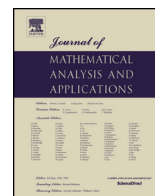




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Regular Articles

Dynamics of the translation semigroup on directed metric trees [☆]Elisabetta Mangino ^{a,*}, Álvaro Vargas-Moreno ^b^a *Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, Via per Arnesano, I-73100 Lecce, Italy*^b *Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, 46022, València, Spain*

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ABSTRACT

The dynamics of the left translation semigroup $\{T_t\}_{t \geq 0}$ on weighted L^p spaces over a directed metric tree $L(G)$ is investigated. Necessary and sufficient conditions on the weight family ρ for the strong continuity of the semigroup are provided. Furthermore, hypercyclicity and weak mixing properties are characterized in terms of the asymptotic decay of ρ along the tree structure. These results generalize classical L^p translation semigroup dynamics to a graph setting.

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0. Introduction

The study of dynamical systems on metric graphs has its roots in the analysis of the transport equation on so-called quantum graphs. In this context, the effects of the flow along the edges are modeled by differential equations and coupled through specific transmission conditions at the vertices. The pioneering works of Kramar and Sikolya [13] and Dorn et al. [4,6,5] initiated a systematic analysis of these problems. The monograph by Mugnolo [16] represents a fundamental reference in this field (see also [1, Chapter 18]). A first approach to the study of hypercyclicity in this setting can be found in [17].

In parallel, the research community turned to the dynamics of operators on graph structures. In particular, the work of Martínez-Avendaño [14] inaugurated the study of the dynamic properties of the backward shift operator on directed trees. More recent efforts by Grosse-Erdmann and Papathanasiou [8,9] have significantly

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advanced the understanding of hypercyclicity and chaos phenomena in this specific dynamical context. Further related results are due to Menet and Papathanasiou [15], Lopez-Martínez and Papathanasiou [11] and Kawamura [12].

Inspired by both the semigroup approach to transport and the rich dynamical analysis of shift operators on trees, this article extends the notion of left translation to the setting of weighted L^p spaces over directed metric trees and studies its dynamics. The first section introduces the structure of metric trees, summarizes the essential definitions of strongly continuous semigroups (C_0 -semigroups) and hypercyclicity, and establishes the theoretical framework for our work.

The central results of this article are twofold. The first, presented in Section 2, establishes the necessary and sufficient conditions on the weight family ρ that guarantee the translation operator generates a strongly continuous semigroup on the weighted spaces L^p_ρ . The second key result, developed in Section 3, provides necessary and sufficient conditions on the weight ρ that ensure the semigroup exhibits hypercyclicity on both rooted and unrooted directed metric trees.

1. Preliminaries

1.1. Strongly continuous semigroups

We recall some basic definitions and results, referring to the monograph [7] for further details.

Definition 1.1. Let X be a Banach space. A one-parameter family $\{T_t\}_{t \geq 0}$ of bounded operators on X is a strongly continuous semigroup (or a C_0 -semigroup) if the following conditions are satisfied:

- (i) $T_0 = I$
- (ii) $T_{t+s} = T_t \circ T_s$ for any $s, t \geq 0$
- (iii) $\lim_{s \rightarrow t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$.

As a consequence of the strong continuity of the semigroup, there is an exponential bound for the norm of the operators, namely there exist $M \geq 1$ and $w \in \mathbb{R}$ such that

$$\|T_t\| \leq M e^{wt}, \quad t \geq 0.$$

We can also express condition (iii) in some equivalent ways.

Proposition 1.2. [7, Ch. 1, Proposition 5.3] Let $\{T_t\}_{t \geq 0}$ be a family of operators on X satisfying (i) and (ii) of Definition 1.1. Then the following assertions are equivalent:

- (i) $\lim_{s \rightarrow t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$;
- (ii) $\lim_{t \rightarrow 0} T_t x = x$ for all $x \in X$.
- (iii) $\{T_t\}_{t \geq 0}$ is locally equicontinuous and there exists a dense subset $X_0 \subset X$ such that $\lim_{t \rightarrow 0} T_t x = x$ for all $x \in X_0$.

1.2. Linear dynamics

The main references for linear dynamics are the monographs [10, 2]. Throughout this section, let X be a separable Banach space. An operator T on X is said to be *hypercyclic* if there is some $x \in X$ whose orbit under T ,

$$\text{Orb}(x, T) := \{x, Tx, T^2x, \dots\},$$

is dense in X . The operator T is said to be *topologically transitive* if for every pair U, V of nonempty open sets there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Furthermore, if $T^k(U) \cap V \neq \emptyset$ for all $k \geq n$, then the operator is said to be *mixing*. For separable Banach spaces, the Birkhoff Transitivity Theorem [10, Theorem 2.19] states that hypercyclicity is equivalent to topological transitivity. Moreover, if the operator defined by

$$(T \oplus T)(x, y) = (Tx, Ty), \quad (x, y) \in X \times X$$

is hypercyclic, then T is called *weakly mixing*. Every weakly mixing operator is hypercyclic, but the converse fails. We refer to [10] for further insights about the equivalence of the weakly mixing property and the Hypercyclicity Criterion.

All these dynamical properties have a counterpart in the context of strongly continuous semigroups. A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on X is said to be *hypercyclic* if there exists some $x \in X$ whose orbit, defined as

$$\text{Orb}(x, \{T_t\}) := \{T_t x : t \geq 0\},$$

is dense in X . Furthermore, the semigroup $\{T_t\}_{t \geq 0}$ is said to be *topologically transitive* if for any pair of nonempty open sets U, V there is some $t_0 \geq 0$ such that $T_{t_0}(U) \cap V \neq \emptyset$, while, if $T_t(U) \cap V \neq \emptyset$ for any $t \geq t_0$ then the semigroup is said to be *mixing*.

Topological transitivity is equivalent to hypercyclicity also in the context of strongly continuous semigroups [10, Ch. 7].

The semigroup $\{T_t\}_{t \geq 0}$ is called *weakly mixing* if $(T_t \oplus T_t)_{t \geq 0}$ is *topologically transitive* in $X \times X$. Every weakly mixing semigroup is hypercyclic, but to the best of the authors' knowledge, whether the converse holds true remains an open problem. A discretization of a semigroup $\{T_t\}_{t \geq 0}$ is a sequence of operators $(T_{t_n})_n$ with $t_n \rightarrow \infty$. A discretization is said to be *autonomous* if there is some t_0 such that $t_n = nt_0$ for every $n \in \mathbb{N}$. The following theorem provides an equivalence between the weak mixing property and the existence of a mixing discretization.

Theorem 1.3. [10, Propositions 7.20, 7.25] *Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup on X . Then the following assertions are equivalent.*

- (i) $\{T_t\}_{t \geq 0}$ is weakly mixing;
- (ii) some discretization of $\{T_t\}_{t \geq 0}$ is mixing;
- (iii) some discretization of $\{T_t\}_{t \geq 0}$ is weakly mixing.
- (iv) some autonomous discretization of $\{T_t\}_{t \geq 0}$ is weakly mixing;
- (v) every autonomous discretization of $\{T_t\}_{t \geq 0}$ is weakly mixing.

Finally, we recall the following important result due to Conejero, Müller and Peris.

Theorem 1.4. [10, Theorem 6.8] *Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup on X . Then the following assertions are equivalent.*

- (i) $\{T_t\}_{t \geq 0}$ is hypercyclic;
- (ii) every operator T_t is hypercyclic for $t > 0$
- (iii) there exists $t_0 > 0$ such that the operator T_{t_0} is hypercyclic.

1.3. Directed metric trees

In this section, we establish the fundamental structure for our investigation, namely the *directed metric tree*. This construction starts with a directed tree, which is then endowed with a continuous metric structure.

We recall that a *directed tree* is a connected graph $G = (V, E)$ without cycles. That is:

- (i) V is a countable set of vertices.
- (ii) $E \subset (V \times V) \setminus \{(v, v) : v \in V\}$ is the set of directed edges (or arcs).
- (iii) For every $u, v \in V$, $u \neq v$ there exists a finite sequence of vertices $v_1, \dots, v_k \in V$ such that $(u, v_1), (v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v) \in E$.
- (iv) There are no closed sequences of vertices $v_1, \dots, v_k \in V$ such that $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v_1) \in E$.
- (v) For each vertex $v \in V$, there is at most one vertex $w \in V$ such that $(w, v) \in E$. In this case, w is the *parent* of v and v is a *child* of w .

A vertex without children is called a *leaf*.

We obtain the *directed metric tree* $L(G)$ by assigning a coordinate $x_e \in [0, 1)$ to every edge $e \in E$, increasing in the direction of the edge. Thus, we identify each edge e with the interval $[0, 1)$. For an edge $e = (v, w)$, v is the *tail* and w is the *head*.

