ON FLAG-TRANSITIVE 2-(k^2, k, λ) DESIGNS WITH $\lambda \mid k$.

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Abstract. It is shown that, apart from the smallest Ree group, a flag-transitive automorphism group G of a 2-(k^2, k, λ) design $\mathcal{D}$, with $\lambda \mid k$, is either an affine group or an almost simple classical group. Moreover, when G is the smallest Ree group, $\mathcal{D}$ is isomorphic either to the 2-$(6^2, 6, 2)$ design or to one of the three 2-$(6^2, 6, 6)$ designs constructed in this paper. All the four 2-designs have the 36 secants of a nondegenerate conic $\mathcal{C}$ of $PG_2(8)$ as a point set and 6-sets of secants in a remarkable configuration as a block set.

1. Introduction and Main Result

A 2-(v, k, λ) *design* $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. We say $\mathcal{D}$ is *nontrivial* if $2 < k < v$. All 2-(v, k, λ) designs in this paper are assumed to be nontrivial. An automorphism of $\mathcal{D}$ is a permutation of the point set which preserves the block set. The set of all automorphisms of $\mathcal{D}$ with the composition of permutations forms a group, denoted by $\mathrm{Aut}(\mathcal{D})$. For a subgroup G of $\mathrm{Aut}(\mathcal{D})$, G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$, and said to be *point-imprimitive* otherwise. A *flag* of D is a pair (x, B) where x is a point and B is a block containing x. If $G \leqslant \mathrm{Aut}(\mathcal{D})$ acts transitively on the set of flags of $\mathcal{D}$, then we say that G is *flag-transitive* and that $\mathcal{D}$ is a *flag-transitive design*.

The 2-(v, k, λ) designs $\mathcal{D}$ admitting a flag-transitive automorphism group G have been widely studied by several authors. In 1990, a classification of those with $\lambda = 1$ and $G \nleq A\Gamma L_1(q)$ was announced by Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl in [13] and proven in [12], [15], [16], [17], [24], [29] and [35]. Since then a special attention was given to the case $\lambda > 1$. A classification of the flag-transitive 2-designs with $\gcd(r, \lambda) = 1$, $\lambda > 1$ and $G \nleq A\Gamma L_1(q)$, where r is the replication number of $\mathcal{D}$, has been announced by Alavi, Biliotti, Daneshkakh, Montinaro, Zhou and their collaborators in [2] and proven in [3], [4], [5], [8], [10], [11], [30], [37], [39], [40], [41], [42], [44], [45] and [46]. Moreover, recently the flag-transitive 2-designs with $\lambda = 2$ have been investigated by Devillers, Liang, Praeger and Xia in [19], where it is shown that apart from the two known symmetric 2-$(16, 6, 2)$ designs, G is primitive of affine or almost simple type. Moreover, a classification is provided when the socle of G is isomorphic to $PSL_n(q) \trianglelefteq G$ and $n \geqslant 3$.

The present paper represents a further contribution to the study of the flag-transitive 2-designs. More precisely, the flag-transitive 2-(k^2, k, λ) designs with $\lambda \mid k$ are investigated. The reason of studying such 2-designs is that they represent a natural generalization of the affine planes in terms of parameters, and also because,

Date: March 18, 2022.
2020 *Mathematics Subject Classification.* 05B05; 05B25; 20B25.
Key words and phrases. 2-design; automorphism group; flag-transitive.
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it is shown in [33] that, the blocks of imprimitivity of a family of flag-transitive, point-imprimitive symmetric 2-designs investigated in [34] have the structure of the 2-designs analyzed here. The following result is obtained:

Theorem 1.1. *Let $\mathcal{D}$ be a 2-(k^2, k, λ), with $\lambda \mid k$, admitting a flag-transitive automorphism group G. Then G is point primitive and one of the following holds:*

(1) G is an affine group.
(2) G is an almost simple classical group.
(3) $\mathcal{D}$ is isomorphic to the 2-$(36, 6, 2)$ design constructed in Example 3.9 and ${}^2G_2(3)' \trianglelefteq G \leqslant {}^2G_2(3)$.
(4) $\mathcal{D}$ is isomorphic to one of the three 2-$(36, 6, 6)$ designs constructed in Example 3.9 and $G \cong {}^2G_3(3)$.

Actually, (3) and (4) are special cases of (2), since ${}^2G_3(3) \cong P\Gamma L_2(8)$. It worth noting that the example in (3) is not contained in [19] and hence it is presumably new. A complete classification of (1) for $G \nleq A\Gamma L_1(q)$, and of (2) are contained in [31] and [32] respectively.

2. Preliminary Reductions

We first collect some useful results on flag-transitive designs.

Lemma 2.1. *Let $\mathcal{D}$ be a 2-(k^2, k, λ) design and let b be the number of blocks of $\mathcal{D}$. Then the number of blocks containing each point of $\mathcal{D}$ is a constant r satisfying the following:*

(1) $r = \lambda(k + 1)$;
(2) $b = \lambda k(k + 1)$;
(3) $(r/\lambda)^2 > k^2$.

Lemma 2.2. *If $\mathcal{D}$ is a 2-(k^2, k, λ) design, with $\lambda \mid k$, admitting a flag-transitive automorphism group G, then the following hold:*

(1) G acts point-primitively on $\mathcal{D}$.
(2) If x is any point of $\mathcal{D}$, then G_x is a large subgroup of G.
(3) $\left|y^{G_x}\right| = (k + 1)\left|B \cap y^{G_x}\right|$ for any point y of $\mathcal{D}$, with $y \neq x$, and for any block B of $\mathcal{D}$ incident with x. In particular, $k + 1$ divides the length of each point-G_x-orbit on $\mathcal{D}$ distinct from $\{x\}$.

Proof. The assertion (1) follows from [18], 2.3.7.c, since $r = (k + 1)\lambda > (k - 3)\lambda$.

The flag-transitivity of G on $\mathcal{D}$ implies $|G| = k^2 |G_x|$, $|G_x| = \lambda (k + 1) |G_{x,B}|$ and hence $|G| < |G_x|^3$, which is the assertion (2).

Let y be any point of $\mathcal{D}$, $y \neq x$, and B be any block of $\mathcal{D}$ incident with x. Since (y^{G_x}, B^{G_x}) is a tactical configuration by [18], 1.2.6, it follows that $\left|y^{G_x}\right| \lambda = r\left|B \cap y^{G_x}\right|$. Hence $\left|y^{G_x}\right| = (k + 1)\left|B \cap y^{G_x}\right|$ as $r = (k + 1)\lambda$. This proves (3). $\square$

The group G is point-primitive on $\mathcal{D}$ by Lemma 2.2(1). The O'Nan-Scott Theorem classifies primitive groups into five types: (i) Affine type; (ii) Almost simple type; (iii) Simple diagonal type; (iv) Product type; (v) Twisted wreath product type (see [27] for details). Hence, the first part of the paper is devoted to prove that only families (i) and (ii) occur. The result is achieved by adapting the techniques developed in [43] to the 2-designs investigated here.

Lemma 2.3. *G is not of simple diagonal type.*

Proof. The proof is essentially that of [43], Propositions 3.1, but we use $r = (k+1)\lambda$ instead of $\lambda \geqslant (r,\lambda)^2$.

Assume that G is of diagonal type. Then

$$G \leqslant W = \{(a_1, ..., a_m)\pi \mid a_i \in Aut(T), \pi \in S_m, a_i \equiv a_j \mod Inn(T) \text{ for all } i, j\}$$

and there is $x \in \mathcal{P}$ such that

$$G_x \leqslant \{(a, ..., a)\pi \mid a \in Aut(T), \pi \in S_m\} \cong Aut(T) \times S_m$$

and $M_x = D = \{(a, ..., a) \mid a \in Inn(T)\}$ is a diagonal subgroup of $M \cong T^m$. Put $\Sigma = \{T_1, ..., T_m\}$, where T_i is identified with $\{(1, ..., t, ..., 1)\pi \mid t \in T\}$ in the i-th position. Then G acts on Σ by [27]. Moreover, the set $\mathcal{P}$ can be identified with the set M/D of the cosets of D in M so that $x = D(1, ..., 1)$, $k^2 = |T|^{m-1}$, since $v = k^2$, and for $y = D(t_1, ..., t_m)$, $\psi = (s_1, ..., s_m) \in M$, $\sigma \in Aut(T)$, $\pi \in S_m$, we have the actions

$$y^\psi = D(t_1 s_1, ..., t_m s_m),\ y^\sigma = D(t_1^\sigma, ..., t_m^\sigma) \text{ and } y^\pi = D(t_{1\pi^{-1}}, ..., t_{m\pi^{-1}}).$$

Since $M \trianglelefteq G$ and G is primitive on $\mathcal{P}$, M is transitive on $\mathcal{P}$. Since $T_1 \trianglelefteq M$, all T_1-orbits on $\mathcal{P}$ have the same length $c > 1$. Let Γ_1 be the T_1-orbit containing x. For any $t_1 = (t, 1..., 1) \in T_1$, we have $x^{t_1} = D(t, 1..., 1)$. So that

$$\Gamma_1 = x^{T_1} = \{D(t, 1..., 1) : t \in T\}$$

and $|\Gamma_1| = \left|x^{T_1}\right| = c$. Similarly, we define $|\Gamma_i| = \left|x^{T_i}\right|$ for $1 \leqslant i \leqslant m$. Clearly, $\Gamma_i \cap \Gamma_j = \{x\}$ for $i \neq j$ provided that $m \geqslant 2$.

Chose a point-G_x-orbit Δ in $\mathcal{P} - \{x\}$ such that $|\Delta \cap \Gamma_1| = d \neq 0$. Let $m_1 = [G_x : N_{G_x}(T_1)]$. Since G_x is isomorphic to a subgroup of $Aut(T) \times S_m$, and G^Σ acts transitively on Σ, it follows that $m_1 \leqslant m$ and hence

$$|\Delta| \leqslant m_1 d \leqslant m\,|T|\,.$$

Then $k+1 \leqslant |\Delta| \leqslant m\,|T|$ by Lemma 2.2(3). Since $v = k^2 = |T|^{m-1}$, we have $|T|^{(m-1)/2} < m\,|T|$ and hence $60^{m-3} \leqslant |T|^{m-3} < m^2$. Therefore, $m \leqslant 3$.

