold of dimension $m$ and $(M, g)$ a lightlike hypersurface of $(\overline{M}, \overline{g})$, where $g$ is the induced degenerate metric on $M$. The intersection of tangent bundle $TM$ and normal bundle $TM^\perp$ is a one-dimensional subbundle, called the *radical distribution* of $M$ and denoted by $RadTM$. Furthermore, there exists a complementary non-degenerate vector bundle of $RadTM$ in $TM$, called a *screen distribution* $S(TM)$ of $M$, such that

$$TM = RadTM \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Following [10], there exists a unique vector bundle $Tr(TM)$ of rank 1 over $M$, called the *lightlike transversal bundle*, such that for any non-zero section $\xi$ of $Rad(TM)$ on a coordinate neighborhood $U$ in $TM$, there exists a unique section $N$ of $Tr(TM)$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, X) = 0, \text{ for all } X \in \Gamma(S(TM)).$$

Thus $T\overline{M}$ splits into

$$T\overline{M} = S(TM) \oplus_{orth} (Rad(TM) \oplus Tr(TM)) = TM \oplus Tr(TM).$$

Let $\xi \in TM$ be a null section, $N$ the corresponding transverse vector field, and $P$ the projection morphism of $TM$ into $S(TM)$. The Gauss and Weingarten formulas in $M$ are then given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{1}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2}$$

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X,PY)\xi, \tag{3}$$

$$\nabla_X \xi = -\overset{*}{A}_\xi X - \tau(X)\xi \tag{4}$$

for any $X,Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M},\overline{g})$, $\nabla$ is the induced connection from $\overline{\nabla}$ on $M$ through the projection along the transverse vector field $N$ and $\overset{*}{\nabla}$ is the induced connection from $\nabla$ on the screen distribution $S(TM)$ through the projection along the null vector field $\xi$.

In the above equations, $B$ is a symmetric bilinear form on $TM$ independent of the choice of the screen distribution and called the *local second fundamental form of* $M$, while $C$ is a bilinear form called the *local second fundamental form of the screen distribution*. We say that $M$ is a *totally geodesic* hypersurface of $\overline{M}$ if any geodesic of $M$ with respect to the induced connection on $M$ is a geodesic of $\overline{M}$ (see [11]).

It is known that $\nabla$ is symmetric, but in general, it is not a metric connection, since

$$(\nabla_X g)(Y,Z) = B(X,Y)\overline{g}(Z,N) + B(X,Z)\overline{g}(Y,N),$$

while $\overset{*}{\nabla}$ is a metric connection on $S(TM)$, which is not necessarily symmetric.

If we denote by $\overline{R}$ and $R$ the Riemannian curvatures with respect to $\overline{\nabla}$ and $\nabla$, we have

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z,PW) =& g(R(X,Y)Z,PW) + \overline{g}(h(X,Z),h^*(Y,PW)) \\ & - \overline{g}(h(Y,Z),h^*(X,PW)), \\ \overline{g}(\overline{R}(X,Y)Z,U) =& \overline{g}((\nabla_X h)(Y,Z),U) - \overline{g}((\nabla_Y h)(X,Z),U), \\ \overline{g}(\overline{R}(X,Y)Z,V) =& \overline{g}(R(X,Y)Z,V), \end{aligned} \tag{5}$$

for any $X,Y,Z,W \in \Gamma(TM)$, $U \in \Gamma(Rad(TM))$, $V \in \Gamma(tr(TM))$.

The non-definite pseudo-Riemannian manifold $\overline{M}$ admits a *quasi-orthonormal local frame field*, i.e., a local frame $\{e_1,\dots,e_r,f_1,\dots f_s,f_1^*,\dots f_s^*\}$ of $\overline{M}$ such that $\overline{g}(e_a,f_i) = \overline{g}(e_i,f_i^*) = 0$, $\overline{g}(e_a,e_b) = \varepsilon_a\delta_{ab}$, $\overline{g}(f_i,f_j) = \overline{g}(f_i^*,f_j^*) = 0$, $\overline{g}(f_i,f_j^*) = \delta_{ij}$, for every $a,b \in \{1,\dots,r\}$ and $i,j \in \{1,\dots,s\}$, where $m = r+2s$ and $\varepsilon_a = \pm 1$. A quasi-orthonormal frame $\{e_1,\dots,e_r,f_1,\dots f_s,f_1^*,\dots f_s^*\}$ on $\overline{M}$ is said a *quasi-orthonormal local frame field along an n-dimensional lightlike submanifold* $M'$ if one of the two following conditions is satisfied:

- $n = s+t$, $1 \le t \le r$, and $\{e_1,\dots,e_t,f_1,\dots f_s\}$ induces a local frame field on $M'$;
- $n \le s$ and $\{f_1,\dots f_n\}$ induces a local frame field on $M'$.

It follows that, in the case of a lightlike hypersurface $M$, a quasi-orthonormal local frame field along $M$ is of the form $\{e_1,\dots,e_{m-2},f,f^*\}$, where $\{e_1,\dots,e_{m-2},f\}$ induces a local frame field on $M$.

Let $S(TM)$ be a screen distribution locally spanned by $\{e_1,\dots,e_{m-2}\}$, so that $\{e_1,\dots,e_{m-2},\xi,N\}$ is a quasi-orthonormal local frame field on $\overline{M}$ and $\{e_1,\dots,e_{m-2},\xi\}$ is the induced local frame field on $M$. The *induced Ricci type tensor* $R^{(0,2)}$ of $M$ is defined for any $X,Y \in \Gamma(TM)$ by

$$R^{(0,2)}(X,Y) = \sum_{j=1}^{m-2} \varepsilon_j g(R(X,e_j)Y,e_j) + \overline{g}(R(X,\xi)Y,N),$$

where $\varepsilon_j = g(e_j, e_j)$. The tensor $R^{(0,2)}$ has a geometrical meaning if $R^{(0,2)}$ is symmetric and its value is independent of the screen distribution, its transversal vector bundle and the null section $\xi$. It is easy to see that it does not depend on the choice of the null section $\xi$, while for the other conditions we have the following results:

- $R^{(0,2)}$ is symmetric if and only if each 1-form $\tau$ induced by $S(TM)$ is closed, i.e., $d\tau = 0$ on any $U \subset M$ [10].
- $R^{(0,2)}$ is symmetric on a lightlike hypersurface whose screen distribution is integrable [18].
- $R^{(0,2)}$ is related to the Ricci tensor of $(\overline{M}, \overline{g})$ by

$$R^{(0,2)}(X,Y) = Ric(X,Y) + B(X,Y)tr(A_N) - g(A_N X, A_\xi Y) - \overline{g}(\overline{R}(\xi, Y)X, N).$$

When the induced Ricci type is not symmetric, C. Atindogbe [19] introduced the *symmetrized induced Ricci tensor* $Ric^{sym}$, defined, for all $X, Y \in \mathfrak{X}(M)$, by

$$Ric^{sym}(X,Y) = \frac{1}{2}[R^{(0,2)}(X,Y) + R^{(0,2)}(Y,X)]. \tag{6}$$

**Definition 1** ([19])**.** *Let $(\overline{M}, \overline{g})$ be a pseudo-Riemannian manifold and $(M, g, S(TM))$ a lightlike hypersurface of $\overline{M}$. The quantity*

$$R = g^{ij} Ric^{sym}_{ij},$$

*where $g^{ij}$ is the pseudo-inverse of $g_{ij}$ (see [20]), is called the* extrinsic scalar curvature *of $(M, g, S(TM))$.*

### *2.2. Geometry of Tangent Bundles*

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\nabla$ the Levi-Civita connection of $g$. The tangent space of $TM$ at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to $\nabla$:

$$(TM)_{(x,u)} = H(x,u) \oplus V(x,u). \tag{7}$$

Given $(x, u) \in TM$, for any vector $X \in T_x M$ there exists a unique vector $X^h \in H(x, u)$ such that $p^* X^h = X$, where $p : TM \longrightarrow M$ is the natural projection. We call $X^h$ the *horizontal lift* of $X$ to the point $(x, u) \in TM$. The *vertical lift* of a vector $X \in T_x M$ to $(x, u) \in TM$ is defined as the vector $X^v \in V(x, u)$ satisfying $X^v(df) = X(f)$, for all functions $f$ on $M$. Here, 1-forms $df$ on $M$ are interpreted as functions on $TM$ (i.e., $(df)(x, u) = u(f)$). Note that the map $X \longrightarrow X^h$ is an isomorphism between the vector spaces $T_x M$ and $H(x, u)$. Similarly, the map $X \longrightarrow X^v$ is an isomorphism between the vector spaces $T_x M$ and $V(x, u)$.

Each tangent vector $Z \in (TM)_{(x,u)}$ can be written in the form

$$Z = X^h + Y^v, \tag{8}$$

where $X, Y \in T_x M$ are uniquely determined vectors. Horizontal and vertical lifts of vector fields on $M$ are defined correspondingly.

Each system of local coordinates $(U; x^i, i = 1, ..., n)$ in $M$ induces on $TM$ a system of local coordinates $(p^{-1}(U); x^i, u^i, i = 1, ..., n)$. Given $x \in U$ and $X \in T_x M$, let $X = \sum X^i (\frac{\partial}{\partial x^i})_x$ be the local expression of $X$ in $(U; x^i, i = 1, ..., n)$. Then, with respect to the induced coordinates, the horizontal lift $X^h$ and the vertical lift $X^v$ of $X$ to $(x, u) \in TM$ are, respectively, expressed by

$$X^h = \sum X^i (\frac{\partial}{\partial x^i})_{(x,u)} - \sum \Gamma^i_{jk} u^j X^k (\frac{\partial}{\partial u^i})_{(x,u)}, \tag{9}$$

$$X^v = \sum X^i (\frac{\partial}{\partial u^i})_{(x,u)}, \tag{10}$$

where $(\Gamma^i_{jk})$ denote the Christoffel's symbols of $g$.

The *canonical vertical vector field* on $TM$ is defined, in terms of local coordinates, by $\mathcal{U} = \sum u^i \frac{\partial}{\partial u^i}$, but it does not depend on the choice of local coordinates and is globally defined on $TM$. For a vector $u = u^i(\frac{\partial}{\partial x^i}))_x \in T_xM$, we see that the vertical lift of $u$ to $(x, u)$ is exactly the value of the canonical vertical vector field at $(x, u)$, i.e., $u^v = \sum u^i(\frac{\partial}{\partial x^i})^v = \mathcal{U}_{(x,u)}$, while the horizontal lift of $u$ to $(x, u)$ is no other than the value at $(x, u)$ of the geodesic vector field $\zeta$ on $TM$, i.e., $u^h = \sum u^i(\frac{\partial}{\partial x^i})^h = \zeta_{(x,u)}$.

It is worth mentioning that the geodesic (resp. canonical vertical) vector field on $TM$ is not a horizontal (resp. vertical) lift of any vector field on $M$. To express it as a horizontal (resp. vertical) lift, we need to introduce lifts of quantities more general than vector fields on $M$. For this, we consider the vector bundle $p^*TM$ induced by the tangent bundle $TM$ and by the natural projection $p : TM \to M$. Any section $s$ of $p^*TM$ is a $C^\infty$-mapping $s : TM \to TM$ such that $p \circ s = p$. The mappings $X \circ p$, where $X \in \mathfrak{X}(M)$, are examples of sections of $p^*TM$.

Sections of $p^*TM$ give rise to special horizontal and vertical vector fields on $TM$: if $s \in \Gamma(p^*TM)$, then we define the *horizontal* (resp. *vertical*) *lift* $s^h$ (resp. $s^v$) as the vector field on $TM$ given by $s^h(u) = (s(u))^h$ (resp. $s^v(u) = (s(u))^v$), for any $u \in TM$, where the lifts are taken at $u$. When $s = X \circ p$, $X \in \mathfrak{X}(M)$, we find the classical definition of horizontal and vertical lifts of vector fields. When $s$ is the identity section, the horizontal (resp. vertical) lift of $s$ is no other than the geodesic vector field $\zeta$ (resp. the canonical vertical vector field $\mathcal{U}$) on $TM$.

Lie brackets of vector fields on $TM$ are described as follows:

**Lemma 1.** *For all vector fields $X, Y$ on $M$:*

*(a)* $[X^h, Y^h] = [X, Y]^h - [R(X \circ p, Y \circ p)\sigma]^v$, *where $\sigma$ is the identity section of $p^*TM$;*
*(b)* $[X^h, Y^v] = \{\nabla_X Y\}^v = \{\nabla_Y X\}^v + [X, Y]^v$;
*(c)* $[X^v, Y^v] = 0$.

To investigate the geometry of tangent bundles of Riemannian manifolds, many (pseudo-)Riemannian metrics have been considered in the literature. The more general class of metrics had been constructed by O. Kowalski and M. Sakizawa [3] using the concept of natural transformations (see [4] for the concept of naturality and associated notions). According to the terminology of [4], we shall call *g-natural* any metric $G$ on $TM$, which comes from $g$ by a first order natural operator $S^2_+T^* \to (S^2T^*)T$. Explicitly, $g$-natural metrics are described as follows (see [5]):

For any $g$-natural metric $G$ on $TM$, there exist six functions $\alpha_i, \beta_i : \mathbb{R}_+ \to \mathbb{R}, i = 1; 2; 3$, such that for every $u, X, Y \in T_xM$:

$$\begin{cases} G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\ & \quad +(\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) &= \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) &= \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u), \end{cases} \tag{11}$$

where $r^2 = \|u\|^2 = g_x(u, u)$. For $dim(M) = 1$, the formulas above hold with $\beta_j = 0$, $j = 1, 2, 3$.

To investigate the properties of $g$-natural metrics, we need the following notations:

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}_+$. Using the notations above, a $g$-natural metric $G$ on the tangent bundle of a Riemannian manifold $(M, g)$ is:

1. non-degenerate if and only if $\alpha(t) \neq 0$ and $\phi(t) \neq 0$ for all $t \in \mathbb{R}_+$;
2. Riemannian if and only if $\alpha_1(t) > 0$, $\alpha(t) > 0$, $\phi_1(t) > 0$ and $\phi(t) > 0$, for all $t \in \mathbb{R}_+$.

We observe explicitly that condition $\alpha_1 = 0$ is not compatible with the Riemannian case. Several well-known (pseudo-)Riemannian metrics on the tangent bundle of a Riemannian manifold are $g$-natural. In particular, in the notations above:

(a) The *Sasaki metric* [21] is the $g$-natural metric given by $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$;
(b) The *Cheeger-Gromoll metric* [22] is obtained for $\alpha_1(t) = \beta_1(t) = \frac{1}{1+t}$ for all $t \in \mathbb{R}_+$, $\alpha_2 = \beta_2 = 0$, $\alpha_1 + \alpha_3 = 1$, $\beta_1 + \beta_3 = 0$;
(c) The *Kaluza–Klein metrics* [23] correspond to conditions $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.

## 3. $g$-Natural Metrics on the Tangent Bundle of a Pseudo-Riemannian Manifold

$g$-natural metrics on the tangent bundle of a pseudo-Riemannian manifold $(M, g)$ can be defined similarly to the Riemannian case:

**Definition 2.** *Let $(M, g)$ be a pseudo-Riemannian manifold. A metric $G$ on $TM$ is $g$-*natural *if there exist six functions $\alpha_i, \beta_i : \mathbb{R} \to \mathbb{R}$, $i = 1; 2; 3$, called* the generating functions of $G$, *such that for every $u, X, Y \in T_xM$:*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(g_x(u,u))g_x(X,Y) + (\beta_1 + \beta_3)(g_x(u,u))g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(g_x(u,u))g_x(X,Y) + \beta_2(g_x(u,u))g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(g_x(u,u))g_x(X,Y) + \beta_2(g_x(u,u))g_x(X,u)g_x(Y,u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(g_x(u,u))g_x(X,Y) + \beta_1(g_x(u,u))g_x(X,u)g_x(Y,u). \end{cases}$$

**Notations.** As in the Riemannian case, we use the following notations:

1. $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
2. $\alpha(t) = \alpha_1(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
3. $\phi(t) = \phi_1(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for $t \in \mathbb{R}$.

**Remark 1.** *Hereafter and unless otherwise stated, when some terms of an expression are evaluated at $(x, u) \in TM$, we make the following conventions:*

- *All the lifts of vectors on $M$ involved in that expression are taken at $(x, u) \in TM$;*
- *All the functions $\alpha_i$, $\beta_i$, $\phi_i$, $\alpha$ and $\phi$ involved in that expression are taken at $g_x(u, u)$.*

### *3.1. Non-Degenerate g-Natural Metrics on the Tangent Bundle of a Pseudo-Riemannian Manifold*

**Proposition 1.** *Let $(M, g)$ be a pseudo-Riemannian manifold of dimension $m \geq 2$. A $g$-natural metric $G$ on $TM$ is non-degenerate if and only if its generating functions satisfy $\alpha(t) \cdot \phi(t) \neq 0$, for every $t \in \mathbb{R}$.*

**Proof.** Denote by $k$ the index of $g$. Let $x \in M$ and $u \in T_xM$. Then, we have the three following cases according to the causal character of $u$:

1. If $u$ is timelike, then $k \geq 1$. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis of $(T_xM, g_x)$, such that $e_1 = \frac{1}{\sqrt{-g(u,u)}}u$, $g(e_i, e_i) = -1$, for $1 \leq i \leq k$ and $g(e_j, e_j) = 1$, for $k+1 \leq j \leq m$. Then, the matrix of $G_{(x,u)}$ with respect to the basis $\{e_1^h, e_2^h, \cdots, e_m^h, e_1^v, e_2^v, \cdots, e_m^v\}$ of $T_{(x,u)}TM$ is given by $P_m(g(u, u))$, where

$$P_m(t) = \begin{pmatrix} -(\phi_1+\phi_3)(t) & 0 & 0 & -\phi_2(t) & 0 & 0 \\ 0 & -(\alpha_1+\alpha_3)(t)I_{k-1} & 0 & 0 & -\alpha_2(t)I_{k-1} & 0 \\ 0 & 0 & (\alpha_1+\alpha_3)(t)I_{m-k} & 0 & 0 & \alpha_2(t)I_{m-k} \\ -\phi_2(t) & 0 & 0 & -\phi_1(t) & 0 & 0 \\ 0 & -\alpha_2(t)I_{k-1} & 0 & 0 & -\alpha_1(t)I_{k-1} & 0 \\ 0 & 0 & \alpha_2(t)I_{m-k} & 0 & 0 & \alpha_1(t)I_{m-k} \end{pmatrix}.$$

It is easy to see that $det(P_m(t)) = \phi(t)\alpha^{m-1}(t)$, so that $G_{(x,u)}$ is non-degenerate on the timelike cone if and only if $\phi(t) \neq 0$ and $\alpha(t) \neq 0$ for all $t \in ]-\infty, 0[$.

2. If $u$ is spacelike, then $m-k \geq 1$. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis of $(T_xM, g_x)$, such that $e_1 = \frac{1}{\sqrt{g(u,u)}}u$, $g(e_i,e_i) = 1$, for $1 \leq i \leq m-k$ and $g(e_j,e_j) = -1$, for $m-k+1 \leq j \leq m$. Then, the matrix of $G_{(x,u)}$ with respect to the basis $\{e_1^h, e_2^h, \cdots, e_m^h, e_1^v, e_2^v, \cdots, e_m^v\}$ of $T_{(x,u)}TM$ is given by $P_m(g(u,u))$, where

$$P_m(t) = \begin{pmatrix} (\phi_1+\phi_3)(t) & 0 & 0 & \phi_2(t) & 0 & 0 \\ 0 & (\alpha_1+\alpha_3)(t)I_{m-k-1} & 0 & 0 & \alpha_2(t)I_{m-k-1} & 0 \\ 0 & 0 & -(\alpha_1+\alpha_3)(t)I_k & 0 & 0 & -\alpha_2(t)I_k \\ \phi_2(t) & 0 & 0 & \phi_1(t) & 0 & 0 \\ 0 & \alpha_2(t)I_{m-k-1} & 0 & 0 & \alpha_1(t)I_{m-k-1} & 0 \\ 0 & 0 & -\alpha_2(t)I_k & 0 & 0 & -\alpha_1(t)I_k \end{pmatrix}.$$

We have $det(P_m(t)) = \phi(t)\alpha^{m-1}(t)$, so that $G_{(x,u)}$ is non-degenerate on the spacelike cone if and only if $\phi(t) \neq 0$ and $\alpha(t) \neq 0$ for all $t \in ]0, \infty[$.

3. If $u$ is lightlike, then either $k \geq 1$ or $m-k \geq 1$. Let $v \in T_xM$ such that $g(v,v) = g(u,u) = 0$ and $g(u,v) = 1$ and let $\{e_i\}_{i=1}^m$ be a basis of $T_xM$, such that $e_1 = u$, $e_2 = v$, $g(e_i,e_j) = 0$, for $3 \leq i \neq j \leq m$ and
   - $g(e_i,e_i) = 1$, for $3 \leq i \leq m$, if $k = 1$;
   - $g(e_i,e_i) = -1$, for $3 \leq i \leq m$, if $m-k = 1$;
   - $g(e_i,e_i) = -1$, for $3 \leq i \leq k+1$, $g(e_i,e_i) = 1$, for $k+2 \leq i \leq m$, if $k \geq 2$ and $m-k \geq 2$.

   As $g(u,u) = 0$, with respect to the basis $\{e_1^h, e_2^h, \cdots, e_m^h, e_1^v, e_2^v, \cdots, e_m^v\}$ of $T_{(x,u)}TM$, the matrix $P_m(0) = P_m(g(u,u))$ of $G_{(x,u)}$ is given by

$$P_m(0) = \begin{pmatrix} 0 & (\alpha_1+\alpha_3)(0) & 0 & 0 & 0 & \alpha_2(0) & 0 & 0 \\ (\alpha_1+\alpha_3)(0) & (\beta_1+\beta_3)(0) & 0 & 0 & \alpha_2(0) & \beta_2(0) & 0 & 0 \\ 0 & 0 & -(\alpha_1+\alpha_3)(0)I_q & 0 & 0 & 0 & -\alpha_2(0)I_q & 0 \\ 0 & 0 & 0 & (\alpha_1+\alpha_3)(0)I_p & 0 & 0 & 0 & \alpha_2(0)I_p \\ 0 & \alpha_2(0) & 0 & 0 & 0 & \alpha_1(0) & 0 & 0 \\ \alpha_2(0) & \beta_2(0) & 0 & 0 & \alpha_1(0) & \beta_1(0) & 0 & 0 \\ 0 & 0 & -\alpha_2(0)I_q & 0 & 0 & 0 & -\alpha_1(0)I_q & 0 \\ 0 & 0 & 0 & \alpha_2(0)I_p & 0 & 0 & 0 & \alpha_1(0)I_p \end{pmatrix},$$

   with $q = k-1$ and $p = m-k-1$. We have $det(P_m(0)) = \alpha^m(0)$ and therefore $G_{(x,u)}$ is non-degenerate if and only if $\alpha(0) \neq 0$.

   Since $\phi(0) = \alpha(0)$, $G_{(x,u)}$ is non-degenerate on $TM$ if and only if $\alpha(0) \cdot \phi(0) \neq 0$.

It follows from the three cases above that $G_{(x,u)}$ is non-degenerate on $TM$ if and only if $\alpha(t) \cdot \phi(t) \neq 0$ for every $t \in \mathbb{R}$. □

### 3.2. Pseudo-Riemannian g-Natural Metrics on the Tangent Bundle of a Pseudo-Riemannian Manifold

To determine the signature of an arbitrary non-degenerate $g$-natural metric on the tangent bundle of a pseudo-Riemannian manifold, we should give at first the signature of its induced metric on the tangent space of the tangent bundle on an arbitrary point $(x,u)$. This leads us to consider three cases corresponding to $(x,u)$ being timelike, spacelike, and lightlike. In Appendix A, we treat in detail the three cases in Appendices A.1, A.2 and A.3, respectively.

Using the discussion in Appendix A, we obtain the following result, which lists all possibilities for the signature of a non-degenerate $g$-natural metric on the tangent bundle of a pseudo-Riemannian manifold $(M,g)$.

**Theorem 1.** *Let $(M,g)$ be a non-definite pseudo-Riemannian manifold of signature $(m-k,k)$ and $G$ be a non-degenerate $g$-natural metric on its tangent bundle. Then, one of the three following non-overlapping situations occurs:*

1. *$\phi(t) > 0$, $\alpha(t) > 0$, $(\phi_1+\phi_3)(t) > 0$ and $(\alpha_1+\alpha_3)(t) > 0$ for all $t \in \mathbb{R}$. In this case, the signature of $G$ is $(2m-2k, 2k)$,*

2. $\phi(t) > 0$, $\alpha(t) > 0$, $(\phi_1 + \phi_3)(t) < 0$ *and* $(\alpha_1 + \alpha_3)(t) < 0$ *for all* $t \in \mathbb{R}$. *In this case, the signature of G is* $(2k, 2m - 2k)$,
3. $\phi(t) < 0$, $\alpha(t) < 0$, *for all* $t \in \mathbb{R}$. *In this case, the signature of G is* $(m, m)$.

**Proof.** Since $g$ is non-definite, there exist timelike, spacelike, and lightlike tangent vectors. Using the fact that the index of a pseudo-Riemannian metric is constant, we deduce from Propositions A1–A3 that the possible signatures of $G$ are either $(2m - 2k, 2k)$ or $(2k, 2m - 2k)$ or $(m, m)$. The same Propositions specify the conditions on the defining function leading to these possible signatures. □

**Remark 2.** *In cases (1) and (2) of Theorem 1, we can replace* $\alpha_1 + \alpha_3 > 0$ *(resp.* $< 0$*) by* $\alpha_1 > 0$ *(resp.* $< 0$*) and* $\phi_1 + \phi_3 > 0$ *( resp.* $< 0$*) by* $\phi_1 > 0$ *( resp.* $< 0$*).*

**Example 1.**

*(i)* *The Sasaki metric on the tangent bundle of a pseudo-Riemannian manifold of signature* $(m - k, k)$ *is a pseudo-Riemannian metric of signature* $(2m - 2k, 2k)$.

*(ii)* *A Kaluza–Klein metric on the tangent bundle of a pseudo-Riemannian manifold of signature* $(m - k, k)$ *is non-degenerate if and only if the function* $\phi \cdot \alpha$ *is positive. Moreover, in this case its signature is*

- $(m, m)$, *if* $\alpha$ *and* $\phi$ *are negative everywhere,*
- $(2m - 2k, 2k)$, *if* $(\alpha_1 + \alpha_3)(t) > 0$ *for all* $t \in \mathbb{R}$,
- $(2k, 2m - 2k)$, *if* $(\alpha_1 + \alpha_3)(t) < 0$ *for all* $t \in \mathbb{R}$.