A directed metric tree is *rooted* if there exists a vertex, called the *root*, that is not the head of any edge (i.e., no edge enters the root). If no such vertex exists, the tree is said to be *unrooted*.

The geometric structure of $L(G)$ can be described by the edge-to-edge incidence matrix $\mathcal{A}$. Indexing the set of edges as $E = \{e_i\}_{i \in I}$, the entries of $\mathcal{A}$ are defined by:

$$\mathcal{A}_{ij} = \begin{cases} 1 & \text{if the head of } e_i \text{ is the tail of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Since the tree structure dictates that each vertex has at most one parent, it holds that

$$\forall j \in I : \quad \text{card}(\{i \in I : \mathcal{A}_{ij} \neq 0\}) \leq 1.$$

That is, each column of $\mathcal{A}$ contains at most one non-zero term.

The powers of $\mathcal{A}$ track paths along the tree: $(\mathcal{A}^n)_{ij} \neq 0$ if and only if there exists a directed path of length n starting with edge e_i and ending with edge e_j . We denote the set of indexes of edges reachable from e_i in $n \in \mathbb{N}$ steps as:

$$M_n(i) = \{j \in I : (\mathcal{A}^n)_{ij} \neq 0\},$$

while we set $M_0(i) := \{i\}$.

Observe that, due to the geometry of the tree, for any $n \in \mathbb{N}$,

$$M_n(i) \cap M_n(j) = \emptyset \quad \text{if } i \neq j. \quad (1.1)$$

We now define the functional space on $L(G)$ on which the dynamics will be studied. A function f on $L(G)$ is identified with the family of its restrictions to the edges, $(f_i)_{i \in I}$, where $f_i = f|_{e_i} : [0, 1) \rightarrow \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

A weight on $L(G)$ is a family $\rho = (\rho_i)_{i \in I}$, where $\rho_i : [0, 1) \rightarrow \mathbb{R}$, $\rho_i \in L^1((0, 1))$ and is a.e. strictly positive.

For a given exponent $p \in [1, \infty)$, we define the weighted space on a single edge e_i as:

$$L_{\rho_i}^p[0, 1) := \{f : [0, 1) \rightarrow \mathbb{R} : f \text{ is measurable and } |f|^p \rho_i \in L^1((0, 1))\},$$

endowed with the norm $\|f\|_{p, \rho_i} = \left(\int_0^1 |f|^p \rho_i dx \right)^{\frac{1}{p}}$.

Naturally, the value at 0 of a function $f \in L_{\rho_i}^p[0, 1)$ carries no measure-theoretic information, but we prefer to include 0 in the interval where these spaces are defined to better highlight the geometric framework.

The space of functions on the entire metric tree, $L_\rho^p(L(G))$, is then defined as:

$$L_\rho^p(L(G)) := \left\{ (f_i)_{i \in I} : f_i \in L_{\rho_i}^p[0, 1) \text{ and } \sum_{i \in I} \|f_i\|_{p, \rho_i}^p < \infty \right\}.$$

This space is a Banach space when equipped with the norm:

$$\|f\|_{p, \rho} := \left(\sum_{i \in I} \|f_i\|_{p, \rho_i}^p \right)^{\frac{1}{p}}, \quad f = (f_i)_{i \in I} \in L_\rho^p(L(G)).$$

It will be useful to observe that condition (1.1) yields for any $f \in L_\rho^p(L(G))$ and for any $n \in \mathbb{N}$

$$\|f\|_{p, \rho}^p = \sum_{i \in I} \|f_i\|_{p, \rho_i}^p \geq \sum_{i \in I} \sum_{j \in M_n(i)} \|f_j\|_{p, \rho_j}^p. \quad (1.2)$$

A straightforward proof gives that

$$\begin{aligned} F_\rho^p &:= \oplus_{i \in I} L_{\rho_i}^p[0, 1) = \\ &= \{(f_i)_{i \in I} : f_i \in L_{\rho_i}^p[0, 1) \text{ and there exists a finite set } F \subseteq I \text{ s.t. } f_i = 0 \text{ if } i \in I \setminus F\} \end{aligned}$$

is dense in $L_\rho^p(L(G))$, and, consequently, $\oplus_{i \in I} C_c(0, 1)$ is dense in $L_\rho^p(L(G))$.

1.4. Left translation semigroups on directed metric trees

We now define the left translation semigroup $\{T_t\}_{t \geq 0}$ on $L_\rho^p(L(G))$. This family of operators describes the flow of a function's value along the directed paths of the tree. For any $s \in [0, 1)$ and $t \geq 0$, let $n(t, s)$ be the integer part of $s + t$, i.e., the unique $n \in \mathbb{N}_0$ such that $n \leq s + t < n + 1$. Keeping in mind that $s \in [0, 1)$, if $n_0 \leq t < n_0 + 1$, then

$$n(t, s) = \begin{cases} n_0 & \text{if } 0 \leq s < n_0 + 1 - t \\ n_0 + 1 & \text{if } n_0 + 1 - t \leq s < 1 \end{cases}. \quad (1.3)$$

For $f \in L_\rho^p(L(G))$ and $t \geq 0$ we define the left translation T_t as

$$(T_t f)(s) := \mathcal{A}^{n(t, s)} f(t + s - n(t, s)), \quad s \in [0, 1).$$

More in detail, for $f = (f_i)_{i \in I} \in L_\rho^p(L(G))$ and $t \geq 0$, the value of the translated function is given by:

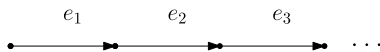
$$(T_t f)_i(s) = \sum_{j \in M_{n(t, s)}(i)} (\mathcal{A}^{n(t, s)})_{ij} \cdot f_j(t + s - n(t, s)), \quad i \in I, \quad s \in [0, 1).$$

The map defined in this way can intuitively be seen as a left translation with several contributions that are described by the matrix $\mathcal{A}$.

Note that if $t = n \in \mathbb{N}$, then $n(t, s) = n$ for every $s \in [0, 1)$ and therefore

$$(T_n f)_i(s) = \sum_{j \in M_n(i)} (\mathcal{A}^n)_{ij} \cdot f_j(s), \quad s \in [0, 1).$$

Remark 1.5. If the tree is a rooted sequence of connected edges of the following type:



it holds that

$$(\mathcal{A})_{ij}^n \neq 0 \Leftrightarrow j = i + n,$$

hence for every $(f_i)_{i \in \mathbb{N}} \in L^p(L(G))$ and every $i \in \mathbb{N}$

$$(T_t f)_i = f_{i+n(t,s)}(t + s - n(t, s)), \quad s \in [0, 1), \quad t \geq 0.$$

Clearly $L_\rho^p(L(G))$ is isometric to $L_\rho^p([0, +\infty))$, with

$$\tilde{\rho}(s) = \rho_i(s - i), \quad s \in [i, i + 1), \quad i \in \mathbb{N}, \quad (1.4)$$

via the isometry

$$\Phi : (f_i)_{i \in I} \mapsto f$$

where $f(u) = f_{[u]}(u - [u])$ for any $u \in [0, +\infty)$.

Under this isometry, the left translation semigroup on the directed metric tree turns into the classical left translation semigroup $(\tilde{T}_t)_{t \geq 0}$ on $L_\rho^p[0, +\infty)$. Indeed, for every $f \in L_\rho^p(L(G))$ and $u \in [0, +\infty)$

$$\begin{aligned} (\Phi(T_t((f_i)_{i \in \mathbb{N}})))(u) &= [T_t((f_i)_{i \in \mathbb{N}})]_{[u]}(u - [u]) \\ &= f_{[u]+n(t, u-[u])}(t + u - [u] - n(t, u - [u])) = \Phi((f_i)_{i \in \mathbb{N}})(t + u) \\ &= \tilde{T}_t(\Phi((f_i)_{i \in \mathbb{N}}))(u). \end{aligned}$$

Analogously, if the tree is an unrooted sequence of connected edges, then $L_\rho^p(L(G))$ is isometric to $L_\rho^p(\mathbb{R})$, where $\tilde{\rho}$ is defined on $\mathbb{R}$ as in (1.4), but with $i \in \mathbb{Z}$. In this case, the left translation semigroup on the directed tree turns into the left translation semigroup on $L_\rho^p(\mathbb{R})$.

2. Strong continuity for the translation semigroup on directed metric trees

In this section we provide a characterization of the weight ρ such that the translation semigroup $\{T_t\}_{t \geq 0}$ is strongly continuous on the weighted space $L_\rho^p(L(G))$.

We begin with the following preparatory result that can be proved by adapting the proof of [8, Lemma 4.3].