Since $r \mid |G_x|$ and G_x is isomorphic to a subgroup of $Aut(T) \times S_m$, it follows that $(k+1)\lambda \mid |T|\,|Out(T)|\,m!$. On the other hand, $k+1 \mid |T|^{m-1} - 1$, as r/λ divides $k^2 - 1$. Thus $k+1 \mid |Out(T)|\,m!$ and hence $|T|^{m-1} = k^2 < |Out(T)|^2 (m!)^2$ with $m \leqslant 3$. At this point the final part of the proof of [43], Propositions 3.1, can be applied to show that no cases occur. □

Lemma 2.4. *G is not of twisted wreath product type*

Proof. We may apply the same argument of [43], Propositions 3.2, to show that there is a point-G_x-orbit Δ in $\mathcal{P} - \{x\}$ such $|\Delta| \leqslant m_1 d \leqslant m\,|T|$ (this is shown in [43], Propositions 3.2, without using the assumption $\lambda \geqslant (r,\lambda)^2$). Then $k+1 \leqslant m\,|T|$ by Lemma 2.2(3). On the other hand, $k+1 > |T|^{m/2}$, since $k^2 = v = |T|^m$. Then $|T|^{m/2} < m\,|T|$ and hence $60^{m-2} \leqslant m$ and $m \leqslant 2$, whereas $m \geqslant 6$ by [27]. □

Theorem 2.5. *G is either of affine type or of almost simple type.*

Proof. The group G is neither of simple diagonal type nor of twisted wreath product type by Lemmas 2.3 and 2.4 respectively. Thus, in order to complete the proof, we need to rule out the case where G has a product action on $\mathcal{P}$. Suppose the contrary. Then there is a group K with a primitive action (of almost simple or diagonal type) on a set Γ of size $v_0 \geqslant 5$, such that $\mathcal{P} = \Gamma^m$ and $G \leqslant K^m \rtimes S_m$, where $m \geqslant 2$. Let $x = (\gamma, ..., \gamma)$ and $y = (\delta, ..., \gamma)$ with $\delta \neq \gamma$ and set $W = K^m$ and $H = W \rtimes S_m$. Then $W_x \cong K_\gamma^m$, $W_{x,y} \cong K_{\gamma,\delta} \times K_\gamma^{m-1}$, $H_x = W_x \rtimes S_m$ and $K_{\gamma,\delta} \times \left(K_\gamma^{m-1} \rtimes S_{m-1}\right) \leqslant H_{x,y}$. Suppose that K has rank s on Γ, $s \geqslant 2$. Then we may choose δ such that $[K_\gamma : K_{\gamma,\delta}] \leqslant \frac{v_0-1}{s-1}$. Hence,

$$\left|x^H\right| = \frac{|K_\gamma|^m \cdot m!}{|K_{\gamma,\delta}|\,|K_\gamma|^{m-1} \cdot (m-1)!} = [K_\gamma : K_{\gamma,\delta}]\, m \leqslant \frac{v_0 - 1}{s-1} m.$$

and, as $x^G \subseteq x^H$, we get

$$v_0^{m/2} = v^{1/2} < k + 1 \leqslant \left|x^G\right| \leqslant \left|x^H\right| \leqslant m\frac{v_0 - 1}{s-1} < mv_0.$$

Then $m = 2, 3$ and $v_0 < 9$, as $m \geqslant 2$. If $m = 3$, then $k^2 = v_0^3$ and hence $v_0 = 4$ and $s = 3$, whereas $v_0 \geqslant 5$. Thus $m = s = 2$. It follows that, K acts 2-transitively on Γ, and $H = K^2 \rtimes S_2$ has rank 3 with H_x-orbits 1, $2(k-1)$ and $(k-1)^2$. Since each H_x-orbit is union G_x-orbit, and since each G_x-orbit on $\mathcal{P} - \{x\}$ has length divisible by $k+1$ by Lemma 2.2(3), we obtain $k + 1 \mid 2(k-1)$ and hence $k = v_0 = 3$. So, we again reach a contradiction as $v_0 \geqslant 5$. □

3. Proof of Theorem 1.1

In this section G is an almost simple group. Hence, $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a non abelian simple group. Moreover X, the socle of G, is either sporadic, or alternating, or an exceptional group of Lie type, or classical. We analyze the first three cases separately. The sporadic one is ruled out simply by filtering the groups listed in [14] with respect to the constraints for X to have a transitive permutation representation of degree k^2, and when this occurs the corresponding stabilizer of a point in X to have the order divisible by $\frac{k+1}{\gcd(k+1, |\mathrm{Out}(X)|)}$ (see Lemma 3.1). The alternating case is settled as follows. We show that X_x, the stabilizer in X of a point x of $\mathcal{D}$, is a large maximal subgroup of X. Hence X_x is listed in Theorem 2 of [9]. Then we combine some group theoretical arguments, in particular those developed in [17], together with some numerical properties of the binomial coefficients to exclude the case. Finally, when G is an exceptional group of Lie type, the reduction to ${}^2G_2(3)$ in its permutation representation of degree 36 is settled by transferring the arguments developed in [3] and in [8] to our context. The key point of the analysis of the 2-designs admitting ${}^2G_2(3)$ as a flag transitive automorphism group is to see that ${}^2G_2(3)$ acts on $PG_2(8)$ preserving a nondegenerate conic $\mathcal{C}$, since ${}^2G_2(3) \cong P\Gamma L_2(8)$. Hence, its permutation representation of degree 36 is equivalent to that on the set of secants to $\mathcal{C}$. Some geometry of $PG_2(8)$ is then used to complete the proof of the case.

Lemma 3.1. *Let $\mathcal{D}$ be a 2-(k^2, k, λ) design, with $\lambda \mid k$, admitting a flag-transitive automorphism group G. If x is any point of $\mathcal{D}$, then $\frac{k+1}{\gcd(k+1, |\mathrm{Out}(X)|)}$ divides $|X_x|$.*

Proof. Let x be any point of $\mathcal{D}$. If y is a point of $\mathcal{D}$, with $y \neq x$, then $\left|y^{X_x}\right| = \frac{\left|B \cap y^{G_x}\right|(k+1)}{\mu}$, where $\mu \left|y^{X_x}\right| = \left|y^{G_x}\right|$, by Lemma 2.2(3), as $X_x \trianglelefteq G_x$. On the other hand, μ divides $|\mathrm{Out}(X)|$, as $\mu = \frac{[G_x:X_x]}{[G_{x,y}:X_{x,y}]}$. Therefore $\frac{k+1}{\gcd(k+1,|\mathrm{Out}(X)|)}$ divides $\left|y^{X_x}\right|$ and hence $|X_x|$. $\square$

Lemma 3.2. *X is not sporadic.*

Proof. Assume that X is sporadic. Then X is listed in [14].

Assume that $X \cong M_i$, where $i = 11, 12, 22, 23$ or 24. Since $[X : X_x] = k^2$, it follows from [26], Table 5.1.C, that $\lambda^2 = 2^{a_1}3^{a_2}$ for some $a_1, a_2 \geqslant 2$. Then $k = 12$ and either $X \cong M_{11}$ and $X_x \cong F_{55}$, or $X \cong M_{12}$ and $X_x \cong PSL_2(11)$ by [14]. However, these cases are ruled out by Lemma 3.1, since $\frac{k+1}{\gcd(k+1,|\mathrm{Out}(X)|)} = 13$ does not divide $|X_x|$.

Assume that $X \cong J_i$, where $i = 1, 2, 3$ or 4. Then k^2 divides 2^2, $2^6 3^2 5^2$, $2^6 3^4$, or $2^{20}3^2 11^2$, respectively, by [26], Table 5.1.C. Then $i = 2$ and either $k = 10$ and $X_x \cong PSU_3(3)$, or $k = 60$ and $X_x \cong PSL_2(7)$ by [14]. However, these cases are ruled out as they contradict Lemma 3.1.

Assume that X is isomorphic to one of the groups HS or McL. By [26], Table 5.1.C, k^2 divides $2^8 3^2 5^2$ or $2^6 3^6 5^2$ respectively. Then either $X \cong HS$, $X_x \cong M_{22}$ and $k = 10$, or $X \cong McL$, $X_x \cong M_{22}$ and $k = 45$. The latter is ruled out by Lemma 3.1, since $\frac{k+1}{\gcd(k+1,|\mathrm{Out}(X)|)} = 23$ does not divide $|X_x|$. The former yields $r = 11\lambda$, where $\lambda = 1, 2, 5$ or 10 as $\lambda \mid k$. If B is any block incident with x, then $[X_x : X_{x,B}]$ divides r. Then $PSL_3(4) \trianglelefteq G_{x,B} \leqslant P\Sigma L_3(4)$, and hence $\lambda = 2$, by [14]. Thus, $|G_B| = 10\,|G_{x,B}|$, since G_B is transitive on B, and hence $b = 44$ or 88. However, $HS \trianglelefteq G \leqslant HS.Z_2$ has no such transitive representation degrees by [14].

It is straightforward to check that the remaining cases are ruled out similarly, as they do not have transitive permutation representations of degree k^2 by [14] and [36]. $\square$

Lemma 3.3. *If $X \cong A_n$, then $n \neq 6$ and $G = X$. Moreover, one of the following holds:*

(1) $X_x = (S_t \times S_{n-t}) \cap A_n$ where $1 \leqslant t < n/2$.
(2) $X_x = (S_t \wr S_h) \cap A_n$ where $n = th$ and $2 \leqslant t \leqslant n/2$.

Proof. Assume that $X \cong A_n$. If $n = 6$, then $k^2 = 3^2$ or 6^2, and the former is ruled out by [14], whereas the latter yields $X_x \cong D_{10}$. However this case cannot occur by Lemma 3.1, since $\frac{k+1}{\gcd(k+1,|\mathrm{Out}(X)|)} = 7$ does not divide $|X_x|$. Thus $n \neq 6$, and hence $|\mathrm{Out}(X)| = 2$ by [26], Theorem 5.1.3.