We now discuss the possible sign of the defining functions starting from a definite metric.

**Proposition 2.** *Let* $(M, g)$ *be a differentiable manifold with a definite metric.*

1. *If g is positive definite, then the signature of a g-natural metric on TM is determined by the following Table 1.*

**Table 1.** The signature of a non-degenerate $g$-natural metric on the tangent bundle of a Riemannian manifold.

| The Signature of $G_{(x,u)}$ | $(\phi_1 + \phi_3)(g(u,u))$ | $(\alpha_1 + \alpha_3)(g(u,u))$ | $\phi(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|---|---|
| $(m-1, m+1)$ | <0 | any | >0 | <0 |
| $(m+1, m-1)$ | >0 | any | >0 | <0 |
| $(m, m)$ | any | any | <0 | <0 |
| $(2m, 0)$ | >0 | >0 | >0 | >0 |
| $(2m-1, 1)$ | any | >0 | <0 | >0 |
| $(2m-2, 2)$ | <0 | >0 | >0 | >0 |
| $(1, 2m-1)$ | any | <0 | <0 | >0 |
| $(2, 2m-2)$ | >0 | <0 | >0 | >0 |
| $(0, 2m)$ | <0 | <0 | >0 | >0 |

2. *If g is negative definite, then the signature of a g-natural metric on TM is given by Table 2.*

**Table 2.** The signature of a non-degenerate $g$-natural metric on the tangent bundle of a negative-definite manifold.

| **The Signature of $G_{(x,u)}$** | $(\phi_1+\phi_3)(g(u,u))$ | $(\alpha_1+\alpha_3)(g(u,u))$ | $\phi(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|---|---|
| $(m-1,m+1)$ | >0 | any | >0 | <0 |
| $(m+1,m-1)$ | <0 | any | >0 | <0 |
| $(m,m)$ | *any* | any | <0 | <0 |
| $(0,2m)$ | >0 | >0 | >0 | >0 |
| $(1,2m-1)$ | any | >0 | <0 | >0 |
| $(2,2m-2)$ | <0 | >0 | >0 | >0 |
| $(2m-2,2)$ | >0 | <0 | >0 | >0 |
| $(2m-1,1)$ | any | <0 | <0 | >0 |
| $(2m,0)$ | <0 | <0 | >0 | >0 |

**Proof.** This follows from the fact that on a Riemannian (resp. negative-definite) manifold, there are only spacelike (resp. timelike) vectors. □

We now list all possible Lorentzian cases.

**Corollary 1.** *Let $(M,g)$ be a pseudo-Riemannian manifold of dimension $m>1$. Then, $(TM,G)$ is a Lorentzian manifold if and only if one of the following conditions holds:*

1. *$(M,g)$ is a Riemannian manifold and*
   - *either $m=2$, $\phi>0$, $\alpha<0$ and $\phi_1+\phi_3>0$, or*
   - *$\alpha_1+\alpha_3>0$, $\phi<0$ and $\alpha>0$.*
2. *$(M,g)$ is a negative definite manifold and*
   - *either $m=2$, $\phi>0$, $\alpha<0$, or*
   - *$\alpha_1+\alpha_3<0$, $\phi<0$ and $\alpha>0$.*

### *3.3. The Levi-Civita Connection of a pseudo-Riemannian $g$-Natural Metric*

We have the following description of the Levi-Civita connection of a pseudo-Riemannian $g$-natural metric, which can be deduced by the same argument used in [9] for the Riemannian case.

**Proposition 3.** *Let $(M,g)$ be a pseudo-Riemannian manifold and $G$ be a pseudo-Riemannian $g$-natural metric on its tangent bundle $TM$. The Levi-Civita connection $\bar{\nabla}$ of $(TM,G)$ is characterized by the following identities,*

$$\begin{aligned}
(\bar{\nabla}_{X^h}Y^h) &= [(\nabla_X Y)+A(X,Y)]^h+[B(X,Y)]^v,\\
(\bar{\nabla}_{X^h}Y^v) &= [C(X,Y)]^h+[(\nabla_X Y)+D(X,Y)]^v,\\
(\bar{\nabla}_{X^v}Y^h) &= [C(Y,X)]^h+[D(Y,X)]^v,\\
(\bar{\nabla}_{X^v}Y^v) &= [E(X,Y)]^h+[F(X,Y)]^v,
\end{aligned}$$

*for all $X,Y\in\mathfrak{X}(M)$, where $A(X,Y)$, $B(X,Y)$, $C(X,Y)$, $D(X,Y)$, $E(X,Y)$ and $F(X,Y)$ are the sections of $p^*TM$ defined, for all $(x,u)\in TM$, by*

$$\begin{aligned}
A(X,Y)(x,u) = &-\frac{\alpha_1\alpha_2}{2\alpha}[R(X_x,u)Y+R(Y_x,u)X_x]\\
&+\frac{\alpha_2(\beta_1+\beta_3)}{2\alpha}\{g(Y_x,u)X+g(X_x,u)Y\}\\
&+\frac{1}{\alpha\phi}\{\alpha_2[\alpha_1(\phi_1(\beta_1+\beta_3)-\phi_2\beta_2)\\
&+\alpha_2(\beta_1\alpha_2-\beta_2\alpha_1)]g(R(X_x,u)Y_x,u)\\
&+\phi_2\alpha(\alpha_1+\alpha_3)'g(X_x,Y_x)\\
&+[\alpha\phi_2(\beta_1+\beta_3)'+(\beta_1+\beta_3)[\alpha_2(\phi_2\beta_2-\phi_1(\beta_1+\beta_3))\\
&+(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)]]g(X_x,u)g(Y_x,u)\}u,
\end{aligned}$$

$$\begin{aligned}
B(X,Y)(x,u) =& \frac{\alpha_2^2}{\alpha} R(X_x,u)Y_x - \frac{\alpha_1(\alpha_1+\alpha_3)}{2\alpha} R(X_x,Y_x)u \\
&- \frac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha}[g(Y_x,u)X_x + g(X_x,u)Y_x] \\
&+ \frac{1}{\alpha\phi}\{\alpha_2[\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) \\
&+ (\alpha_1+\alpha_3)(\beta_2\alpha_1 - \beta_1\alpha_2)]g(R(X_x,u)Y_x,u) \\
&- \alpha(\phi_1+\phi_3)(\alpha_1+\alpha_3)' g(X_x,Y_x) \\
&+ [-\alpha(\phi_1+\phi_3)(\beta_1+\beta_3)' \\
&+ (\beta_1+\beta_3)[(\alpha_1+\alpha_3)[-\phi_2\beta_2 + (\phi_1+\phi_3)\beta_1] \\
&+ \alpha_2[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2]]]g(X_x,u)g(Y_x,u)\}u, \\
C(X,Y)(x,u) =& -\frac{\alpha_1^2}{2\alpha} R(Y_x,u)X_x + \frac{\alpha_1(\beta_1+\beta_3)}{2\alpha} g(X_x,u)Y_x \\
&+ \frac{1}{\alpha}[\alpha_1(\alpha_1+\alpha_3)' - \alpha_2(\alpha_2' - \frac{\beta_2}{2})]g(Y_x,u)X_x \\
&+ \frac{1}{\alpha\phi}\{\frac{\alpha_1}{2}[\alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) \\
&+ \alpha_1(\phi_1(\beta_1+\beta_3) - \phi_2\beta_2)]g(R(X_x,u)Y_x,u) \\
&+ \alpha[\frac{\phi_1}{2}(\beta_1+\beta_3) + \phi_2(\alpha_2' - \frac{\beta_2}{2})]g(X_x,Y_x) \\
&+ [\alpha\phi_1(\beta_1+\beta_3)' + [\alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1) + \alpha_1(\phi_2\beta_2 \\
&- (\beta_1+\beta_3)\phi_1)][(\alpha_1+\alpha_3)' + \frac{\beta_1+\beta_3}{2}] \\
&+ [\alpha_2(\beta_1(\phi_1+\phi_3) - \beta_2\phi_2) - \alpha_1(\beta_2(\alpha_1+\alpha_3) \\
&- \alpha_2(\beta_1+\beta_3)](\alpha_2' - \frac{\beta_2}{2})]g(X_x,u)g(Y_x,u)\}u,
\end{aligned}$$

$$\begin{aligned}
D(X,Y)(x,u) =& \frac{1}{\alpha}\{\frac{\alpha_1\alpha_2}{2} R(Y_x,u)X_x - \frac{\alpha_2(\beta_1+\beta_3)}{2} g(X_x,u)Y_x \\
&+ [(\alpha_1+\alpha_3)(\alpha_2' - \frac{\beta_2}{2}) - \alpha_2(\alpha_1+\alpha_3)']g(Y_x,u)X_x\} \\
&+ \frac{1}{\alpha\phi}\{\frac{\alpha_1}{2}[(\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1) \\
&+ \alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3))]g(R(X_x,u)Y_x,u) \\
&- \alpha[\frac{\phi_2}{2}(\beta_1+\beta_3) + (\phi_1+\phi_3)(\alpha_2' - \frac{\beta_2}{2})]g(X_x,Y_x) \\
&+ [-\alpha\phi_2(\beta_1+\beta_3)' + [(\alpha_1+\alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2) \\
&+ \alpha_2(\phi_1(\beta_1+\beta_3) - \phi_2\beta_2)][(\alpha_1+\alpha_3)' + \frac{\beta_1+\beta_3}{2}] \\
&+ [(\alpha_1+\alpha_3)(\beta_2\phi_2 - \beta_1(\phi_1+\phi_3)) \\
&+ \alpha_2(\beta_2(\alpha_1+\alpha_3) - \alpha_2(\beta_1+\beta_3))](\alpha_2' - \frac{\beta_2}{2})]g(X_x,u)g(Y_x,u)\}u,
\end{aligned}$$

$$\begin{aligned}E(X,Y)(x,u) =&\frac{1}{\alpha}[\alpha_1(\alpha_2' + \frac{\beta_2}{2}) - \alpha_2\alpha_1'][g(X_x,u)Y_x + g(Y_x,u)X_x] \\ &+ \frac{1}{\alpha\phi}\{\alpha[\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1')]g(X_x,Y_x) \\ &+ [\alpha(2\phi_1\beta_2' - \phi_2\beta_1') + 2\alpha_1'[\alpha_1(\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)) \\ &+ \alpha_2(\beta_1(\phi_1+\phi_3) - \beta_2\phi_2)] + (2\alpha_2' + \beta_2)[\alpha_1(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) \\ &+ \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)]]g(X_x,u)g(Y_x,u)\}u, \\ F(X,Y)(x,u) =&\frac{1}{\alpha}[\alpha_1'(\alpha_1+\alpha_3) - \alpha_2(\alpha_2' + \frac{\beta_2}{2})][g(X_x,u)Y_x + g(Y_x,u)X_x] \\ &+ \frac{1}{\alpha\phi}\{\alpha[(\phi_1+\phi_3)(\beta_1 - \alpha_1') - \phi_2\beta_2]g(X_x,Y_x) \\ &+ [\alpha(\beta_1'(\phi_1+\phi_3) - 2\beta_2'\phi_2) + 2\alpha_1'[\alpha_2(\beta_2(\alpha_1+\alpha_3) - \alpha_2(\beta_1+\beta_3)) \\ &+ (\alpha_1+\alpha_3)(\beta_2\phi_2 - \beta_1(\phi_1+\phi_3))] + (2\alpha_2' + \beta_2)[\alpha_2(\phi_1(\beta_1+\beta_3) \\ &- \phi_2\beta_2) + (\alpha_1+\alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2)]]g(X_x,u)g(Y_x,u)\}u.\end{aligned}$$

## 4. The Null Tangent Bundle

Let $(M,g)$ be a non-definite pseudo-Riemannian manifold. We call *the null tangent bundle* of $(M,g)$ the subset of $TM$ given by

$$T_0M = \{(x,u) \in TM : \ g_x(u,u) = 0 \ and \ u \neq 0\}.$$

### *4.1. Differentiable Structure on the Null Tangent Bundle*

**Proposition 4.** *$T_0M$ is an imbedded submanifold of $TM$ of dimension $2m-1$. Furthermore, if we denote by $p_0$ the restriction to $T_0M$ of the projection $p: TM \to M$, then $(T_0M, p_0, M)$ is a subbundle over $M$ with fiber $p_0^{-1}(x)$ at $x \in M$ diffeomorphic to the null cone $\Lambda_x$ at $x$.*

**Proof.** We consider the function

$$\begin{array}{rrcl} q: & TM \setminus \{0_x, x \in M\} & \to & \mathbb{R} \\ & (x,u) & \mapsto & g_x(u,u). \end{array}$$

Then, $q$ is a smooth function and $T_0M = q^{-1}(0)$. Suppose that there exists a point $(x,u) \in q^{-1}(0)$ such that $d_{(x,u)}q = 0$ and let $\tilde{q}_x$ be the restriction of $q$ to the fiber $T_xM \setminus \{0_x\}$ and $i: T_xM \hookrightarrow TM$ the canonical inclusion. We then have

$$d_{(x,u)}q \circ i = d_u\tilde{q}_x = 2g_x(u,.) = 0.$$

Since $g$ is non-degenerate, we conclude that $u = 0$, which is a contradiction. Therefore, we deduce that 0 is a regular value of $q$ and hence $T_0M$ is an imbedded submanifold of $TM$. □

**Proposition 5.** *For any $(x,u) \in T_0M$, the tangent space of $T_0M$ at $(x,u)$ is given by*

$$T_{(x,u)}T_0M = \Big\{X^h + Y^v : X, Y \in T_xM, g(Y,u) = 0\Big\}.$$

*Moreover, $T_0M$ is an orientable submanifold of $TM$.*

**Proof.** We know that, for each $(x,u) \in TM$, we have the decomposition $T_{(x,u)}TM = H_{(x,u)}TM \oplus V_{(x,u)}TM$. Since $T_0M$ is a subbundle of $TM$, its tangent bundle at a point $(x,u) \in T_0M$ can be decomposed as $T_{(x,u)}T_0M = H_{(x,u)}TM \oplus T_{(x,u)}\Lambda_x$. Using the notations of the proof of Proposition 4, we have

$$T_{(x,u)}\Lambda_x = \ker d_{(x,u)}\tilde{q}_x = \{X^v \in V_{(x,u)}TM, g(X,u) = 0\}.$$

Here we used the isomorphism between $V_{(x,u)}$ and $T_{(x,u)}T_xM$. This proves the first part of the Proposition. On the other hand, $TM$ is an orientable manifold, and the geodesic vector field $\zeta$ defined by $\zeta_{(x,u)} = u^h$ induces a nowhere vanishing vector field on $T_0M$. So, $T_0M$ is an orientable submanifold. □

*4.2. Induced g-Natural Metrics on the Null Tangent Bundle*

Taking into account Proposition 5, the induced metric $\tilde{G}$ on $T_0M$ of a $g$-natural metric $G$ on $TM$ is completely determined by

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^v) = bg_x(X,Y), \\ \tilde{G}_{(x,u)}(X^v, Y^v) = ag_x(X,Y), \end{cases} \quad (12)$$

where $a = \alpha_1(0)$, $c = \alpha_3(0)$, $d = \beta_3(0)$ and $b = \alpha_2(0)$.

**Remark 3.** *Hereafter and without loss of generality, we shall consider, in the study of the geometry of null tangent bundle, only g-natural metrics on TM given by (12). Note that, by virtue of Theorem 1,* $a \neq 0$, $a + c \neq 0$ *and* $a(a+c) - b^2 \neq 0$. *Furthermore, if g is of signature* $(m-k, k)$, *then we have:*

- $a > 0$, $a + c > 0$ *and* $a(a+c) - b^2 > 0$ *if G is of signature* $(2m-2k, 2k)$;
- $a < 0$, $a + c < 0$ *and* $a(a+c) - b^2 > 0$ *if G is of signature* $(2k, 2m-2k)$;
- $a(a+c) - b^2 < 0$ *if G is of signature* $(m, m)$.

**Proposition 6.** *Let* $(M, g)$ *be a pseudo-Riemannian manifold and G a pseudo-Riemannian g-natural metric on TM.* $T_0M$ *is a lightlike hypersurface of* $(TM, G)$ *and the radical of* $T_{(x,u)}T_0M$ *at a point* $(x,u) \in T_0M$ *is given by*

$$Rad(T_{(x,u)}(T_0M)) = Span\{-bu^h + (a+c)u^v\}.$$

*Thus,* $Rad(T_0M)$ *is spanned by the vector field induced on* $T_0M$ *by the vector field* $\xi = -b\zeta + (a+c)\mathcal{U}$ *on TM, where* $\zeta$ *and* $\mathcal{U}$ *are, respectively, the geodesic vector field and the canonical vertical vector field on TM.*

**Proof.** Let $(x,u) \in TM$ such that $g(u,u) = 0$ and $v \in T_xM$ such that $g(v,v) = g(u,u) = 0$ and $g(u,v) = 1$. Consider a basis $\{e_i\}_{i=1}^m$ of $T_xM$ such that $e_1 = u$, $e_2 = v$, $g(u,e_i) = g(v,e_i) = g(e_i,e_j) = 0$, for $3 \leq i \neq j \leq m$ and

- $g(e_i,e_i) = 1$, for $3 \leq i \leq m$, if $k = 1$;
- $g(e_i,e_i) = -1$, for $3 \leq i \leq m$, if $m - k = 1$;
- $g(e_i,e_i) = -1$, for $3 \leq i \leq k+1$, $g(e_i,e_i) = 1$, for $k+2 \leq i \leq m$, if $k \geq 2$ and $m - k \geq 2$.

The matrix of $\tilde{G}_{(x,u)}$ with respect to the basis $\{e_1^h, e_2^h, \cdots, e_m^h, e_1^v, e_3^v, \cdots, e_m^v\}$ of $T_{(x,u)}T_0M$ is given by

$$\tilde{G}_{(x,u)} = \begin{pmatrix} 0 & a+c & 0 & 0 & 0 & 0 & 0 \\ a+c & d & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -(a+c)I_{k-1} & 0 & 0 & -bI_{k-1} & 0 \\ 0 & 0 & 0 & (a+c)I_{m-k-1} & 0 & 0 & bI_{m-k-1} \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -bI_{k-1} & 0 & 0 & -aI_{k-1} & 0 \\ 0 & 0 & 0 & bI_{m-k-1} & 0 & 0 & aI_{m-k-1} \end{pmatrix},$$

whose determinant is 0. Therefore, the induced metric $\tilde{G}$ on $T_0M$ is degenerate.
Since $T_0M$ is a degenerate hypersurface of $TM$, $Rad(T_{(x,u)}(T_0M))$ is one-dimensional, and we can easily see that $-bu^h + (a+c)u^v \in Rad(T_{(x,u)}(T_0M))$. □

*4.3. A Screen Distribution on the Null Tangent Bundle*

Let $(x,u) \in T_0M$. If we consider the basis $\{e_i\}_{i=1}^m$ of $T_xM$ constructed in the proof of Proposition 6, it is easy to check that a complementary vector subspace of $Rad(T_{(x,u)}T_0M)$ on $T_{(x,u)}T_0M$ is given by

$$S(T_{(x,u)}T_0M) = Span\{au^h - bu^v, v^h, e_3^h, \cdots, e_m^h, e_3^v, \cdots, e_m^v\}, \tag{13}$$

and that

$$S(T_{(x,u)}T_0M)^\perp = Span\{-bu^h + (a+c)u^v, v^v\} \quad \text{and} \quad tr(T_{(x,u)}T_0M) = Span\{v^v\}.$$

Our aim is to construct a (local) screen distribution $S(T(T_0M))$ so that its fiber at $(x,u) \in T_0M$ is expressed as $S(T_{(x,u)}T_0M)$ and, consequently, its corresponding lightlike transversal vector bundle is vertical.

As mentioned in Proposition 6, $\xi$ is a non-zero section of $Rad(T(T_0M))$. Then, there is locally a unique section $N$ of $tr(T(T_0M))$ such that $G(N,\xi) = 1$, $G(N,N) = G(N,W) = 0$, for any local section $W$ of $S(T(T_0M))$ (cf. [11]). Since $tr(T_{(x,u)}T_0M)$ is a vertical vector of $T_xM$, then $N$ is vertical at any point where it is defined.

To construct explicitly $N$, we consider the pull-back vector bundle $p_0^*TM$ induced by the tangent bundle $TM$ and the restriction $p_0 : T_0M \to M$ to $T_0M$ of the natural projection $p : TM \to M$. The induced metric on $p_0^*TM$ by $g$ is none other than the metric induced from the bundle metric $\hat{g}$ on $p^*TM$; hence, we shall denote it by the same symbol $\hat{g}$. It is then defined by

$$\hat{g}_{(x,u)}(X,Y) := g_x(X_{(x,u)}, Y_{(x,u)}),$$

for any $X, Y \in \Gamma(p_0^*TM)$ and $(x,u) \in T_0M$. Furthermore, $\hat{g}_{(x,u)}$ has the same signature as $g_x$.

The restriction to $T_0M$ of the identity section $\sigma$ of $p^*TM$ can be considered as a section of the vector bundle $p_0^*TM$, which we also denote by $\sigma$. Then, around any vector of $T_0M$ there is an open neighborhood $U$ in $T_0M$ and a local section $V$ of $p_0^*TM$ defined on $U$, such that $\hat{g}(\sigma|_U, V) = 1$ and $\hat{g}(V,V) = 0$. From now on, we shall denote $\sigma|_U$ by $\sigma$. Taking a smallest $U$ if necessary, we can find $m-2$ sections $e_i$, $i = 3, ..., m$, of $\Gamma((p_0^*TM)|_U)$ such that $\hat{g}(\sigma, e_i) = \hat{g}(V, e_i) = \hat{g}(e_i, e_j) = 0$, for $3 \le i \ne j \le m$ and

- $\hat{g}(e_i, e_i) = 1$, for $3 \le i \le m$, if $k = 1$;
- $\hat{g}(e_i, e_i) = -1$, for $3 \le i \le m$, if $m - k = 1$;
- $\hat{g}(e_i, e_i) = -1$, for $3 \le i \le k+1$, $\hat{g}(e_i, e_i) = 1$, for $k+2 \le i \le m$, if $k \ge 2$ and $m - k \ge 2$.

It is easy to check that the transverse vector field $N$ of the screen distribution $S(T(T_0M))$ on $U$ is given by

$$N_{(x,u)} = \frac{1}{\alpha}(V(u))^v, \quad \text{for all} \quad (x,u) \in U,$$

where the vertical lift is taken at $(x,u)$.

On the other hand, if we define on $U$ the vector fields $W$, $E_i$, $E_{m+i}$, $i = 3, ..., m$, by

$$W_{(x,u)} := (V(u))^h, \quad E_i(x,u) := (e_i(x,u))^h \quad \text{and} \quad E_{m+i}(x,u) := (e_i(x,u))^v,$$

where the lifts are taken at $(x,u)$, then the screen distribution $S(T(T_0M))$ is locally generated (on $U$) by $a\,\zeta - b\,\mathcal{U}, W, E_i, E_{m+i}$, $i = 3, \ldots, m$ and the lightlike transversal vector bundle $tr(T(T_0M))$ with respect to $S(T(T_0M))$ is locally generated (on $U$) by $N$.

Now, by virtue of Proposition 5, the restriction to $T_0M$ of any horizontal lift of a vector field on $M$ is a vector field on $T_0M$, but this is not the case for vertical lifts. So to go further in our study, we need to construct a new lift that gives rise to (local) vector fields on $T_0M$. We define then the *tangential lift* $X^t$ of a vector field $X$ on $M$, with respect to the screen distribution $S(T(T_0M))$, as the vector field defined by

$$X^t_{(x,u)} := X^v - g(X,u)(V(u))^v, \quad \text{for all} \quad (x,u) \in U,$$

where the lifts are taken at $(x,u)$. By Proposition 5, $X^t_{(x,u)} \in T_{(x,u)}T_0M$, so that $X^t \in \mathfrak{X}(U)$. Furthermore, we can check easily the following result.

**Lemma 2.** *For any $(x,u) \in U$, we have*

$$u^t_{(x,u)} = u^v_{(x,u)} \qquad \text{and} \qquad (V(u))^t_{(x,u)} = 0.$$

As for the case of sections of $p^*TM$, we can define horizontal and tangential lifts of sections of $p_0^*TM$, to obtain horizontal and tangential vector fields on $T_0M$. For any section $s \in \Gamma(p_0^*TM)$, the horizontal (resp. tangential) lift of $s$ is the horizontal vector field on $T_0M$ (resp. on $U$) defined by $s^h(u) = (s(u))^h$ (resp. $s^t(u) = (s(u))^t$), for all $u \in T_0M$ (resp. $u \in U$).

For example, the vector fields $\xi$, $N$, $W$, $E_i$ and $E_{m+i}$, $i = 3, ..., m$ on $U$ can be expressed by means of lifts of sections as follows:

$$\xi = -b\sigma^h + (a+c)\sigma^v, \qquad N = V^v, \qquad W = V^h, \qquad E_i = e_i^h, \qquad E_{m+i} = e_i^v. \tag{14}$$

The tangential lift of a section $s \in \Gamma(p_0^*TM|_U)$ can be seen as the vertical lift of an auxiliary section $\tilde{s} \in \Gamma(p_0^*TM|_U)$ defined, for all $u \in U$, by

$$\tilde{s}(u) = s(u) - g(u, s(u))V(u),$$

i.e.,

$$\tilde{s} = s \circ j|_U - \hat{g}(\sigma, s)V.$$

It is easy to check that this operation satisfies the following properties:

**Lemma 3.** *For any $s, s' \in \Gamma(p_0^*TM|_U)$ and $X \in \mathfrak{X}(M)$, we have*

1. $\tilde{\sigma} = \sigma$;
2. $\tilde{V} = 0$;
3. $\tilde{X} := \widetilde{X \circ p_0} = X \circ p_0 - \hat{g}(X, \sigma)V$;
4. $\hat{g}(\tilde{s}, \sigma) = 0$;
5. $\hat{g}(\tilde{s}, V) = \hat{g}(s, V)$;
6. $\hat{g}(\tilde{s}, \tilde{s'}) = \hat{g}(s, s') - \hat{g}(s, \sigma)\hat{g}(\hat{s'}, V) - \hat{g}(s, V)\hat{g}(\hat{s'}, \sigma)$;
7. $\tilde{s}^h = s^h - \hat{g}(\sigma, s)W$;
8. $\tilde{s}^t = \tilde{s}^v = s^t$;
9. $\tilde{X}^t = \tilde{X}^v = X^t$.