Lemma 2.1. (see [8, Lemma 4.3]) Let $E \subseteq \mathbb{R}^d$ be a Lebesgue measurable set, I a finite or countable set, $\rho = (\rho_i)_{i \in I}$ a family of positive measurable functions on E , and define the set

$$\mathcal{V} = \left\{ (v_i)_{i \in I} : v_i \text{ measurable on } E \text{ for every } i \in I, \sum_{i \in I} |v_i(s)| = 1 \text{ for all } s \in E \right\}.$$

Then, for every $s \in E$

$$\inf_{(v_i)_{i \in I} \in \mathcal{V}} \left(\sum_{i \in I} |v_i(s)|^p \rho_i(s) \right) = \left(\sum_{i \in I} \frac{1}{\rho_i(s)^{1/(p-1)}} \right)^{1-p}, \quad 1 < p < \infty.$$

Proposition 2.2. Let $\rho = (\rho_i)_{i \in I}$ be a weight on the directed metric graph $L(G)$ and let $M \geq 1$ and $w \in \mathbb{R}$.

(a) The following assertions are equivalent:

- (i) The family $\{T_t\}_{t \geq 0}$ is a strongly continuous semigroup on $L_\rho^1(L(G))$ such that $\|T_t\| \leq Me^{wt}$ for all $t \geq 0$;
- (ii) for all $i \in I$, $t \geq 0$, and a.e. $s \in [0, 1]$

$$\rho_i(s) \leq Me^{wt} \inf_{j \in M_{n(t,s)}(i)} \rho_j(s + t - n(t, s)).$$

(b) Let $1 < p < \infty$. Then the following assertions are equivalent:

- (i) The family $\{T_t\}_{t \geq 0}$ is a strongly continuous semigroup on $L_\rho^p(L(G))$ such that $\|T_t\| \leq Me^{wt}$ for all $t \geq 0$;
- (ii) for all $i \in I$, $t \geq 0$, and a.e. $s \in [0, 1]$

$$\left(\sum_{j \in M_{n(t,s)}(i)} \frac{1}{\rho_j(s + t - n(t, s))^{1/(p-1)}} \right)^{p-1} \leq M^p e^{pwt} \frac{1}{\rho_i(s)}.$$

Proof. (a) (ii) $\Rightarrow$ (i): We first show that for every $t > 0$ the map T_t is a continuous operator on $L_\rho^1(L(G))$ and $\|T_t\| \leq Me^{wt}$.

Fix $t_0 > 0$ and let $n_0 \in \mathbb{N} \cup \{0\}$ be such that $n_0 \leq t_0 < n_0 + 1$ and $f \in L_\rho^1(L(G))$. Taking into account (1.2) and (1.3), we compute the norm of $T_{t_0}f$ and apply condition (ii), so that:

$$\begin{aligned} \|T_{t_0}f\|_{1,\rho} &= \\ &= \sum_{i \in I} \int_0^{n_0+1-t_0} |(\mathcal{A}^{n_0}f)_i(s+t_0-n_0)| \rho_i(s) ds \\ &\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 |(\mathcal{A}^{n_0+1}f)_i(s+t_0-n_0-1)| \rho_i(s) ds \\ &= \sum_{i \in I} \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i)} f_j(s+t_0-n_0) \right| \rho_i(s) ds \\ &\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 \left| \sum_{j \in M_{n_0+1}(i)} f_j(s+t_0-n_0-1) \right| \rho_i(s) ds \\ &\leq \sum_{i \in I} \int_0^{n_0+1-t_0} \sum_{j \in M_{n_0}(i)} |f_j(s+t_0-n_0)| \rho_i(s) ds \end{aligned}$$

$$\begin{aligned}
& + \sum_{i \in I} \int_{n_0+1-t_0}^1 \sum_{j \in M_{n_0+1}(i)} |f_j(s+t_0-n_0-1)| \rho_i(s) ds \\
& \leq M e^{wt_0} \sum_{i \in I} \int_0^{n_0+1-t_0} |f_i(s+t_0-n_0)| \rho_i(s+t_0-n_0) ds \\
& \quad + M e^{wt_0} \sum_{i \in I} \int_{n_0+1-t_0}^1 |f_i(s+t_0-n_0-1)| \rho_i(s+t_0-n_0-1) ds \\
& = M e^{wt_0} \sum_{i \in I} \int_0^1 |f_i(s)| \rho_i(s) ds = M e^{wt_0} \|f\|_{1,\rho}.
\end{aligned}$$

As a consequence, the family $\{T_t\}_{t \geq 0}$ is locally equicontinuous on $L_\rho^1(L(G))$.

The strong continuity of the semigroup is established by showing that

$$\lim_{t \rightarrow 0} \|T_t g - g\|_{1,\rho} = 0$$

holds for g in a dense subset of $L_\rho^1(L(G))$, specifically for $g \in \oplus_{i \in I} C_c(0, 1)$ (see Proposition 1.2).

To this end, let $g \in \oplus_{i \in I} C_c(0, 1)$, $I_g := \{i \in I \mid g_i \neq 0\}$ and $\epsilon > 0$, and observe that g_i is uniformly continuous for each $i \in I_g$. Since I_g is finite, there exists some $0 < \bar{t} \leq 1$ such that

$$|g_i(s+t) - g_i(s)| < \frac{\epsilon}{2 \sum_{i \in I_g} \int_0^1 \rho_i(s) ds}$$

for all $s \in [0, 1)$, $0 < t \leq \bar{t}$ and $i \in I_g$. Then, for $0 < t \leq \bar{t}$, we obtain

$$\sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds = \sum_{i \in I_g} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds < \epsilon/2.$$

On the other hand, there exists some $0 < t_0 \leq \bar{t}$ such that for all $t \leq t_0$,

$$M e^{wt} \sum_{i \in I} \int_0^t |g_i(s)| \rho_i(s) ds = M e^{wt} \sum_{i \in I_g} \int_0^t |g_i(s)| \rho_i(s) ds < \epsilon/4$$

and also

$$\sum_{i \in I} \int_{1-t}^1 |g_i(s)| \rho_i(s) ds = \sum_{i \in I_g} \int_{1-t}^1 |g_i(s)| \rho_i(s) ds < \epsilon/4.$$

As a consequence, for $t \leq t_0$, we obtain:

$$\begin{aligned}
& \|T_t g - g\|_{1,\rho} = \\
& = \sum_{i \in I_g} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds + \sum_{i \in I} \int_{1-t}^1 |(\mathcal{A}g)_i(s+t-1) - g_i(s)| \rho_i(s) ds
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds \\
&\quad + \sum_{i \in I} \int_{1-t}^1 |(\mathcal{A}g)_i(s+t-1)| \rho_i(s) ds + \sum_{i \in I} \int_{1-t}^1 |g_i(s)| \rho_i(s) ds \\
&\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds \\
&\quad + \sum_{i \in I} \int_{1-t}^1 \left| \sum_{j \in M_1(i)} g_j(s+t-1) \right| \rho_i(s) ds + \sum_{i \in I} \int_{1-t}^1 |g_i(s)| \rho_i(s) ds \\
&\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds \\
&\quad + Me^{wt} \sum_{i \in I} \int_{1-t}^1 \sum_{j \in M_1(i)} |g_j(s+t-1)| \rho_j(s+t-1) ds \\
&\quad + \sum_{i \in I} \int_{1-t}^1 |g_i(s)| \rho_i(s) ds \\
&\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)| \rho_i(s) ds \\
&\quad + Me^{wt} \sum_{i \in I} \int_0^t |g_i(s)| \rho_i(s) ds \\
&\quad + \sum_{i \in I} \int_{1-t}^1 |g_i(s)| \rho_i(s) ds < \epsilon,
\end{aligned}$$

where we applied (1.2) to get the last inequality.