Let $\mu = [G_x : X_x]$. Since $G = G_x X$, it follows that $G_x/X_x \cong G/X \leqslant \mathrm{Out}(X)$ and hence $\mu \leqslant 2$. Assume that $\mu = 2$. Let M be a maximal subgroup of X containing X_x. Then x^M is a block of imprimitivity for X and hence $\left|x^M\right| \mid k^2$. Since $x^M - \{x\}$ is union of X_x-orbit, and each X_x-orbit distinct from $\{x\}$ is of length divisible by $\frac{k+1}{\gcd(k+1,2)}$ Lemma 3.1, it follows that $\frac{k+1}{\gcd(k+1,2)} \mid \left|x^M\right| - 1$. Then $\left|x^M\right| = c\frac{k+1}{\gcd(k+1,2)} + 1$, for some $c \geqslant 1$, and hence $k^2 = d\left(c\frac{k+1}{\gcd(k+1,2)} + 1\right)$ for some $d \geqslant 1$. Thus

$$dc\frac{k+1}{\gcd(k+1,2)} + d - 1 = k^2 - 1 \tag{3.1}$$

and hence

$$d = \theta\frac{k+1}{\gcd(k+1,2)} + 1 \tag{3.2}$$

for some $\theta \geqslant 1$. Now, substituting (3.2) in (3.1) we obtain

$$\theta c\,(k+1)+\theta\gcd(k+1,2)<(k-1)\,(\gcd(k+1,2))^2$$

and hence $\theta c<(\gcd(k+1,2))^2$. Therefore, k is odd and $(\theta, c)=(1,1),(1,2),(1,3)$ or $(3,1)$, which, substituted in (3.1) and (3.2), yield $(k,d,c)=(15,9,3)$ or $(3,3,1)$. Then $n\leqslant[A_n:M]=9,3$, respectively. Actually, $n=9$ by [14], as $n\geqslant 5$ and $n\neq 6$. However, A_9 has no transitive permutation representations of degree 15^2. Thus $\mu=1$, $G=X$, and hence X_x is a large, maximal subgroup of X by Lemma 2.2(2). The last part of Lemma's statement follows from [9], Theorem 2, since $[X:X_x]=k^2$. □

Lemma 3.4. *Case (2) of Lemma 3.3 cannot occur.*

Proof. Assume that $X_x=(S_t\wr S_h)\cap A_n$, where $n=th$ and $2\leqslant t\leqslant n/2$. Then

$$k^2=[X:X_x]=\frac{th!}{(t!)^h\,h!}. \tag{3.3}$$

By [17], there is a G_x-orbit (namely a 2-cycle) of length either $h(h-1)$ or $t^2\frac{h(h-1)}{2}$ according to whether $t=2$ or $t>2$ respectively. Then $k+1$ divides the length of such a orbit by Lemma 2.2(3), as $G=X$ by Lemma 3.3. Thus, in both cases we have $\frac{th!}{(t!)^h h!}<(th)^2$. The inequality

$$\frac{h^{ht}}{(ht)^h\cdot h}=\frac{e\left(\frac{th}{e}\right)^{th}}{e^h t^h\left(\frac{t}{e}\right)^{th}\cdot eh\left(\frac{h}{e}\right)^h}\leqslant\frac{th!}{(t!)^h\,h!}<(th)^2 \tag{3.4}$$

is determined by using the known bound $e\left(\frac{f}{e}\right)^f\leqslant f!\leqslant ef\left(\frac{f}{e}\right)^n$ for $f\in\mathbb{N}$, where e is the Napier's constant. Thus $h^{ht}<(ht)^{h+2}\cdot h<(ht)^{h+3}$.

Assume that $h^t\geqslant(ht)^2$, then $(ht)^{2h}\leqslant h^{ht}<(ht)^{h+3}$ and hence $h=2$, as $2\leqslant t\leqslant n/2$. Then (3.4) becomes $2^t\leqslant\binom{2t}{t}=\frac{2t!}{(t!)^2}<8t^2$ and so $t\leqslant 9$. However, (3.3) is not fulfilled for $h=2$ and any of these values of t.

Assume that $h^t<(ht)^2$. Then $2^{t-2}\leqslant h^{t-2}<t^2$ and hence either $t=2$, or $3\leqslant t\leqslant 8$ and $h\leqslant 9$. Actually, the pairs (h,t) in latter case do not fulfill (3.3). Hence $t=2$. Since $h!\geqslant 2^h$, being $h\geqslant 2$, and (3.4) yields $2^h\leqslant\binom{2h}{h}\leqslant\frac{2h!}{2^h h!}<(2h)^2$ and hence $h\leqslant 8$. However, (3.3) is not fulfilled for $t=2$ and any of these values of h. □

Lemma 3.5. *X is not isomorphic to A_n.*

Proof. In order to prove the assertion we need to rule out case (1) of Lemma 3.3, since case (2) has been ruled out in Lemma 3.4. Hence, assume that $X_x=(S_t\times S_{n-t})\cap A_n$ where $1\leqslant t<n/2$. Then the action on the point set of $\mathcal{D}$ and on the t-subsets of $\{1,\dots,n\}$ are equivalent. Thus $k^2=\binom{n}{t}$. Then either $t\leqslant 2$, or $n=50$ and $t=3$ by [1], Chapter 3, since $1\leqslant t<n/2$.

Assume that $t=1$. Then $k^2=n\geqslant 9$, $X_x\cong A_{n-1}$ and hence X acts point-2-transitively on $\mathcal{D}$. Then $X_{x,B}$ is a subgroup of X_x of index $r=\lambda(\sqrt{n}+1)$, where $\lambda\mid\sqrt{n}$. Since X_x has no subgroups of index less than $n-1$, as $n\geqslant 9$, it follows that $\lambda=\sqrt{n}$. Then $r<\binom{n-1}{2}$ and hence $X_{x,B}$ is one of the subgroups of A_{n-1} listed in [20], Theorem 5.2.A, as $n\geqslant 9$. Assume that $X_{x,B}$ preserves an s-subset of

$\{1, ..., n-1\}$, where $s = 1, 2$. Then $\binom{n-1}{s} = \sqrt{n}(\sqrt{n}+1)$, a contradiction. Then n is odd and $r = [X_x : X_{x,B}] = \frac{1}{2}\binom{n-1}{n/2}$, since $G = X$ by Lemma 3.3. Therefore

$$2^{n/2-1} \leqslant \frac{1}{2}\binom{n-1}{n/2} = \sqrt{n}(\sqrt{n}+1) < 2n$$

and hence $n = 9$, $X_x \cong A_8$ and $X_{x,B} \cong AGL_3(2)$, since n is a square and n is odd. However, this case is ruled out since $r \neq 15$.

Assume that $t = 2$. Then G has rank 3 and, if x is any point of $\mathcal{D}$, the G_x-orbits, say $\mathcal{O}_i$, $i = 1, 2, 3$, have length $1, 2n-4$ and $\binom{n-2}{2}$, respectively (see [17]). Then $k+1$ divides the length of each of such orbits by Lemma 2.2(3), as $G = X$ by Lemma 3.3. Then $2\,(n-2) = c(k+1)$ for some $c \geqslant 1$ and hence

$$\left(\frac{2\,(n-2)}{c} - 1\right)^2 = \frac{n(n-1)}{2}. \tag{3.5}$$

Then (3.5) yields $c = 2$, $n = 9$, $k = 6$ and $X_x \cong S_7$. Note that, $\lambda > 1$ since there are no affine planes of order 6. Also $\lambda \neq 6$, since S_7 has no transitive permutation representations of degree 63 by [14]. Thus, either $\lambda = 2$, $X_{x,B} \cong A_6$ and $X_B \cong (A_6 \times Z_3) : Z_2$, or $\lambda = 3$, $X_{x,B} \cong Z_2 \times S_5$ and $X_B \cong S_4 \times S_5$ again by [14], since $\lambda \mid k$. Then the actions on the point-set and on the block-set of $\mathcal{D}$ are equivalent to the actions on the sets of 2-subsets and $(\lambda+1)$-subsets of $\mathbb{N}_9 = \{1, ..., 9\}$ respectively. Then we may identify the point-set and the block-set of $\mathcal{D}$ with these sets, respectively, in a way that the incidence relation is the set-theoretic inclusion, as $X_{x,B}$ is isomorphic either to A_6 or to $Z_2 \times S_5$ according to whether $\lambda = 2$ or 3 correspondingly. So, $k \leqslant \lambda + 1 \leqslant 4$ and we reach a contradiction as $k = 6$.

Assume that $t = 3$ and $n = 50$. Then $X_x \cong (S_3 \times S_{47}) \cap A_{50}$, $k = 140$ and $r = 141\lambda$, where $\lambda \mid 140$ and $b = 141 \cdot 140 \cdot \lambda$. Then $b \leqslant 141 \cdot 140^2 < \binom{50}{6}$ and $b \neq \frac{1}{2}\binom{50}{25}$ and hence $A_{50-\ell} \leqslant X_B \leqslant (S_\ell \times S_{50-\ell}) \cap A_{50}$, where $\ell < 6$, by [20], Theorem 5.2.A. Moreover $|X_B|$ is coprime to 47, as $b = 141 \cdot 140 \cdot \lambda$, and hence $\ell = 4, 5$. Thus $f\binom{50}{\ell} = 141 \cdot 140 \cdot \lambda$, where f is the index of X_B in $(S_\ell \times S_{50-\ell}) \cap A_{50}$. If $\ell = 5$, then 23 divides $\binom{50}{5}$ and hence λ, whereas $\lambda \mid 140$. Therefore, $\ell = 4$, $A_{46} \leqslant X_B \leqslant (D_8 \times S_{46}) \cap A_{50}$ and hence $f = 3\mu$ and $\lambda = 35\mu$ with $\mu \leqslant 4$. Then X_B preserves a 4-set Y of $\mathbb{N}_{50} = \{1, ..., 50\}$, whereas X_x preserves a 3-set Z of $\mathbb{N}_{50}$. Then $X_{x,B}$ preserves $Z \cup Y$. Set $w = |Z \cup Y|$, then $4 \leqslant w \leqslant 7$, and $\binom{50}{w}$ must divide $[X : X_{x,B}]$, which is equal to $\binom{50}{3} \cdot 141 \cdot 35 \cdot \mu$. Thus $w = 4$ and hence $Z \subseteq Y$. Then $A_{46} \leqslant X_{x,B}$, and so $k \mid 8$, as $A_{46} \leqslant X_B \leqslant (D_8 \times S_{46}) \cap A_{50}$. This is a contradiction, as $k = 6$. □

Lemma 3.6. *If X is isomorphic to socle a finite exceptional group of Lie type, then $\mathcal{D}$ is a 2-$(36, 6, \lambda)$ design, where $\lambda \mid 6$ and $\lambda > 1$, and X is isomorphic either to $G_2(2)'$ or to ${}^2G_2(3)'$.*

Proof. Recall that an exceptional group of Lie type is simple apart from ${}^2B_2(2)$, $G_2(2)$, ${}^2G_2(3)$, or ${}^2F_4(2)$ by [26], Theorem 5.1.1. Thus, either X is isomorphic to an exceptional simple group of Lie type, or X is isomorphic to one of the groups ${}^2B_2(2)$, $G_2(2)$, ${}^2G_2(3)$, or ${}^2F_4(2)$. If the latter occurs, since G has a primitive permutation representation of degree k^2, then the unique admissible cases to be analyzed are either $G_2(2)$ and $k = 6$, or ${}^2G_2(3)$ and $k = 3, 6$ by [14]. Suppose that $k = 3$. Then $\lambda = 1, 3$ as $\lambda \mid k$. If $\lambda = 1$, then $\mathcal{D} \cong AG_2(3)$ and hence $G \leqslant AGL_2(3)$, which is impossible. Then $\lambda = 3$, $r = 12$ and hence $G \cong P\Gamma L_2(8)$ and $G_x \cong F_{56} : Z_3$. So

$|G_{x,B}| = 14$ and $G_{x,B} \leqslant F_{56}$, a contradiction. Thus $k = 6$ and hence $\lambda = 1, 2, 3$ or 6 as $\lambda \mid k$. Also, $\lambda > 1$ since there are no affine planes of order 6. Therefore, $k = 6$, $\lambda \mid 6$ and $\lambda > 1$, and X is isomorphic either to $G_2(2)'$ or to ${}^2G_2(3)'$.