Finally, we note that the tangent space at $(x,u) \in U$ of $T_0M$ is expressed in terms of horizontal and tangential lifts as follows:

$$T_{(x,u)}T_0M = \{X^h + Y^t, X, Y \in T_xM\},$$

where the lifts are taken at $(x,u)$. We deduce from (12) the following characterization.

**Lemma 4.** *The induced metric $\tilde{G}$ on $T_0M$ of a g-natural metric $G$ on $TM$ given by (12) is completely determined on $U$ by*

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^t) = b[g_x(X,Y) - g_x(Y,u)g_x(X,V(u))], \\ \tilde{G}_{(x,u)}(X^t, Y^t) = a[g_x(X,Y) - g_x(X,u)g_x(Y,V(u)) - g_x(Y,u)g_x(X,V(u))], \end{cases}$$

*for all $(x,u) \in U$ and $X, Y \in T_xM$.*

### 4.4. The Induced Connection on the Null Tangent Bundle Associated to the Screen Distribution

As for the case of the tangent bundle, to make calculations on covariant derivatives on the null tangent bundle, we need to introduce the induced connection on the vector bundle $p_0^*TM$ induced by the tangent bundle $TM$ and the projection $p_0 : T_0M \to M$. Remarking that the vector bundle $p_0^*TM$ is also the vector bundle induced from $p^*TM$ and the inclusion map $j : T_0M \hookrightarrow TM$ (since $p \circ j = p_0$), we deduce that the covariant derivative associated to induced connection on $p_0^*TM$ is the restriction of $\hat{\nabla}$ and hence we denote it in the same way. More precisely, let $(e_1, \ldots, e_m)$ be a moving frame on an open set $\bar{U}$ of $M$. Then, $(e_1 \circ p_0, \ldots, e_m \circ p_0)$ is a moving frame on $(\Gamma(p_0^*TM))|_{p^{-1}(\bar{U})}$ and, for every section $s \in \Gamma(p_0^*TM)$, we have $s|_{p_0^{-1}(\bar{U})} = \sum_{i=1}^{m} s^i e_i \circ p_0$, where $s^i \in C^\infty(p_0^{-1}(\bar{U}))$, $i = 1, \ldots, m$.

For $Z \in \mathfrak{X}(T_0M)$, $\hat{\nabla}_Z s$ is given on $p_0^{-1}(\bar{U})$ by the expression

$$\hat{\nabla}_Z s|_{p_0^{-1}(\bar{U})} = \sum_{i=1}^{m} \Big[ Z(s^i) e_i \circ p_0 + s^i (\nabla_{(p_0)_* Z} e_i) \circ p_0 \Big].$$

In particular, for any $Z \in \mathfrak{X}(T_0M)$ and $Y \in \mathfrak{X}(M)$, we have

$$\hat{\nabla}_Z (Y \circ p_0) := (\nabla_{(p_0)_* Z} Y) \circ p_0.$$

If $Z$ is either a horizontal or a tangential lift of a vector field $X \in \mathfrak{X}(M)$, then we have

$$\hat{\nabla}_{X^h} s|_{p_0^{-1}(\bar{U})} = \sum_{i=1}^{m} \Big[ X^h(s^i) e_i \circ p_0 + s^i (\nabla_X e_i) \circ p_0 \Big], \tag{15}$$

$$\hat{\nabla}_{Y^t} s|_{U \cap p_0^{-1}(\bar{U})} = \sum_{i=1}^{m} \Big[ Y^v(s^i) - \hat{g}(Y \circ p_0, \sigma) V^v(s^i) \Big] e_i \circ p_0. \tag{16}$$

We can define in the same way $\hat{\nabla}_Z s$ pointwise, i.e., when $Z \in TT_0M$.

It is obvious that the connection $\hat{\nabla}$ on $p_0^*TM$ is compatible with the induced metric $\hat{g}$. Consequently, we have the following.

**Lemma 5.** *For any $X, Y \in \mathfrak{X}(M)$, we have*
1. $\hat{\nabla}_{X^h}(Y \circ p_0) = (\nabla_X Y) \circ p_0$;
2. $\hat{\nabla}_{X^t}(Y \circ p_0) = 0$;
3. $\hat{\nabla}_{X^h} \sigma = 0$;
4. $\hat{\nabla}_{X^t} \sigma = X \circ p_0 - \hat{g}(X \circ p_0, \sigma) V$.
5. $\hat{g}(\hat{\nabla}_{X^h} V, \sigma) = 0$,
6. $\hat{g}(\hat{\nabla}_{X^t} V, \sigma) = -\hat{g}(X \circ p_0, V)$,

*where $\sigma$ denotes the identity section on $p_0^*TM$.*

**Proof.** The four first identities are obvious consequences of (15) and (16). For the fifth identity, using the fact that $\hat{g}(V, \sigma) = 1$, the compatibility of $\hat{\nabla}$ with $\hat{g}$ and the first identity, we have

$$\hat{g}(\hat{\nabla}_{X^h} V, \sigma) = X^h(\hat{g}(V, \sigma)) - \hat{g}(\hat{\nabla}_{X^h} \sigma, V) = 0.$$

The last identity can be proved in a similar way. □

Using Lemma 5 and the compatibility of $\hat{\nabla}$ with $\hat{g}$, we obtain the following.

**Lemma 6.** *For all $s \in \Gamma(p_0^*TM)$, $X, Y, Z \in \mathfrak{X}(M)$, we have*
1. $X^h(\hat{g}(s, \sigma)) = \hat{g}(\hat{\nabla}_{X^h} s, \sigma)$;
   *In particular,* $X^h(\hat{g}(Y \circ p_0, \sigma)) = \hat{g}((\nabla_X Y) \circ p_0, \sigma)$
2. $X^h(\hat{g}(Y \circ p_0, Z \circ p_0)) = X(g(Y, Z)) \circ p_0$;

3. $X^h(f(\hat{g}(\sigma,\sigma))) = 2f'(\hat{g}(\sigma,\sigma))\hat{g}(X\circ p_0,\sigma)$.
4. $X^h(\hat{g}(s,V))|_U = \hat{g}(\hat{\nabla}_{X^h}s,V) + \hat{g}(s,\hat{\nabla}_{X^h}V)$;
   *In particular,* $X^h(\hat{g}(Y\circ p_0,V))|_U = \hat{g}((\nabla_X Y)\circ p_0,V) + \hat{g}(Y\circ p_0,\hat{\nabla}_{X^h}V)$;
5. $X^t((Y\circ p_0, Z\circ p_0)) = 0$;
6. $X^t(\hat{g}(s,V)) = \hat{g}(s,\hat{\nabla}_{X^t}V)$;
   *In particular,* $X^t(\hat{g}(Y\circ p_0,V)) = \hat{g}(Y\circ p_0,\hat{\nabla}_{X^t}V)$;
7. $X^t(\hat{g}(s,\sigma)) = g(\tilde{X},s) + \hat{g}(\sigma,,\hat{\nabla}_{X^t}s)$;
   *In particular,*
$$X^t(\hat{g}(Y\circ p_0,\sigma)) = \hat{g}(\tilde{X},Y\circ p_0) = \hat{g}(Y,X)\circ p_0 - \hat{g}(X\circ p_0,\sigma)\hat{g}(Y\circ p_0,V);$$
8. $X^t(f(\hat{g}(\sigma,\sigma))) = 0$.

**Lemma 7.** *The Lie bracket on $T_0M$ satisfies the following identities on $U$:*
1. $[X^h,Y^h] = [X,Y]^h - [R(X\circ p_0,Y\circ p_0)\sigma]^t$,
2. $[X^h,Y^t] = \left[\nabla_X Y - \hat{g}(Y\circ p_0,\sigma)\hat{\nabla}_{X^h}V\right]^t$,
3. $[X^t,Y^t] = \left[\hat{g}(X\circ p_0,\sigma)\hat{\nabla}_{Y^t}V - \hat{g}(Y\circ p_0,\sigma)\hat{\nabla}_{X^t}V\right]^t$

*for all $X,Y\in\mathfrak{X}(M)$.*

**Proof.** To prove the first identity, it suffices to use the first identity of Lemma 1 and the fact that
$$[R(X\circ p_0,Y\circ p_0)\sigma]^t = [R(X\circ p_0,Y\circ p_0)\sigma]^v - \hat{g}(R(X\circ p_0,Y\circ p_0)\sigma,\sigma)V^v = [R(X\circ p_0,Y\circ p_0)\sigma]^v.$$
To prove the second identity, we use the second identity of Lemmas 1 and 6, to obtain
$$\begin{aligned}[X^h,Y^t] =&[X^h,Y^v] - X^h(\hat{g}(Y\circ p_0,\sigma))V^v - \hat{g}(Y\circ p_0,\sigma)[X^h,V^v]\\ =&(\nabla_X Y)^v - \hat{g}((\nabla_X Y)\circ p_0,\sigma)V^v - \hat{g}(Y\circ p_0,\sigma)[X^h,V^v]\\ =&(\nabla_X Y)^t - \hat{g}(Y\circ p_0,\sigma)[X^h,V^v].\end{aligned}$$
On the other hand, using the local expression of $V$ and the definition of $\hat{\nabla}$, we can check that
$$[X^h,V^v] = \left[\hat{\nabla}_{X^h}V\right]^v = \left[\hat{\nabla}_{X^h}V\right]^t,$$
since $\hat{g}(\hat{\nabla}_{X^h}V,\sigma) = 0$ by the fifth identity of Lemma 5. This completes the proof of the second identity of the Lemma. The third identity of the Lemma is proved in a similar way by using the third identity of Lemma 1, the last identity of Lemmas 5 and 6. □

The following result will be used in the calculations:

**Lemma 8.** *For all $s\in\Gamma(p^*TM)$, we have*
1. $G(s^h,\xi) = 0$ *and* $G(s^v,\xi) = \alpha\hat{g}(s,\sigma)$;
2. $G(s^h,N) = \frac{b}{\alpha}\hat{g}(s,V)$ *and* $G(s^v,N) = \frac{a}{\alpha}\hat{g}(s,V)$ *on $U$.*

According to Remark 3, to study the geometry of $T_0M$ equipped with the induced metric of a $g$-natural metric $G$ on $TM$, we can assume that $G$ is given by (12). In this case, using Proposition 3, the Levi-Civita connection of $(TM,G)$ reduces to the form given in the following.

**Proposition 7.** *Let* $(M, g)$ *be a pseudo-Riemannian manifold and* $G$ *be a pseudo-Riemannian* $g$*-natural metric given by* (12) *on its tangent bundle* $TM$. *The Levi-Civita connection* $\bar{\nabla}$ *of* $(TM, G)$ *is characterized by the following identities:*

$$\begin{aligned}
(\bar{\nabla}_{X^h}Y^h) &= [(\nabla_X Y) + A(X,Y)]^h + [B(X,Y)]^v, \\
(\bar{\nabla}_{X^h}Y^v) &= [C(X,Y)]^h + [(\nabla_X Y) + D(X,Y)]^v, \\
(\bar{\nabla}_{X^v}Y^h) &= [C(Y,X)]^h + [D(Y,X)]^v, \\
(\bar{\nabla}_{X^v}Y^v) &= [E(X,Y)]^h + [F(X,Y)]^v,
\end{aligned}$$

*for all* $X, Y \in \mathfrak{X}(M)$, *where* $A(X,Y)$, $B(X,Y)$, $C(X,Y)$, $D(X,Y)$, $E(X,Y)$ *and* $F(X,Y)$ *are the sections of* $p^*TM$ *defined by*

$$\begin{aligned}
A(X,Y)(x,u) =& -\frac{ab}{2\alpha}[R(X_x,u)Y_x + R(Y_x,u)X_x] + \frac{abd}{\alpha^2}[ag(R(X_x,u)Y_x,u) \\
& - dg(X_x,u)g(Y_x,u)]u + \frac{bd}{2\alpha}[g(Y_x,u)X_x + g(X_x,u)Y_x], \\
B(X,Y)(x,u) =& -\frac{d(a+c)}{2\alpha}[g(Y_x,u)X_x + g(X_x,u)Y_x] - \frac{a(a+c)}{2\alpha}R(X_x,Y_x)u \\
& + \frac{b^2}{\alpha}R(X_x,u)Y_x + \frac{b^2 d}{\alpha^2}[dg(X_x,u)g(Y_x,u) - ag(R(X_x,u)Y_x,u)]u, \\
C(X,Y)(x,u) =& \frac{ad}{2\alpha}[g(X_x,Y_x)u + g(X_x,u)Y_x] - \frac{a^2}{2\alpha}R(Y_x,u)X_x \\
& - \frac{a^2 d}{2\alpha^2}[dg(X_x,u)g(Y_x,u) - ag(R(Y_x,u)X_x,u)]u, \\
D(X,Y)(x,u) =& -\frac{bd}{2\alpha}[g(X_x,Y_x)u + g(X_x,u)Y_x] + \frac{ab}{2\alpha}R(Y_x,u)X_x \\
& + \frac{abd}{2\alpha^2}[dg(X_x,u)g(Y_x,u) - ag(R(Y_x,u)X_x,u)]u, \\
E(X,Y)(x,u) =& F(X,Y)(x,u) = 0,
\end{aligned}$$

*for all* $X, Y \in \mathfrak{X}(M)$ *and* $(x,u) \in TM$.

Before giving the induced connection on $T_0M$ associated with the screen distribution $S$, we shall give the local second fundamental form.

**Proposition 8.** *Let* $(M, g)$ *be a non-definite pseudo-Riemannian manifold and* $G$ *a pseudo-Riemannain* $g$*-natural metric on its tangent bundle. The local second fundamental form* $B_0$ *of* $T_0M$ *associated with the screen distribution* $S(T(T_0M))$, *given by Equation* (13), *is characterized on* $U$ *by*

$$B_0(X^h_{(x,u)}, Y^h_{(x,u)}) = -d(a+c)g(X_x,u)g(Y_x,u) + b^2 g(R(X_x,u)Y_x,u),$$

$$\begin{aligned}
B_0(X^h_{(x,u)}, Y^t_{(x,u)}) =& \frac{ab}{2}[g(R(X_x,u)Y_x,u) - g(Y_x,u)g(R(V(u),u)X_x,u)] \\
=& \frac{ab}{2}g(R(X_x u,u)\tilde{Y}_u,u), \\
B_0(X^t_{(x,u)}, Y^h_{(x,u)}) =& \frac{ab}{2}[g(R(X_x,u)Y_x,u) - g(X_x,u)g(R(V(u),u)Y_x,u)] \\
=& \frac{ab}{2}g(R(\tilde{X}_u,u)Y_x,u), \\
B_0(X^t_{(x,u)}, Y^t_{(x,u)}) =& \alpha[-g(X_x,Y_x) + g(X_x,u)g(Y_x,V(u)) + g(Y_x,u)g(X_x,V(u))] \\
=& -\alpha g(\tilde{X}_u, \tilde{Y}_u),
\end{aligned}$$

*for all* $(x,u) \in U$ *and* $X, Y \in T_x M$.

**Proof.** For $(x,u) \in U$, we have by Proposition 7 and Lemma 8

$$\begin{aligned} B_0(X^h, Y^h) =& G(\bar{\nabla}_{X^h} Y^h, \xi_{)} = G([(\nabla_X Y) + A(X,Y)]^h + [B(X,Y)]^v, \xi) \\ =& \alpha \hat{g}(B(X,Y), \sigma). \end{aligned} \tag{17}$$

Using the expression of $B(X,Y)$, a routine calculation yields the first identity of the Proposition. A similar calculation gives

$$\begin{aligned} B_0(X^t, Y^h) =& G(\bar{\nabla}_{X^t} Y^h, \xi_{)} = G(\bar{\nabla}_{\tilde{X}^v} Y^h, \xi_{)} \\ =& G([C(Y,\tilde{X})]^h + [D(Y,\tilde{X})]^v, \xi) = \alpha \hat{g}(D(Y,\tilde{X}), \sigma). \end{aligned} \tag{18}$$

From the expression of $D(X,Y)$, we obtain the third identity of the Proposition. The second identity follows from the symmetry of $B_0$.
Now, let us establish the last identity of the Proposition. Using Lemmas A1 and A2, it is easy to check that

$$\begin{aligned} (\bar{\nabla}_{X^t} Y^t) =& \bar{\nabla}_{X^t} Y^v - X^t(\hat{g}(Y \circ p_0, \sigma)) V^v - \hat{g}(Y \circ p_0, \sigma) \bar{\nabla}_{X^t} V^v \\ =& \bar{\nabla}_{X^t} Y^v - \hat{g}(\tilde{X}, Y \circ p_0) V^v - \hat{g}(Y \circ p_0, \sigma) \bar{\nabla}_{X^t} V^v \\ =& - \hat{g}(\tilde{X}, Y \circ p_0) V^v - \hat{g}(Y \circ p_0, \sigma) \bar{\nabla}_{X^t} V^v, \end{aligned}$$

since, from Proposition 7, we have $\bar{\nabla}_{X^t} Y^v = \bar{\nabla}_{\tilde{X}^v} Y^v = 0$. On the other hand, by the fourth identity of Lemma A3, we have

$$\bar{\nabla}_{X^t} V^v = \bar{\nabla}_{\tilde{X}^v} V^v = \left[\hat{\nabla}_{\tilde{X}^v} V\right]^v = \left[\hat{\nabla}_{X^t} V\right]^v$$

and hence,

$$(\bar{\nabla}_{X^t} Y^t) = -\left[\hat{g}(Y \circ p_0, \sigma) \hat{\nabla}_{X^t} V + \hat{g}(\tilde{X}, Y \circ p_0) V\right]^v. \tag{19}$$

Using again Lemmas 5 and 8, we obtain

$$\begin{aligned} B_0(X^t, Y^t) =& - \alpha \hat{g}(\hat{g}(Y \circ p_0, \sigma) \hat{\nabla}_{X^t} V + \hat{g}(\tilde{X}, Y \circ p_0) V, \sigma) \\ =& - \alpha \{\hat{g}(Y \circ p_0, \sigma) \hat{g}(\hat{\nabla}_{X^t} V, \sigma) + \hat{g}(\tilde{X}, Y \circ p_0) \hat{g}(V, \sigma)\} \\ =& - \alpha \{-\hat{g}(Y \circ p_0, \sigma) \hat{g}(X \circ p_0, V) + \hat{g}(\tilde{X}, Y \circ p_0)\} \\ =& - \alpha \{-\hat{g}(Y \circ p_0, \sigma) \hat{g}(\tilde{X}, V) + \hat{g}(\tilde{X}, Y \circ p_0)\} \\ =& - \alpha \hat{g}(\tilde{X}, \tilde{Y}). \end{aligned}$$

□

**Corollary 2.** *Let* $(M, g)$ *be a non-definite pseudo-Riemannian manifold and* $G$ *a pseudo-Riemannian* $g$*-natural metric on* $TM$. *Then,* $(T_0 M, \tilde{G}, S(T(T_0 M)))$ *is not umbilical at any point of* $U$. *Consequently, it is never totally umbilical.*

**Proof.** Suppose that $(T_0 M, \tilde{G}, S(T(T_0 M)))$ is umbilical at $(x,u) \in U$. Then there is $\mu \in \mathbb{R}$ such that $B_0(Z, W) = \mu \tilde{G}(Z, W)$, for any $Z, W \in T_{(x,u)}(T_0 M)$. In particular, we have

$$B_0(X^t_{(x,u)}, Y^t_{(x,u)}) = \mu \tilde{G}_{(x,u)}(X^t, Y^t),$$

for any $X, Y \in T_x M$. Using the fourth identity of Proposition 8 and Lemma 4, we find that $a \neq 0$ and $\mu = -\frac{\alpha}{a}$. As a consequence, we obtain

$$B_0(X^h_{(x,u)}, Y^h_{(x,u)}) = -\frac{\alpha}{a} \tilde{G}_{(x,u)}(X^h, Y^h).$$

In particular, if we take $X = u$ and $Y = V(u)$, we obtain, using the first identity of Proposition 8 and Lemma 4, $a + c = 0$. Then, $b \neq 0$ and

$$g(R(X_x, u)Y_x, u) = -\frac{\alpha d}{ab^2} g(X_x, u) g(Y_x, u).$$

Substituting from the last equation into the second identity of Proposition 8, we obtain $B_0(X^h_{(x,u)}, Y^t_{(x,u)}) = 0$. We deduce that $-\frac{\alpha}{a}\tilde{G}_{(x,u)}(X^h, Y^h) = 0$, for all $X, Y \in T_x M$, which contradicts the fact that $\alpha \neq 0$. □

As a corollary of Proposition 8, we obtain the following result.

**Theorem 2.** *Let $(M, g)$ be a non-definite pseudo-Riemannian manifold and $G$ a pseudo-Riemannian $g$-natural metric on its tangent bundle. The induced connection $\tilde{\nabla}$ on $T_0 M$ corresponding to the screen distribution $S(T(T_0 M))$ is determined on $U$ by*

$$\begin{aligned}
\tilde{\nabla}_{X^h} Y^h &= [\hat{\nabla}_{X^h} Y + A(X, Y)]^h + [B(X, Y)]^t \\
\tilde{\nabla}_{X^h} Y^t &= [C(X, \tilde{Y})]^h + [\hat{\nabla}_{X^h} \tilde{Y} + D(X, \tilde{Y})]^t \\
\tilde{\nabla}_{X^t} Y^h &= [C(Y, \tilde{X})]^h + [D(Y, \tilde{X})]^t \\
\tilde{\nabla}_{X^t} Y^t &= [\hat{\nabla}_{X^t} \tilde{Y}]^t,
\end{aligned}$$

*for all $X, Y \in \mathfrak{X}(T_0 M)$.*

**Proof.** Using Proposition 7 and (17) and the fact that $N = \frac{1}{\alpha} V^v$, we have

$$\begin{aligned}
\tilde{\nabla}_{X^h} Y^h &= \bar{\nabla}_{X^h} Y^h - B_0(X^h, Y^h) N \\
&= [(\nabla_X Y) + A(X, Y)]^h + [B(X, Y)]^v - \hat{g}(B(X, Y), \sigma) V^v \\
&= [(\nabla_X Y) + A(X, Y)]^h + [B(X, Y) - \hat{g}(B(X, Y), \sigma) V]^v \\
&= [\hat{\nabla}_{X^h} Y + A(X, Y)]^h + [B(X, Y)]^t.
\end{aligned}$$

In the same way, using Proposition 7 and (18) and the fact that $N = \frac{1}{\alpha} V^v$, we have

$$\begin{aligned}
\tilde{\nabla}_{X^t} Y^h &= \bar{\nabla}_{X^t} Y^h - B_0(X^t, Y^h) N = \bar{\nabla}_{\tilde{X}^v} Y^h - B_0(X^t, Y^h) N \\
&= [C(Y, \tilde{X})]^h + [D(Y, \tilde{X})]^v - \hat{g}(D(X, YY, \tilde{X}), \sigma) V^v \\
&= [C(Y, \tilde{X})]^h + [D(Y, \tilde{X}) - \hat{g}(D(Y, \tilde{X}), \sigma) V]^v \\
&= [C(Y, \tilde{X})]^h + [D(Y, \tilde{X})]^t.
\end{aligned}$$

The second identity of the Theorem follows from the third one and from Lemma 7, using the vanishing of the torsion of $\bar{\nabla}$. The last identity follows from (19) and the last identity of Proposition 8. □

As a consequence of Theorem 2, we have the following result, which will play a fundamental role in the calculation of the Riemannian curvature of $(T_0 M, \tilde{G})$. The proof uses the same arguments as those in the proof of Lemma A3.