(i) $\Rightarrow$ (ii): Let us now assume that $\{T_t\}_{t \geq 0}$ is a strongly continuous semigroup in $L^1_\rho(L(G))$, such that $\|T_t\| \leq Me^{wt}$ for all $t \geq 0$. Assume that (ii) does not hold. Then there exist $t_0 > 0$ and $i_0 \in I$ such that the set

$$B := \{s \in [0, 1) : \rho_{i_0}(s) > Me^{wt_0} \inf_{j \in M_{n(t_0, s)}(i_0)} \rho_j(s+t_0-n(t_0, s))\}$$

has Lebesgue measure $\lambda(B) > 0$. Let $n_0 \in \mathbb{N} \cup \{0\}$ such that $n_0 \leq t_0 < n_0 + 1$. Then $B = B_1 \cup B_2$, where

$$\begin{aligned}
B_1 &:= \left\{ s \in [0, n_0 + 1 - t_0) : \rho_{i_0}(s) > Me^{wt_0} \inf_{j \in M_{n_0}(i_0)} \rho_j(s+t_0-n_0) \right\}, \\
B_2 &:= \left\{ s \in [n_0 + 1 - t_0, 1) : \rho_{i_0}(s) > Me^{wt_0} \inf_{j \in M_{n_0+1}(i_0)} \rho_j(s+t_0-n_0-1) \right\}.
\end{aligned}$$

Clearly either $\lambda(B_1) > 0$ or $\lambda(B_2) > 0$. Assume without loss of generality that $\lambda(B_1) > 0$ (the argument for $\lambda(B_2) > 0$ is entirely analogous). Since

$$B_1 \subseteq \bigcup_{j \in M_{n_0}(i_0)} \{s \in [0, n_0 + 1 - t_0] : \rho_{i_0}(s) > Me^{wt_0} \rho_j(s + t - n_0)\}$$

we get that there exists $j_0 \in M_{n_0}(i_0)$ such that the set

$$B_1^{j_0} = \{s \in [0, n_0 + 1 - t_0] : \rho_{i_0}(s) > Me^{wt_0} \rho_{j_0}(s + t - n_0)\}$$

has strictly positive Lebesgue measure. Define the function $f := (f_i)_{i \in I}$ as

$$f_i(u) := \begin{cases} \frac{1}{\rho_{j_0}(u)} & \text{if } i = j_0 \text{ and } u \in B_1^{j_0} + t_0 - n_0; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f \in L^1_\rho(L(G))$ and

$$\|f\|_{1,\rho} = \sum_{i \in I} \int_0^1 |f_i(s)| \rho_i(s) ds = \lambda(B_1^{j_0}) > 0.$$

Nevertheless,

$$\begin{aligned} \|T_{t_0} f\| &= \sum_{i \in I} \int_0^{n_0+1-t_0} |(\mathcal{A}^{n_0} f)_i(s + t_0 - n_0)| \rho_i(s) ds + \\ &+ \sum_{i \in I} \int_{n_0+1-t_0}^1 |(\mathcal{A}^{n_0+1} f)_i(s + t_0 - n_0 - 1)| \rho_i(s) ds \\ &= \sum_{i \in I} \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i)} f_j(s + t_0 - n_0) \right| \rho_i(s) ds \\ &+ \sum_{i \in I} \int_{n_0+1-t_0}^1 \left| \sum_{j \in M_{n_0+1}(i)} f_j(s + t_0 - n_0 - 1) \right| \rho_i(s) ds \\ &\geq \sum_{i \in I} \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i)} f_j(s + t_0 - n_0) \right| \rho_i(s) ds = \int_0^{n_0+1-t_0} |f_{j_0}(s + t_0 - n_0)| \rho_{i_0}(s) ds \\ &= \int_{B_1^{j_0}} |f_{j_0}(s + t_0 - n_0)| \rho_{i_0}(s) ds \\ &> Me^{wt_0} \int_{B_1^{j_0}} |f_{j_0}(s + t_0 - n_0)| \rho_{j_0}(s + t_0 - n_0) ds = Me^{wt_0} \lambda(B_1^{j_0}) = Me^{wt_0} \|f\|_{1,\rho}, \end{aligned}$$

which contradicts (i). Hence (a.ii) must hold and the conclusion follows.

(b) (ii) $\Rightarrow$ (i): Consider $t_0 > 0$, $n_0 \in \mathbb{N} \cup \{0\}$ such that $n_0 \leq t_0 < n_0 + 1$ and $f \in L^p_\rho(L(G))$. Let p^* be the conjugate exponent of p . By observing that $p^*/p = 1/(p-1)$, by (ii), Hölder's inequality and (1.2), we obtain

$$\|T_{t_0} f\|_{p,\rho}^p =$$

$$\begin{aligned}
&= \sum_{i \in I} \int_0^{n_0+1-t_0} |(\mathcal{A}^{n_0} f)_i(s+t_0-n_0)|^p \rho_i(s) ds + \\
&\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 |(\mathcal{A}^{n_0+1} f)_i(s+t_0-n_0-1)|^p \rho_i(s) ds \\
&= \sum_{i \in I} \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i)} f_j(s+t_0-n_0) \right|^p \rho_i(s) ds + \\
&\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 \left| \sum_{j \in M_{n_0+1}(i)} f_j(s+t_0-n_0-1) \right|^p \rho_i(s) ds \\
&= \sum_{i \in I} \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i)} f_j(s+t_0-n_0) \frac{\rho_j(s+t_0-n_0)^{1/p}}{\rho_j(s+t_0-n_0)^{1/p}} \right|^p \rho_i(s) ds + \\
&\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 \left| \sum_{j \in M_{n_0+1}(i)} f_j(s+t_0-n_0-1) \frac{\rho_j(s+t_0-n_0-1)^{1/p}}{\rho_j(s+t_0-n_0-1)^{1/p}} \right|^p \rho_i(s) ds \\
&\leq \sum_{i \in I} \int_0^{n_0+1-t_0} \left(\sum_{j \in M_{n_0}(i)} |f_j(s+t_0-n_0)|^p \rho_j(s+t_0-n_0) \right) \times \\
&\quad \times \left(\sum_{j \in M_{n_0}(i)} \frac{1}{\rho_j(s+t_0-n_0)^{1/(p-1)}} \right)^{p-1} \rho_i(s) ds \\
&\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 \left(\sum_{j \in M_{n_0+1}(i)} |f_j(s+t_0-n_0-1)|^p \rho_j(s+t_0-n_0-1) \right) \times \\
&\quad \times \left(\sum_{j \in M_{n_0+1}(i)} \frac{1}{\rho_j(s+t_0-n_0-1)^{1/(p-1)}} \right)^{p-1} \rho_i(s) ds \\
&\leq M^p e^{pwt_0} \sum_{i \in I} \int_0^{n_0+1-t_0} \sum_{j \in M_{n_0}(i)} |f_j(s+t_0-n_0)|^p \rho_j(s+t_0-n_0) ds + \\
&\quad + M^p e^{pwt_0} \sum_{i \in I} \int_{n_0+1-t_0}^1 \sum_{j \in M_{n_0+1}(j)} |f_j(s+t_0-n_0-1)|^p \rho_j(s+t_0-n_0-1) ds \\
&\leq M^p e^{pwt_0} \|f\|_{p,\rho}^p.
\end{aligned}$$

Hence, it follows that the family $\{T_t\}_{t \geq 0}$ is locally equicontinuous.

Now we prove that the semigroup $\{T_t\}_{t \geq 0}$ is strongly continuous. Let $g \in \oplus_{i \in I} C_c(0, 1)$, $I_g := \{i \in I \mid g_i \neq 0\}$ and $\epsilon > 0$. As in case (a), it is easy to see that there is some $t_0 \in (0, 1]$ such that for $0 < t \leq t_0$:

$$\sum_{j \in I} \int_0^{1-t} |g_j(s+t) - g_j(s)|^p \rho_j(s) ds = \sum_{j \in I_g} \int_0^{1-t} |g_j(s+t) - g_j(s)|^p \rho_j(s) ds < \epsilon/2,$$

$$\sum_{j \in I_{1-t}} \int_0^1 |g_j(s)|^p \rho_j(s) ds = \sum_{j \in I_{g_{1-t}}} \int_0^1 |g_j(s)|^p \rho_j(s) ds < \frac{\epsilon}{2^{p+1}},$$

$$M^p e^{pwt} \sum_{i \in I} \int_0^t |g_i(s)|^p \rho_i(s) ds = M^p e^{pwt} \sum_{i \in I_g} \int_0^t |g_i(s)|^p \rho_i(s) ds < \frac{\epsilon}{2^{p+1}}.$$