Assume that X is isomorphic to an exceptional simple group of Lie type. Suppose that G_x is not parabolic. Then G_x is one of the groups listed in [8], Theorem 1.6, or equivalently in Tables 2 and 3 of [3], since G_x is a large maximal subgroup of G by Lemma 2.2(2). In [3], Alavi points out that the first and the second column of Table 2 contains X and X_x, respectively, the third one contains a lower bound ℓ_v for $v = [X : X_x]$, and the fourth one contains an upper bound u_r for r, determined by using the fact that r is a common divisor of $v - 1$, of $|G_x|$ and of the subdegrees of G_x by his Lemma 4. Then, the author shows that $u_r^2 < \ell_v$ for each case contained in Table 2, hence $r^2 < v$, and so all the cases in Table 2 are ruled out in his paper.

Our aim is to transfer Alavi's argument in order to rule out the possibility for G_x to not be a parabolic subgroup of G. Clearly, the first three columns of Table 2 have the same meaning as in our paper, where $v = k^2$ for us. The role of r and of Lemma 4 of [3] are played by $r/\lambda = k + 1$ and by our Lemma 2.2(3) respectively. Thus the upper bound u_r for r in [3] becomes an upper bound for r/λ in our context. Therefore the inequality $u_r^2 < \ell_v$ implies $(r/\lambda)^2 < k^2$ but this is impossible in our context, since $r/\lambda = k+1$. Thus all the groups listed in Tables 2 of [3] cannot occur.

It is even easier to rule out the groups listed in Tables 3 of [3] as they are filtered with respect to the property that $v = k^2$. Indeed, we obtain the following admissible cases:

(1) $X \cong G_2(3)$, $X_x \cong 2^3 : PSL_3(2)$ and $k^2 = 3528 = 2^3 3^2 7^2$;
(2) $X \cong G_2(4)$, $X_x \cong PSL_2(13)$ and $k^2 = 230400 = 2^{10} 3^2 5^2$.

Then $k + 1$ is 43 or 481 respectively, but none of these divides the order of the corresponding G_x. So, these cases violate Lemma 2.2(3) and hence they are ruled out.

Assume that G_x is a maximal parabolic subgroup of G. Assume that $E_6(q)$ is not contained in G. Then G has a subdegree of order p^t (e.g. see [3], Lemma 3, or [35], Lemma 2.6). Then $\frac{r}{\lambda} \mid p^t$ and so $k + 1 = p^s$ for some $s \leqslant t$. Then $k = p^s - 1$ and hence $k^2 = (p^s - 1)^2$. Then $s \leqslant \zeta_p(G)$, where $\zeta_p(G)$ is defined in [26] (5.2.4), and is determined in Proposition 5.2.17.(i) and Table 5.2.C. If $s = \zeta_p(G)$, then $\left(p^{\zeta_p(G)} - 1\right)^2 \mid |X|$. On the other hand, $|X|$ is listed in [26], Table 5.1.B, and hence none of these groups admits $\left(p^{\zeta_p(G)} - 1\right)^2$ as a divisor. Then $s < \zeta_p(G)$. Then G_x contains a Sylow u-subgroup G, where u is a primitive prime divisor of $p^{\zeta_p(G)} - 1$, since $(p, \zeta_p(G)) \neq (2, 6)$ being $X \ncong G_2(2)'$. On the other hand, G_x can be obtained by deleting the i-th node in the Dynkin diagram of X, and we see that none of these groups is of order divisible by u. Indeed, for instance, if $F_4(q) \trianglelefteq G$, $q = p^f$, then $\zeta_p(G) = 12f$ and hence G_x contains a Sylow u-subgroup of G, where u is a primitive prime divisor of $p^{12f} - 1$, whereas the maximal parabolic subgroups are of type $B_3(p^f)$, $C_3(p^f)$ or $A_1(p^f) \times A_2(p^f)$ and none of these is divisible by u. As stressed out in [3], Remark 1, even in the case $E_6(q)$, when G contains a graph automorphism or G_x is parabolic of type $1, 2$ or 4, then G has a subdegree of order p^t, and hence these cases are excluded by the above argument. For the remaining maximal parabolic subgroups we may use the same argument as [3] at pp.1012–1013, with r/λ and Lemma 2.2(3) in the role of r and Lemma 4 of [3], respectively, to see that $(r/\lambda)^2 > v = k^2$ is violated. Hence $E_6(q) \trianglelefteq G$ cannot occur and the proof is thus completed. □

Lemma 3.7. *X is not isomorphic to $G_2(2)'$.*

Proof. Suppose that $\mathcal{D}$ is a 2-$(36, 6, \lambda)$ design, where $\lambda \mid 6$ and $\lambda > 1$, admitting a flag-transitive automorphism group $X \trianglelefteq G \leqslant Aut(X)$, where $X \cong G_2(2)'$. Therefore $PSL_2(7) \trianglelefteq G_x \leqslant PGL_2(7)$ by [14], and $\lambda = 2, 3$ or 6.

Assume that $\lambda = 2$. Then $|G_{x,B}| = 12$ and hence either $G = X$ and $G_{x,B} \cong A_4$, or $G \neq X$ and $G_{x,B} \cong A_4$, D_{12} by [14]. Then 72 divides $|G_B|$ and hence $G_B \leqslant (U : Z_8) .Z_2$, where U is a Sylow 3-subgroup of G. Then $U : Z_8 \leqslant G_B$ since 36 divides $|G_B \cap (U : Z_8)|$ and since $U/Z(U) : Z_8$ is a Frobenius group (for instance, see [23], Satz II.10.12, since $G_2(2)' \cong PSU_3(3)$). So 9 divides $|G_{x,B}|$, a contradiction.

Assume that $\lambda = 3$ or 6. Then $4 \mid |X_{x,B}|$ as $r = 21$ or 63. If $G \neq X$, then the set of points of $\mathcal{D}$ distinct from x is partitioned into two G_x-orbits, say $\mathcal{O}_1$, $\mathcal{O}_2$, of length 14 and 21 respectively by [38]. If $G = X$, then $\mathcal{O}_1$ is split into two X_x-orbits $\mathcal{O}_{1j}$, $j = 1, 2$, each of length 7, whereas $\mathcal{O}_2$ is also a X_x-orbit. Hence, $X_{x,y} \cong D_8$ for $y \in \mathcal{O}_2$.

Let B be any block incident with x. Then $|\mathcal{O}_i| = 7\,|\mathcal{O}_i \cap B|$, where $i = 1, 2$, by Lemma 2.2(3) and hence $|\mathcal{O}_1 \cap B| = 2$ and $|\mathcal{O}_2 \cap B| = 3$. Then $X_{x,B}$ fixes $\mathcal{O}_1 \cap B$ pointwise, since $|\mathcal{O}_{1j} \cap B| = 1$ for each $j = 1, 2$. Then there is a non trivial subgroup W of $X_{x,B}$ of index at most 2, such that $B \subseteq Fix(W)$, since $4 \mid |X_{x,B}|$.

If $\lambda = 3$, then $r = 21$, $b = 126$ and $X_{x,B} \cong D_8$. Hence, W is isomorphic to Z_4, E_4 or to D_8, and $N_{X_x}(W)$ is isomorphic to D_8, S_4 or to D_8 respectively. The number of points in $\mathcal{O}_2$ fixed by W is given by the well known formula $|N_{X_x}(W)|\,\left|X_{x,y} \cap W^X\right| / |X_{x,y}|$. Thus, it is easy to see that, W fixes 6 or 1 points in $\mathcal{O}_2$ according to whether W is or is not isomorphic to E_4 respectively, since $X_{x,y} \cong D_8$. Therefore, $W \cong E_4$, since $B \subseteq Fix(W)$ and $|\mathcal{O}_2 \cap B| = 3$. Then W lies in the kernel N of the action of G_B on B. Also $Fix(W) \neq Fix(X_{x,B})$ as $X_{x,B} \cong D_8$ fixes exactly one point on $\mathcal{O}_2$, and hence $N \cap X_B = W$. Therefore $W \trianglelefteq G_B$ and hence $G_B \leqslant N_G(W)$, where $N_G(W)$ is isomorphic either to $(Z_4)^2 .S_3$ or to $(Z_4)^2 .D_{12}$ according as $G = X$ or $G \neq X$, respectively, by [14]. Note that, $[N_G(W) : G_B] = 2$, since $[G : N_G(W)] = 63$ and $b = 126$. Thus, $G_B \lhd N_G(W)$ and 3 divides $|N_{X_x}(W) \cap G_B|$ as $N_{X_x}(W) \cong S_4$. Hence, $G_{x,B}$ contains a Sylow 3-subgroup of $N_G(W)$, since these are cyclic of order 3, but this contradicts $[G_B : G_{x,B}] = 6$.

If $\lambda = 6$, then $r = 42$, $b = 252$ and hence either $X_{x,B} \cong Z_4$ or $X_{x,B} \cong E_4$. Therefore, W is isomorphic to Z_2, E_4 or to Z_4. Actually, $W \ncong Z_4$ since Z_4 fixes one point on $\mathcal{O}_2 \cap B$, whereas $B \subseteq Fix(W)$ and $|\mathcal{O}_2 \cap B| = 3$.

Assume that $W \cong E_4$. Then $W = X_{x,B}$ and hence $W = X_B \cap N$, where N is defined above. Therefore $W \trianglelefteq G_B$ and hence $G_B \leqslant N_G(W)$. Moreover, $[N_G(W) : G_B] = 4$, since $[G : N_G(W)] = 63$ and $b = 252$. So, $G \cong G_2(2)$, $N_G(W) \cong (Z_4)^2 .D_{12}$ and $G_B \cong (Z_4)^2 .Z_3$ by [14]. Then $G_B \lhd N_G(W)$, 3 divides $|N_{X_x}(W) \cap G_B|$ and we reach a contradiction as above.