**Lemma 9.** *Let $\tilde{\nabla}$ be induced connection on $T_0M$ corresponding to the screen distribution $S(T(T_0M))$. For all $X \in \mathfrak{X}(M)$ and $s \in \Gamma(p^*(TM))$, we have*

$$\begin{aligned}\tilde{\nabla}_{X^h}s^h &= \left[\hat{\nabla}_{X^h}s + A(X,s)\right]^h + [B(X,s)]^t\\ \tilde{\nabla}_{X^h}s^t &= [C(X,\tilde{s})]^h + \left[\hat{\nabla}_{X^h}\tilde{s} + D(X,\tilde{s})\right]^t\\ \tilde{\nabla}_{X^t}s^h &= \left[\hat{\nabla}_{X^t}s + C(s,\tilde{X})\right]^h + \left[D(s,\tilde{X})\right]^t\\ \tilde{\nabla}_{X^t}s^t &= \left[\hat{\nabla}_{X^t}\tilde{s}\right]^t.\end{aligned}$$

If we denote by $P$ the projection morphism of $\Gamma(T(T_0M))$ on $\Gamma(S(T(T_0M)))$ with respect to the decomposition

$$T(T_0M) = RadT(T_0M) \oplus_{\text{orth}} S(T(T_0M)),$$

then, for any $X \in \mathfrak{X}(M)$, we have

$$PX^h = X^h - \frac{b}{\alpha}\hat{g}(X \circ p_0, V)\xi, \quad \text{and} \quad PX^t = X^t - \frac{a}{\alpha}\hat{g}(X \circ p_0, V)\xi.$$

**Proposition 9.** *The local screen second fundamental form $C$ of the screen distribution $S(T(T_0M))$ is characterized at any $(x,u) \in U$ by*

$$\begin{aligned}C(X^h, PY^h)_{(x,u)} =& -\frac{a}{2\alpha}g(R(X_x,Y_x)u, V(u)) - \frac{b}{\alpha}g(Y_x, \hat{\nabla}_{X^h_{(x,u)}}V)\\ &- \frac{d}{2\alpha}[g(Y_x,u)g(X_x,V(u)) + g(X_x,u)g(Y_x,V(u))]\\ &+ \frac{b^2}{2\alpha^2}[dg(X_x,u) - ag(R(X_x,u)u,V(u))]g(Y_x,V(u)),\\ C(X^h, PY^t)_{(x,u)} =& -\frac{a}{\alpha}g(Y_x, \hat{\nabla}_{X^h_{(x,u)}}V)\\ &- \frac{ab}{2\alpha^2}[ag(R(X_x,u)u,V(u)) - dg(X_x,u)]g(Y_x,V(u)),\\ C(X^t, PY^h)_{(x,u)} =& -\frac{b}{\alpha}[g(Y_x, \hat{\nabla}_{X^t_{(x,u)}}V) + g(Y_x,V(u))g(X_x,V(u))],\\ C(X^t, PY^t)_{(x,u)} =& -\frac{a}{\alpha}[g(Y_x, \hat{\nabla}_{X^t_{(x,u)}}V) + g(Y_x,V(u))g(X_x,V(u))].\end{aligned}$$

**Proof.** Recall that the transverse vector field $N$ of the screen distribution $S(T(T_0M))$ is given on $U$ by $N_{(x,u)} = \frac{1}{\alpha}(V(u))^v$. From $C(X^h, PY^h) = G(\tilde{\nabla}_{X^h}PY^h, N)$, we deduce from Lemma 6 that

$$\begin{aligned}C(X^h, PY^h) =& G(\tilde{\nabla}_{X^h}Y^h, N) - \frac{b}{\alpha}[\hat{g}((\nabla_X Y) \circ p_0, V)\\ &+ \hat{g}(Y \circ p_0, \hat{\nabla}_{X^h}V)] - \frac{b}{\alpha}\hat{g}(Y \circ p_0, V)G(\tilde{\nabla}_{X^h}\xi, N).\end{aligned}$$

Taking into account Theorem 2, calculations yield

$$\begin{aligned}G(\tilde{\nabla}_{X^h}Y^h, N) =& \frac{b}{\alpha}\hat{g}((\nabla_X Y), V) - \frac{a}{2\alpha}\hat{g}(R(X,Y)\sigma, V)\\ &- \frac{d}{2\alpha}[\hat{g}(Y \circ p_0, \sigma)\hat{g}(X \circ p_0, V) + \hat{g}(X \circ p_0, \sigma)\hat{g}(Y \circ p_0, V)],\end{aligned}$$

and

$$G(\tilde{\nabla}_{X^h}\xi, N) = \frac{ab}{2\alpha}\hat{g}(R(X\circ p_0,\sigma)\sigma, V) - \frac{bd}{2\alpha}\hat{g}(X\circ p_0,\sigma),$$

which gives the first identity of the Proposition. To prove the three other identities of the Proposition, we use the same arguments as before and the following consequences of Theorem 2

$$G(\tilde{\nabla}_{X^h}Y^t, N) = \frac{a}{\alpha}\hat{g}((\nabla_X Y)\circ p_0, V),$$

$$G(\tilde{\nabla}_{X^t}Y^h, N) = G(\tilde{\nabla}_{X^t}Y^t, N) = 0,$$

and

$$G(\tilde{\nabla}_{X^t}\xi, N) = \hat{g}(X\circ p_0, V).$$

□

**Corollary 3.** *Let $G$ be a Kaluza–Klein metric on $TM$. The screen distribution $S(T(T_0M))$ has a symmetric second fundamental form (and hence is integrable) if and only if*

1. $R(V,\sigma) = 0,$
2. $\hat{\nabla}_{X^h}V = 0$ *for all* $X\in\mathfrak{X}(M)$ *and*
3. $\hat{g}(\hat{\nabla}_{PX^t}V, Y\circ p_0) = \hat{g}(\hat{\nabla}_{PY^t}V, X\circ p_0)$ *for all* $X, Y\in\mathfrak{X}(M)$.

**Proof.** Since $G$ is a Kaluza–Klein metric, we have $b = d = 0$. Using Proposition 9, it is then easy to check that

$$\begin{aligned}
C(PX^h, PY^h) &= -\frac{a}{2\alpha}\hat{g}(R(X\circ p_0, Y\circ p_0)\sigma, V),\\
C(PX^h, PY^t) &= -\frac{a}{2\alpha}\hat{g}(Y\circ p_0, \hat{\nabla}_{X^h}V),\\
C(PX^t, PY^h) &= 0,\\
C(PX^t, PY^t) &= -\frac{a}{\alpha}\hat{g}(Y\circ p_0, \hat{\nabla}_{PX^t}V),
\end{aligned}$$

for all $X, Y\in\mathfrak{X}(M)$. The screen second fundamental form is symmetric if and only if $C$ restricted to $\Gamma(S(T(T_0M)))\times\Gamma(S(T(T_0M)))$ is symmetric, which completes the proof. □

It is worthwhile to note that, while umbilical screen distributions on the null tangent bundle do not exist, screen distributions can be integrable, as the example below shows.

**Example 2.** *Let $M = \mathbb{R}^2$ and $g$ the metric given by $g = dx_1\otimes dx_2 + dx_2\otimes dx_1$. Let $TM\simeq\mathbb{R}^4$ be endowed with any pseudo-Riemannian Kaluza–Klein g-natural metric. We shall construct a vector field $V$ on $T_0\mathbb{R}^2$, which satisfies the three conditions of Corollary 3, which shows that the screen distribution $S(T(T_0M))$ has a symmetric second fundamental form (and hence is integrable). We denote by $(e_1, e_2)$ the canonical frame field on $\mathbb{R}^2$ given by $e_1(x) = (1,0)\in T_xM\simeq\mathbb{R}^2$, where $x = (x_1, x_2)$ and $e_2(x) = (0,1)$. It is easy to see that $g(e_1,e_1) = g(e_2,e_2) = 0$ and $g(e_1,e_2) = 1$, so that $(x,e_1),(x,e_2)\in T_0M$. Furthermore, for any $u\in T_xM$ such that $(x,u)\in T_0M$, there are $\lambda,\mu\in\mathbb{R}$, $(\lambda,\mu)\neq(0,0)$ such that $u = \lambda e_1(x) + \mu e_2(x)$ and $g(u,u) = 0$, i.e., $2\lambda\mu = 0$, and hence either $\lambda = 0$ or $\mu = 0$. Then, we deduce that*

$$T_0M = \{(x, \lambda e_i(x)), \lambda\in\mathbb{R}^*, i = 1,2 \text{ and } x\in\mathbb{R}^2\}.$$

*For any $u = \lambda e_i(x)\in T_0M$, we set $V(u) = \mu e_j(x)\in T_0M$. Since we must have $g(u, V(u)) = 1$, we obtain $i\neq j$ and $\mu = \frac{1}{\lambda}$. We deduce that $V(\lambda e_i(x)) = \frac{1}{\lambda}e_j(x)$, $j\neq i$, i.e., $V(\lambda e_i) = \frac{1}{\lambda}\sum_{j=1}^{2}(1-\delta_{ij})e_j$. For $\varepsilon = 1$ or $0$, if we denote $e_\varepsilon := \varepsilon e_1 + (1-\varepsilon)e_2$, then*

$$T_0M = \{(x, \lambda e_\varepsilon(x)), \lambda\in\mathbb{R}^*, \varepsilon = 1 \text{ or } 0 \text{ and } x\in\mathbb{R}^2\},$$

*and* $V(\lambda e_\varepsilon) = \frac{1}{\lambda} e_{1-\varepsilon}$.
*The manifold* $(\mathbb{R}^2, g)$ *being flat,* $V$ *satisfies automatically the first condition of Corollary 3. We also have* $\hat{\nabla}_{X^h_{(x,u)}} V = X^h_{(x,u)}(V^i)e_i$ *for any* $X \in \mathfrak{X}(\mathbb{R}^2)$. *Let* $U_1 = \mathbb{R}^2 \times \mathbb{R}^* \times \{0\}$ *and* $U_2 = \mathbb{R}^2 \times \{0\} \times \mathbb{R}^*$, *so that* $T_0M = U_1 \cup U_2$. *For any* $(x_1, x_2, u_1, 0) \in U_1$, $(x_1, x_2, 0, u_2) \in U_2$, *we have*

$$V(x_1, x_2, u_1, 0) = (0, \frac{1}{u_1}), \quad V(x_1, x_2, 0, u_2) = (\frac{1}{u_2}, 0).$$

*Thus,*

$$\hat{\nabla}_{e_i^h} V = e_i^h(V^j)e_j = 0,$$

*which gives the second condition of Corollary 3. In order to check whether the third condition is satisfied, we calculate* $\hat{g}(\hat{\nabla}_{Pe_i^t} V, e_j \circ p_0)$ *for* $i = 1, 2$ *and* $j \neq i$. *On* $U_2$, *we have* $V(x_1, x_2, 0, u_2) = (\frac{1}{u_2}, 0)$. *Hence,* $\hat{g}(V, e_1 \circ p_0) = 0$ *and*

$$\begin{aligned}
\hat{\nabla}_{Pe_1^t} V = \hat{\nabla}_{e_1^t} V &= \hat{\nabla}_{e_1^v} V - \hat{g}(e_1 \circ p_0, \sigma)\hat{\nabla}_{V^v} V \\
&= \hat{\nabla}_{e_1^v} V - u_2 g(e_1, e_2)\hat{\nabla}_{V^v} V \\
&= \hat{\nabla}_{e_1^v} V - u_2 \hat{\nabla}_{V^v} V \\
&= \sum_i [e_1^v(V^i)e_i - u_2 \sum_j V^j e_j^v(V^i)e_i] \\
&= \sum_i [e_1^v(V^i)e_i - u_2 V^1 e_1^v(V^i)e_i] \\
&= 0.
\end{aligned}$$

*On* $U_1$, *we have* $V(x_1, x_2, u_1, 0) = (0, \frac{1}{u_1})$, *then* $\hat{g}(e_1 \circ p_0, V) = \frac{1}{u_1} g(e_1, e_2) = \frac{1}{u_1}$ *and so,*

$$\hat{\nabla}_{Pe_1^t} V = \hat{\nabla}_{e_1^t} V - \frac{1}{u_1} u_1 \hat{\nabla}_{e_1^t} V = 0.$$

*Using similar arguments, we show that* $\hat{\nabla}_{Pe_2^t} V = 0$.

**Lemma 10.** *The one-form* $\tau$ *corresponding by* $G$ *to the null vector field* $\xi$ *is characterized by*

$$\tau(X^h)_{(x,u)} = -\frac{b}{2\alpha}[dg(X_x, u) - ag(R(X_x, u)V(u), u)], \qquad \tau(X^t)_{(x,u)} = -g(X_x, V(u)),$$

*for every* $X \in \Gamma(T_0M)$ *and* $(x, u) \in T_0M$.

**Proof.** Calculations, using Proposition 7 and Lemma 5, yield on $U$

$$\begin{aligned}
\tau(X^h) =& G(\bar{\nabla}_{X^h} N, \xi) = \frac{1}{\alpha} G(\bar{\nabla}_{X^h} V^v, \xi) \\
=& -\frac{b}{2\alpha}[d\hat{g}(X \circ p_0, \sigma) - a\hat{g}(R(X \circ p_0, \sigma)V, \sigma)]
\end{aligned}$$

and

$$\tau(X^t) = G(\bar{\nabla}_{X^t} N, \xi) = \frac{1}{\alpha} G(\bar{\nabla}_{X^t} V^v, \xi) = \hat{g}(\hat{\nabla}_{X^t} V, \sigma) = -\hat{g}(X \circ p_0, V).$$

□

The following two Lemmas are obtained by calculations and arguments similar to the ones above.

**Lemma 11.** *The shape operator $A_N$ of $(T_0M, \tilde{G}, S(T(T_0M)))$ is characterized by*

$$\begin{aligned}
A_N(X^h) =& \Big\{ -\frac{ad}{2\alpha^2}[\hat{g}(X \circ p_0, V)\sigma + \hat{g}(X \circ p_0, \sigma)V] + \frac{a^2}{2\alpha^2} R(V,\sigma)X \circ p_0 \\
& + \frac{a^2 d}{2\alpha^3}[d\hat{g}(X \circ p_0, \sigma) - a\hat{g}(R(V,\sigma)X, \sigma)]\sigma \Big\}^h \\
& + \Big\{ -\frac{1}{\alpha}\hat{\nabla}_{X^h} V + \frac{bd}{2\alpha^2}\hat{g}(X \circ p_0, V)\sigma - \frac{ab}{2\alpha^2} R(V,\sigma)X \circ p_0 \\
& - \frac{abd}{2\alpha^3}[d\hat{g}(X \circ p_0, \sigma) - a\hat{g}(R(V,\sigma)X, \sigma)]\sigma \Big\}^t, \\
A_N(X^t) =& -\frac{1}{\alpha}\{\hat{\nabla}_{X^t} V\}^t.
\end{aligned}$$

**Lemma 12.** *The following formulas characterize the shape operator $A_\xi$ of the screen distribution $S(T(T_0M))$ of $(T_0M, \tilde{G})$:*

$$\begin{aligned}
A_\xi(X^h) =& \Big\{ \frac{ab^2}{2\alpha} R(X \circ p_0, \sigma)\sigma + d\hat{g}(X \circ p_0, \sigma)\sigma + \frac{ab^2}{2\alpha}\hat{g}(R(X \circ p_0, \sigma)V, \sigma)\sigma \Big\}^h \\
& + \Big\{ \frac{b(\alpha - b^2)}{2\alpha} R(X \circ p_0, \sigma)\sigma - \frac{ab(a+c)}{2\alpha}\hat{g}(R(X \circ p_0, \sigma)V, \sigma)\sigma \Big\}^t, \\
A_\xi(X^t) =& -b\Big\{ X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma \\
& - \frac{a^2}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X \circ p_0, \sigma)R(V,\sigma)\sigma] \Big\}^h \\
& + \Big\{ (a+c)[X + \hat{g}(X \circ p_0, V)\sigma] \\
& - \frac{ab^2}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X \circ p_0, \sigma)R(V,\sigma)\sigma] \Big\}^t.
\end{aligned}$$

**Theorem 3.** *Let $(M, g)$ be a non-definite pseudo-Riemannian manifold and $G$ a g-natural metric on $TM$. Then, $(T_0M, \tilde{G}, S(T(T_0M)))$ is never a screen conformal lightlike submanifold of $(TM, G)$.*

**Proof.** Suppose that $(T_0M, \tilde{G}, S(T(T_0M)))$ is screen conformal. Then, there is a non-vanishing smooth function $\varphi$ on $T_0M$ such that

$$A_N(X^h) = \varphi A_\xi(X^h), \quad \text{and} \quad A_N(X^t) = \varphi A_\xi(X^t).$$

Using Lemmas 11 and 12, the second equation becomes

$$\begin{aligned}
0 =& -b\varphi\{X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma \\
& - \frac{a^2}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X \circ p_0, \sigma)R(V,\sigma)\sigma] \Big\}
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
-\frac{1}{\alpha}\hat{\nabla}_{X^t} V =& \varphi(a+c)[X + \hat{g}(X \circ p_0, V)\sigma] \\
& - \frac{ab^2}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X\circ, \sigma)R(V,\sigma)\sigma].
\end{aligned} \tag{21}$$

We deduce from (20) that either $b = 0$ or

$$\frac{a^2}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X \circ p_0, \sigma)R(V, \sigma)\sigma] = X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma,$$

for any $X \in \mathfrak{X}(M)$.
If we assume that $b = 0$, Equation (21) becomes

$$-\frac{1}{\alpha}\hat{\nabla}_{X^t}V = \varphi(a + c)[X + \hat{g}(X \circ p_0, V)\sigma]$$

hence,

$$0 = g(\hat{\nabla}_{X^t}V, V) = -2a(a + c)^2\varphi(x, u)\hat{g}(X \circ p_0, V),$$

for any $X \in T_xM$.
In particular, for $X = u$, we find $\varphi(x, u) = 0$, which cannot occur. So $b \neq 0$ and

$$\frac{a^2}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X \circ p_0, \sigma)R(V, \sigma)\sigma] = X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma.$$

If $a = 0$, Equation (20) is equivalent to

$$-b\varphi[X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma] = 0.$$

If we take $X = u$ in the third equation, we obtain $-2b\varphi(x, u)u = 0$, which is a contradiction. Hence, $a \neq 0$ and

$$\frac{a}{2\alpha}[R(X \circ p_0, \sigma)\sigma - \hat{g}(X \circ p_0, \sigma)R(V, \sigma)\sigma] = \frac{1}{a}[X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma].$$

Using Equation (21), we find that

$$(a + c)\varphi[X + \hat{g}(X \circ p_0, V)\sigma] - \frac{b^2}{a}[X - \hat{g}(X \circ p_0, \sigma)V + \hat{g}(X \circ p_0, V)\sigma] = -\frac{1}{\alpha}\hat{\nabla}_{X^t}V, \quad (22)$$

that is,

$$\frac{1}{a}\Big\{(a(a + c)\varphi - b^2)[X + \hat{g}(X \circ p_0, V)\sigma] + b^2\hat{g}(X \circ p_0, \sigma)V\Big\} = -\frac{1}{\alpha}\hat{\nabla}_{X^t}V,$$

which yields

$$(a + c)\varphi\hat{g}(X \circ p_0, \sigma) = \frac{1}{\alpha}\hat{g}(X \circ p_0, V).$$

If we take $X = V$ in the last equation, we find that $a + c = 0$. Then, by (22),

$$-\frac{2b^2}{a}\hat{g}(X \circ p_0, V) = 0,$$

for every $X \in \mathfrak{X}(M)$, yielding a contradiction. □

### *4.5. Some Geometric Properties of the Null Tangent Bundle Related to the Curvature*

As a first consequence of Proposition A4, we have the following:

**Theorem 4.** *$(T_0M, \tilde{G}, S(T(T_0M)))$ is never flat.*

**Proof.** Suppose that $\tilde{R}(X^t_{(x,u)}, Y^t_{(x,u)})Z^t_{(x,u)} = 0$, for any $(x, u) \in U$ and $X, Y, Z \in T_xM$. Then, from the corresponding identity in Proposition A4, we have

$$g(\tilde{Y}(u), \tilde{Z}(u))\hat{\nabla}_{X^t_{(x,u)}}V - g(\tilde{X}(u), \tilde{Z}(u))\hat{\nabla}_{Y^t_{(x,u)}}V = 0,$$

whence it follows that $g(\tilde{Y}(u), \tilde{Z}(u))g(\hat{\nabla}_{X^t_{(x,u)}} V, u) - g(\tilde{X}(u), \tilde{Z}(u))g(\hat{\nabla}_{Y^t_{(x,u)}} V, u) = 0$, that is,

$$g(\tilde{Y}(u), \tilde{Z}(u))g(X, V(u)) - g(\tilde{X}(u), \tilde{Z}(u))g(Y, V(u)) = 0.$$

In particular, if we take $X = u$ and $Y \neq 0$ orthogonal to both $u$ and $V(u)$, we find that $g(Y, Z) = 0$, for every $Z \in T_x M$, which contradicts the fact that $g$ is a non-degenerate metric on $M$. □

Concerning the symmetry of the Ricci type tensor, we have the following corollary of Proposition A5, in the case of Kaluza–Klein type metrics on the null tangent bundle:

**Proposition 10.** *Let $G$ denote a pseudo-Riemannian metric on $TM$, either of Kaluza–Klein type ($b = 0$) or such that $a = 0$. Then, the Ricci type tensor of $(T_0M, \tilde{G}, S(T(T_0M)))$ is symmetric if and only if*

1. *$R(\sigma, V) = 0$, and*
2. *$\hat{\nabla}_{X^h} V = 0$, for all $X \in \mathfrak{X}(M)$.*

**Proof.** From the first identity of Proposition A5, we have $R^{(0,2)}_{(x,u)}(X^h, Y^h) = R^{(0,2)}_{(x,u)}(Y^h, X^h)$ if and only if

$$\begin{aligned} 0 =& g(R(X, Y)u, V(u)) + \frac{ab}{2\alpha}[g((\nabla_u R)(X, V(u))Y, u) - g((\nabla_u R)(Y, V(u))X, u)] \\ &+ \frac{ab}{2\alpha}[g(R(Y, u)u, \hat{\nabla}_{X^h_{(x,u)}} V) - g(R(X, u)u, \hat{\nabla}_{Y^h_{(x,u)}} V)], \end{aligned}$$

while, using the second and third identities of Proposition A5, we deduce that $R^{(0,2)}_{(x,u)}(X^h, Y^t) = R^{(0,2)}_{(x,u)}(Y^t, X^h)$ is equivalent to

$$\begin{aligned} 0 =& \frac{ab}{2\alpha}[g(R(\tilde{Y}(u), X)u, V(u)) - g(R(X, u)\tilde{Y}(u), V(u)) \\ &- g(Y, V(u))g(R(V(u), u)X, u)] + g(\hat{\nabla}_{X^h_{(x,u)}} V, Y). \end{aligned}$$

Assume now that either $b = 0$ or $a = 0$. Then, the Ricci type tensor of $(T_0M, \tilde{G}, S(T(T_0M)))$ is symmetric if and only if $g(R(X, Y)u, V(u)) = 0$ and $g(\hat{\nabla}_{X^h_{(x,u)}} V, Y) = 0$, for any $(x, u) \in U$ and $X, Y \in T_x M$. □

**Remark 4.** *Example 2 gives a situation where the Ricci type tensor of $(T_0M, \tilde{G}, S(T(T_0M)))$ is symmetric.*

**Corollary 4.** *If $M$ is a non-definite pseudo-Riemannian metric of constant curvature $k$ and $G$ is a $g$-natural metric on $TM$, then the Ricci type tensor of $(T_0M, \tilde{G}, S(T(T_0M)))$ is symmetric if and only if*

1. *$k = 0$, and*
2. *$\hat{\nabla}_{X^h_{(x,u)}} V = 0$, for any $(x, u) \in U$ and $X \in T_x M$.*

**Proof.** From the first identity of Proposition A5, we have $R^{(0,2)}_{(x,u)}(X^h, Y^h) = R^{(0,2)}_{(x,u)}(Y^h, X^h)$ if and only if $g(R(X, Y)u, V(u)) = 0$, that is

$$k[g(X, V(u))g(Y, u) - g(X, u)g(Y, V(u))] = 0.$$

If we take $X = V(u)$ and $Y = u$, we find that $k = 0$.
The second condition follows directly from $R^{(0,2)}_{(x,u)}(X^h, Y^t) = R^{(0,2)}_{(x,u)}(Y^t, X^h)$ in Proposition A5. □

Now, as a consequence of Proposition A6, we have a simpler expression of the extrinsic scalar curvature in the case when the base manifold is of constant sectional curvature, as the following result shows:

**Proposition 11.** *If $M$ is a non-definite pseudo-Riemannian metric of constant sectional curvature $k$ and $G$ is a g-natural metric on $TM$, then the extrinsic scalar curvature of $(T_0M, \tilde{G}, S(T(T_0M)))$ is given by*

$$\tilde{R}_{(x,u)} = -\frac{mak}{\alpha} + \frac{(3m-5)ab^2k}{2\alpha^2} + \frac{(m-1)d}{\alpha^2}[\alpha + (m-1)b^2] - \frac{b}{\alpha}\sum_{i=3}^{m} \varepsilon_i g(\hat{\nabla}_{e_i^h} V, e_i)$$

*for all $(x,u) \in U$.*

The following corollary deals with a special case when the Ricci type tensor is symmetric.

**Corollary 5.** *Let $(M, g)$ be a non-definite semi-Riemannian manifold such that $R(\sigma, V) = 0$. Suppose that $\hat{\nabla}_{X^h} V = 0$ for all $X \in \mathfrak{X}(M)$ and let $G$ be a Kaluza–Klein pseudo-Riemannian g-natural metric on $TM$ ($b = d = 0$). Then, the extrinsic scalar curvature of $(T_0M, \tilde{G}, S(T(T_0M)))$ is given by*

$$\begin{aligned}\tilde{R}_{(x,u)} =& \frac{a}{\alpha}R_x + \sum_{i=3}^{m} \varepsilon_i \{ \frac{a^3}{\alpha^2}[3g(R(u,e_i)u, R(V(u),e_i)u) - g(R(u,e_i)V(u), R(u,e_i)u)] \\ &+ \frac{a^3}{4\alpha^2}\sum_{j=3}^{m} \varepsilon_j [3g(R(e_j,e_i)u, R(e_j,e_i)u) - 2g(R(e_i,u)e_j, R(e_i,u)e_j)]\},\end{aligned}$$

*for all $(x,u) \in U$. Moreover, if $(M, g)$ has a constant sectional curvature, then the extrinsic scalar curvature of $(T_0M, \tilde{G}, S(T(T_0M)))$ is a constant, given by*

$$\tilde{R} = -\frac{mk}{a+c}.$$

For the sign of the extrinsic scalar curvature, we have

**Corollary 6.** *Let $(M, g)$ be a non-definite semi-Riemannian manifold such that $R(\sigma, V) = 0$. Suppose that $\hat{\nabla}_{X^h} V = 0$ for all $X \in \mathfrak{X}(M)$. Assume that $G$ is a pseudo-Riemannian g-natural metric on $TM$ such that $a = 0$. Then, $(T_0M, \tilde{G}, S(T(T_0M)))$ has*

(*i*) *A positive extrinsic scalar curvature if and only if $Ric(u,u) > (m-1)(m-2)\frac{d}{2b^2}$, for all $(x,u) \in T_0M$.*
(*ii*) *A vanishing extrinsic scalar curvature if and only if $Ric(u,u) = (m-1)(m-2)\frac{d}{2b^2}$, for all $(x,u) \in T_0M$.*
(*iii*) *A negative extrinsic scalar curvature if and only if $Ric(u,u) < (m-1)(m-2)\frac{d}{2b^2}$, for all $(x,u) \in T_0M$.*

**Proof.** Suppose that $a = 0$. We deduce from the Proposition A6 that the extrinsic scalar curvature of $(T_0M, \tilde{G}, S(T(T_0M)))$ is equal to

$$\tilde{R}_{(x,u)} = 2b^2 Ric(u,u) - (m-1)(m-2)d. \tag{23}$$

The result is then a direct consequence of Equation (23). □

**Author Contributions:** All authors contributed equally to this work with regard to conceptualization, methodology, validation, formal analysis, investigation, resources, data curation, writing—original draft preparation, writing—review and editing, visualization. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** All data supporting reported results are included in the paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A. The Signature of Non-Degenerate $g$-Natural Metrics on the Tangent Bundle of a Pseudo-Riemannian Manifold

### *Appendix A.1. On the Timelike Cone*

In this case, we have $k \leq 1$. Let $x \in M$ and $u \in T_xM$ such that $g_x(u,u) < 0$. Put $e_1 := \frac{1}{\sqrt{-g(u,u)}} u$ and let $e_i \in T_xM$, $i = 2, \ldots, m$ such that $g(e_i, e_i) = -1$ for $1 \leq i \leq k$ and $g(e_i, e_i) = 1$ for $k+1 \leq i \leq m$.