Therefore, by condition (b.ii), for $t \leq t_0$ we have:

$$\begin{aligned} & \|T_t g - g\|_{p,\rho}^p = \\ &= \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)|^p \rho_i(s) ds + \sum_{i \in I_{1-t}} \int_0^1 |(\mathcal{A}g)_i(s+t-1) - g(s)|^p \rho_i(s) ds \\ &\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)|^p \rho_i(s) ds \\ &\quad + 2^{p-1} \sum_{i \in I_{1-t}} \int_0^1 |(\mathcal{A}g)_i(s+t-1)|^p \rho_i(s) ds + 2^{p-1} \sum_{i \in I_{1-t}} \int_0^1 |g_i(s)|^p \rho_i(s) ds \\ &\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)|^p \rho_i(s) ds \\ &\quad + 2^{p-1} \sum_{i \in I_{1-t}} \int_0^1 \left| \sum_{j \in M_1(i)} g_j(s+t-1) \frac{\rho_j(s+t-1)^{1/p}}{\rho_j(s+t-1)^{1/p}} \right|^p \rho_i(s) ds + 2^{p-1} \sum_{i \in I_{g_{1-t}}} \int_0^1 |g_i(s)|^p \rho_i(s) ds \\ &\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)|^p \rho_i(s) ds \\ &\quad + 2^{p-1} \sum_{i \in I_{1-t}} \int_0^1 \left(\sum_{j \in M_1(i)} |g_j(s+t-1)|^p \rho_j(s+t-1) \right) \left(\sum_{j \in M_1(i)} \frac{1}{\rho_j(s+t-1)^{1/(p-1)}} \right)^{p-1} \rho_i(s) ds \\ &\quad + 2^{p-1} \sum_{i \in I_{g_{1-t}}} \int_0^1 |g_i(s)|^p \rho_i(s) ds \\ &\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)|^p \rho_i(s) ds \\ &\quad + M^p e^{pwt} 2^{p-1} \sum_{i \in I_{1-t}} \int_0^1 \sum_{j \in M_1(i)} |g_j(s+t-1)|^p \rho_j(s+t-1) ds + 2^{p-1} \sum_{i \in I_{g_{1-t}}} \int_0^1 |g_i(s)|^p \rho_i(s) ds \\ &\leq \sum_{i \in I} \int_0^{1-t} |g_i(s+t) - g_i(s)|^p \rho_i(s) ds \\ &\quad + M^p e^{pwt} 2^{p-1} \sum_{i \in I} \int_0^t |g_i(u)|^p \rho_i(u) du + 2^{p-1} \sum_{i \in I_{1-t}} \int_0^1 |g_i(s)|^p \rho_i(s) ds < \epsilon, \end{aligned}$$

where we applied (1.2) to get the second-last inequality. Hence the conclusion follows by Theorem 1.2.

(i) $\Rightarrow$ (ii). Assume that (b.ii) is not satisfied. Then there exist some $t_0 > 0$, $i_0 \in I$, such that the set

$$B := \left\{ s \in [0, 1) : \left(\sum_{j \in M_{n(t_0, s)}(i_0)} \frac{1}{\rho_j(s + t_0 - n(t_0, s))^{1/(p-1)}} \right)^{1-p} < \frac{\rho_{i_0}(s)}{M e^{wt_0}} \right\}$$

has strictly positive Lebesgue measure. Let $n_0 \in \mathbb{N}$ such that $n_0 \leq t_0 < n_0 + 1$. Then $B = B_1 \cup B_2$ where

$$B_1 := \left\{ s \in [0, n_0 + 1 - t_0) : \left(\sum_{j \in M_{n_0}(i_0)} \frac{1}{\rho_j(s + t_0 - n_0)^{1/(p-1)}} \right)^{1-p} < \frac{\rho_{i_0}(s)}{M^p e^{pwt_0}} \right\},$$

$$B_2 := \left\{ s \in [n_0 + 1 - t_0, 1) : \left(\sum_{j \in M_{n_0+1}(i_0)} \frac{1}{\rho_j(s + t_0 - n_0 - 1)^{1/(p-1)}} \right)^{1-p} < \frac{\rho_{i_0}(s)}{M^p e^{pwt_0}} \right\},$$

hence B_1 or B_2 have strictly positive Lebesgue measure. Assume without loss of generality that $\lambda(B_1) > 0$.

By Lemma 2.1, there exists a sequence $(v_j(\cdot + t_0 - n_0))_{j \in M_{n_0}(i_0)}$ such that every $v_j(\cdot + t_0 - n_0)$ is measurable on B_1 , for every $s \in B_1$

$$\sum_{j \in M_{n_0}(i_0)} |v_j(s + t_0 - n_0)| = 1,$$

and also

$$\sum_{j \in M_{n_0}(i_0)} |v_j(s + t_0 - n_0)|^p \rho_j(s + t_0 - n_0) < \frac{\rho_{i_0}(s)}{M^p e^{pwt_0}}.$$

Define the function f on $L(G)$ such that

$$f_j(t) := \begin{cases} v_j(t) & \text{if } j \in M_{n_0}(i_0) \text{ and } t \in B_1 + t_0 - n_0; \\ 0 & \text{otherwise.} \end{cases}$$

We have the following.

$$\begin{aligned} \|T_{t_0} f\|_{p, \rho}^p &= \\ &= \sum_{i \in I} \int_0^{n_0+1-t_0} |(\mathcal{A}^{n_0} f)_i(s + t_0 - n_0)|^p \rho_i(s) ds + \\ &\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 |(\mathcal{A}^{n_0+1} f)_i(s + t_0 - n_0 - 1)|^p \rho_i(s) ds \\ &= \sum_{i \in I} \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i)} f_j(s + t_0 - n_0) \right|^p \rho_i(s) ds + \\ &\quad + \sum_{i \in I} \int_{n_0+1-t_0}^1 \left| \sum_{j \in M_{n_0+1}(i_0)} f_j(s + t_0 - n_0 - 1) \right|^p \rho_i(s) ds \end{aligned}$$

$$= \int_0^{n_0+1-t_0} \left| \sum_{j \in M_{n_0}(i_0)} f_j(s+t_0-n_0) \right|^p \rho_{i_0}(s) ds = \int_{B_1} \rho_{i_0}(s) ds.$$

Nevertheless,

$$\begin{aligned} \|f\|_{p,\rho}^p &= \sum_{i \in I} \int_0^1 |f_i(s)|^p \rho_i(s) ds = \int_0^1 \sum_{i \in I} |f_i(s)|^p \rho_i(s) ds \\ &= \int_{B_1} \sum_{j \in M_{n_0}(i_0)} |f_j(s+t_0-n_0)|^p \rho_j(s+t_0-n_0) ds \\ &= \int_{B_1} \sum_{j \in M_{n_0}(i_0)} |v_j(s+t_0-n_0)|^p \rho_j(s+t_0-n_0) ds \\ &< \int_{B_1} \frac{\rho_{i_0}(s)}{M^p e^{pwt_0}} ds = \frac{\|T_{t_0} f\|^p}{M^p e^{pwt_0}}, \end{aligned}$$

which contradicts the assumption, and therefore (b.ii) must hold. $\square$

Remark 2.3. When considering the specific tree structure described in Remark 1.5, conditions (a)(ii) and (b)(ii) become:

$$\frac{\rho_i(s)}{\rho_{i+n(t,s)}(s+t-n(t,s))} \leq M^p e^{pwt}$$

for all $i \in \mathbb{N}$, $t \geq 0$ and almost every $s \in [0, 1)$. By translating this property to the weight $\tilde{\rho}$ on the half-line $[0, +\infty)$, we get that:

$$\tilde{\rho}(u) \leq M^p e^{pwt} \tilde{\rho}(u+t), \quad t \geq 0, \text{ a.e. } u \geq 0.$$

This inequality is the precise condition that guarantees the strong continuity of the classical left translation semigroup on the space $L_{\tilde{\rho}}^p([0, +\infty))$ (see [10, Example 7.4]).

In the following, and in order to avoid some technical problems, we will consider a slightly stronger condition for the weight ρ .

Definition 2.4. Let $\rho = (\rho_i)_{i \in I}$ be a weight on the directed metric tree $L(G)$ such that for every $i \in I$ and for every $s \in [0, 1)$ it holds that $\rho_i(s) > 0$. Then ρ is said to be a p -admissible weight for $L(G)$ if there exists $M \geq 1$ and $w \in \mathbb{R}$ such that for all $i \in I$, $s \in [0, 1)$ and $t \geq 0$:

$$\rho_i(s) \leq M e^{wt} \inf_{j \in M_{n(t,s)}(i)} \rho_j(s+t-n(t,s)) \quad \text{if } p = 1 \quad (2.5)$$

$$\left(\sum_{j \in M_{n(t,s)}(i)} \frac{1}{\rho_j(s+t-n(t,s))^{1/(p-1)}} \right)^{p-1} \leq M^p e^{pwt} \frac{1}{\rho_i(s)}, \quad \text{if } 1 < p < \infty. \quad (2.6)$$

Remark 2.5. Observe that under conditions (2.5) and (2.6), it holds that, for every $i \in I$, $s, t \geq 0$ such that $0 \leq s+t < 1$

$$\rho_i(s) \leq M e^{wt} \rho_i(s+t)$$

and therefore, $0 < \rho_i(0) \leq Me^{|w|}\rho_i(t)$ for every $t \in [0, 1)$. Hence we get that $\inf_{s \in [0, 1)} \rho_i(s) > 0$. Moreover,

$$\begin{aligned} \rho_i(s) &\leq Me^{w(1-s)} \inf_{j \in M_1(i)} \rho_j(0) && \text{if } p = 1 \\ \rho_i(s) &\leq Me^{w(1-s)} \left(\sum_{j \in M_1(i)} \rho_j(0)^{-1/(p-1)} \right)^{1-p} && \text{if } 1 < p < \infty. \end{aligned}$$

In any case, ρ_i is bounded and bounded away from 0 on $[0, 1)$.