Assume $W \cong Z_2$, $|Fix(W) \cap \mathcal{O}_2| = 5$ and $B \subset Fix(W)$. As above, W lies in the kernel N of the action of G_B on B. Then $Fix(W) \neq Fix(X_{x,B})$. Indeed, it is true for $X_{x,B} \cong Z_4$, as any Z_4 fixes exactly one point on $\mathcal{O}_2$, and it is still true for $X_{x,B} \cong E_4$ as it follows from the above arguments where it is shown that a subgroup isomorphic to E_4 cannot fix B pointwise. Consequently, we have that $N \cap X_B = W$. Thus $G_B \leqslant C_G(W)$, where $C_G(W)$ is isomorphic either to $GU_2(3)$

or to $\Gamma U_2(3)$ according as $G = X$ or $G \neq X$, respectively, by [14]. Moreover, $[C_G(W) : G_B] = 4$, as $b = 252$, and hence $|X_B| = 24$. Therefore, $X_B \cong Z.S_3$ or $X_B \cong SL_2(3)$ since $C_X(W) \cong (Z \circ SL_2(3)).Z_2$, where Z is the center of $GU_2(3)$. Assume the former occurs. Since $[X_B : X_{x,B}]$ divides 6, it follows that $Z \leqslant X_{x,B}$. Then Z fixes B pointwise, as Z is normal in G_B, and hence $Z \leqslant N \cap X_B = W$. This is impossible, since $W \cong Z_2$ whereas $Z \cong Z_4$. Thus $X_B \cong SL_2(3)$. Therefore, either $G \cong G_2(2)'$ and hence $G_B \cong SL_2(3)$, or $G \cong G_2(2)$ and $G_B \cong GL_2(3)$, $Z_4 \circ SL_2(3)$, as $[C_G(W) : G_B] = 4$. However, it is easily seen with the aid of [47] that, no 2-designs occur. □

Before analyzing the remaining case involving $G_3(3)$. Recall some useful facts about the action of this group in the desarguesian plane of order 8.

Let $G \cong G_3(3)$. It is well known that $G \cong P\Gamma L_2(8)$ acts on $PG_2(8)$ preserving a regular hyperoval, namely a 10-arc, consisting of a nondegenerate conic $\mathcal{C}$ and it nucleus N (see [21], Section 8). Moreover, G acts primitively on the set $\mathcal{S}$ of the 36 secants to $\mathcal{C}$ by [14]. If ℓ is any secant to $\mathcal{C}$ and $\{P_1, P_2\} = \ell \cap \mathcal{C}$, then $G_\ell \cong F_{42}$ and the set $\mathcal{S}(\ell)$ consisting of the 14 secant lines to $\mathcal{C}$ incident with P_1 or P_2 is a G_ℓ-orbit, as G acts 3-transitively on $\mathcal{C}$. The complementary set $\mathcal{S} - \mathcal{S}(\ell)$ consisting of 21 secants intersecting ℓ in a point different from P_1, P_2 is also a G_ℓ-orbit.

By [14], G contains two conjugacy classes of subgroups of order 3, one contained in G' the other in $G - G'$. If $\langle \eta \rangle$ and $\langle \gamma \rangle$ are the representatives of such classes, then $C_G(\eta) \cong S_3$ and $C_G(\gamma) \cong Z_3 \times S_3$. Moreover, $C_G(\eta) = C_{G'}(\eta)$ and $C_{G'}(\gamma) \cong S_3$. We may choose η and γ to belong to the same Sylow 3-subgroup of G in a way that $C_G(\gamma) = \langle \gamma \rangle \times C_G(\eta)$. Note that, $C_G(\eta) = \langle \eta, \sigma \rangle$, where σ is an involutory elation of $PG_2(8)$ with center C not in $\mathcal{C}$ and axis a tangent to $\mathcal{C}$ (a detailed description of the collineations of the desarguesian plane can be found in [21] and in [22]).

Set $K = \langle \gamma, \sigma \rangle$. Then $K \cong Z_6$ is a self-normalizing subgroup of G by [14]. Moreover, K fixes exactly one point F on $\mathcal{C}$, and $\langle \gamma \rangle$ fixes two further points switched by σ, say W and W^σ. Then $C = W^\sigma W \cap F_0 N$, $a = F_0 N$. The set $\{F, W, W^\sigma\}$ is a $C_G(\gamma)$-orbit on $\mathcal{C}$. The set $\mathcal{C} - \{F, W, W^\sigma\}$ is split into two $\langle \gamma \rangle$-orbits, $\left\{P, P^\gamma, P^{\gamma^2}\right\}$ and $\left\{P^\sigma, P^{\sigma\gamma}, P^{\sigma\gamma^2}\right\}$, and these are also $\langle \gamma, \eta \rangle$-orbits. Moreover, $\left\{P, P^\gamma, P^{\gamma^2}, P^\sigma, P^{\sigma\gamma}, P^{\sigma\gamma^2}\right\}$ is both a K-orbit and a $C_G(\gamma)$-orbit on $\mathcal{C}$.

Lemma 3.8. *The following hold:*

(1) $(PP^\gamma)^{C_G(\gamma)}$ *is the unique* $C_G(\gamma)$*-orbit on* $\mathcal{S}$ *of length* 6.

(2) $(PP^\gamma)^{C_G(\eta)}$ *and* $\left(FP^{\gamma^i}\right)^{C_G(\eta)}$*, where* $i = 0, 1, 2$*, are the unique* $C_G(\eta)$*-orbits on* $\mathcal{S}$ *of length* 6.

(3) $(P^\gamma P)^K$, $(P^\gamma P^\sigma)^K$, $(PW)^K$, $(P^\sigma W)^K$*, and* $(PF)^K$ *are the unique* K*-orbits on* $\mathcal{S}$ *of length* 6.

In particular, $(PP^\gamma)^{C_G(\gamma)} = (PP^\gamma)^{C_G(\eta)} = (P^\gamma P)^K$.

Proof. Since $\left\{P, P^\gamma, P^{\gamma^2}\right\}$ and $\left\{P^\sigma, P^{\sigma\gamma}, P^{\sigma\gamma^2}\right\}$ are $\langle \gamma, \eta \rangle$-orbits, it follows that $\left|(PP^\gamma)^{C_G(\gamma)}\right| \mid 6$. On the other hand $\left|(PP^\gamma)^K\right| = 6$ and $(PP^\gamma)^K \subseteq (PP^\gamma)^{C_G(\gamma)}$. Thus $(PP^\gamma)^{C_G(\gamma)}$ is a $C_G(\gamma)$-orbit on $\mathcal{S}$ of length 6.

Since $\{F, W, W^{\sigma}\}$ is a $C_G(\gamma)$-orbit on $\mathcal{C}$. It follows that $\left|(FW)^{C_G(\gamma)}\right| = 3$. Moreover, $\left|(FP)^{K}\right| = 6$ and hence $\left|(FP)^{C_G(\gamma)}\right| = 18$, as $\left\{P, P^{\gamma}, P^{\gamma^2}, P^{\sigma}, P^{\sigma\gamma}, P^{\sigma\gamma^2}\right\}$ is both a K-orbit and a $C_G(\gamma)$-orbit on $\mathcal{C}$.

Since σ fixes PP^{σ}, it follows that $(PP^{\sigma})^{C_G(\gamma)}$ is of odd length. Assume that $\left|(PP^{\sigma})^{C_G(\gamma)}\right| = 3$. Then there is an element ϑ in $\langle \gamma, \eta \rangle$ preserving PP^{σ} and hence fixing both P and P^{σ}. Then ϑ fixes $\mathcal{C} - \{F, W, W^{\sigma}\}$ pointwise, since $\langle \gamma, \eta \rangle$ is an elementary abelian group of order 9 acting transitively both on $\left\{P, P^{\gamma}, P^{\gamma^2}\right\}$ and on $\left\{P^{\sigma}, P^{\sigma\gamma}, P^{\sigma\gamma^2}\right\}$. However, this is impossible. Hence $\left|(PP^{\sigma})^{C_G(\gamma)}\right| = 9$. As $(FW)^{C_G(\gamma)} \cup (FP)^{C_G(\gamma)} \cup (PP^{\sigma})^{C_G(\gamma)}$ covers $\mathcal{C} - (PP^{\gamma})^{C_G(\gamma)}$, the assertion (1) follows.

Since $C_G(\eta) \trianglelefteq C_G(\gamma)$, each $C_G(\eta)$-orbit on $\mathcal{S}$ of length 6 lies either in $(PP^{\gamma})^{C_G(\gamma)}$ or in $(FP)^{C_G(\gamma)}$. Since $C_G(\eta) = C_{G'}(\eta)$, and $\langle \eta \rangle$ acts semiregularly on $\mathcal{C}$, it results that $\langle \eta \rangle$ does not fix secants to $\mathcal{C}$. On the other hand, since $C_G(\gamma)_{PP^{\gamma}} \cong Z_3$, $C_G(\gamma)_{PP^{\gamma}} \cap G' = 1$ and $C_G(\gamma)_{FP} = 1$, it follows that $(PP^{\gamma})^{C_G(\eta)} = (PP^{\gamma})^{C_G(\gamma)}$ and that $(FP)^{C_G(\gamma)}$ is split into three $C_G(\eta)$-orbits each of length 6, namely, $\left(FP^{\gamma^i}\right)^{C_G(\eta)}$ where $i = 0, 1, 2$.

Finally, it is easy to check that $W^{\sigma}W$, WF, $P^{\sigma}P$, PF, PW, $P^{\sigma}W$, $P^{\gamma}P$ and $P^{\gamma}P^{\sigma}$ are representatives of all K-orbits on $\mathcal{S}$ and these have lengths 1, 2, 3, 6, 6, 6, 6, 6 respectively. Thus (3) holds. $\square$

Example 3.9. Let $\ell_1 = PP^{\gamma}$, $\ell_2 = P^{\gamma}P^{\sigma}$, $\ell_3 = PW$ and $\ell_4 = P^{\sigma}W$, and let $B_1 = \ell_1^{C_G(\gamma)}$ and $B_i = \ell_i^{K}$ for $i = 2, 3, 4$. Then the following hold:

(1) $\mathcal{D}_1 = (\mathcal{S}, B_1^{G})$ is a 2-$(36, 6, 2)$ design admitting G as the full flag-transitive automorphism group. Moreover, G' acts flag-transitively on $\mathcal{D}_1$.
(2) $\mathcal{D}_i = (\mathcal{S}, G_i^{G})$, where $i = 2, 3, 4$, is a 2-$(36, 6, 6)$ design admitting G as the unique flag-transitive automorphism group.
(3) $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$, are pairwise non isomorphic.