1. If $(\phi_1 + \phi_3)(g(u,u)) \neq 0$ and $(\alpha_1 + \alpha_3)(g(u,u)) \neq 0$, we set

$$E_1 = -\phi_2(g(u,u))e_1^h + (\phi_1 + \phi_3)(g(u,u))e_1^v, \quad E_{m+1} = e_1^h,$$
$$E_i = -\alpha_2(g(u,u))e_i^h + (\alpha_1 + \alpha_3)(g(u,u))e_i^v, \quad E_{m+i} = e_i^h,$$

$i = 2, \ldots, m$. The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} -\phi(\phi_1+\phi_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha(\alpha_1+\alpha_3)I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(\alpha_1+\alpha_3)I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\phi_1+\phi_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_{m-k} \end{pmatrix}$$

where the functions in the matrix are evaluated at $g(u,u)$. We deduce that the signature $G_{(x,u)}$ is determined by the following Table A1.

**Table A1.** $(\phi_1 + \phi_3)(g(u,u)) \neq 0$ and $(\alpha_1 + \alpha_3)(g(u,u)) \neq 0$ (timelike case)

| The signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $(\alpha_1+\alpha_3)(g(u,u))$ | $\phi$(g(u, u)) | $\alpha(g(u,u))$ |
|---|---|---|---|---|
| $(m-1, m+1)$ | $> 0$ | $\neq 0$ | $> 0$ | $< 0$ |
| $(m+1, m-1)$ | $< 0$ | $\neq 0$ | $> 0$ | $< 0$ |
| $(m, m)$ | $\neq 0$ | $\neq 0$ | $< 0$ | $< 0$ |
| $(2m-2k, 2k)$ | $> 0$ | $> 0$ | $> 0$ | $> 0$ |
| $(2m-2k+1, 2k-1)$ | $\neq 0$ | $> 0$ | $< 0$ | $> 0$ |
| $(2m-2k+2, 2k-2)$ | $< 0$ | $> 0$ | $> 0$ | $> 0$ |
| $(2k-2, 2m-2k+2)$ | $> 0$ | $< 0$ | $> 0$ | $> 0$ |
| $(2k-1, 2m-2k+1)$ | $\neq 0$ | $< 0$ | $< 0$ | $> 0$ |
| $(2k, 2m-2k)$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ |

2. If $(\phi_1 + \phi_3)(g(u,u)) = 0$ and $(\alpha_1 + \alpha_3)(g(u,u)) \neq 0$, then $\phi(g(u,u)) = -\phi_2^2(g(u,u))$. Being $G$ non-degenerate, we deduce that $\phi_2(g(u,u)) \neq 0$. We have two situations:
   (i) $\phi_1(g(u,u)) = 0$: in this case, we set

$$E_1 = \frac{1}{\sqrt{2}}(e_1^h + e_1^v), \quad E_{m+1} = \frac{1}{\sqrt{2}}(e_1^h - e_1^v),$$
$$E_i = -\alpha_2(g(u,u))e_i^h + (\alpha_1 + \alpha_3)(g(u,u))e_i^v, \quad E_{m+i} = e_i^h, \quad i = 2, \ldots, m.$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} -\phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha(\alpha_1+\alpha_3)I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(\alpha_1+\alpha_3)I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_{m-k} \end{pmatrix}$$

where the functions in the matrix are evaluated at $g(u,u)$. Then, the signature of $G_{(x,u)}$ is given by the following Table A2.

**Table A2.** $(\phi_1+\phi_3)(g(u,u)) = \phi_1(g(u,u)) = 0$ and $(\alpha_1+\alpha_3)(g(u,u)) \neq 0$ (timelike case).

| The signature of $G_{(x,u)}$ | $\alpha(g(u,u))$ | $(\alpha_1+\alpha_3)(g(u,u))$ |
|---|---|---|
| $(m,m)$ | $<0$ | $\neq 0$ |
| $(2m-2k+1, 2k-1)$ | $>0$ | $>0$ |
| $(2k-1, 2m-2k+1)$ | $>0$ | $<0$ |

(ii) $\phi_1(g(u,u)) \neq 0$: we put

$$\begin{aligned} E_1 &= \phi_1(g(u,u))e_1^h - \phi_2(g(u,u))e_1^v, \quad E_{m+1} = e_1^v, \\ E_i &= -\alpha_2(g(u,u))e_i^h + (\alpha_1+\alpha_3)(g(u,u))e_i^v, \quad E_{m+i} = e_i^h, \end{aligned}$$

$i = 2, \ldots, m$. The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \phi_1\phi_2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha(\alpha_1+\alpha_3)I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(\alpha_1+\alpha_3)I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_{m-k} \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. Hence, the signature of $G_{(x,u)}$ is the same as in Table A2.

3. If $(\phi_1+\phi_3)(g(u,u)) = 0$ and $(\alpha_1+\alpha_3)(g(u,u)) = 0$, then $\phi_2(g(u,u)) \neq 0$ and $\alpha_2(g(u,u)) \neq 0$. One of the following cases occurs:

(i) $\phi_1(g(u,u)) = 0$ and $\alpha_1(g(u,u)) = 0$. In this case, we set

$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad i = 1, \ldots, m.$$

The matrix of $G$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2 I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k} \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. We deduce that $G_{(x,u)}$ is of signature $(m,m)$.

(ii) $\phi_1(g(u,u)) \neq 0$ and $\alpha_1(g(u,u)) \neq 0$. We set

$$\begin{aligned} E_1 &= \phi_1(g(u,u))e_1^h - \phi_2(g(u,u))e_1^v, \quad E_{m+1} = e_1^v, \\ E_i &= \alpha_1(g(u,u))e_i^h - \alpha_2(g(u,u))e_i^v, \quad E_{m+i} = e_i^v, \quad 2 \leq i \leq m. \end{aligned}$$

Then, the matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$G = \begin{pmatrix} -\phi_1\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1\alpha I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1\alpha I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 I_{m-k} \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. The signature of $G_{(x,u)}$ is then $(m,m)$.

(iii) If $\phi_1(g(u,u)) = 0$ and $\alpha_1(g(u,u)) \neq 0$, then $\phi_2(g(u,u)) \neq 0$. We put

$$\begin{aligned} E_1 =& \frac{1}{\sqrt{2}}(e_1^h + e_1^v), \quad E_{m+1} = \frac{1}{\sqrt{2}}(e_1^h - e_1^v), \\ E_i =& \alpha_1(g(u,u))e_i^h - \alpha_2(g(u,u))e_i^v, \quad E_{m+i} = e_i^v, \quad 2 \leq i \leq m. \end{aligned}$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1\alpha I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1\alpha I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 I_{m-k} \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. Since $\alpha(g(u,u)) = -\alpha_2^2(g(u,u)) < 0$, the signature of $G_{(x,u)}$ is $(m,m)$.

(iv) If $\phi_1(g(u,u)) \neq 0$ and $\alpha_1(g(u,u)) = 0$, we put

$$\begin{aligned} E_1 =& \phi_1(g(u,u))e_1^h - \phi_2(g(u,u))e_1^v, \quad E_{m+1} = e_1^v, \\ E_i =& \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 2 \leq i \leq n. \end{aligned}$$

Then, the matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\phi_1\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2 I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k} \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. We deduce that the signature of $G_{(x,u)}$ is $(m,m)$.

4. If $(\alpha_1 + \alpha_3)(g(u,u)) = 0$ and $(\phi_1 + \phi_3)(g(u,u)) \neq 0$, then in particular $\alpha_2(g(u,u)) \neq 0$. We have one of the following cases:

(i) $\alpha_1(g(u,u)) = 0$. In this case, we put

$$\begin{aligned} E_1 =& -\phi_2(g(u,u))e_1^h + (\phi_1 + \phi_3)(g(u,u))e_1^v, \quad E_{m+1} = e_1^h, \\ E_i =& \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 2 \leq i \leq m. \end{aligned}$$

So, the matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} -\phi(\phi_1+\phi_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2 I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\phi_1+\phi_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k} \end{pmatrix}.$$

where the functions in the matrix are evaluated at $g(u,u)$. Therefore the signature of $G_{(x,u)}$ is determined as in Table A3.

**Table A3.** $(\alpha_1+\alpha_3)(g(u,u)) = \alpha_1(g(u,u)) = 0$, $(\phi_1+\phi_3)(g(u,u)) \neq 0$ (timelike case).

| The signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $\phi(g(u,u))$ |
|---|---|---|
| $(m+1, m-1)$ | $< 0$ | $> 0$ |
| $(m-1, m+1)$ | $> 0$ | $> 0$ |
| $(m, m)$ | $\neq 0$ | $< 0$ |

(ii) $\alpha_1(g(u,u)) \neq 0$, we put

$$\begin{aligned} E_1 &= -\phi_2(g(u,u))e_1^h + (\phi_1+\phi_3)(g(u,u))e_1^v, \quad E_{m+1} = e_1^h, \\ E_i &= \alpha_1(g(u,u))e_i^h - \alpha_2(g(u,u))e_i^v, \quad E_{m+i} = e_i^v, \quad 2 \leq i \leq m. \end{aligned}$$

On this basis, the matrix of $G_{(x,u)}$ takes the form

$$\begin{pmatrix} -\phi(\phi_1+\phi_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1\alpha I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1\alpha I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\phi_1+\phi_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_1 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_1 I_{m-k} \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. Then, the signature is determined by Table A4.

**Table A4.** $(\alpha_1+\alpha_3)(g(u,u)) = 0$, $\alpha_1(g(u,u)) \neq 0$, $(\phi_1+\phi_3)(g(u,u)) \neq 0$ (timelike case).

| The Signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $\phi(g(u,u))$ |
|---|---|---|
| $(m-1, m+1)$ | $> 0$ | $> 0$ |
| $(m+1, m-1)$ | $< 0$ | $> 0$ |
| $(m, m)$ | $\neq 0$ | $< 0$ |

Summarizing the above discussion, we proved the following result.

**Proposition A1.** *Let $(M,g)$ be a pseudo-Riemannian manifold and $G$ be a non-degenerate g-natural metric on its tangent bundle. Then, for any timelike vector $(x,u) \in TM$, the signature of $G_{(x,u)}$ is given in Table A5.*

**Table A5.** Signature at timelike vectors.

| The Signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $(\alpha_1+\alpha_3)(g(u,u))$ | $\phi(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|---|---|
| $(m-1,m+1)$ | $>0$ | any | $>0$ | $<0$ |
| $(m+1,m-1)$ | $<0$ | any | $>0$ | $<0$ |
| $(m,m)$ | any | any | $<0$ | $<0$ |
| $(2m-2k,2k)$ | $>0$ | $>0$ | $>0$ | $>0$ |
| $(2m-2k+1,2k-1)$ | any | $>0$ | $<0$ | $>0$ |
| $(2m-2k+2,2k-2)$ | $<0$ | $>0$ | $>0$ | $>0$ |
| $(2k-2,2m-2k+2)$ | $>0$ | $<0$ | $>0$ | $>0$ |
| $(2k-1,2m-2k+1)$ | any | $<0$ | $<0$ | $>0$ |
| $(2k,2m-2k)$ | $<0$ | $<0$ | $>0$ | $>0$ |

*Appendix A.2. On the Spacelike Cone*

In this case, we have $m-k\leq 1$. Let $x\in M$ and $u\in T_xM$ such that $g_x(u,u)>0$. Put $e_1:=\frac{1}{\sqrt{g(u,u)}}u$ and let $e_i\in T_xM$, $i=2,\dots,m$ such that $g(e_i,e_i)=1$ for $1\leq i\leq m-k$ and $g(e_i,e_i)=-1$ for $m-k+1\leq i\leq m$.

1. If $(\phi_1+\phi_3)(g(u,u))\neq 0$ and $(\alpha_1+\alpha_3)(g(u,u))\neq 0$, we set

$$\begin{aligned} E_1 &= -\phi_2(g(u,u))e_1^h+(\phi_1+\phi_3)(g(u,u))e_1^v, \quad E_{m+1}=e_1^h,\\ E_i &= -\alpha_2(g(u,u))e_i^h+(\alpha_1+\alpha_3)(g(u,u))e_i^v, \quad E_{m+i}=e_i^h, \quad i=2,\dots,m. \end{aligned}$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} \phi(\phi_1+\phi_3) & 0 & 0 & 0 & 0 & 0\\ 0 & \alpha(\alpha_1+\alpha_3)I_{m-k-1} & 0 & 0 & 0 & 0\\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_k & 0 & 0 & 0\\ 0 & 0 & 0 & (\phi_1+\phi_3) & 0 & 0\\ 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_{m-k-1} & 0\\ 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_k \end{pmatrix}$$

where the functions in the matrix are evaluated at $g(u,u)$. So, the signature of $G_{(x,u)}$ is determined by Table A6.

**Table A6.** $(\phi_1+\phi_3)(g(u,u))\neq 0$ and $(\alpha_1+\alpha_3)(g(u,u))\neq 0$ (spacelike case).

| The Signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $(\alpha_1+\alpha_3)(g(u,u))$ | $\phi(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|---|---|
| $(m-1,m+1)$ | $<0$ | $\neq 0$ | $>0$ | $<0$ |
| $(m+1,m-1)$ | $>0$ | $\neq 0$ | $>0$ | $<0$ |
| $(m,m)$ | $\neq 0$ | $\neq 0$ | $<0$ | $<0$ |
| $(2m-2k,2k)$ | $>0$ | $>0$ | $>0$ | $>0$ |
| $(2m-2k-1,2k+1)$ | $\neq 0$ | $>0$ | $<0$ | $>0$ |
| $(2m-2k-2,2k+2)$ | $<0$ | $>0$ | $>0$ | $>0$ |
| $(2k+1,2m-2k-1)$ | $\neq 0$ | $<0$ | $<0$ | $>0$ |
| $(2k+2,2m-2k-2)$ | $>0$ | $<0$ | $>0$ | $>0$ |
| $(2k,2m-2k)$ | $<0$ | $<0$ | $>0$ | $>0$ |

2. If $(\phi_1+\phi_3)(g(u,u))=0$ and $(\alpha_1+\alpha_3)(g(u,u))\neq 0$, we consider separately two cases:

(i) $\phi_1(g(u,u)) = 0$: we put

$$E_1 = \frac{1}{\sqrt{2}}(e_1^h + e_1^v), \quad E_{m+1} = \frac{1}{\sqrt{2}}(e_1^h - e_1^v),$$
$$E_i = -\alpha_2(g(u,u))e_i^h + (\alpha_1 + \alpha_3)(g(u,u))e_i^v, \quad E_{m+i} = e_i^h,$$

$i = 2, \dots, m$. The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha(\alpha_1+\alpha_3)I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. We deduce that the signature of $G_{(x,u)}$ is given by Table A7.

**Table A7.** $(\phi_1 + \phi_3)(g(u,u)) = \phi_1(g(u,u)) = 0$ and $(\alpha_1 + \alpha_3)(g(u,u)) \neq 0$ (spacelike case).

| The signature of $G_{(x,u)}$ | $(\alpha_1 + \alpha_3)(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|
| $(2m-2k-1, 2k+1)$ | $> 0$ | $> 0$ |
| $(2k+1, 2m-2k-1)$ | $< 0$ | $> 0$ |
| $(m,m)$ | $\neq 0$ | $< 0$ |

(ii) $\phi_1(g(u,u)) \neq 0$: let

$$E_1 = \phi_1(g(u,u))e_1^h - \phi_2(g(u,u))e_1^v, \quad E_{m+1} = e_1^v,$$
$$E_i = -\alpha_2(g(u,u))e_i^h + (\alpha_1 + \alpha_3)(g(u,u))e_i^v, \quad E_{m+i} = e_i^h,$$

$i = 2, \dots, m$. The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} -\phi_1\phi_2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha(\alpha_1+\alpha_3)I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. As a consequence, the signature of $G_{(x,u)}$ is described in Table A8.

**Table A8.** $(\phi_1 + \phi_3)(g(u,u)) = 0$, $\phi_1(g(u,u)) \neq 0$ and $(\alpha_1 + \alpha_3)(g(u,u)) \neq 0$ (spacelike case).

| The signature of $G_{(x,u)}$ | $(\alpha_1 + \alpha_3)(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|
| $(2m-2k-1, 2k+1)$ | $> 0$ | $> 0$ |
| $(2k+1, 2m-2k-1)$ | $< 0$ | $> 0$ |
| $(m,m)$ | $\neq 0$ | $< 0$ |

3. If $(\phi_1 + \phi_3)(g(u,u)) = 0$ and $(\alpha_1 + \alpha_3)(g(u,u)) = 0$, then we have in particular $\alpha_2(g(u,u)) \neq 0$ and $\phi_2(g(u,u)) \neq 0$. We have one of the four following cases:

(i) $\phi_1(g(u,u)) = 0$ and $\alpha_1(g(u,u)) = 0$. We put

$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad i = 1, \dots, m.$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. So, the signature of $G_{(x,u)}$ is $(m,m)$.

(ii) $\phi_1(g(u,u)) = 0$ and $\alpha_1(g(u,u)) \neq 0$, we put

$$E_1 = \frac{1}{\sqrt{2}}(e_1^h + e_1^v), \quad E_{m+1} = \frac{1}{\sqrt{2}}(e_1^h - e_1^v),$$
$$E_i = \alpha_1(g(u,u))e_i^h - \alpha_2(g(u,u))e_i^v, \quad E_{m+i} = e_i^v, \quad 2 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} \phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1\alpha I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. Thus the signature of $G_{(x,u)}$ is $(m,m)$.

(iii) $\phi_1(g(u,u)) \neq 0$ and $\alpha_1(g(u,u)) = 0$, we put

$$E_1 = \phi_1(g(u,u))e_1^h - \phi_2(g(u,u))e_1^v, \quad E_{m+1} = e_1^v,$$
$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 2 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \phi_1\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$, and the signature of $G_{(x,u)}$ is $(m,m)$.

(iv) $\phi_1(g(u,u)) \neq 0$ and $\alpha_1(g(u,u)) \neq 0$, we put

$$E_1 = \phi_1(g(u,u))e_1^h - \phi_2(g(u,u))e_1^v, \quad E_{m+1} = e_1^v,$$
$$E_i = \alpha_1(g(u,u))e_i^h - \alpha_2(g(u,u))e_i^v, \quad E_{m+i} = e_i^v, \quad 2 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ is then

$$\begin{pmatrix} \phi_1\phi & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1\alpha I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$, so the signature of $G_{(x,u)}$ is $(m,m)$.

4. If $(\alpha_1+\alpha_3)(g(u,u)) = 0$ and $(\phi_1+\phi_3)(g(u,u)) \neq 0$, we have one of the following cases

 (i) $\alpha_1(g(u,u)) = 0$. In this case, we consider the basis $\{E_i\}_{i=1}^{2m}$, where

$$\begin{aligned} E_1 =& -\phi_2(g(u,u))e_1^h + (\phi_1+\phi_3)(g(u,u))e_1^v, \quad E_{m+1} = e_1^h, \\ E_i =& \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 2 \leq i \leq m, \end{aligned}$$

and we find that the matrix of $G_{(x,u)}$ with respect to this basis is

$$\begin{pmatrix} \phi(\phi_1+\phi_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & (\phi_1+\phi_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$. Thus, the signature of $G_{(x,u)}$ is determined by Table A9.

**Table A9.** $(\alpha_1+\alpha_3)(g(u,u)) = \alpha_1(g(u,u)) = 0$ and $(\phi_1+\phi_3)(g(u,u)) \neq 0$ (spacelike case).

| The Signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $\phi(g(u,u))$ |
|---|---|---|
| $(m+1, m-1)$ | $> 0$ | $> 0$ |
| $(m-1, m+1)$ | $< 0$ | $> 0$ |
| $(m,m)$ | $\neq 0$ | $< 0$ |

 (ii) $\alpha_1(g(u,u)) \neq 0$, we put

$$\begin{aligned} E_1 =& -\phi_2(g(u,u))e_1^h + (\phi_1+\phi_3)(g(u,u))e_1^v, \quad E_{m+1} = e_1^h, \\ E_i =& \alpha_1(g(u,u))e_i^h - \alpha_2(g(u,u))e_i^v, \quad E_{m+i} = e_i^v, \quad 2 \leq i \leq m. \end{aligned}$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \phi(\phi_1+\phi_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1\alpha I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & (\phi_1+\phi_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 I_{m-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_k \end{pmatrix},$$

where the functions in the matrix are evaluated at $g(u,u)$, and the signature is determined by Table A10.

**Table A10.** $(\alpha_1+\alpha_3)(g(u,u))=0$, $\alpha_1(g(u,u))\neq 0$ and $(\phi_1+\phi_3)(g(u,u))\neq 0$ (spacelike case).

| The Signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $\phi(g(u,u))$ |
|---|---|---|
| $(m+1,m-1)$ | $>0$ | $>0$ |
| $(m-1,m+1)$ | $<0$ | $>0$ |
| $(m,m)$ | $\neq 0$ | $<0$ |

In summary, in the previous discussion we proved the following.

**Proposition A2.** *Let $(M,g)$ be a pseudo-Riemannian manifold and G be a non-degenerate g-natural metric on its tangent bundle. Then, for any spacelike vector $(x,u)\in TM$, the signature of $G_{(x,u)}$ is given as in Table A11.*

**Table A11.** Signature at spacelike vectors.

| The Signature of $G_{(x,u)}$ | $(\phi_1+\phi_3)(g(u,u))$ | $(\alpha_1+\alpha_3)(g(u,u))$ | $\phi(g(u,u))$ | $\alpha(g(u,u))$ |
|---|---|---|---|---|
| $(m-1,m+1)$ | $<0$ | any | $>0$ | $<0$ |
| $(m+1,m-1)$ | $>0$ | any | $>0$ | $<0$ |
| $(m,m)$ | any | any | $<0$ | $<0$ |
| $(2m-2k,2k)$ | $>0$ | $>0$ | $>0$ | $>0$ |
| $(2m-2k-1,2k+1)$ | any | $>0$ | $<0$ | $>0$ |
| $(2m-2k-2,2k+2)$ | $<0$ | $>0$ | $>0$ | $>0$ |
| $(2k+1,2m-2k-1)$ | any | $<0$ | $<0$ | $>0$ |
| $(2k+2,2m-2k-2)$ | $>0$ | $<0$ | $>0$ | $>0$ |
| $(2k,2m-2k)$ | $<0$ | $<0$ | $>0$ | $>0$ |

*Appendix A.3. On the Lightlike Cone*

In this case, we have $k\leq 1$ and $m-k\geq 1$. Let $(x,u)\in TM$ such that $g(u,u)=0$ and $v\in T_xM$ such that $g(v,v)=g(u,u)=0$ and $g(u,v)=1$. Consider a basis $\{e_i\}_{i=1}^m$ of $T_xM$ such that $e_1=u$, $e_2=v$, $g(u,e_i)=g(v,e_i)=g(e_i,e_j)=0$, for $3\leq i\neq j\leq m$ and

- $g(e_i,e_i)=1$, for $3\leq i\leq m$, if $k=1$;
- $g(e_i,e_i)=-1$, for $3\leq i\leq m$, if $m-k=1$;
- $g(e_i,e_i)=-1$, for $3\leq i\leq k+1$, $g(e_i,e_i)=1$, for $k+2\leq i\leq m$, if $k\geq 2$ and $m-k\geq 2$.

We set $q=k-1$ and $p=m-k-1$.

1. If $(\alpha_1+\alpha_3)(0)\neq 0$, $(\beta_1+\beta_3)(0)\neq 0$, we treat separately two cases.
   (i) $\gamma := \big(2\alpha_2\beta_2(\alpha_1+\alpha_3)-\beta_1(\alpha_1+\alpha_3)^2-\alpha_2^2(\beta_1+\beta_3)\big)(0)\neq 0$: we take

$$E_1=u^h-\frac{\alpha_1+\alpha_3}{\beta_1+\beta_3}(0)v^h,\ E_2=v^h,$$

$$E_{m+2}=\frac{\alpha_2(\beta_1+\beta_3)-\beta_1(\alpha_1+\alpha_3)}{(\alpha_1+\alpha_3)^2}(0)u^h-\frac{\alpha_2}{\alpha_1+\alpha_3}(0)v^h+v^v,$$

$$E_{m+1}=u^v-\frac{\alpha_2}{\alpha_1+\alpha_3}(0)u^h+\frac{\alpha(\alpha_1+\alpha_3)}{\gamma}(0)E_{m+2},$$

$$E_i=-\alpha_2(0)e_i^h+(\alpha_1+\alpha_3)(0)e_i^v,\quad E_{m+i}=e_i^h,\quad 3\leq i\leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\frac{(\alpha_1+\alpha_3)^2}{\beta_1+\beta_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1+\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(\alpha_1+\alpha_3)I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha^2}{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-\gamma}{(\alpha_1+\alpha_3)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_p \end{pmatrix}$$

where the functions in the matrix are evaluated at 0. We deduce that the signature of $G_{(x,u)}$ is determined by Table A12.

**Table A12.** $(\alpha_1+\alpha_3)(0) \neq 0$, $(\beta_1+\beta_3)(0) \neq 0$ and $\gamma \neq 0$ (lightlike case)

| **The Signature of $G_{(x,u)}$** | $(\alpha_1+\alpha_3)(0)$ | $\alpha(0)$ |
|---|---|---|
| $(2m-2k, 2k)$ | $>0$ | $>0$ |
| $(m, m)$ | $\neq 0$ | $<0$ |
| $(2k, 2m-2k)$ | $<0$ | $>0$ |

(ii) $\gamma = 0$: we set

$$E_1 = u^h - \frac{\alpha_1+\alpha_3}{\beta_1+\beta_3}(0)v^h, \quad E_2 = v^h,$$

$$E_{m+1} = \frac{1}{\sqrt{2}}\left[u^v + v^v - \frac{\alpha_2}{\alpha_1+\alpha_3}(0)v^h + \frac{\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)}{(\alpha_1+\alpha_3)^2}(0)u^h\right],$$

$$E_{m+2} = \frac{1}{\sqrt{2}}\left[u^v - v^v + \frac{\alpha_2}{\alpha_1+\alpha_3}(0)v^h - \frac{\alpha_2(\beta_1+\beta_3) - \beta_2(\alpha_1+\alpha_3)}{(\alpha_1+\alpha_3)^2}(0)u^h\right],$$

$$E_i = -\alpha_2(0)e_i^h + (\alpha_1+\alpha_3)(0)e_i^v, \quad E_{m+i} = e_i^h, \quad i = 3, \ldots, m.$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} -\frac{(\alpha_1+\alpha_3)^2}{\beta_1+\beta_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1+\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(\alpha_1+\alpha_3)I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{\alpha_1+\alpha_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-\alpha}{\alpha_1+\alpha_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. This yields that the signature of $G_{(x,u)}$ is determined by Table A13.