3. Hypercyclicity for the left translation semigroup on directed metric trees

In this section we provide necessary and sufficient conditions for a p -admissible weight sequence $(\rho_i)_{i \in I}$ that ensure hypercyclicity for the translation semigroup on directed metric trees. We begin by observing that, if the directed tree G has a leaf, then the left translation semigroup cannot be hypercyclic in any admissible space $L_p^p(L(G))$. Indeed, let $f \in L_p^p(L(G))$ and $e = (v_{i_0}, v_{j_0}) \in E$ be such that v_{j_0} is a leaf of G . Then it is clear that $(T_t f)_e(s) = 0$ for all $t > 1$ and $0 \leq s < 1$, and therefore the semigroup cannot be hypercyclic.

Theorem 3.1. *Let $L(G)$ be the directed metric tree associated with $G = (V, E)$. Assume that $L(G)$ is rooted and that G is without leaves.*

(a) *The following assertions are equivalent:*

- (i) *the translation semigroup is hypercyclic on $L_p^1(L(G))$;*
- (ii) *the translation semigroup is weakly mixing on $L_p^1(L(G))$;*
- (iii) *there exists some increasing sequence $(n_h)_{h \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for all $i \in I$:*

$$\lim_{h \rightarrow \infty} \inf_{j \in M_{n_h}(i)} \inf_{s \in [0, 1)} \rho_j(s) = 0.$$

(b) *If $1 < p < \infty$, the following assertions are equivalent:*

- (i) *the translation semigroup is hypercyclic on $L_p^p(L(G))$;*
- (ii) *the translation semigroup is weakly mixing on $L_p^p(L(G))$;*
- (iii) *there exists some increasing sequence $(n_h)_{h \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for all $i \in I$:*

$$\lim_{h \rightarrow \infty} \sup_{s \in [0, 1)} \left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) = \infty.$$

Proof. We will only provide the proof of assertion (b), as the proof of (a) is analogous.

The equivalence of (ii) and (i) follows from the fact that F_p^p is a dense subspace such that the orbit of each $f \in F_p^p$ is bounded (see [10, Theorem 2.48]).

Along the proof, it will be useful to observe that the limit in (iii) can be reformulated as

$$\lim_{h \rightarrow \infty} \inf_{s \in [0, 1)} \left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{-1} = 0.$$

We will prove that (iii) $\Rightarrow$ (i), by showing that given $\epsilon > 0$, and $f_1, f_2 \in F_p^p$, there is some $g \in L_p^p(L(G))$ and $t > 0$ such that

$$\|f_1 - g\|_{p,\rho} < \epsilon \quad \text{and} \quad T_t g = f_2,$$

which by density of F_ρ^p implies that $\{T_t\}_{t \geq 0}$ is topologically transitive and consequently the translation semigroup will be hypercyclic.

Observe that since $f_1 \in F^p$, there is some $N_0 \in \mathbb{N}$ such that

$$T_n f_1 = 0$$

for all $n \geq N_0$. Now, take $M \geq 1$ and $w \in \mathbb{R}$ such that ρ satisfies the condition (2.6). Furthermore, define $I_1 := \{i \in I : (f_1)_i \neq 0\}$, $I_2 := \{i \in I : (f_2)_i \neq 0\}$ and the constants:

$$C_1 := \min_{i \in I_2} \inf_{s \in [0,1)} \rho_i(s) > 0, \quad C_2 := \max_{i \in I_2} \sup_{s \in [0,1)} \rho_i(s) > 0.$$

Given $\delta > 0$, by (iii) we have that there is some $n_h \in \mathbb{N}$ such that $n_h > N_0$ and for all $i \in I_2$ there exist some $s_{i,n_h} \in [0, 1)$ with

$$\left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(s_{i,n_h})^{1/(p-1)}} \right)^{1-p} < \delta. \quad (3.7)$$

By Lemma 2.1, for each $i \in I_2$ there exists a sequence $(\nu_j^i)_{j \in M_{n_h}(i)}$, $\nu_j^i \geq 0$ such that

$$\sum_{j \in M_{n_h}(i)} \nu_j^i = 1 \quad (3.8)$$

and also

$$\sum_{j \in M_{n_h}(i)} |\nu_j^i|^p \rho_j(s_{i,n_h}) < \delta. \quad (3.9)$$

Define the function $g \in L_\rho^p(L(G))$ such that

$$g_i(s) := \begin{cases} (f_1)_i(s) & \text{if } i \in I_1; \\ (f_2)_l(s) \nu_i^l & \text{if } i \in M_{n_h}(l) \text{ for some } l \in I_2; \\ 0 & \text{otherwise.} \end{cases}$$

Since $n_h > N_0$ we have by (3.8) that if $i \in I_2$ then

$$(T_{n_h} g)_i(s) = \sum_{j \in M_{n_h}(i)} (f_2)_i(s) \nu_j^i = (f_2)_i(s).$$

On the other hand, if $i \notin I_2$ then $(T_{n_h} g)_i = 0$, and consequently $T_{n_h} g = f_2$.

Now we prove that $\|f_1 - g\|_{p,\rho} < \epsilon$. By equation (3.9) we have:

$$\begin{aligned} \|f_1 - g\|_{p,\rho}^p &= \sum_{i \in I} \int_0^1 |(f_1)_i(s) - g_i(s)|^p \rho_i(s) ds \\ &= \sum_{l \in I_2} \sum_{j \in M_{n_h}(l)} \int_0^1 |(f_2)_l(s) \nu_j^l|^p \rho_j(s) ds \end{aligned}$$

$$\begin{aligned}
&= \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p \sum_{j \in M_{n_h}(l)} |\nu_j^l|^p \rho_j(s) ds \\
&\leq \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p \sum_{j \in M_{n_h}(l)} |\nu_j^l|^p \frac{C_2}{C_1} \rho_j(s_{l,n_h}) ds \leq \frac{C_2}{C_1} \delta \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p ds \\
&\leq \frac{C_2}{C_1^2} \delta \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p \rho_l(s) ds = \frac{C_2}{C_1^2} \delta \|f_2\|_{p,\rho}^p.
\end{aligned}$$

We obtain the assertion by choosing $\delta = \frac{C_1^2}{C_2} \|f_2\|_{p,\rho}^{-p} \epsilon^p$.

We will now prove that (ii) $\Rightarrow$ (iii). Assume that the translation semigroup is weakly mixing but ρ does not satisfy condition (iii). By Theorem 1.3 every autonomous discretization is weakly mixing, which holds in particular for the sequence $(T_n)_{n \in \mathbb{N}}$. As a consequence, the sequence $(T_n)_n$ is hereditarily transitive (see e.g. [10, Exercise 1.6.4]), namely there exists an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $(T_{m_k})_{k \in \mathbb{N}}$ is transitive for every subsequence $(m_k)_{k \in \mathbb{N}}$ of $(n_k)_{k \in \mathbb{N}}$.

Assume that there exists some $i_0 \in I$ such that

$$\lim_{k \rightarrow \infty} \inf_{s \in [0,1)} \left(\sum_{j \in M_{n_k}(i_0)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{-1} \neq 0.$$

Then there exists a subsequence $(n_{k(\xi)})_{\xi \in \mathbb{N}}$ and $\varepsilon_0 > 0$ such that

$$\inf_{s \in [0,1)} \left(\sum_{j \in M_{n_{k(\xi)}}(i_0)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{1-p} > \varepsilon_0.$$

Taking Remark 2.5 into account, we get that there exists some $K \geq 0$ such that

$$\rho_{i_0}(s) \leq K \left(\sum_{j \in M_{n_{k(\xi)}}(i_0)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{1-p} \quad (3.10)$$

for all $\xi \in \mathbb{N}$ and $s \in [0,1)$. Now, consider $g \in L_\rho^p(L(G))$ defined as

$$g_i(s) := \begin{cases} \frac{2^{1-1/p}}{\rho_{i_0}(s)^{1/p}} & \text{if } i = i_0; \\ 0 & \text{otherwise.} \end{cases}$$