Proof. Let Z denote $C_G(\gamma)$ for $i = 1$ and K for $i > 1$. Clearly, $Z \leqslant G_{B_i}$. Assume that $Z \neq G_{B_i}$. Then $1 \neq G_{\ell_i, B_i} \leqslant F_{42}$ and G_{B_i} does not contain elements of order 7, as $|B_i| = 6$ and as the unique element of G fixing more than three points on $\mathcal{C}$ is the identity. Thus $G_{\ell_i, B_i} \leqslant Z_6$. Suppose that $|G_{\ell_i, B_i}|$ is even. Then either $36 \mid |G_{B_1}|$, or $12 \mid |G_{B_i}|$ for $i > 1$. In the former case we obtain $G = G_{B_1}$ by [14], but this is impossible. Hence $i > 1$ and either $G_{B_i} \cong A_4$ or G_{B_i} contains a Sylow 2-subgroup of G. Both are ruled out. Indeed, the former cannot occur since $Z_6 \cong K \leqslant G_{B_i}$ but A_4 does not contain such groups. In the latter case G_{B_i} contains an involution α fixing $\ell_i \cap B_i$ pointwise. However, this is impossible since the involutions are elations of $PG_2(8)$ and their unique fixed point is the tangency point of their axis, whereas ℓ_i is a secant to $\mathcal{C}$. Thus $G_{\ell_i, B_i} \cong Z_3$ and hence $|G_{B_i}| = 18$. Therefore, $G_{B_i} = C_G(\gamma)$. This is clear for $i = 1$, whereas, for $i > 1$ it follows from $Z_6 \cong K \leqslant G_{B_i}$ and $K \cap G_{\ell_i, B_i} = 1$. However, $G_{B_1} = C_G(\gamma)$ contradicts the assumption. Thus $i > 1$ and hence $B_i = B_1$ by Lemma 3.8(1), but we still obtain a contradiction since B_1 is also a K-orbit by Lemma 3.8(3). Thus $G_{B_i} = Z$ for each $i = 1, 2, 3, 4$. Therefore, by [18], 1.2.6, $\mathcal{D}_i = (\mathcal{S}, B_i^{G})$ is a flag-transitive tactical configuration with parameters $(v, b, k, r) = (36, 84, 6, 14)$ or $(36, 252, 6, 42)$ according as $i = 1$ or $i = 2, 3, 4$ respectively.

In order to prove that $\mathcal{D}_i$ is a 2-design with $\lambda = 2$ for $i = 1$ and $\lambda = 6$ for $i > 1$, bearing in mind that G acts transitively on $\mathcal{S}$, and for any $\ell \in \mathcal{S}$ the group G_ℓ acts transitively both on $\mathcal{S}(\ell)$ and on $\mathcal{S} - \mathcal{S}(\ell)$, it is enough to prove that there are precisely λ elements of B^G containing ℓ_i and any $m_i \in B_i - \{\ell_i\}$, i fixed.

(i). $\mathcal{D}_1 = (\mathcal{S}, B_1^G)$ **is a** 2-$(36, 6, 2)$ **design admitting** G **as the full flag-transitive automorphism group of** $\mathcal{D}_1$**. Moreover,** G' **acts flag-transitively on** $\mathcal{D}_1$**.**

Let $m_1 \in B_1$. Assume that $\ell_1 \cap m_1 \in \mathcal{C}$. Then $m_1 \in \mathcal{S}(\ell_1)$ and hence $\left|m_1^{G_{\ell_1}}\right| = 14$. Clearly $\left|B_1^{G_{\ell_1}}\right| = 14$ as $G_{\ell_1,B_1} \cong Z_3$. Moreover, $\left|B_1 \cap m_1^{G_{\ell_1}}\right| = 2$ and hence $\left(m_1^{G_{\ell_1}}, B_1^{G_{\ell_1}}\right)$ is a tactical configuration with parameters $(v_1, b_1, k_1, r_1) = (14, 14, 2, 2)$ by [18], 1.2.6. Hence, the number of secants in $B_1^{G_{\ell_1}}$ containing both ℓ_1 and m_1 is 2. Then the number of secants in B_1^G containing both ℓ_1 and m_1 is 2, as $\mathcal{D}_1$ is a flag-transitive tactical configuration.

If $\ell_1 \cap m_1 \notin \mathcal{C}$. Then $m_1 \in \mathcal{S} - \mathcal{S}(\ell_1)$ and hence $\left|m_1^{G_{\ell_1}}\right| = 21$. Moreover, $\left|B_1^{G_{\ell_1}}\right| = 14$ and $\left|B_1 \cap m_1^{G_{\ell_1}}\right| = 3$. Indeed, $B_1 \cap m_1^{G_{\ell_1}} = m_1^{\langle \gamma \rangle}$ and hence $\left(m_1^{G_{\ell_1}}, B_1^{G_{\ell_1}}\right)$ is a tactical configuration with parameters $(v_1, b_1, k_1, r_1) = (21, 14, 3, 2)$. Hence, the number of elements in B^G containing both ℓ_1 and m_1 is 2, as $\mathcal{D}_1$ is a flag-transitive tactical configuration. Thus, there are precisely 2 elements of B^G containing both ℓ_1 and m_1 regardless $\ell_1 \cap m_1$ lies or does not lie in $\mathcal{C}$. Therefore, $\mathcal{D}_1 = (\mathcal{S}, B_1^G)$ is a 2-$(36, 6, 2)$ design admitting G as a flag-transitive automorphism group.

Note that, $\mathrm{Aut}(\mathcal{D}_1) = G$ as a consequence of the O'Nan-Scott Theorem (e.g. see [20], Theorem 4.1A), since $v = 2^2 \cdot 3^2$, $G = \mathrm{Aut}(G)$ and $G \leqslant \mathrm{Aut}(\mathcal{D}_1)$. Thus G is the full flag-transitive automorphism group of $\mathcal{D}_1$.

Since $r = 14$, $(G')_\ell \cong D_{14}$, and $(G')_{\ell,B_1} \leqslant G_{\ell,B_1} \cong Z_3$, it follows that $(G')_{\ell,B_1} = 1$. Therefore, $\left[(G')_\ell : (G')_{\ell,B_1}\right] = 14$ and hence $G' \cong PSL_2(8)$ acts flag-transitively on $\mathcal{D}_1$.

(ii). $\mathcal{D}_2 = (\mathcal{S}, B_2^G)$ **is a** 2-$(36, 6, 6)$ **design admitting** G **as the unique flag-transitive automorphism group.**

Let $m_2 \in B_2$. Then $\left(m_2^{G_{\ell_2}}, B_2^{G_{\ell_2}}\right)$ is a tactical configuration with parameters (v_2, b_2, k_2, r_2) equal either to $(14, 42, 2, 6)$ or to $(21, 42, 3, 6)$ according as $\ell_2 \cap m_2$ lies or does not lie in $\mathcal{C}$ respectively. Therefore, $\mathcal{D}_2$ is a 2-$(36, 6, 6)$ design admitting G as a flag-transitive automorphism group.

Arguing as in (i), we see that G is the full flag-transitive automorphism group of $\mathcal{D}_2$. Let H be the minimal flag-transitive automorphism group of $\mathcal{D}_2$. Then $H \leqslant G$ and hence $H = G$ by [14], since $2^3 \cdot 3^3 \cdot 7 \mid [G : G_{\ell_2,B_2}]$. Thus G is the unique flag-transitive automorphism group of $\mathcal{D}_2$.

(iii). $\mathcal{D}_i = (\mathcal{S}, B_i^G)$**,** $i = 3, 4$**, is a** 2-$(36, 6, 6)$ **design admitting** G **as the unique flag-transitive automorphism group.**

Let $m_3 \in B_3$ and suppose that $\ell_3 \cap m_3 \in \mathcal{C}$. Since G is 3-transitive on $\mathcal{C}$, we may assume that $\ell_3 \cap m_3 = \{P\}$. Then G_{ℓ_3,m_3} fixes the vertices of the triangle inscribed in $\mathcal{C}$ having ℓ_3, m_3 as two of its three sides. Hence $G_{\ell_3,m_3} \cong Z_3$ since G is 3-transitive on $\mathcal{C}$. Thus, $\left|m_3^{G_{\ell_3}}\right| = 14$. Since G_{B_1} acts regularly on B_1 it follows that $\left|B^{G_{\ell_3}}\right| = 42$. Finally, B contains exactly 3 lines of $m_3^{G_{\ell_3}}$ including ℓ_3. Indeed, G_{ℓ_3} contains a cyclic group of order 7 acting regularly on $\mathcal{C} - \ell_3$. Thus $\left|B \cap m_3^{G_{\ell_3}}\right| = 2$ and hence $\left(m_3^{G_{\ell_3}}, B^{G_{\ell_3}}\right)$ is a tactical configuration with parameters $(v_3, b_3, k_3, r_3) = (14, 42, 2, 6)$. Thus the number of blocks containing both ℓ_3 and m_3 is 6 as $\mathcal{D}_3$ is a flag-transitive tactical configuration.

Suppose that $\ell_3 \cap m_3 \notin \mathcal{C}$. We may assume that $W \in \ell_3$ and $W^\sigma \in m_3$. Then G_{ℓ_3, m_3} is generated by the unique elation of $PG_2(8)$ lying in G and with center $\ell_3 \cap m_3$. Thus $G_{\ell_3, m_3} \cong Z_2$ and hence $\left|m_3^{G_{\ell_3}}\right| = 21$. As above $\left|B^{G_{\ell_3}}\right| = 42$. Also $\left|B \cap m_3^{G_{\ell_3}}\right| = 3$. Indeed $G_{\ell_3, W^\sigma} \cong Z_6$ acts regularly on $\mathcal{C} - (\ell_3 \cup \{W^\sigma\})$. Therefore $\left(m_3^{G_{\ell_3}}, B^{G_{\ell_3}}\right)$ is a tactical configuration with parameters $(v_3', b_3', k_3', r_3') = (21, 42, 3, 6)$. Thus the number of blocks containing both ℓ_3 and m_3 is 6 as $\mathcal{D}_3$ is a flag-transitive tactical configuration. Therefore, $\mathcal{D}_3 = (\mathcal{S}, B_2^G)$ is a 2-(36, 6, 6) design admitting G as a flag-transitive automorphism group. Arguing as in (i) and (ii), we see that G is the unique flag-transitive automorphism group of $\mathcal{D}_3$. The statement (iii) for $\mathcal{D}_4$ is proven similarly.