**Table A13.** $(\alpha_1+\alpha_3)(0) \neq 0$, $(\beta_1+\beta_3)(0) \neq 0$ and $\gamma = 0$ (lightlike case).

| **The Signature of $G_{(x,u)}$** | $(\alpha_1+\alpha_3)(0)$ | $\alpha(0)$ |
|---|---|---|
| $(m, m)$ | $\neq 0$ | $<0$ |
| $(2m-2k, 2k)$ | $>0$ | $>0$ |
| $(2k, 2m-2k)$ | $>0$ | $<0$ |

2. If $(\alpha_1+\alpha_3)(0) = 0$, (hence, $(\beta_1+\beta_3)(0) = \gamma$) and $(\beta_1+\beta_3)(0) \neq 0$, then we have one of the following cases:

(i) $\alpha_1(0) = 0$,

- if $\beta_1(0) \neq 0$, we put

$$E_1 = u^h - \frac{\alpha_2}{\beta_1}(0)v^v + \frac{\beta_2}{\beta_1}(0)u^v, \quad E_2 = v^h,$$

$$E_{m+1} = u^v - \frac{\alpha_2}{\beta_1+\beta_3}(0)v^h, \quad E_{m+2} = v^v - \frac{\beta_2}{\alpha_2}(0)u^v,$$

$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\frac{\alpha_2^2}{\beta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1+\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\alpha_2^2}{\beta_1+\beta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. So, the signature of $G_{(x,u)}$ is $(m,m)$.

- if $\beta_1(0) = 0$, we set

$$E_1 = \frac{1}{\sqrt{2}}\left[u^h - v^v + \frac{\beta_2}{\alpha_2}(0)u^v\right], \quad E_2 = v^h,$$

$$E_{m+1} = u^v - \frac{\alpha_2}{\beta_1+\beta_3}(0)v^h, \quad E_{m+2} = \frac{1}{\sqrt{2}}\left[u^h + v^v - \frac{\beta_2}{\alpha_2}(0)u^v\right],$$

$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1+\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_{k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_{m-k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\alpha_2^2}{\beta_1+\beta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 I_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_{m-k-1} \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. Then the signature of $G_{(x,u)}$ is $(m,m)$.

(ii) $\alpha_1(0) \neq 0$,

- if $A := \beta_1(0) + \frac{\alpha_1^2(\beta_1+\beta_3)}{\alpha_2^2}(0) - \frac{2\alpha_1\beta_2}{\alpha_2}(0) \neq 0$, we put

$$E_2 = v^h, \quad E_{m+2} = v^v - \frac{\alpha_1}{\alpha_2}(0)v^h + \frac{\alpha_1(\beta_1+\beta_3) - \beta_2\alpha_2}{\alpha_2^2}(0)u^v,$$

$$E_1 = u^h - \frac{\alpha_2}{A}(0)E_{m+2}, \quad E_{m+1} = u^v - \frac{\alpha_2}{\beta_1+\beta_3}(0)v^h,$$

$$E_i = \alpha_1(0)e_i^h - \alpha_2(0)e_i^v, \quad E_{m+i} = e_i^v, \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \frac{\alpha}{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1+\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1\alpha I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{\beta_1+\beta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. We deduce that the signature of $G$ is $(m, m)$.

- if $A = 0$, we put

$$E_1 = \frac{1}{\sqrt{2}}\left[u^h + v^v - \frac{\alpha_1}{\alpha_2}(0)v^h + \frac{\alpha_1(\beta_1+\beta_3) - \beta_2\alpha_2}{\alpha_2^2}(0)u^v\right],$$

$$E_2 = v^h, \quad E_{m+1} = u^v - \frac{\alpha_2}{\beta_1+\beta_3}(0)v^h,$$

$$E_{m+2} = \frac{1}{\sqrt{2}}\left[u^h - v^v + \frac{\alpha_1}{\alpha_2}(0)v^h - \frac{\alpha_1(\beta_1+\beta_3) - \beta_2\alpha_2}{\alpha_2^2}(0)u^v\right],$$

$$E_i = \alpha_1(0)e_i^h - \alpha_2(0)e_i^v, \quad E_{m+i} = e_i^v, \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1+\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1\alpha I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{\beta_1+\beta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. In this case, the signature of $G_{(x,u)}$ is $(m, m)$.

3. If $(\alpha_1+\alpha_3)(0) = 0$ and $(\beta_1+\beta_3)(0) = 0$
   (i) $\alpha_1(0) = 0$,
   - if $\beta_1(0) \neq 0$, we put

$$E_1 = u^h - \frac{\alpha_2}{\beta_1}(0)v^v + \frac{\beta_2}{\beta_1}(0)u^v, \quad E_2 = \frac{1}{\sqrt{2}}(v^h + u^v),$$

$$E_{m+1} = \frac{1}{\sqrt{2}}(v^h - u^v), \quad E_{m+2} = v^v - \frac{\beta_2}{\alpha_2}(0)u^v,$$

$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\frac{\alpha_2^2}{\beta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. Then, the signature of $G_{(x,u)}$ is $(m, m)$.

- if $\beta_1(0) = 0$ we set

$$E_1 = \frac{1}{\sqrt{2}}\left[u^h - v^v + \frac{\beta_2}{\alpha_2}(0)u^v\right], \quad E_2 = \frac{1}{\sqrt{2}}(v^h + u^v),$$

$$E_{m+1} = \frac{1}{\sqrt{2}}(v^h - u^v), \quad E_{m+2} = \frac{1}{\sqrt{2}}\left[u^h + v^v - \frac{\beta_2}{\alpha_2}(0)u^v\right],$$

$$E_i = \frac{1}{\sqrt{2}}(e_i^h + e_i^v), \quad E_{m+i} = \frac{1}{\sqrt{2}}(e_i^h - e_i^v), \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} -\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0, and the signature of $G_{(x,u)}$ is $(m, m)$.

(ii) $\alpha_1(0) \neq 0$,

- if $(\beta_1\alpha_2 - 2\beta_2\alpha_1)(0) \neq 0$ we put

$$E_2 = \frac{1}{\sqrt{2}}(v^h + u^h), \quad E_{m+2} = v^v - \frac{\alpha_1}{\alpha_2}(0)v^h - \frac{\beta_2}{\alpha_2}(0)u^v,$$

$$E_1 = u^h - \frac{\alpha_2^2}{\beta_1\alpha_2 - 2\beta_2\alpha_1}(0)E_{m+2}, \quad E_{m+1} = \frac{1}{\sqrt{2}}(v^h - u^v),$$

$$E_i = \alpha_1(0)e_i^h - \alpha_2(0)e_i^v, \quad E_{m+i} = e_i^v, \quad 3 \leq i \leq m.$$

The matrix of $G_{(x,u)}$ in the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} -\frac{\alpha_2^3}{\beta_1\alpha_2-2\beta_2\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1\alpha I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta_1\alpha_2-2\beta_2\alpha_1}{\alpha_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. This implies that the signature of $G_{(x,u)}$ is $(m,m)$.

- if $(\beta_1\alpha_2 - 2\beta_2\alpha_1)(0) = 0$, we set

$$E_1 = \frac{1}{\sqrt{2}}\left[u^h + v^v - \frac{\alpha_1}{\alpha_2}(0)v^h - \frac{\beta_2}{\alpha_2}(0)u^v\right],$$

$$E_2 = \frac{1}{\sqrt{2}}(v^h + u^v), \quad E_{m+1} = \frac{1}{\sqrt{2}}(v^h - u^v),$$

$$E_{m+2} = \frac{1}{\sqrt{2}}\left[u^h - \left(v^v - \frac{\alpha_1}{\alpha_2}(0)v^h - \frac{\beta_2}{\alpha_2}(0)u^v\right)\right],$$

$$E_i = \alpha_1(0)e_i^h - \alpha_2(0)e_i^v, \quad E_{m+i} = e_i^v, \quad 3 \le i \le m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is

$$\begin{pmatrix} \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1\alpha I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1\alpha I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. Then, the signature of $G_{(x,u)}$ is $(m,m)$.

4. If $(\alpha_1 + \alpha_3)(0) \neq 0$ and $(\beta_1 + \beta_3)(0) = 0$, then $\gamma = (\alpha_1 + \alpha_3)(0)[2\alpha_2\beta_2 - \beta_1(\alpha_1 + \alpha_3)](0)$.
   (a) If $\gamma \neq 0$, we set

$$E_1 = \frac{1}{\sqrt{2}}(u^h + v^h), \quad E_2 = \frac{1}{\sqrt{2}}(u^h - v^h),$$

$$E_{m+2} = v^v - \frac{\alpha_2}{\alpha_1 + \alpha_3}(0)v^h - \frac{\beta_2}{\alpha_1 + \alpha_3}(0)u^h,$$

$$E_{m+1} = u^v - \frac{\alpha_2}{\alpha_1 + \alpha_3}(0)u^h + \frac{\alpha(\alpha_1 + \alpha_3)}{\gamma}(0)E_{m+2},$$

$$E_i = -\alpha_2(0)e_i^h + (\alpha_1 + \alpha_3)(0)e_i^v, \quad E_{m+i} = e_i^h, \quad i = 3, \dots, m.$$

The matrix of $G_{(x,u)}$ with respect to the basis $\{E_i\}_{i=1}^{2m}$ is given by

$$\begin{pmatrix} \alpha_1+\alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\alpha_1+\alpha_3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(\alpha_1+\alpha_3)I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha^2}{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-\gamma}{(\alpha_1+\alpha_3)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_p \end{pmatrix}$$

where the functions in the matrix are evaluated at 0. Therefore, the signature is determined by Table A14.

**Table A14.** $(\alpha_1+\alpha_3)(0) \neq 0$, $(\beta_1+\beta_3)(0) = 0$ and $\gamma \neq 0$ (lightlike case).

| The Signature of $G_{(x,u)}$ | $(\alpha_1+\alpha_3)(0)$ | $\alpha(0)$ |
|---|---|---|
| $(m, m)$ | $\neq 0$ | $< 0$ |
| $(2m-2k, 2k)$ | $> 0$ | $> 0$ |
| $(2k, 2m-2k)$ | $> 0$ | $< 0$ |

(b) If $\gamma = 0$, we set

$$E_1 = \frac{1}{\sqrt{2}}(u^h + v^h), \quad E_2 = \frac{1}{\sqrt{2}}(u^h - v^h),$$

$$E_{m+1} = \frac{1}{\sqrt{2}}\left[u^v + v^v - \frac{\alpha_2}{\alpha_1+\alpha_3}(0)v^h - \frac{\alpha_2+\beta_2}{\alpha_1+\alpha_3}(0)u^h\right],$$

$$E_{m+2} = \frac{1}{\sqrt{2}}\left[u^v - v^v + \frac{\alpha_2}{\alpha_1+\alpha_3}(0)v^h + \frac{\beta_2-\alpha_2}{\alpha_1+\alpha_3}(0)u^h\right],$$

$$E_i = -\alpha_2(0)e_i^h + (\alpha_1+\alpha_3)(0)e_i^v, \quad E_{m+i} = e_i^h, \quad i = 3, \ldots, m.$$

The matrix of $G_{(x,u)}$, in the basis $\{E_i\}_{i=1}^{2m}$, is given by

$$\begin{pmatrix} \alpha_1+\alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\alpha_1+\alpha_3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(\alpha_1+\alpha_3)I_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha(\alpha_1+\alpha_3)I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{\alpha_1+\alpha_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-\alpha}{\alpha_1+\alpha_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\alpha_1+\alpha_3)I_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\alpha_1+\alpha_3)I_p \end{pmatrix},$$

where the functions in the matrix are evaluated at 0. Thus the signature of $G_{(x,u)}$ is given by Table A15.

**Table A15.** $(\alpha_1+\alpha_3)(0) \neq 0$, $(\beta_1+\beta_3)(0) = 0$ and $\gamma = 0$ (lightlike case).

| The Signature of $G_{(x,u)}$ | $(\alpha_1+\alpha_3)(0)$ | $\alpha(0)$ |
|---|---|---|
| $(m, m)$ | $\neq 0$ | $< 0$ |
| $(2m-2k, 2k)$ | $> 0$ | $> 0$ |
| $(2k, 2m-2k)$ | $> 0$ | $< 0$ |

Thus, we obtained the following:

**Proposition A3.** *Let $(x,u) \in TM$ such that $u$ is a lightlike vector with respect to $g$. The signature $(r,s)$ of the g-natural metric in $(x,u)$ is determined as follows:*

- *If $(\alpha_1+\alpha_3)(0) \neq 0$ and*
  - *(i) $\alpha(0) < 0$, then $(r,s) = (m,m)$,*

(*ii*) $\alpha(0) > 0$ *and* $(\alpha_1 + \alpha_3)(0) > 0$, *then* $(r,s) = (2m - 2k, 2k)$,
(*iii*) $\alpha(0) > 0$ *and* $(\alpha_1 + \alpha_3)(0) < 0$, *then* $(r,s) = (2k, 2m - 2k)$.

- *If* $(\alpha_1 + \alpha_3)(0) = 0$ *then* $(r,s) = (m,m)$.

## Appendix B. Curvatures on $T_0M$ Associated to the Screen Distribution

We shall now proceed to calculate the Riemannian curvature of $(TM, G)$. For this, we need to know the covariant derivative of horizontal (resp. vertical) lifts of sections that are involved in the identities of Proposition 3. We will start by recalling some facts about some needed induced connections.

It is well known that the Levi-Civita connection $\nabla$ on $(M, g)$ induces a connection $\hat{\nabla}$ on the vector bundle $p^*TM$ induced from the tangent bundle $TM$ and its projection map $p : TM \to M$. Remark that for any $X \in \mathfrak{X}(M)$, we have $X \circ p \in \Gamma(p^*TM)$, but sections in $\Gamma(p^*TM)$ are not necessarily of this form. To define $\hat{\nabla}$, we proceed in two steps:

1. For any $Z \in \mathfrak{X}(TM)$ and $Y \in \mathfrak{X}(M)$,

$$\hat{\nabla}_Z(Y \circ p) := (\nabla_{p_* Z} Y) \circ p.$$

2. To define $\hat{\nabla}_Z s$, for any $s \in \Gamma(p^*TM)$, we proceed locally: Let $(e_1, \cdots, e_m)$ be a moving frame on an open set $\bar{U}$ of $M$. Then, $(e_1 \circ p, \cdots, e_m \circ p)$ is a moving frame on $(\Gamma(p^*TM))|_{p^{-1}(\bar{U})}$. In particular, $s|_{p^{-1}(\bar{U})}$ is expressed as $s|_{p^{-1}(\bar{U})} = \sum_{i=1}^{m} s^i e_i \circ p$, when $s^i \in C^\infty(p^{-1}(\bar{U}))$, $i = 1, \cdots, m$. We define $\hat{\nabla}_Z s$ on $p^{-1}(\bar{U})$ as:

$$\begin{aligned} \hat{\nabla}_Z s|_{p^{-1}(\bar{U})} &:= \sum_{i=1}^{m} \Big[ Z(s^i) e_i \circ p + s^i \hat{\nabla}_Z (e_i \circ p) \Big] \\ &= \sum_{i=1}^{m} \Big[ Z(s^i) e_i \circ p + s^i (\nabla_{p_* Z} e_i) \circ p \Big] \end{aligned}$$

In particular, if $Z$ is either a horizontal or a vertical lift of $X \in \mathfrak{X}(M)$, then we have

$$\hat{\nabla}_{X^h} s = \sum_{i=1}^{m} \Big[ X^h(s^i) e_i \circ p + s^i (\nabla_X e_i) \circ p \Big], \qquad \hat{\nabla}_{X^v} s = \sum_{i=1}^{m} X^v(s^i) e_i. \tag{A1}$$

We can define in the same manner $\hat{\nabla}_Z s$ pointwise, i.e., when $z \in TTM$. It is easy to check, using (A1), that the following result holds.

**Lemma A1.** *For any* $X, Y \in \mathfrak{X}(M)$, *we have*

1. $\hat{\nabla}_{X^h}(Y \circ p) = (\nabla_X Y) \circ p$;
2. $\hat{\nabla}_{X^v}(Y \circ p) = 0$;
3. $\hat{\nabla}_{X^h} \sigma = 0$
4. $\hat{\nabla}_{X^v} \sigma = X \circ p$,

*where* $\sigma$ *denotes the identity section of* $p^*TM$.

The metric $g$ on $M$ induces naturally a bundle metric $\hat{g}$ on $p^*TM$, defined by

$$\hat{g}_{(x,u)}(X, Y) := g_x(X_{(x,u)}, Y_{(x,u)}),$$

for any $X, Y \in \Gamma(p^*TM)$ and $(x,u) \in T_0M$. Furthermore, it is easy to check that $\hat{g}_{(x,u)}$ has the same signature as $g_x$ and that the induced connection $\hat{\nabla}$ is compatible with it. Using this compatibility and Lemma A1, we obtain the identities stated in the following result.

**Lemma A2.** *For all $X, Y, Z \in \mathfrak{X}(M)$, we have*

1. $X^h(\hat{g}(Y \circ p, \sigma)) = \hat{g}((\nabla_X Y) \circ p, \sigma)$;
2. $X^h(\hat{g}(Y \circ p, Z \circ p)) = X(g(Y, Z)) \circ p$;
3. $X^v(\hat{g}(Y \circ p, \sigma)) = \hat{g}(Y, X) \circ p$;
4. $X^v(\hat{g}(Y \circ p, Z \circ p)) = 0$.

Next, we state the following useful Lemma, giving the covariant derivative of the horizontal and vertical lifts of sections of $p^*TM$.

**Lemma A3.** *Denote by $\bar{\nabla}$ and $\hat{\nabla}$ the Levi-Civita connection of $(TM, G)$ and the induced connection on $p^*TM$, respectively. For any $X \in \mathfrak{X}(M)$ and $s \in \Gamma(p^*TM)$, we have*

(i) $\bar{\nabla}_{X^h} s^h = \left[\hat{\nabla}_{X^h} s + A(X, s)\right]^h + [B(X, s)]^v$;
(ii) $\bar{\nabla}_{X^h} s^v = [C(X, s)]^h + \left[\hat{\nabla}_{X^h} s + D(X, s)\right]^v$;
(iii) $\bar{\nabla}_{X^v} s^h = \left[\hat{\nabla}_{X^v} s + C(s, X)\right]^h + [D(s, X)]^v$;
(iv) $\bar{\nabla}_{X^v} s^v = \left[\hat{\nabla}_{X^v} s\right]^v$.

**Proof.** We shall prove the first identity, the proof of the others being similar. Expressing $s$ locally as $s|_{p^{-1}(\bar{U})} = \sum_{i=1}^{m} s^i e_i \circ p$ and using the first identity of Proposition 3 and the linearity of $A(X, Y)$ and $B(X, Y)$ with respect to $X$ and $Y$, we have on $\bar{U}$

$$\begin{aligned}
\bar{\nabla}_{X^h} s^h &= \sum_{i=1}^{m} \left[X^h(s^i)(e_i \circ p)^h + s^i \bar{\nabla}_{X^h} (e_i \circ p)^h\right] \\
&= \sum_{i=1}^{m} \left[X^h(s^i)(e_i \circ p)^h + s^i \left(\bar{\nabla}_{X^h} e_i^h\right) \circ p\right] \\
&= \sum_{i=1}^{m} \left\{ \left[X^h(s^i) e_i \circ p + s^i (\nabla_X e_i) \circ p + s^i A(X, e_i) \circ p\right]^h + \left[s^i B(X, e_i) \circ p\right]^v \right\} \\
&= \left[\hat{\nabla}_{X^h} s + A(X, s)\right]^h + [B(X, s)]^v.
\end{aligned}$$

□

*Appendix B.1. The Induced Curvature Tensor on $T_0M$ Associated to the Screen Distribution*

**Proposition A4.** *Let $(M, g)$ be a non-definite pseudo-Riemannian manifold. Denote by $\bar{R}$ and $\hat{R}$ the curvature tensor fields of $\bar{\nabla}$ and $\hat{\nabla}$, respectively. Then, the curvature tensor $\tilde{R}$ on $T_0M$ associated with $\tilde{\nabla}$ is characterized, for all vector fields $X, Y, Z \in \mathfrak{X}(M)$ and $(x, u) \in U$, by the following identities:*

$$\begin{aligned}
&\tilde{R}(X^h_{(x,u)}, Y^h_{(x,u)})Z^h_{(x,u)} = \\
&= \Big\{ R(X_x, Y_x)Z_x + \frac{ab}{2\alpha}[(\nabla_u R)(X_x, Y_x)Z_x - (\nabla_{X_x} R)(Z_x, u)Y_x \\
&+ (\nabla_{Y_x} R)(Z_x, u)X_x] - \frac{a^2 bd}{\alpha^2} g_x((\nabla_u R)(X_x, Y_x)Z_x, u)u \\
&- \frac{d^2(\alpha + 2b^2)}{4\alpha^2}[g(X_x, u)Y_x - g(Y_x, u)X_x]g(Z_x, u) \\
&- \frac{ad(\alpha - b^2)}{4\alpha^2}[g(X_x, u)R(Y_x, u)Z_x - g(Y_x, u)R(X_x, u)Z_x \\
&- g(Z_x, u)R(X_x, Y_x)u] + \frac{a^2 b^2}{4\alpha^2}[R(X_x, u)R(Y_x, u)Z_x \\
&- R(Y_x, u)R(X_x, u)Z_x + R(X_x, u)R(Z_x, u)Y_x - R(Y_x, u)R(Z_x, u)X_x]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{a^2b^2d^2}{2\alpha^3}[g(Y_x,u)R(X_x,u)u - g(X_x,u)R(Y_x,u)u]g(Z_x,u) \\
&- \frac{a^3b^2d}{2\alpha^3}[g(R(Y_x,u)Z_x,u)R(X_x,u)u - g(R(X_x,u)Z_x,u)R(Y_x,u)u] \\
&+ \frac{a^2}{4\alpha}[R(R(Y_x,Z_x)u,u)X_x - R(R(X_x,Z_x)u,u)Y_x - 2R(R(X_x,Y_x)u,u)Z_x] \\
&- \frac{ab^2d}{2\alpha^2}[g(R(Y_x,u)Z_x,u)g(V(u),X_x)u - g(R(X_x,u)Z_x,u)g(V(u),Y_x)u] \\
&- \frac{ad^2(a+c)}{4\alpha^2}[g(Y_x,u)g(X_x,Z_x) - g(X_x,u)g(Y_x,Z_x)]u \\
&- \frac{3a^2d(a+c)}{4\alpha^2}g(R(X_x,Y_x)Z_x,u)u \\
&+ \frac{ad^2(a+c)}{2\alpha^2}[g(Y_x,u)g(X_x,V(u)) - g(Y_x,V(u))g(X_x,u)]g(Z_x,u)u \\
&+ \frac{a^3b^2d}{4\alpha^3}[g(R(X_x,u)u,R(Z_x,u)Y_x + R(Y_x,u)Z_x) \\
&- g(R(Y_x,u)u,R(Z_x,u)X_x + R(X_x,u)Z_x)]u \\
&+ \frac{a^3d}{4\alpha^2}[g(R(X_x,u)u,R(Y_x,Z_x)u) - g(R(Y_x,u)u,R(X_x,Z_x)u) \\
&- 2g(R(X_x,Y_x)u,R(Z_x,u)u)]u \\
&- \frac{a^3b^2d}{2\alpha^3}[g(R(Y_x,u)Z_x,u)g(R(X_x,u)V(u),u) \\
&- g(R(X_x,u)Z_x,u)g(R(Y_x,u)V(u),u)]u \\
&+ \frac{a^3d^2(a+c)}{4\alpha^3}[g(R(Y_x,u)\tilde{Z}(u),u)g(X_x,u)u \\
&- g(R(X_x,u)\tilde{Z}(u),u)g(Y_x,u)u] \\
&+ \frac{ab^2d}{2\alpha^2}[g(R(X_x,u)Z_x,u)Y_x - g(R(Y_x,u)Z_x,u)X_x] \\
&+ \frac{a^3d^2(a+c)}{4\alpha^3}[g(Y_x,u)g(R(V(u),u)X_x,u) \\
&- g(X_x,u)g(R(V(u),u)Y_x,u)]g(Z_x,u)u \\
&+ \frac{a^2d(a+c)}{2\alpha^2}[g(X_x,u)R(V(u),u)Y_x - g(Y_x,u)R(V(u),u)X_x]g(Z_x,u) \\
&- \frac{ab^2d}{2\alpha^2}[g(X_x,u)g(R(Y_x,u)Z_x,u) - g(Y_x,u)g(R(X_x,u)Z_x,u)]V(u) \\
&+ \frac{a^2b^2}{2\alpha^2}[g(R(Y_x,u)Z_x,u)R(V(u),u)X_x - g(R(X_x,u)Z_x,u)R(V(u),u)Y_x]\Big\}^h \\
&+ \Big\{\frac{a(a+c)}{2\alpha}((\nabla_{Z_x}R)(X_x,Y_x)u) + \frac{ab^2d}{\alpha^2}g(((\nabla_u R)(X_x,Y_x)Z_x),u)u \\
&- \frac{b^2}{\alpha}((\nabla_u R)(X_x,Y_x)Z_x) - \frac{3abd(a+c)}{4\alpha^2}g(R(X_x,Y_x)u,Z_x)u \\
&+ \frac{abd(a+c)}{2\alpha^2}[g(R(Y_x,u)Z_x,u)X_x - g(R(X_x,u)Z_x,u)Y_x] \\
&+ \frac{bd^2(a+c)}{2\alpha^2}[g(X_x,u)Y_x - g(Y_x,u)X_x]g(Z_x,u) \\
&+ \frac{bd^2(a+c)}{4\alpha^2}[g(Y_x,u)g(X_x,Z_x) - g(X_x,u)g(Y_x,Z_x)]u \\
&+ \frac{bd^2(a+c)}{2\alpha^2}[g(X_x,u)g(Y_x,V(u)) - g(Y_x,u)g(X_x,V(u))]g(Z_x,u)u \\
&- \frac{abd^2}{2\alpha^2}[g(X_x,u)g(R(Z_x,u)Y_x,u) - g(Y_x,u)g(R(Z_x,u)X_x,u)]u \\
&- \frac{ab^3}{2\alpha^2}[g(R(Y_x,u)Z_x,u)R(V(u),u)X_x - g(R(X_x,u)Z_x,u)R(V(u),u)Y_x]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{3bd}{4\alpha}-\frac{b^3d}{4\alpha^2}\right][g(X_x,u)R(Y_x,u)Z_x\\
&-g(Y_x,u)R(X_x,u)Z_x-g(Z_x,u)R(X_x,Y_x)u]\\
&+\frac{a^2bd(b^2-\alpha)}{2\alpha^3}[g(R(Y_x,u)Z_x,u)R(X_x,u)u-g(R(X_x,u)Z_x,u)R(Y_x,u)u]\\
&+\frac{b^3d}{2\alpha^2}[g(R(Y_x,u)Z_x,u)g(X_x,V(u))-g(R(X_x,u)Z_x,u)g(Y_x,V(u))]u\\
&-\frac{abd^2(b^2-\alpha)}{2\alpha^3}[g(Y_x,u)R(X_x,u)u-g(X_x,u)R(Y_x,u)u]g(Z_x,u)\\
&-\frac{ab}{4\alpha}[R(R(Y_x,Z_x)u,u)X_x-R(R(X_x,Z_x)u,u)Y_x\\
&-2R(R(X,Y)u,u)Z-R(X,R(Y,u)Z)u-R(X,R(Z,u)Y)\\
&+R(Y,R(X,u)Z)u+R(Y,R(Z,u)X)u]\\
&+\frac{ab^3}{4\alpha^2}[R(X,u)(R(Y,u)Z+R(Z,u)Y)-R(Y,u)(R(X,u)Z+R(Z,u)X)]\\
&-\frac{a^2bd^2(a+c)}{4\alpha^3}[g(R(V(u),u)X_x,u)g(Y_x,u)\\
&-g(R(V(u),u)Y_x,u)g(X_x,u)]g(Z_x,u)u\\
&+\frac{d(a+c)}{\alpha}[g(Y_x,u)(\hat{\nabla}_{X^h}V)_{(x,u)}-g(X_x,u)(\hat{\nabla}_{Y^h}V)_{(x,u)}]g(Z_x,u)\\
&-\frac{b^2}{\alpha}[g(R(Y_x,u)Z_x,u)(\hat{\nabla}_{X^h}V)_{(x,u)}-g(R(X_x,u)Z_x,u)(\hat{\nabla}_{Y^h}V)_{(x,u)}]\\
&-\frac{a^2b^3d}{2\alpha^3}[g(R(Y_x,u)Z_x,u)g(R(X_x,u)u,V(u))\\
&-g(R(X_x,u)Z_x,u)g(R(Y_x,u)u,V(u))]u\\
&+\frac{abd(a+c)}{2\alpha^2}[g(Y_x,u)R(V(u),u)X_x-g(X_x,u)R(V(u),u)Y_x]g(Z_x,u)\\
&+\frac{a^2bd^2(a+c)}{4\alpha^3}[g(Y_x,u)g(R(\tilde{Z}(u),u)X_x,u)-g(X_x,u)g(R(\tilde{Z}(u),u)Y_x,u)]u\\
&-\frac{a^2b^3d}{4\alpha^3}[g(R(Y_x,u)Z_x+R(Z_x,u)Y_x,R(X_x,u)u)\\
&-g(R(X_x,u)Z_x+R(Z_x,u)X_x,R(Y_x,u)u)]\\
&-\frac{a^2bd}{4\alpha^2}[g(R(X_x,u)u,R(Y_x,Z_x)u)-g(R(Y_x,u)u,R(X_x,Z_x)u)\\
&-2g(R(X_x,Y_x)u,R(Z_x,u)u)]u\}^t,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{R}(X^h_{(x,u)},Y^h_{(x,u)})Z^t_{(x,u)}=\\
&\quad=\Big\{-\frac{a^2bd}{4\alpha^2}[g(Y_x,\tilde{Z}(u))R(X_x,u)u-g(X_x,\tilde{Z}(u))R(Y_x,u)u\\
&\quad+g(Y_x,u)R(X_x,\tilde{Z}(u))u-g(X_x,u)R(Y_x,\tilde{Z}(u))u]\\
&\quad+\frac{a^3b}{2\alpha^2}[g_x((\nabla_{X_x}R)(Z_x,u)Y_x,u)-g(Z_x,u)[g(\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Y_x),u)\\
&\quad-g(R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Y,u)-g(R(V(u),u)\nabla_{X_x}Y,u)]]u\\
&\quad-\frac{a^3b}{2\alpha^2}[g_x((\nabla_{Y_x}R)(Z_x,u)X_x,u)-g(Z_x,u)[g(\hat{\nabla}_{Y^h_{(x,u)}}(R(V,\sigma)X_x),u)\\
&\quad-g(R(\hat{\nabla}_{Y^h_{(x,u)}}V,u)X_x,u)-g(R(V(u),u)\nabla_{Y_x}X,u)]]u\\
&\quad+\frac{a^3b}{2\alpha^2}[g(R(\tilde{Z}(u),u)Y_x,u)R(V(u),u)X_x-g(R(\tilde{Z}(u),u)X_x,u)R(V(u),u)Y_x]\\
&\quad+\frac{a^4bd}{4\alpha^3}[g(R(X_x,u)u,R(\tilde{Z}(u),u)Y_x)-g(R(Y_x,u)u,R(\tilde{Z}(u),u)X_x)]u
\end{aligned}
$$