By equation (3.10) and Hölder's inequality, for any $f \in L_\rho^p(L(G))$ such that $\|f\|_{p,\rho}^p < \frac{1}{2K}$ and any $\xi \in \mathbb{N}$ we obtain:

$$\begin{aligned}
\|T_{n_{k(\xi)}} f - g\|_{p,\rho}^p &= \sum_{i \in I} \int_0^1 |(\mathcal{A}^{n_{k(\xi)}} f)_i(s) - g_i(s)|^p \rho_i(s) ds \\
&\geq \int_0^1 |(\mathcal{A}^{n_{k(\xi)}} f)_{i_0}(s) - g_{i_0}(s)|^p \rho_{i_0}(s) ds \geq \int_0^1 (2^{1-p} |g_{i_0}(s)|^p - |(\mathcal{A}^{n_{k(\xi)}} f)_{i_0}(s)|^p) \rho_{i_0}(s) ds
\end{aligned}$$

$$\begin{aligned}
&= 1 - \int_0^1 |(\mathcal{A}^{n_{k(\xi)}} f)_{i_0}(s)|^p \rho_{i_0}(s) ds = 1 - \int_0^1 \left| \sum_{j \in M_{n_{k(\xi)}}(i_0)} f_j(s) \right|^p \rho_{i_0}(s) ds \\
&\geq 1 - \int_0^1 \left(\sum_{j \in M_{n_{k(\xi)}}(i_0)} |f_j(s)|^p \rho_j(s) \right) \left(\sum_{j \in M_{n_{k(\xi)}}(i_0)} \frac{1}{\rho_j(s)^{p^*/p}} \right)^{p-1} \rho_{i_0}(s) ds \\
&\geq 1 - K \int_0^1 \sum_{j \in M_{n_{k(\xi)}}(i_0)} |f_j(s)|^p \rho_j(s) ds \\
&\geq 1 - K \|f\|_{p,\rho}^p \geq 1/2.
\end{aligned}$$

Hence, the sequence of operators $(T_{n_{k(\xi)}})_{\xi \in \mathbb{N}}$ is not transitive, which is a contradiction. Therefore, if the translation semigroup is weakly mixing, condition (iii) must hold. $\square$

Remark 3.2. A slight modification of the proof of Theorem 3.1 allow to show that, either on $L_\rho^1(L(G))$ or on $L_\rho^p(L(G))$, the translation semigroup is mixing if and only if condition (iii) on the weight of the space holds for the full sequence $(n)_{n \in \mathbb{N}}$.

We will now characterize hypercyclicity for the unrooted case. In order to do so, given a directed metric tree $L(G)$, where $G = (V, E)$ is a directed tree, and $E = (e_i)_{i \in I}$, we define $K_n(j) \in I$ as the index of the unique edge such that $j \in M_n(K_n(j))$. The edge $e_{K_n(j)}$ can be considered as an ancestor of e_j . Moreover, we define the set

$$G(j) = \bigcup_{n=1}^{\infty} M_n(K_n(j)),$$

and observe that if $j_1 \in G(j)$ then $G(j_1) = G(j)$.

Furthermore, we say that $j_1 \sim_G j_2$ if $j_1 \in G(j_2)$, so that $\sim_G$ defines an equivalence relation on E .

Theorem 3.3. Let G be an unrooted directed tree without leaves and let $L(G)$ be the directed metric tree associated with G .

(a) The following assertions are equivalent:

- (i) the left translation semigroup is hypercyclic on $L_\rho^1(L(G))$;
- (ii) the left translation semigroup is weakly mixing on $L_\rho^1(L(G))$;
- (iii) there exists an increasing sequence $(n_h)_h \in \mathbb{N}^{\mathbb{N}}$ such that for all $i \in I$

$$\lim_{h \rightarrow \infty} \inf_{s \in [0,1)} \inf_{j \in M_{n_h}(i)} \rho_j(s) = 0,$$

and

$$\lim_{h \rightarrow \infty} \inf_{s \in [0,1)} \min \left\{ \rho_{K_{n_h}(i)}(s), \inf_{j \in M_{n_h}(K_{n_h}(i))} \rho_j(s) \right\} = 0.$$

(b) For $1 < p < \infty$, the following assertions are equivalent:

- (i) the left translation semigroup is hypercyclic on $L_\rho^p(L(G))$;
- (ii) the left translation semigroup is weakly mixing on $L_\rho^p(L(G))$;

(iii) there exists an increasing sequence $(n_h)_h \in \mathbb{N}^{\mathbb{N}}$ such that for all $i \in I$

$$\lim_{h \rightarrow \infty} \sup_{s \in [0,1)} \left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) = \infty$$

and

$$\lim_{h \rightarrow \infty} \sup_{s \in [0,1)} \left(\frac{1}{\rho_{K_{n_h}(i)}(s)^{1/(p-1)}} + \sum_{j \in M_{n_h}(K_{n_h}(i))} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) = \infty$$

Proof. Since the proofs for both cases are similar, we only present the proof of case (b). We turn first our attention to the implication $(i) \Rightarrow (iii)$.

We will prove that given a finite subset $F \subseteq I$, for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that for every $i \in F$:

$$\sup_{s \in [0,1)} \left(\sum_{j \in M_n(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) > \frac{1}{\epsilon},$$

$$\sup_{s \in [0,1)} \left(\frac{1}{\rho_{K_n(i)}(s)^{1/(p-1)}} + \sum_{j \in M_n(K_n(i))} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) > \frac{1}{\epsilon}.$$

The assertion will follow by taking an increasing sequence $(F_h)_{h \in \mathbb{N}}$ of finite subsets of I such that $\bigcup_h F_h = I$ and a sequence $(\epsilon_h)_{h \in \mathbb{N}}$ of positive numbers tending to 0. Then, we can construct an increasing sequence $(n_h)_{h \in \mathbb{N}}$ of natural numbers such that

$$\forall h \in \mathbb{N} \quad \forall i \in F_h \quad \sup_{s \in [0,1)} \left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) > \frac{1}{\epsilon_h},$$

$$\sup_{s \in [0,1)} \left(\frac{1}{\rho_{K_{n_h}(i)}(s)^{1/(p-1)}} + \sum_{j \in M_{n_h}(K_{n_h}(i))} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) > \frac{1}{\epsilon_h}.$$

If $i \in I$, then $i \in F_h$ for any $h \geq h_0$ for some $h_0 \in \mathbb{N}$, and therefore

$$\forall h \in \mathbb{N}, h \geq h_0 : \sup_{s \in [0,1)} \left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) > \frac{1}{\epsilon_h},$$

$$\sup_{s \in [0,1)} \left(\frac{1}{\rho_{K_{n_h}(i)}(s)^{1/(p-1)}} + \sum_{j \in M_{n_h}(K_{n_h}(i))} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) > \frac{1}{\epsilon_h}$$

and the assertion follows.

So, consider a finite subset $F \subset I$ and a subset $H \subset F$ such that for all $i \in F$ there is some $i_1 \in H$ such that $i_1 \sim_G i$ and if $i_1, i_2 \in H$ then $i_1 \not\sim_G i_2$ (namely H is a set of representatives with respect to the equivalence relation $\sim_G$ in F). Define the functions

$$(f_F)_i(s) := \begin{cases} 1 & \text{if } i \in F, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } s \in [0,1),$$

$$(f_H)_i(s) := \begin{cases} 1 & \text{if } i \in H, \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } s \in [0, 1).$$

Since the semigroup $\{T_t\}_{t \geq 0}$ is hypercyclic, by Theorem 1.4, it follows that the operator T_1 is also hypercyclic. Hence, given $\varepsilon \in]0, 4^{\frac{1}{p}} \min_{i \in F} \inf \rho_i[$ and $N \in \mathbb{N}$, there exist $f \in L^p_\rho(L(G))$ and $n \geq N$ such that

$$\|f - f_H\|_{p,\rho} < \frac{\varepsilon}{4^{1/p}}, \quad \|T_n f - f_F\|_{p,\rho} < \frac{\varepsilon}{4^{1/p}}, \quad (3.11)$$

and we can choose n large enough so that

$$H \cap M_n(F) = \emptyset, \quad K_n(H) \cap F = \emptyset \quad (3.12)$$

and

$$i, j \in F, i \sim_G j \Rightarrow K_n(i) = K_n(j). \quad (3.13)$$

Claim 1: for all $i \in F$

$$\sup_{s \in [0,1)} \left(\sum_{j \in M_n(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) \geq \left(\frac{4}{\varepsilon^p \sup \rho_i} \right)^{\frac{1}{p-1}} \left((\inf \rho_i)^{\frac{1}{p}} - \frac{\varepsilon}{4^{\frac{1}{p}}} \right)^{\frac{p-1}{p}}$$

Indeed, by (3.11) and (3.12), for each $i \in F$

$$\sum_{j \in M_n(i)} \int_0^1 |f_j(s)|^p \rho_j(s) ds < \frac{\varepsilon^p}{4},$$

and

$$\left(\int_0^1 |(T_n f)_i(s) - 1|^p \rho_i(s) ds \right)^{\frac{1}{p}} \leq \|T_n f - f_F\|_{p,\rho} < \frac{\varepsilon}{4^{1/p}}$$

and therefore

$$\left(\int_0^1 |(T_n f)_i(s)|^p \rho_i(s) ds \right)^{\frac{1}{p}} > \left(\int_0^1 \rho_i(s) ds \right)^{\frac{1}{p}} - \frac{\varepsilon}{4^{1/p}} \geq (\inf \rho_i)^{\frac{1}{p}} - \frac{\varepsilon}{4^{\frac{1}{p}}}.$$