(iv). **$\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$, are pairwise non isomorphic.**

Since $G_{B_i} = K$ is self-normalizing in G, where $i = 1, 2, 3$, it follows that B_i is the unique block of $\mathcal{D}_i$ preserved by G_{B_i}. Clearly, $\mathcal{D}_2 \ncong \mathcal{D}_3$ and $\mathcal{D}_2 \ncong \mathcal{D}_4$, since none of the 6 secants lying in B_2 contains a point fixed by $\langle \gamma \rangle$, whereas B_3 and B_4 do.

Suppose that Φ is an isomorphism from $\mathcal{D}_3$ onto $\mathcal{D}_4$. Since G acts point-transitively on $\mathcal{D}_i$, $i = 3, 4$, we may assume that Φ fixes F. Also $G^\Phi = G$, since G is the full flag-transitive automorphism group of $\mathcal{D}_i$. Then $[\Phi, G] = 1$, since $\mathrm{Aut}(G) = G$, and hence $\Phi = 1$ as Φ fixes F. Then $\mathcal{D}_3 = \mathcal{D}_4$. Then there is $\delta \in G$ such that $B_1^\delta = B_2$. Hence $\delta \in N_G(G_{B_3})$, as $G_{B_3} = G_{B_4} = K$. Then $\delta \in G_{B_3}$, since K is self-normalizing in G, and hence $B_3 = B_4$, a contradiction. Thus $\mathcal{D}_2 \ncong \mathcal{D}_3$. □

Theorem 3.10. *The following hold:*

(1) If $\mathcal{D}$ is a 2-$(36^2, 6, \lambda)$ design, with $\lambda \mid 6$, admitting G as a flag-transitive automorphism group, then either $\lambda = 2$ and $\mathcal{D}$ is isomorphic to $\mathcal{D}_1$, or $\lambda = 6$ and $\mathcal{D}$ is isomorphic to one of the $\mathcal{D}_i$, where $i = 2$, 3 or 4.

(2) If $\mathcal{D}$ is a 2-$(36^2, 6, 2)$ design admitting G' as a flag-transitive automorphism group, then $\mathcal{D}$ is isomorphic to $\mathcal{D}_1$.

Proof. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a 2-$(36^2, 6, \lambda)$, with $\lambda \mid 6$, admitting G a flag-transitive automorphism group. Then G acts point-primitively on $\mathcal{D}$ by Lemma 2.2(1). On the other hand, G has a unique permutation representation of degree 36 by [14] and this one is equivalent to that on $\mathcal{S}$, the set of secants to the nondegenerate conic $\mathcal{C}$ of $PG_2(8)$ preserved by G. Hence, we may identify the point set of $\mathcal{D}$ with $\mathcal{S}$. Thus, any block B of $\mathcal{D}$ consists of 6 secants to $\mathcal{C}$.

Assume that $\lambda = 2$ and that G' acts flag-transitively on $\mathcal{D}$. Set $J = G'$. Then $|J_B| = 6$ and hence J_B is a J-conjugate of $C_G(\eta)$ (recall that $C_G(\eta) = C_J(\eta)$). Without loss, we may assume that $J_B = C_G(\eta)$. Then B is one of the following $C_G(\eta)$-orbits $(PP^\gamma)^{C_G(\eta)}$ and $\left(FP^{\gamma^i}\right)^{C_G(\eta)}$, where $i = 0, 1, 2$, by Lemma 3.8(3).

Assume that $B = (FP)^{C_G(\eta)}$. Set $\ell = FP$. Since $C_G(\eta)$ acts transitively on $\{F, W, W^\sigma\}$, it follows that $B = \left\{\ell, \ell^\sigma, \ell^\eta, \ell^{\sigma\eta}, \ell^{\eta^2}, \ell^{\sigma\eta^2}\right\}$, where $\ell \cap \ell^\sigma = \{F\}$, $\ell^\eta \cap \ell^{\sigma\eta} = \{W\}$ and $\ell^{\eta^2} \cap \ell^{\sigma\eta^2} = \{W^\sigma\}$. Moreover, as $C_G(\eta)$ acts regularly on $\mathcal{C} - \{F, W, W^\sigma\}$, any two secants lying in B do not intersect in $\mathcal{C} - \{F, W, W^\sigma\}$. Thus, through any point of $\mathcal{C} - \{F, W, W^\sigma\}$ there is exactly one secant to $\mathcal{C}$ lying in B and incident with the point.

Let $m \in B$, $m \neq \ell$. Assume that $\ell \cap m \in \mathcal{C}$. Then $m = \ell^\sigma \in \mathcal{S}(\ell)$ and hence $\left|m^{J_\ell}\right| = 12$ as $\mathcal{S}(\ell)$ is also a J_ℓ-orbit, being $J_\ell \cong D_{14}$. Clearly $\left|B^{J_\ell}\right| = 14$ as $J_{\ell,B} = 1$. If there is $e \in B \cap m^{J_\ell}$, with $e \neq m$, then $e \in \mathcal{S}(\ell)$ and hence $e \cap \ell \in \mathcal{C}$. Then $e \cap \ell \in \mathcal{C} - \{F, W, W^\sigma\}$, as $e \neq m = \ell^\sigma$, and we obtain a contradiction, since through any point of $\mathcal{C} - \{F, W, W^\sigma\}$ there is exactly one secant to $\mathcal{C}$ lying in B and incident with the point. Thus $\left|B \cap m^{J_\ell}\right| = 1$ and hence $\left(m^{J_\ell}, B^{J_\ell}\right)$ is a tactical configuration with parameters $(v', b', k', r') = (14, 14, 1, 1)$ by [18], 1.2.6. Hence the number of elements in B^{J_ℓ} containing both ℓ_1 and m_1 is 1. Then the number of B^J containing both ℓ_1 and m_1 is 1, as $\mathcal{D}$ is flag-transitive by our assumption. However, that is impossible as it contradicts the assumption $\lambda = 2$. The cases $B = \left(FP^{\gamma^i}\right)^{C_G(\eta)}$, with $i = 1, 2$, are similarly ruled out. Then $B = (PP^\gamma)^{C_G(\eta)}$ and hence $B = (P^\gamma P)^{C_G(\gamma)}$ by Lemma 3.8. Thus, $\mathcal{D} \cong \mathcal{D}_1$ by Example 3.9(1).

Assume that $\lambda = 2$ and that G acts flag-transitively on $\mathcal{D}$. Let $\ell \in B$, then $G_{\ell,B} \cong Z_3$ and hence $|G_B| = 18$. Moreover, $G_{\ell,B} \cap G' = 1$ and $G_B \cap G' \cong S_3$, since $G' \cong PSL_2(8)$, and hence $G_B \cong Z_3 \times S_3$. Since the centralizer in G of any subgroup of order 3 of G' is of order 27 by [14], it follows that $G_B = C_G(\rho)$ for some element ρ of order 3 lying in $G - G'$. We may assume that $G_B = C_G(\gamma)$, since the subgroups of order 3 of G intersecting G' in 1 lies in one conjugacy class under G again by [14]. Thus, $B = B_1$ by Lemma 3.8(1) and so $\mathcal{D} \cong \mathcal{D}_1$ by Example 3.9(1).

Assume that $\lambda = 3$. Then $G_{\ell,B} \cong Z_2$ and $|G_B| = 12$, as $k = 6$, and hence $G_B \cong A_4$ by [14]. Let $\alpha, \beta, \delta \in G_B$ such that $\langle \alpha, \beta \rangle \cong E_4$ and $o(\delta) = 3$. Since G_B preserves $\mathcal{C}$, the group $\langle \alpha, \beta \rangle$ consists of elations of $PG_2(8)$ having the same axis a tangent to $\mathcal{C}$ and distinct centers C_α, C_β and $C_{\alpha\beta}$. Furthermore, $\langle \delta \rangle$ fixes a and permutes transitively $\{C_\alpha, C_\beta, C_{\alpha\beta}\}$. Then the block B consists of two secants incident with C_α, two ones incident with C_β and two ones incident with $C_{\alpha\beta}$. We may assume that $C_\alpha \in \ell$. Let $m \in B - \{\ell\}$ such that $C_\alpha \in m$. Then $\left(m^{G_\ell}, B^{G_\ell}\right)$ is a tactical configuration with parameters $(v'', b'', k'', r'') = (21, 21, 1, 1)$ by [18], 1.2.6. Hence the number of elements in B^{G_ℓ} containing both ℓ and m is 1. Then the number of B^G containing both ℓ and m is 1, as $\mathcal{D}$ is flag-transitive. However, that is impossible as it contradicts the assumption $\lambda = 3$.

Assume that $\lambda = 6$. Then $|G_B| = 6$. If $G_B \leqslant G'$, then $G_B \cong S_3$ and hence is a G-conjugate of $C_G(\eta)$ by [14]. Without loss, we may assume that $G_B = C_G(\eta)$. Then B is one of the $C_G(\eta)$-orbits $(PP^\gamma)^{C_G(\eta)}$ and $\left(FP^{\gamma^t}\right)^{C_G(\eta)}$, where $t = 0, 1, 2$, by Lemma 3.8(2). If $B = (PP^\gamma)^{C_G(\eta)}$, then $B = (P^\gamma P)^{C_G(\gamma)}$ again by Lemma 3.8, and hence $\mathcal{D} \cong \mathcal{D}_1$ by Example 3.9(1), whereas $\lambda = 6$. So, $B \neq (PP^\gamma)^{C_G(\eta)}$ and hence $B_t = \left(FP^{\gamma^t}\right)^{C_G(\eta)}$ for some $t = 0, 1, 2$.

Let $t = 0$ and let $m = \ell^\sigma$. Then $m \cap \ell \in \mathcal{C}$. A similar argument to that used for the case $\lambda = 2$ shows that $\left(m^{G_\ell}, B^{G_\ell}\right)$ is a tactical configuration with parameters $(v''', b''', k''', r''') = (14, 14, 1, 1)$ by [18], 1.2.6, since through any point of

$\mathcal{C} - \{F, W, W^\sigma\}$ there is exactly one secant to $\mathcal{C}$ lying in B and incident with the point. Hence the number of elements in B^{G_ℓ} containing both ℓ and m is 1. Then the number of B^G containing both ℓ and m is 1, as $\mathcal{D}$ is flag-transitive. So $\lambda = 1$, but this contradicts our assumption. The cases $t = 1, 2$ are excluded similarly.