$$- \frac{a^4bd}{4\alpha^3}[g(R(\tilde{Z}(u),u)Y_x,u)R(X_x,u)u - g(R(\tilde{Z}(u),u)X_x,u)R(Y_x,u)u]$$
$$+ \frac{a^4bd}{4\alpha^3}[g(R(Z_x,u)Y_x,u)g(R(X_x,u)u,V) - g(R(Z_x,u)X_x,u)g(R(Y_x,u)u,V))]u$$
$$- \frac{a^2bd}{4\alpha^2}[g(R(\tilde{Z}(u),u)Y_x,u)X_x - g(R(\tilde{Z}(u),u)X_x,u)Y_x]$$
$$+ \frac{abd^2}{2\alpha^2}[g(X_x,u)g(Y_x,\tilde{Z}(u)) - g(Y_x,u)g(X_x,\tilde{Z}(u))]u$$
$$+ \frac{a^3b}{4\alpha^2}[R(X_x,u)R(\tilde{Z}(u),u)Y_x - R(Y_x,u)R(\tilde{Z}(u),u)X_x]$$
$$- \frac{a^2}{2\alpha}[(\nabla_X R)(Z,u)Y - g(Z,u)[\hat{\nabla}_{X^h}(R(V,u)Y) - R(\hat{\nabla}_{X^h}V,u)Y - R(V,u)\nabla_X Y]]$$
$$+ \frac{a^2}{2\alpha}[(\nabla_Y R)(Z,u)X - g(Z,u)[\hat{\nabla}_{Y^h}(R(V,u)X) - R(\hat{\nabla}_{Y^h}V,u)X - R(V,u)\nabla_Y X]]$$
$$- \frac{a^2bd}{4\alpha^2}[g(R(\tilde{Z}(u),u)Y,u)[g(X,u)V + g(X,V)u]$$
$$- g(R(\tilde{Z}(u),u)X,u)[g(Y,u)V + g(Y,V)u]]$$
$$\left. + \frac{bd}{4\alpha}[g(R(X_x,\tilde{Z}(u))Y_x,u) - g(R(Y_x,\tilde{Z}(u))X_x,u)]u] \right\}^h$$
$$+ \left\{ R(X_x,Y_x)\tilde{Z}(u) + \frac{ad(b^2-\alpha)}{4\alpha^2}[g(\tilde{Z}(u),Y_x)R(X_x,u)u - g(\tilde{Z}(u),X_x)R(Y_x,u)u] \right.$$
$$+ \frac{ad(b^2-\alpha)}{4\alpha^2}[g(Y_x,u)R(X_x,u)\tilde{Z}(u) - g(X_x,u)R(Y_x,u)\tilde{Z}(u)]$$
$$+ \frac{ab^2d}{2\alpha^2}[g(R(Y_x,u)X_x,\tilde{Z}(u)) - g(R(X_x,u)Y_x,\tilde{Z}(u))]u$$
$$+ \frac{a^2d(a+c)}{4\alpha^2}[g(R(\tilde{Z}(u),u)Y_x,u)X_x - g(R(\tilde{Z}(u),u)X_x,u)Y_x]$$
$$- \frac{d^2}{4\alpha}[g(Y_x,\tilde{Z}(u))g(X_x,u)u - g(X_x,\tilde{Z}(u))g(Y_x,u)u]$$
$$+ \frac{a^2}{4\alpha}[R(X_x,R(\tilde{Z}(u),u)Y_x)u - R(Y_x,R(\tilde{Z}(u),u)X_x)u]$$
$$- \frac{a^2d^2}{4\alpha^2}[g(X_x,u)g(R(\tilde{Z}(u),u)Y_x,u) - g(Y,u)g(R(\tilde{Z}(u),u)X_x,u)]u$$
$$+ \frac{a^3d(b^2-\alpha)}{4\alpha^3}[g(R(\tilde{Z}(u),u)Y_x,u)R(X_x,u)u - g(R(\tilde{Z}(u),u)X_x,u)R(Y_x,u)u]$$
$$- \frac{a^3b^2d}{4\alpha^3}[g(R(\tilde{Z}(u),u)Y_x,u)g(R(X_x,u)u,V) - g(R(\tilde{Z}(u),u)X_x,u)g(R(Y_x,u)u,V)]u$$
$$- \frac{a^3b^2d}{4\alpha^3}[g(R(X_x,u)u,R(\tilde{Z}(u),u)Y_x) - g(R(Y_x,u)u,R(\tilde{Z}(u),u)X_x)]u$$
$$- \frac{a^2b^2}{2\alpha^2}[R(X_x,u)R(\tilde{Z}(u),u)Y_x - R(Y_x,u)R(\tilde{Z}(u),u)X_x]$$
$$+ \frac{ab}{2\alpha}[(\nabla_{X_x}R)(Z_x,u)Y_x - g(Z_x,u)[\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Y)$$
$$- R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Y_x - R(V(u),u)\nabla_{X_x}Y]]$$
$$- \frac{ab}{2\alpha}[(\nabla_{Y_x}R)(Z_x,u)X_x - g(Z_x,u)[\hat{\nabla}_{Y^h_{(x,u)}}(R(V,\sigma)X)$$
$$- R(\hat{\nabla}_{Y^h_{(x,u)}}V,u)X_x - R(V(u),u)\nabla_{Y_x}X]]$$
$$- \frac{a^2bd}{2\alpha^2}[g((\nabla_{X_x}R)(Z_x,u)Y_x,u) - g(Z_x,u)[g(\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Y),u)$$
$$- g(R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Y_x,u) - g(R(V(u),u)\nabla_{X_x}Y,u)]]u$$
$$+ \frac{a^2bd}{2\alpha^2}[g((\nabla_{Y_x}R)(Z_x,u)X_x,u) - g(Z,u)[g(\hat{\nabla}_{Y^h_{(x,u)}}(R(V,\sigma)X),u)$$

$$
\begin{aligned}
&- g(R(\hat{\nabla}_{Y^h_{(x,u)}}V,u)X_x,u) - g(R(V(u),u)\nabla_{Y_x}X,u)]]u \\
&- \frac{a^2b^2}{4\alpha^2}[g(R(\tilde{Z}(u),u)Y_x,u)R(V(u),u)X_x - g(R(\tilde{Z}(u),u)X_x,u)R(V(u),u)Y_x] \\
&+ \frac{ab^2d}{4\alpha^2}[g(R(\tilde{Z}(u),u)Y_x,u)g(X_x,V(u)) - g(R(\tilde{Z}(u),u)X_x,u)g(Y_x,V(u))]u \\
&- \frac{ab}{2\alpha}[g(R(\tilde{Z}_x,u)Y_x,u)\hat{\nabla}_{X^h_{(x,u)}}V - g(R(\tilde{Z}(u),u)X_x,u)\hat{\nabla}_{Y^h_{(x,u)}}V]\Big\}^t,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{R}(X^h_{(x,u)},Y^t_{(x,u)})Z^h_{(x,u)} = \\
&= \Big\{-\frac{a^2bd}{4\alpha^2}[g(\tilde{Y}(u),Z_x)R(X_x,u)u + g(Z_x,u)R(X_x,\tilde{Y}(u))u - g(X_x,u)R(\tilde{Y}(u),u)Z_x] \\
&+ \frac{abd^2}{2\alpha^2}[g(Z_x,u)g(\tilde{Y}(u),X_x)u + g(X_x,u)g(\tilde{Y}(u),Z_x)u + g(X_x,u)g(Z_x,u)\tilde{Y}(u)] \\
&- \frac{a^2bd}{4\alpha^2}[g(R(\tilde{Y}(u),u)Z_x,u)X_x + 2g(R(X_x,u)Z_x,u)\tilde{Y}(u)] \\
&- \frac{a^2}{2\alpha}[(\nabla_{X_x}R)(Y_x,u)Z_x - g(Y_x,u)[\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Z) \\
&- R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Z_x - R(V(u),u)\nabla_{X_x}Z]] \\
&- \frac{a^2bd}{2\alpha^2}[g(R(Z_x,\tilde{Y}(u))X_x,u)u + 2g(R(X_x,\tilde{Y}(u))Z_x,u)u] \\
&+ \frac{a^3b}{4\alpha^2}[R(X_x,u)R(\tilde{Y}(u),u)Z_x - R(\tilde{Y}(u),u)R(X_x,u)Z_x \\
&- R(\tilde{Y}(u),u)R(Z_x,u)X_x + g(R(\tilde{Y}(u),u)Z_x,u)R(V(u),u)X_x] \\
&+ \frac{a^4bd}{4\alpha^3}[g(R(X_x,u)u,R(\tilde{Y}(u),u)Z_x) - g(R(\tilde{Y}(u),u)u,R(X_x,u)Z_x + R(Z_x,u)X_x)]u \\
&+ \frac{a^4bd}{4\alpha^3}g(R(X_x,u)u,V(u))g(R(\tilde{Y}(u),u)Z_x,u)u \\
&- \frac{a^2bd}{4\alpha^2}g(R(\tilde{Y}(u),u)Z_x,u)[g(X_x,u)V(u) + g(X_x,V(u))u] \\
&- \frac{a^4bd}{4\alpha^3}[g(R(\tilde{Y}(u),u)Z_x,u)R(X_x,u)u - 2g(R(X_x,u)Z_x,u)R(\tilde{Y}(u),u)u] \\
&+ \frac{a^3b}{2\alpha^2}[g_x((\nabla_{X_x}R)(Y_x,u)Z_x,u) - g(Y_x,u)[g(\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Z),u) \\
&- g(R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Z_x,u) - g(R(V(u),u)\nabla_{X_x}Z,u)]]u \\
&- \frac{a^3bd^2}{2\alpha^3}g(X_x,u)g(Z_x,u)R(\tilde{Y}(u),u)u \\
&+ \frac{ab}{2\alpha}[R(X_x,\tilde{Y}(u))Z_x + R(Z_x,\tilde{Y}_x)X_x] - \frac{bd}{2\alpha}[g(Z_x,\tilde{Y}(u))X_x + g(X_x,\tilde{Y}(u))Z_x]\Big\}^h \\
&+ \Big\{\frac{ad(b^2-\alpha)}{4\alpha^2}[g(\tilde{Y}(u),Z_x)R(X_x,u)u - g(X_x,u)R(\tilde{Y}(u),u)Z_x \\
&+ g(Z_x,u)R(X_x,\tilde{Y}(u))u] + \frac{a^3b^2d}{4\alpha^3}[g(R(\tilde{Y}(u),u)u,R(Z_x,u)X_x \\
&+ R(X_x,u)Z_x) - g(R(X_x,u)u,R(\tilde{Y}(u),u)Z_x)]u \\
&- \frac{a^3b^2d}{4\alpha^3}g(R(\tilde{Y}(u),u)Z_x,u)g(R(X_x,u)u,V(u))u \\
&- \frac{a^2d^2}{4\alpha^2}g(X_x,u)g(R(\tilde{Y}(u),u)Z_x,u)u \\
&- \frac{a^2b^2}{4\alpha^2}[R(X_x,u)R(\tilde{Y}(u),u)Z - R(\tilde{Y}(u),u)R(X_x,u)Z_x - R(\tilde{Y}(u),u)R(Z_x,u)X_x]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{a^3b^2d}{2\alpha^3} g(R(X_x,u)Z_x,u)R(\tilde{Y}(u),u)u + \frac{a^2b^2d^2}{2\alpha^3} g(Z_x,u)g(X_x,u)R(\tilde{Y}(u),u)u \\
&+ \frac{ab}{2\alpha}[(\nabla_{X_x}R)(Y_x,u)Z_x - g(Y_x,u)[\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Z) - R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Z_x \\
&- R(V(u),u)\nabla_{X_x}Z] - g(R(\tilde{Y}(u),u)Z,u)(\hat{\nabla}_{X^h_{(x,u)}}V)] \\
&- \frac{a^2bd}{2\alpha^2}[g((\nabla_{X_x}R)(Y_x,u)Z_x,u) - g(Y_x,u)[g(\hat{\nabla}_{X^h_{(x,u)}}(R(V,\sigma)Z),u) \\
&- g(R(\hat{\nabla}_{X^h_{(x,u)}}V,u)Z_x,u) - g(R(V(u),u)\nabla_{X_x}Z,u)]]u \\
&- \frac{a^2b^2}{4\alpha^2} g(R(\tilde{Y}_x,u)Z_x,u)R(V(u),u)X_x \\
&+ [\frac{b^2}{\alpha} g(R(X_x,u)Z_x,u) + \frac{d(a+c)}{\alpha} g(Z_x,u)g(X_x,u)]\hat{\nabla}_{Y^t_{(x,u)}}V \\
&+ \frac{a^2}{4\alpha} R(X_x,R(\tilde{Y}(u),u)Z_x)u + \frac{a^2d(a+c)}{4\alpha^2} g(R(\tilde{Y}(u),u)Z_x,u)X_x \\
&- \frac{d^2(\alpha+2b^2)}{4\alpha^2}[g(\tilde{Y}(u),Z_x)u + g(Z_x,u)Y_x]g(X_x,u) \\
&+ \frac{ab^2d}{4\alpha^2} g(R(\tilde{Y}(u),u)Z_x,u)g(V(u),X_x)u \\
&+ \frac{ab^2d}{2\alpha^2}[g(R(Z_x,u)X_x,u)Y_x + g(R(X_x,u)Z_x,\tilde{Y}(u))u + 2g(R(Z_x,u)X_x,\tilde{Y}(u))u] \\
&- \frac{a^3d(\alpha-b^2)}{4\alpha^3} g(R(\tilde{Y}(u),u)Z_x,u)R(X_x,u)u - \frac{b^2d^2}{2\alpha^2} g(X_x,\tilde{Y}(u))g(Z_x,u)u \\
&+ \frac{d(a+c)}{2\alpha}[g(X_x,\tilde{Y}(u))Z_x + g(Z_x,\tilde{Y}(u))X_x] \\
&+ \frac{a(a+c)}{2\alpha} R(X_x,Z_x)\tilde{Y}(u) - \frac{b^2}{\alpha} R(X_x,\tilde{Y}(u))Z_x \Big\}^t,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{R}(X^h_{(x,u)},Y^t_{(x,u)})Z^t_{(x,u)} = \\
&\quad = \{aM(u;X_x,Y_x,Z_x)\}^h - \{bM(u;X_x,Y_x,Z_x) - g(\tilde{Z}(u),\tilde{Y}(u))\hat{\nabla}_{X^h_{(x,u)}}V \\
&\quad + g(Z,u)\hat{R}(X^h,Y^t)V - \frac{ab}{2\alpha} g(R(\tilde{Z}(u),u)X_x,u)\hat{\nabla}_{Y^t_{(x,u)}}V\}^t \\
&\tilde{R}(X^t_{(x,u)},Y^t_{(x,u)})Z^h_{(x,u)} = \\
&\quad = a\{N(u;X_x,Y_x,Z_x)\}^h - b\{N(u;X_x,Y_x,Z_x) \\
&\quad - \frac{a}{2\alpha}[g(R(\tilde{Y}(u),u)Z_x,u)\hat{\nabla}_{X^t_{(x,u)}}V - g(R(\tilde{X}(u),u)Z_x,u)\hat{\nabla}_{Y^t_{(x,u)}}V]\}^t \\
&\tilde{R}(X^t_{(x,u)},Y^t_{(x,u)})Z^t_{(x,u)} = \Big\{g(\tilde{Y}(u),\tilde{Z}(u))\hat{\nabla}_{X^t_{(x,u)}}V - g(\tilde{X}(u),\tilde{Z}(u))\hat{\nabla}_{Y^t_{(x,u)}}V\Big\}^t,
\end{aligned}
$$

*where*

$$
\begin{aligned}
&M(u;X_x,Y_x,Z_x) = \\
&\quad = -\frac{d}{2\alpha}[g(X_x,\tilde{Z}(u))\tilde{Y}(u) + g(X_x,\tilde{Y}(u))\tilde{Z}(u)] \\
&\quad + \frac{d}{2\alpha}[g(X_v,V(u))u + g(X_x,u)V(u)]g(\tilde{Z}(u),\tilde{Y}(u)) \\
&\quad + \frac{a}{2\alpha}[R(\tilde{Z}(u),\tilde{Y}(u))X_x - g(\tilde{Z}(u),\tilde{Y}(u))R(V(u),u)X_x] \\
&\quad - \frac{ad^2}{4\alpha^2} g(\tilde{Z}(u),\tilde{Y}(u))g(X_x,u)u + \frac{a^2d}{4\alpha^2}[g(X_x,\tilde{Z}(u))R(\tilde{Y}(u),u)u \\
&\quad + g(X_x,u)R(\tilde{Y}(u),u)\tilde{Z}(u) + g(\tilde{Y}(u),\tilde{Z}(u))g(R(V(u),u)X_x,u)u] \\
&\quad + \frac{a^4d}{4\alpha^3}[g(R(\tilde{Z}(u),u)X_x,u)R(\tilde{Y}(u),u)u - g(R(\tilde{Y}(u),u)u,R(\tilde{Z}(u),u)X_x)u]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{a^3 d^2}{2\alpha^3} g(R(\tilde{Z}(u),u)\tilde{Y}(u),u) g(X_x,u)u - \frac{a^3}{4\alpha^2} R(\tilde{Y}(u),u)R(\tilde{Z}(u),u)X_x \\
&- \frac{a}{2\alpha} g(Z_x,u)[\hat{\nabla}_{Y^t_{(x,u)}}(R(V,\sigma)X) - R(\hat{\nabla}_{Y^t_{(x,u)}}V,u)X_x - R(V(u),\tilde{Y}(u))X_x] \\
&+ \frac{a^2 d}{2\alpha^2} g(Z_x,u)[g(\hat{\nabla}_{Y^t_{(x,u)}}(R(V,\sigma)X),u) - g(R(\hat{\nabla}_{Y^t_{(x,u)}}V,u)X_x,u) \\
&- g(R(V(u),\tilde{Y}(u))X_x,u)]u - \frac{a^2 d}{4\alpha^2}[g(R(\tilde{Z}(u),u)X_x,\tilde{Y}(u))u \\
&+ g(R(\tilde{Z}(u),u)X_x,u)\tilde{Y}(u) + 2g(R(\tilde{Z}(u),\tilde{Y}(u))X_x,u)u],
\end{aligned}
$$

$$
\begin{aligned}
N(u;&X_x,Y_x,Z_x) = \\
&= \frac{a}{\alpha} R(\tilde{X}(u),\tilde{Y}(u))Z_x - \frac{a^2 d}{4\alpha^2}[g(\tilde{Y}(u),Z_x)R(\tilde{X}(u),u)u \\
&- g(\tilde{X}(u),Z_x)R(\tilde{Y}(u),u)u + g(Z_x,u)R(\tilde{X}(u),\tilde{Y}(u))u] \\
&- \frac{a^4 d}{4\alpha^3}[g(R(\tilde{Y}(u),u)Z_x,u)R(\tilde{X}(u),u)u - g(R(\tilde{X}(u),u)Z_x,u)R(\tilde{Y}(u),u)u] \\
&+ \frac{a^3}{4\alpha^2}[R(\tilde{X}(u),u)R(\tilde{Y}(u),u)Z_x - R(\tilde{Y}(u),u)R(\tilde{X}(u),u)Z_x] \\
&+ \frac{a^4 d}{4\alpha^3}[g(R(\tilde{X}(u),u)u,R(\tilde{Y}(u),u)Z_x) - g(R(\tilde{Y}(u),u)u,R(\tilde{X}(u),u)Z_x)]u \\
&+ \frac{a}{2\alpha} g(Y_x,u)[\hat{\nabla}_{X^t_{(x,u)}}(R(V,\sigma)Z) - R(\hat{\nabla}_{X^t_{(x,u)}}V,u)Z_x - R(V(u),\tilde{X}(u))Z_x] \\
&- \frac{a}{2\alpha} g(X_x,u)[\hat{\nabla}_{Y^t_{(x,u)}}(R(V,\sigma)Z) - R(\hat{\nabla}_{Y^t_{(x,u)}}V,u)Z_x - R(V(u),\tilde{Y}(u))Z_x] \\
&- \frac{a^2 d}{2\alpha^2} g(Y,u)[g(\hat{\nabla}_{X^t_{(x,u)}}(R(V,\sigma)Z),u) - g(R(\hat{\nabla}_{X^t_{(x,u)}}V,u)Z_x,u) \\
&- g(R(V(u),\tilde{X}(u))Z_x,u)]u + \frac{a^2 d}{2\alpha^2} g(X,u)[g(\hat{\nabla}_{Y^t_{(x,u)}}(R(V,\sigma)Z),u) \\
&- g(R(\hat{\nabla}_{Y^t_{(x,u)}}V,u)Z_x,u) - g(R(V(u),\tilde{Y}(u))Z_x,u)]u + \frac{a^2 d}{4\alpha^2}[g(R(\tilde{Y}(u),u)Z_x,\tilde{X}(u)) \\
&- g(R(\tilde{X}(u),u)Z_x,\tilde{Y}(u)) - 4g(R(\tilde{X}(u),\tilde{Y}(u))Z_x,u)]u \\
&+ \frac{a^2 d}{4\alpha^2}[g(R(\tilde{Y}(u),u)Z_x,u)\tilde{X}(u) - g(R(\tilde{X}(u),u)Z_x,u)\tilde{Y}(u)].
\end{aligned}
$$

**Proof.** We shall prove the last identity, the others are calculated in the same way using Theorem 2 and Lemmas 5, 7 and 9. By definition, we have

$$\tilde{R}(X^t,Y^t)Z^t = \tilde{\nabla}_{X^t}\tilde{\nabla}_{Y^t}Z^t - \tilde{\nabla}_{Y^t}\tilde{\nabla}_{X^t}Z^t - \tilde{\nabla}_{[X^t,Y^t]}Z^t.$$

But

$$
\begin{aligned}
\tilde{\nabla}_{X^t}\tilde{\nabla}_{Y^t}Z^t =& \tilde{\nabla}_{X^t}\left\{-\hat{g}(Z\circ p_0,\sigma)\hat{\nabla}_{Y^t}V\right\}^t \\
=& - X^t(\hat{g}(Z\circ p_0,\sigma))\hat{\nabla}_{Y^t}V - \hat{g}(Z\circ p_0,\sigma)\tilde{\nabla}_{X^t}(\hat{\nabla}_{Y^t}V)^t \\
=& - [g(X,Z) - \hat{g}(X\circ p_0,\sigma)\hat{g}(Z\circ p_0,V)]\hat{\nabla}_{Y^t}V \\
&- \hat{g}(Z\circ p_0,\sigma)\left\{\hat{\nabla}_{X^t}\widetilde{\hat{\nabla}_{Y^t}V}\right\}^t \\
=& - \hat{g}(\tilde{X},Z\circ p_0)\hat{\nabla}_{Y^t}V - \hat{g}(Z\circ p_0,\sigma)\left\{\hat{\nabla}_{X^t}\hat{\nabla}_{Y^t}V + g(Y\circ p_0,V)\hat{\nabla}_{X^t}V\right\}^t
\end{aligned}
$$

and

$$[X^t,Y^t] = \{\hat{g}(X\circ p_0,\sigma)\hat{\nabla}_{Y^t}V - \hat{g}(Y\circ p_0,\sigma)\hat{\nabla}_{X^t}V\}^t.$$

Then,

$$\tilde{R}(X^t,Y^t)Z^t = \left\{-\hat{g}(\tilde{X},\tilde{Z})\hat{\nabla}_{Y^t}V + \hat{g}(\tilde{Y},\tilde{Z})\hat{\nabla}_{X^t}V - \hat{g}(Z\circ p_0,\sigma)\hat{R}(X^t,Y^t)V\right\}^t.$$

But $\hat{R}(X^t,Y^t)V = R(p_*X^t,p_*Y^t)V = 0$, thus

$$\tilde{R}(X^t,Y^t)Z^t = \{-\hat{g}(\tilde{X},\tilde{Z})\hat{\nabla}_{Y^t}V + \hat{g}(\tilde{Y},\tilde{Z})\hat{\nabla}_{X^t}V\}^t.$$

□

*Appendix B.2. The Ricci Type Tensor on $T_0M$ Associated to the Screen Distribution*

By contraction of the curvature tensor described in Proposition A4 and the definition of the Ricci type tensor, we obtain the following:

**Proposition A5.** *Let $(M,g)$ be a non-definite pseudo-Riemannian manifold and $G$ be a pseudo-Riemannian g-natural metric on $TM$. The Ricci type tensor of $(T_0M,\tilde{G},S(T(T_0M)))$ is characterized by*

$$\begin{aligned}
R^{(0,2)}_{(x,u)}(X^h,Y^h) =& \frac{\alpha-b^2}{2\alpha}[2Ric(X,Y) - g(R(X,V(u))Y,u)] \\
&+ \frac{b^2+\alpha}{2\alpha}g(R(X,u)Y,V(u)) - \frac{a^2bd}{\alpha^2}g((\nabla_u R)(X,u)Y,u) \\
&+ \frac{ab}{2\alpha}[g((\nabla_u R)(X,u)Y,V(u)) + g((\nabla_{V(u)}R)(Y,u)X,u) \\
&+ 2g((\nabla_u R)(X,V(u))Y,u)] + \frac{a^3d(b^2-\alpha)}{2\alpha^3}g(R(X,u)u,R(Y,u)u) \\
&+ \frac{3a^2}{4\alpha}[g(R(Y,u)u,R(X,V)u) + g(R(Y,V)u,R(X,u)u)] \\
&- \frac{a^2b^2}{4\alpha^2}[g(R(X,u)u,R(V(u),u)Y + R(Y,u)V(u)) \\
&- g(R(V(u),u)u,R(X,u)Y + R(Y,u)X) \\
&+ g(R(X,u)V + R(V(u),u)X,R(Y,u)u) \\
&- g(R(Y,u)V(u),u)g(R(X,u)V(u),u)] + \frac{(m+1)ab^2d}{2\alpha^2}g(R(X,u)Y,u) \\
&- \frac{a^2d(a+c)}{2\alpha^2}[g(X,u)g(R(Y,u)V(u),u) + g(Y,u)g(R(X,u)V(u),u)] \\
&- \frac{d^2[(2m-1)\alpha+4mb^2]}{4\alpha^2}g(X,u)g(Y,u) \\
&+ \frac{d(a+c)}{2\alpha}[2g(X,Y) - g(X,u)g(Y,V(u)) - g(Y,u)g(X,V(u))] \\
&+ \frac{ab}{2\alpha}g(R(Y,u)u,\hat{\nabla}_{X^h_{(x,u)}}V) + \sum_{i=3}^{m}\varepsilon_i\{\frac{ab}{2\alpha}[g((\nabla_{e_i}R)(X,u)Y,e_i) \\
&+ g((\nabla_{e_i}R)(Y,u)X,e_i)] - \frac{a^2b^2}{4\alpha^2}g(R(X,u)e_i,R(Y,u)e_i) \\
&+ \frac{a^2}{4\alpha}[3g(R(X,e_i)u,R(Y,e_i)u) - g(R(u,e_i)Y,R(u,e_i)X)]\} \\
R^{(0,2)}_{(x,u)}(X^h,Y^t) =& -\frac{ab}{2\alpha}Ric(\tilde{Y}(u),X) - \frac{a^3b}{2\alpha^2}[g((\nabla_u R)(Y,u)X,u) \\
&- g(Y,u)[g(\hat{\nabla}_{u^h}(R(V,\sigma)\bar{X}),u) - g(R(\hat{\nabla}_{u^h}V,u)X,u) \\
&- g(R(V(u),u)\nabla_u\bar{X},V(u))] + \frac{a^2}{2\alpha}[g((\nabla_{V(u)}R)(Y,u)X,u) \\
&- g(Y,u)[g(\hat{\nabla}_{V^h_{(x,u)}}(R(V,\sigma)\bar{X}),u) - g(R(\hat{\nabla}_{V^h_{(x,u)}}V,u)X,u) \\
&- g(R(V,u)\nabla_{V(u)}\bar{X},u)] + g((\nabla_u R)(Y,u)X,V(u)) \\
&- g(Y,u)[g(\hat{\nabla}_{u^h}(R(V,\sigma)\bar{X}),V) - g(R(\hat{\nabla}_{u^h}V,u)X,V(u)) \\
&- g(R(V(u),u)\nabla_u\bar{X},V(u))]] + \frac{ab}{2\alpha}[g(\hat{\nabla}_{u^t}(R(V,\sigma)\bar{X}),V(u)) \\
&- g(R(\hat{\nabla}_{u^t}V,u)X,V(u)) - g(R(V(u),u)X,V(u))]g(Y,u)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{a^2bd}{2\alpha^2}[g(\hat{\nabla}_{u^t}(R(V,\sigma)\bar{X}),u) - g(R(\hat{\nabla}_{u^t}V,u)X,u) \\
&- g(R(V(u),u)X,u)]g(Y,u) - \frac{a^3b}{4\alpha^2}[g(R(\tilde{Y}(u),u)u,R(X,u)V(u) \\
&+ R(V(u),u)X) + g(R(X,u)u,R(\tilde{Y}(u),u)V(u)) \\
&- g(R(V(u),u)u,R(\tilde{Y}(u),u)X)] + \frac{a^4bd}{2\alpha^3}g(R(X,u)u,R(\tilde{Y}(u),u)u) \\
&+ \frac{mbd}{2\alpha}g(X,\tilde{Y}(u)) + \frac{(2m+3)a^2bd}{4\alpha^2}g(R(X,u)\tilde{Y}(u),u) \\
&+ \frac{ab}{2\alpha}[g(R(\tilde{Y}(u),u)X,V(u)) - g(Y,V(u))g(R(V(u),u)X,u)] \\
&+ g(\hat{\nabla}_{X^h_{(x,u)}}V,\tilde{Y}(u)) + \sum_{i=3}^{m}\varepsilon_i\{\frac{a^2}{2\alpha}[g((\nabla_{e_i}R)(Y,u)X,e_i) \\
&- g(Y,u)[g(\hat{\nabla}_{e_i^h}(R(V,\sigma)\bar{X}),e_i) - g(R(\hat{\nabla}_{e_i^h}V,u)X,e_i) \\
&- g(R(V(u),u)\nabla_{e_i}\bar{X},e_i)]] + \frac{ab}{2\alpha}g(Y,u)[g(\hat{\nabla}_{e_i^t}(R(V,\sigma)\bar{X}),e_i) \\
&- g(R(\hat{\nabla}_{e_i^t}V,u)X,e_i) - g(R(V(u),e_i)X,e_i)] \\
&- \frac{a^3b}{4\alpha^2}g(R(X,u)e_i,R(\tilde{Y}(u),u)e_i)\},
\end{aligned}
$$

$$
\begin{aligned}
R^{(0,2)}_{(x,u)}(X^t,Y^h) = &- \frac{ab}{2\alpha}Ric(\tilde{X}(u),Y) - \frac{a^3b}{2\alpha^2}[g((\nabla_u R)(X,u)Y,u) \\
&- g(X,u)[g(\hat{\nabla}_{u^h}(R(V,\sigma)\bar{Y}),u) - g(R(\hat{\nabla}_{u^h}V,u)Y,u) \\
&- g(R(V(u),u)\nabla_u\bar{Y},V(u))] + \frac{a^2}{2\alpha}[g((\nabla_{V(u)}R)(X,u)Y,u) \\
&- g(X,u)[g(\hat{\nabla}_{V^h_{(x,u)}}(R(V,\sigma)\bar{Y}),u) - g(R(\hat{\nabla}_{V^h_{(x,u)}}V,u)Y,u) \\
&- g(R(V(u),u)\nabla_{V(u)}\bar{Y},u)] + g((\nabla_u R)(X,u)Y,V(u)) \\
&- g(X,u)[g(\hat{\nabla}_{u^h}(R(V,\sigma)\bar{Y}),V(u)) - g(R(\hat{\nabla}_{u^h}V,u)Y,V(u)) \\
&- g(R(V(u),u)\nabla_u\bar{Y},V)]] + \frac{ab}{2\alpha}[g(\hat{\nabla}_{u^t}(R(V,\sigma)\bar{Y}),V(u)) \\
&- g(R(\hat{\nabla}_{u^t}V,u)Y,V(u)) - g(R(V(u),u)Y,V(u))]g(X,u) \\
&- \frac{a^2bd}{2\alpha^2}[g(\hat{\nabla}_{u^t}(R(V,\sigma)\bar{Y}),u) - g(R(\hat{\nabla}_{u^t}V,u)Y,u) \\
&- g(R(V(u),u)Y,u)]g(X,u) - \frac{a^3b}{4\alpha^2}[g(R(\tilde{X}(u),u)u,R(Y,u)V(u) \\
&+ R(V(u),u)Y) + g(R(\tilde{X}(u),u)V(u),R(Y,u)u) \\
&- g(R(V(u),u)u,R(\tilde{X}(u),u)Y)] + \frac{a^4bd}{2\alpha^3}g(R(\tilde{X}(u),u)u,R(Y,u)u) \\
&+ \frac{(2m+3)a^2bd}{4\alpha^2}g(R(\tilde{X}(u),u)Y,u) + \frac{ab}{\alpha}g(R(\tilde{X}(u),V(u))Y,u) \\
&+ \frac{mbd}{2\alpha}g(\tilde{X}(u),Y) + \sum_{i=3}^{m}\varepsilon_i\{\frac{a^2}{2\alpha}[g((\nabla_{e_i}R)(X,u)Y,e_i) \\
&- g(X,u)[g(\hat{\nabla}_{e_i^h}(R(V,\sigma)\bar{Y}),e_i) - g(R(\hat{\nabla}_{e_i^h}V,u)Y,e_i) \\
&- g(R(V(u),u)\nabla_{e_i}\bar{Y},e_i)]] + \frac{ab}{2\alpha}g(X,u)[g(\hat{\nabla}_{e_i^t}(R(V,\sigma)\bar{Y},e_i) \\
&- g(R(\hat{\nabla}_{e_i^t}V,u)Y,e_i) - g(R(V(u),e_i)Y,e_i)] \\
&- \frac{a^3b}{4\alpha^2}g(R(\tilde{X}(u),u)e_i,R(\tilde{Y}(u),u)e_i))\},
\end{aligned}
$$

$$
\begin{aligned}
R^{(0,2)}_{(x,u)}(X^t,Y^t) = &-\frac{a^4}{4\alpha^2}[g(R(\tilde{X}(u),u)u,R(\tilde{Y}(u),u)V(u)) \\
&+g(R(\tilde{Y}(u),u)u,R(\tilde{X}(u),u)V(u))] \\
&+\frac{a^5 d}{2\alpha^3}g(R(\tilde{Y}(u),u)u,R(\tilde{X}(u),u)u) \\
&-\frac{a^4}{4\alpha^2}\sum_{i=3}^{m}\varepsilon_i g(R(\tilde{X}(u),u)e_i,R(\tilde{Y}(u),u)e_i),
\end{aligned}
$$

*for all $(x,u) \in U$ and $X,Y \in T_xM$, where $\{e_i, i=3,\dots,m\}$ is an orthonormal family of $T_xM$ such that $g(e_i,u)=g(e_i,V(u))=0$, and $\tilde{X},\tilde{Y} \in \mathfrak{X}(M)$ are any arbitrary extensions of $X,Y$, respectively.*

*Appendix B.3. The Extrinsic Scalar Curvature on $T_0M$ Associated to the Screen Distribution*

Finally, by contraction of symmetrized induced Ricci tensor from the Ricci type tensor described in Proposition A5 and the definition of the extrinsic scalar curvature, we obtain the following:

**Proposition A6.** *Let $(M,g)$ be a non-definite semi-Riemannian manifold. The extrinsic scalar curvature of $(T_0M,\tilde{G},S(T(T_0M)))$, where $\tilde{G}$ is the metric on $T_0M$ induced from a g-natural metric $G$ on $TM$, is given by*

$$
\begin{aligned}
\tilde{R}_{(x,u)} = &\frac{a}{\alpha}R_x + [2b^2 - \frac{a^2 d}{2\alpha^3}((m+2)b^2+2\alpha)]Ric(u,u) \\
&-\frac{ab^2}{2\alpha^2}[3Ric(u,V(u))+2g(R(V(u),u)V(u),u)] \\
&+\frac{a^3(b^2-6\alpha)}{4\alpha^3}g(R(V(u),u)u,R(V(u),u)u) \\
&+\frac{a^2 b}{2\alpha^2}[g(R(V(u),u)u,\hat{\nabla}_{u^h}V)-3g((\nabla_u R)(V(u),u)V(u),u)] \\
&+\frac{(m-1)d}{\alpha^2}[\alpha+(m-1)b^2]+\sum_{i=3}^{m}\varepsilon_i\{[\frac{a^2b^2}{2\alpha}-\frac{a^4 d}{2\alpha^3}]g(R(e_i,u)u,R(e_i,u)u) \\
&+\frac{a^3}{\alpha^2}[3g(R(u,e_i)u,R(V(u),e_i)u)-g(R(u,e_i)V(u),R(u,e_i)u)] \\
&+\frac{a^2 b}{2\alpha^2}[g_x(R(e_i,u)u,(\hat{\nabla}_{e_i^h}V))-3g((\nabla_{e_i}R)(V(u),u)u,e_i)]-\frac{b}{\alpha}g(\hat{\nabla}_{e_i^h}V,e_i) \\
&+\frac{a^3}{4\alpha^2}\sum_{j=3}^{m}\varepsilon_j[3g(R(e_j,e_i)u,R(e_j,e_i)u)-2g(R(e_i,u)e_j,R(e_i,u)e_j)]\},
\end{aligned}
$$

*for all $(x,u) \in U$, where $R$ is the scalar curvature of $M$.*

Finally, if we restrict ourselves to the Sasaki metric on $TM$ in Propositions 9 and A4–A6, we obtain the following:

**Example A1.** *Let $(M,g)$ be a non-definite pseudo-Riemannian manifold and let $TM$ be equipped with the Sasaki metric. Then*

1. *The local screen second fundamental form $C$ of the screen distribution $S(T(T_0M))$ is characterized at any $(x,u) \in U$ by*

$$
\begin{aligned}
C(X^h,PY^h)_{(x,u)} = &-\frac{1}{2}g(R(X_x,Y_x)u,V(u)), \\
C(X^h,PY^t)_{(x,u)} = &-g(Y_x,\hat{\nabla}_{X^h_{(x,u)}}V), \\
C(X^t,PY^h)_{(x,u)} = &0, \\
C(X^t,PY^t)_{(x,u)} = &-[g(Y_x,\hat{\nabla}_{X^t_{(x,u)}}V)+g(Y_x,V(u))g(X_x,V(u))].
\end{aligned}
$$

2. *The curvature tensor $\tilde{R}$ on $T_0M$ associated to $\tilde{\nabla}$ is characterized, for all vector fields $X, Y, Z \in \mathfrak{X}(M)$ and $(x,u) \in U$, by the following identities:*

$$\begin{aligned}\tilde{R}(X^h_{(x,u)}, Y^h_{(x,u)})Z^h_{(x,u)} =& \Big\{ R(X_x, Y_x)Z_x + \frac{1}{4}[R(R(Y_x, Z_x)u, u)X_x \\ & - R(R(X_x, Z_x)u, u)Y_x - 2R(R(X_x, Y_x)u, u)Z_x] \Big\}^h \\ & + \frac{1}{2}\Big\{ (\nabla_{Z_x} R)(X_x, Y_x)u \Big\}^t,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^h_{(x,u)}, Y^h_{(x,u)})Z^t_{(x,u)} =& \Big\{ -\frac{1}{2}[(\nabla_X R)(Z, u)Y - g(Z, u)[\hat{\nabla}_{X^h}(R(V, u)Y) \\ & - R(\hat{\nabla}_{X^h} V, u)Y - R(V, u)\nabla_X Y]] \Big\}^h + \Big\{ R(X_x, Y_x)\tilde{Z}(u) \\ & + \frac{1}{4}[R(X_x, R(\tilde{Z}(u), u)Y_x)u - R(Y_x, R(\tilde{Z}(u), u)X_x)u] \Big\}^t,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^h_{(x,u)}, Y^t_{(x,u)})Z^h_{(x,u)} =& -\frac{1}{2}\Big\{ (\nabla_{X_x} R)(Y_x, u)Z_x - g(Y_x, u)[\hat{\nabla}_{X^h_{(x,u)}}(R(V, \sigma)Z) \\ & - R(\hat{\nabla}_{X^h_{(x,u)}} V, u)Z_x - R(V(u), u)\nabla_{X_x} Z] \Big\}^h \\ & + \frac{1}{4}\{ R(X_x, R(\tilde{Y}(u), u)Z_x)u + 2R(X_x, Z_x)\tilde{Y}(u) \}^t,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^h_{(x,u)}, Y^t_{(x,u)})Z^t_{(x,u)} =& \Big\{ \frac{1}{2}[R(\tilde{Z}(u), \tilde{Y}(u))X_x - g(\tilde{Z}(u), \tilde{Y}(u))R(V(u), u)X_x] \\ & - \frac{1}{2}g(Z_x, u)[\hat{\nabla}_{Y^t_{(x,u)}}(R(V, \sigma)X) - R(\hat{\nabla}_{Y^t_{(x,u)}} V, u)X_x - R(V(u), \tilde{Y}(u))X_x] \Big\}^h \\ & + \Big\{ g(\tilde{Z}(u), \tilde{Y}(u))\hat{\nabla}_{X^h_{(x,u)}} V - g(Z, u)\hat{R}(X^h, Y^t)V \Big\}^t,\end{aligned}$$

$$\begin{aligned}\tilde{R}(X^t_{(x,u)}, Y^t_{(x,u)})Z^h_{(x,u)} =& \Big\{ R(\tilde{X}(u), \tilde{Y}(u))Z_x + \frac{1}{4}[R(\tilde{X}(u), u)R(\tilde{Y}(u), u)Z_x - R(\tilde{Y}(u), u)R(\tilde{X}(u), u)Z_x] \\ & + \frac{1}{2}g(Y_x, u)[\hat{\nabla}_{X^t_{(x,u)}}(R(V, \sigma)Z) - R(\hat{\nabla}_{X^t_{(x,u)}} V, u)Z_x - R(V(u), \tilde{X}(u))Z_x] \\ & - \frac{1}{2}g(X_x, u)[\hat{\nabla}_{Y^t_{(x,u)}}(R(V, \sigma)Z) - R(\hat{\nabla}_{Y^t_{(x,u)}} V, u)Z_x - R(V(u), \tilde{Y}(u))Z_x] \Big\}^h \\ & - \frac{1}{2}\Big\{ g(R(\tilde{Y}(u), u)Z_x, u)\hat{\nabla}_{X^t_{(x,u)}} V - g(R(\tilde{X}(u), u)Z_x, u)\hat{\nabla}_{Y^t_{(x,u)}} V] \Big\}^t,\end{aligned}$$

$$\tilde{R}(X^t_{(x,u)}, Y^t_{(x,u)})Z^t_{(x,u)} = \Big\{ g(\tilde{Y}(u), \tilde{Z}(u))\hat{\nabla}_{X^t_{(x,u)}} V - g(\tilde{X}(u), \tilde{Z}(u))\hat{\nabla}_{Y^t_{(x,u)}} V \Big\}^t,$$

3. *The Ricci type tensor of $(T_0M, \tilde{G}, S(T(T_0M)))$ is characterized by*

$$\begin{aligned}R^{(0,2)}_{(x,u)}(X^h, Y^h) =& \frac{1}{2}[2Ric(X, Y) - g(R(X, V(u))Y, u) + g(R(X, u)Y, V(u))] \\ & + \frac{3}{4}[g(R(Y, u)u, R(X, V)u) + g(R(Y, V)u, R(X, u)u)] \\ & + \sum_{i=3}^{m} \frac{\varepsilon_i}{4}[3g(R(X, e_i)u, R(Y, e_i)u) - g(R(u, e_i)Y, R(u, e_i)X)]\end{aligned}$$

$$
\begin{aligned}
R^{(0,2)}_{(x,u)}(X^h, Y^t) =& \frac{1}{2}[g((\nabla_{V(u)}R)(Y,u)X,u) \\
&- g(Y,u)[g(\hat{\nabla}_{V^h_{(x,u)}}(R(V,\sigma)\bar{X}),u) - g(R(\hat{\nabla}_{V^h_{(x,u)}}V,u)X,u) \\
&- g(R(V,u)\nabla_{V(u)}\bar{X},u)] + g((\nabla_u R)(Y,u)X,V(u)) \\
&- g(Y,u)[g(\hat{\nabla}_{u^h}(R(V,\sigma)\bar{X}),V) - g(R(\hat{\nabla}_{u^h}V,u)X,V(u)) \\
&- g(R(V(u),u)\nabla_u\bar{X},V(u))]] + \sum_{i=3}^{m}\frac{\varepsilon_i}{2}\{g((\nabla_{e_i}R)(Y,u)X,e_i) \\
&- g(Y,u)[g(\hat{\nabla}_{e_i^h}(R(V,\sigma)\bar{X}),e_i) - g(R(\hat{\nabla}_{e_i^h}V,u)X,e_i) \\
&- g(R(V(u),u)\nabla_{e_i}\bar{X},e_i)]\}, \\
R^{(0,2)}_{(x,u)}(X^t, Y^h) =& \frac{1}{2}[g((\nabla_{V(u)}R)(X,u)Y,u) - g(X,u)[g(\hat{\nabla}_{V^h_{(x,u)}}(R(V,\sigma)\bar{Y}),u) \\
&- g(R(\hat{\nabla}_{V^h_{(x,u)}}V,u)Y,u) - g(R(V(u),u)\nabla_{V(u)}\bar{Y},u)] \\
&+ g((\nabla_u R)(X,u)Y,V(u)) - g(X,u)[g(\hat{\nabla}_{u^h}(R(V,\sigma)\bar{Y}),V(u)) \\
&- g(R(\hat{\nabla}_{u^h}V,u)Y,V(u)) - g(R(V(u),u)\nabla_u\bar{Y},V)]] \\
&+ \sum_{i=3}^{m}\frac{\varepsilon_i}{2}\{g((\nabla_{e_i}R)(X,u)Y,e_i) - g(X,u)[g(\hat{\nabla}_{e_i^h}(R(V,\sigma)\bar{Y}),e_i) \\
&- g(R(\hat{\nabla}_{e_i^h}V,u)Y,e_i) - g(R(V(u),u)\nabla_{e_i}\bar{Y},e_i)]\}, \\
R^{(0,2)}_{(x,u)}(X^t, Y^t) =& -\frac{1}{4}[g(R(\tilde{X}(u),u)u,R(\tilde{Y}(u),u)V(u)) + g(R(\tilde{Y}(u),u)u,R(\tilde{X}(u),u)V(u))] \\
&- \frac{1}{4}\sum_{i=3}^{m}\varepsilon_i g(R(\tilde{X}(u),u)e_i,R(\tilde{Y}(u),u)e_i),
\end{aligned}
$$

*for all* $(x,u) \in U$ *and* $X, Y \in T_xM$, *where* $\{e_i, i = 3, ..., m\}$ *is an orthonormal family of* $T_xM$ *such that* $g(e_i,u) = g(e_i,V(u)) = 0$, *and* $\bar{X}, \bar{Y} \in \mathfrak{X}(M)$ *are any arbitrary extensions of* $X, Y$, *respectively.*

4. *The extrinsic scalar curvature of* $(T_0M, \tilde{G}, S(T(T_0M)))$ *is given by*

$$
\begin{aligned}
\tilde{R}_{(x,u)} =& R_x + 2Ric(u,u) - \frac{3}{2}g(R(V(u),u)u,R(V(u),u)u) \\
&+ \sum_{i=3}^{m}\varepsilon_i\{[3g(R(u,e_i)u,R(V(u),e_i)u) - g(R(u,e_i)V(u),R(u,e_i)u)] \\
&+ \frac{1}{4}\sum_{j=3}^{m}\varepsilon_j[3g(R(e_j,e_i)u,R(e_j,e_i)u) - 2g(R(e_i,u)e_j,R(e_i,u)e_j)]\},
\end{aligned}
$$

*for all* $(x,u) \in U$, *where R is the scalar curvature of M.*

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