Assume that $G_B \not\leq G'$. Then $G_B \cong Z_6$ and hence is a G-conjugate of K. Thus, without loss, we may assume that $G_B = K$. Then B is one of the following K-orbits on $\mathcal{S}$: $(P^\gamma P)^K$, $(P^\gamma P^\sigma)^K$, $(PW)^K$, $(P^\sigma W)^K$, and $(PF)^K$ by Lemma 3.8(3). Then $B = (P^\gamma P)^K = (P^\gamma P)^{C_G(\gamma)}$ implies $\mathcal{D} \cong \mathcal{D}_1$ by Example 3.9(1), and we again reach a contradiction as $\lambda = 6$. Thus, $B \neq (P^\gamma P)^K$. Also $B \neq (PF)^K$, otherwise all the secants lying in any block of $\mathcal{D}$ intersects in a point, whereas for any two secants s and s' to $\mathcal{C}$ such that $s \cap s' \notin \mathcal{C}$, then there are no blocks of $\mathcal{D}$ incident with them. Thus, B is either $(P^\gamma P^\sigma)^K$ or $(PW)^K$, or $(P^\sigma W)^K$, and we obtain $\mathcal{D} \cong \mathcal{D}_i$, where $i = 2$, 3 or 4, respectively, by Example 3.9(2). □

Proof of Theorem 1.1. By Theorem 2.5, $Soc(G)$, the socle of G, is either an elementary abelian p-group for some prime p or a non abelian simple group.

Assume that the latter occurs. Then X is neither sporadic nor an alternating group by Lemmas 3.2 and 3.5 respectively. If X is classical, then assertion (1) is immediate, but also (2b)–(2c) follow by Theorem 3.10, since $PSL_2(8) \cong {}^2G_2(3)'$. Finally, if X is isomorphic to the socle a finite exceptional group of Lie type, then 2-$(36, 6, \lambda)$ design, where $\lambda = 2, 3$ or 6, and $X \cong {}^2G_2(3)'$ by Lemmas 3.6 and 3.7. Then the assertions (2b)–(2c) follow again from Theorem 3.10. This completes the proof. □

References

[1] M. Aigner, G. M. Ziegler, Proofs from THE BOOK, (5th ed.). Berlin, New York: Springer-Verlag. 2014 6

[2] S. H. Alavi , M. Bayat, M. Biliotti, A. Daneshkhah, E. Francot, H. Guan, A. Montinaro, F. Mouseli, P. Rizzo, D. Tian, Y. Wang, X. Zhan, Y. Zhang, S. Zhou, Y. Zhu, Block designs with flag-transitive automorphism groups. submitted 1

[3] S. H. Alavi, Flag-Transitive Block Designs and Finite Simple Exceptional Groups of Lie Type, Graphs and Combinatorics **36** (2020) 1001–1014. 1, 4, 8

[4] S. H. Alavi, A note on two families of 2-designs arose from Suzuki-Tits ovoid, Algebra and Discrete Math. Accepted. 1

[5] S. H. Alavi, M. Bayat, A. Daneshkhah, Symmetric designs admitting flag-transitive and point-primitive automorphism groups associated to two dimensional projective special groups, Designs, Codes and Cryptography **79** (2016) 337–351. 1

[6] S. H. Alavi, M. Bayat, A. Daneshkhah, Flag-transitive block designs and unitary groups, Monatshefte für Mathematik **193** (2020) 535–553.

[7] S. H. Alavi, M. Bayat, A. Daneshkhah, Correction to: Flag-transitive block designs and unitary groups, Monatshefte für Mathematik **195** (2021) 371–376.

[8] S. H. Alavi, M. Bayat, and A. Daneshkhah, Finite exceptional groups of Lie type and symmetric designs. Submitted arxiv:1702.01257. 1, 4, 8

[9] S. H. Alavi, T. C. Burness, Large subgroups of simple groups, J. Algebra **421** (2015) 187–233. 4, 6

[10] M. Biliotti, A. Montinaro, On flag-transitive symmetric designs of affine type. J. Combin. Des. **25** (2017) 85–97. 1

[11] M. Biliotti, A. Montinaro, P. Rizzo, 2-(v, k, λ) designs, with $(r, \lambda) = 1$, admitting a solvable flag-transitive automorphism group of affine type. J. Combin. Des **27** (2019) 784–800. 1

[12] F. Buekenhout, A. Delandtsheer, J. Doyen, Finite linear spaces with flag-transitive sporadic groups, Unpublished note. 1

[13] F. Buekenhout, A. Delantsheer, J. Doyen, P. B. Kleidman, M. Liebeck, and J. Saxl, Linear Spaces with flag-transitive automorphism groups, Geom. Dedicata **36** (1990) 89-94. 1
[14] J. H. Conway, R. T. Curtis, R. A. Parker, R. A. Wilson, An Atlas of Finite Groups, Clarendon Press, Oxford, 1985. 4, 5, 6, 7, 9, 10, 11, 12, 13, 14
[15] D. Davies, Automorphisms of designs, Phd thesis, University of East Anglia (1987). 1
[16] A. Delandtsheer, Flag-transitive finite simple groups, Arch. Math. (Basel) **47** (1986) 395–400. 1
[17] A. Delandtsheer, Finite Flag-transitive Linear Spaces with Alternating Socle, pp. 79–88 in Algebraic combinatorics and Applications, Eds. A. Betten, A Kohnert, R. Laue, A. Wassermann, Proc.of theEuroconference ALCOMA, held in Göβwienstein, Germany, Springer, 1999. 1, 4, 6, 7
[18] P. Dembowski, Finite Geometries, Springer, Berlin, Heidelberg, New York, 1968. 2, 11, 12, 14
[19] A. Deviller, H. Liang, C. E. Praeger, B. Xia, OnGalg-transitive 2-$(v, k, 2)$ designs, J. Combin. Theory Ser. A **177** (2021) 105309. 1, 2
[20] J. D. Dixon, B. Mortimer, Permutation Groups. Springer Verlag, New York (1966). 6, 7, 12
[21] J. W. P. Hirshfeld, Projective Geometry Over Finite Fields. Second Edition. Clarendon Press, Oxford, 1998. 10
[22] D. R. Hughes, F.C. Piper, Projective Planes. Springer, New York, Berlin (1973). 10
[23] B. Huppert, Endliche gruppen I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin–New York, 1967. 9
[24] P. B. Kleidman, The finite flag-transitive linear spaces with an exceptional automorphism group, 461 in: Finite geometries and combinatorial designs (Lincoln, NE, 1987), vol. 111 of Contemp. Math., 462 Amer. Math. Soc., Providence, RI, 1990, pp. 117–136. 1
[25] W. M. Kantor, Classification of 2-transitive symmetric designs, Graphs Combin. **1** (1985) 165–450.
[26] P. Kleidman, M. Liebeck, The subgroup structure of the finite classical groups. London Mathematical Society Lecture Note Series, 129. Cambridge University Press, Cambridge, 1990. 5, 7, 8
[27] M. W. Liebeck, C. E. Praeger, J. Saxl, The O'Nan Scott theorem for finite primitve permutation groups, J. Austral. Math. Soc. **44** (1988) 389–396. 2, 3
[28] M. W. Liebeck, J. Saxl, and G. M. Seitz, On the overgroups of irreduciblensubgroups of the finite classical groups, Proc. London Math. Soc. **55** (1987) 507–537.
[29] M. W. Liebeck, The classification of finite linear spaces with flag-transitive automorphism groups of affine type, J. Combin. Theory Ser. A **84** (1998) 196–235. 1
[30] A. Montinaro, M. Biliotti, E. Francot, Classification of $2-(v, k, \lambda)$ designs with $(r, \lambda) = 1$ and $\lambda > 1$, admitting a non-solvable flag-transitive automorphism group of affine type, J. Algebr. Comb. (2021). https://doi.org/10.1007/s10801-021-01075-1. 1
[31] A. Montinaro, Classification of the 2-(k^2, k, λ) design, with $\lambda \mid k$, admitting a flag-transitive automorphism group G of affine type. Submitted. 2
[32] A. Montinaro, Classification of the 2-(k^2, k, λ) design, with $\lambda \mid k$, admitting a flag-transitive automorphism group G of almost simple type. Submitted. 2
[33] A. Montinaro, Flag-transitive, point-imprimitive symmetric 2-(v, k, λ) designs with $k > \lambda(\lambda - 3)/2$. Submitted. 2
[34] C. E. Praeger, S. Zhou, Imprimitive flag-transitive symmetric designs. J. Comb. Theory Ser. A, **113** (2006) 1381–1395. 2
[35] J. Saxl. On nite linear spaces with almost simple flag-transitive automorphism groups. J. Combin. Theory Ser. A, **100** (2002) 322–348. 1, 8
[36] R. A. Wilson, Maximal subgroups of sporadic groups, arXiv:1701.02095v2 5
[37] D. Tian, S. Zhou, Flag-transitive $2-(v, k, \lambda)$ symmetric designs with sporadic socle, Journal of Combinatorial Designs **23** (2015) 140–150. 1
[38] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, R. Abbott, An Atlas of Finite Groups Representation- Version 3, available at http://brauer.maths.qmul.ac.uk/Atlas/v3/ 9
[39] P. H. Zieschang, Flag transitive automorphism groups of 2-designs with $(r, \lambda) = 1$, Journal of Algebra **118** (1988) 369–375. 1

[40] X. Zhan, S. Ding, Comments on "Flag-transitive block designs and unitary groups", Monatshefte für Mathematik **195** (2021) 177–180. 1
[41] X. Zhan, S. Zhou, Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and sporadic socle, Des. Codes Cryptography **81** (2016) 481–487. 1
[42] X. Zhan, S. Zhou, Non-symmetric 2-designs admitting a two-dimensional projective linear group, Designs, Codes and Cryptography **86** (12) (2018) 2765–2773. 1
[43] S. Zhou, X. Zhan, Flag-transitive automorphism groups of 2-designs with $\lambda \geqslant (r, \lambda)^2$ and an application to symmetric designs, Ars Math. Contemp. **14** (2018) 187–195. 2, 3
[44] Y. Zhang, S. Zhou, Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and Exceptional Groups of Lie type, The Electronic Journal of Combinatorics **27** (2020) P2.9. 1
[45] S. Zhou, Y. Wang, Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and alternating socle, Electron. J. Combin. **22** (2015) Paper 2.6, 15. 1
[46] Y. Zhu, H. Guan, S. Zhou, Flag-transitive 2-(v, k, λ) symmetric designs with $(r, \lambda) = 1$ and alternating socle, Frontiers of Mathematics in China **10** (2015) 1483–1496. 1
[47] The GAP Group, GAP "Groups, Algorithms, and Programming, Version 4.11.0, 2020, available at http://www.gap.system.org

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