Hence, we get:

$$\begin{aligned} & (\inf \rho_i)^{\frac{1}{p}} - \frac{\varepsilon}{4^{\frac{1}{p}}} < \\ & \left(\int_0^1 |(T_n f)_i(s)|^p \rho_i(s) ds \right)^{\frac{1}{p}} = \left(\int_0^1 \left| \sum_{j \in M_n(i)} f_j(s) \right|^p \rho_i(s) ds \right)^{\frac{1}{p}} \\ & \leq \left(\int_0^1 \left(\sum_{j \in M_n(i)} |f_j(s)|^p \rho_j(s) \right) \left(\sum_{j \in M_n(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{p-1} \rho_i(s) ds \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
&\leq (\sup \rho_i)^{\frac{1}{p}} \cdot \sup_{s \in [0,1)} \left(\sum_{j \in M_n(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{\frac{p-1}{p}} \left(\sum_{j \in M_n(i)} \int_0^1 |f_j(s)|^p \rho_j(s) ds \right)^{\frac{1}{p}} \\
&\leq \frac{\epsilon}{4^{1/p}} (\sup \rho_i)^{\frac{1}{p}} \cdot \sup_{s \in [0,1)} \left(\sum_{j \in M_n(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right)^{\frac{p-1}{p}}.
\end{aligned}$$

Consequently:

$$\sup_{s \in [0,1)} \left(\sum_{j \in M_n(i)} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) \geq \left(\frac{4}{\epsilon^p \sup \rho_i} \right)^{\frac{1}{p-1}} \left((\inf \rho_i)^{\frac{1}{p}} - \frac{\epsilon}{4^{\frac{1}{p}}} \right)^{\frac{p-1}{p}}$$

Claim 2: For each $i \in F$

$$\sup_{s \in [0,1)} \left(\frac{1}{\rho_{K_n(i)}(s)^{1/(p-1)}} + \sum_{j \in M_n(K_n(i))} \frac{1}{\rho_j(s)^{1/(p-1)}} \right) \geq \frac{1}{(2\epsilon)^{p/(p-1)}}$$

Observe that, by the definition of H and (3.13), it is enough to prove Claim 2 for any $i \in H$. To this aim, fix $i \in H$ and consider the function g defined by

$$g_j(s) = \begin{cases} f_j(s) - \delta_{ij} & \text{if } j \in M_n(K_n(i)) \\ 0 & \text{otherwise} \end{cases}, \quad s \in [0,1).$$

By (3.11) we have

$$\begin{aligned}
&\sum_{j \in M_n(K_n(i))} \int_0^1 |g_j(s)|^p \rho_j(s) ds \\
&= \int_0^1 |f_i(s) - 1|^p \rho_i(s) ds + \sum_{j \in M_n(K_n(i)) \setminus \{i\}} \int_0^1 |f_j(s)|^p \rho_j(s) ds \\
&\leq \|f - f_H\|^p < \frac{\epsilon^p}{4},
\end{aligned}$$

by observing that $(M_n(K_n(i)) \setminus \{i\}) \cap H = \emptyset$. Therefore

$$\begin{aligned}
&\lambda \left(\left\{ s \in [0,1) : \sum_{j \in M_n(K_n(i))} |g_j(s)|^p \rho_j(s) \geq \epsilon^p \right\} \right) \\
&\leq \frac{1}{\epsilon^p} \int_0^1 \sum_{j \in M_n(K_n(i))} |g_j(s)|^p \rho_j(s) ds < \frac{1}{4}.
\end{aligned}$$

Hence there exists some set $E_1 \subseteq [0,1)$ with $\lambda(E_1) > 3/4$ such that for all $s \in E_1$

$$\sum_{j \in M_n(K_n(i))} |g_j(s)|^p \rho_j(s) < \epsilon^p.$$

By (3.11) and (3.12), for each $i \in H$ we have

$$\begin{aligned} \int_0^1 |(T_n f)_{K_n(i)}|^p \rho_{K_n(i)}(s) ds &= \int_0^1 \left| \sum_{j \in M_n(K_n(i))} f_j(s) \right|^p \rho_{K_n(i)}(s) ds \\ &= \int_0^1 \left| \left(1 + \sum_{j \in M_n(K_n(i))} g_j(s) \right) \right|^p \rho_{K_n(i)}(s) ds < \frac{\epsilon^p}{4}, \end{aligned}$$

and, again we conclude that there exists a set $E_2 \subseteq [0, 1)$ such that $\lambda(E_2) > 3/4$ and for all $s \in E_2$

$$\rho_{K_n(i)}(s) \left| \left(1 + \sum_{j \in M_n(K_n(i))} g_j(s) \right) \right|^p ds \leq \epsilon^p.$$

Define now $E := E_1 \cap E_2$. Since $\lambda(E_1), \lambda(E_2) > 3/4$, it is clear that $\lambda(E_1 \cap E_2) > 0$. Let $s_0 \in E$. If $\rho_{K_n(i)}(s_0) \leq (2\epsilon)^p$, then Claim 2 is satisfied. If $\rho_{K_n(i)}(s_0) \geq (2\epsilon)^p$, then

$$\left| 1 + \sum_{j \in M_n(K_n(i))} g_j(s_0) \right|^p \leq \frac{1}{2^p}.$$

This yields that

$$\sum_{j \in M_n(K_n(i))} |g_j(s_0)| \geq \left| \sum_{j \in M_n(K_n(i))} g_j(s_0) \right| \geq \frac{1}{2}.$$

For any $j \in M_n(K_n(i))$, consider the functions

$$v_j(s_0) = \frac{g_j(s_0)}{\sum_{j \in M_n(K_n(i))} |g_j(s_0)|} \chi_{E \cap \{\rho_{K_n(i)} \geq (2\epsilon)^p\}}, \quad s_0 \in [0, 1).$$

Every v_j is measurable on E and $\sum_{j \in M_n(K_n(i))} |v_j(s_0)| = 1$ for any $s \in E \cap \{\rho_{K_n(i)} \geq (2\epsilon)^p\}$. Therefore, by Lemma 2.1,

$$\left(\sum_{j \in M_n(K_n(i))} \frac{1}{\rho_j(s_0)^{1/(p-1)}} \right)^{1-p} \leq \sum_{j \in M_n(K_n(i))} |v_j(s_0)|^p \rho_j(s_0) < (2\epsilon)^p$$

for any $s_0 \in E \cap \{\rho_{K_n(i)} \geq (2\epsilon)^p\}$.

Hence for any $s_0 \in E$

$$\sum_{j \in M_n(K_n(i))} \frac{1}{\rho_j(s_0)^{1/(p-1)}} \geq \frac{1}{(2\epsilon)^{p/(p-1)}},$$

and Claim 2 follows.

To prove the implication (iii) $\Rightarrow$ (i), it is enough to show that, given $f_1, f_2 \in F_\rho^p$ and $\epsilon > 0$, there is some $g \in L_\rho^p(L(G))$ and $\bar{t} > 0$ such that

$$\|f_1 - g\|_{p,\rho} < \epsilon \quad \text{and} \quad \|T_{\bar{t}}g - f_2\|_{p,\rho} < \epsilon.$$

Without loss of generality, we can assume that, for $k = 1, 2$, if $(f_k)_i \neq 0$ for some $i \in I$, then $(f_k)_h = 0$ for all $h \sim_G i$, $h \neq i$. Indeed, any $f \in F_\rho^p$ is a linear combination of functions satisfying this condition.

Define $I_1 := \{i \in I : (f_1)_i \neq 0\}$, $I_2 := \{i \in I : (f_2)_i \neq 0\}$ and the constants:

$$C_1 := \min_{i \in I_1 \cup I_2} \inf_{s \in [0,1]} \rho_i(s) > 0 \quad C_2 := \max_{i \in I_1 \cup I_2} \sup_{s \in [0,1]} \rho_i(s) > 0$$

Clearly, there is some $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$

$$\left(\bigcup_{i \in I_2} M_n(i) \right) \cap I_1 = \emptyset, \quad \left(\bigcup_{i \in I_1} K_n(i) \right) \cap I_2 = \emptyset \quad (3.14)$$

for all $n \geq N_0$.

Given fixed $\delta_1, \delta_2 > 0$, by (iii) there exists $n_h \in \mathbb{N}$, $n_h > N_0$, such that for all $i \in I_1$ there is $s_{i,n_h} \in [0, 1)$ for which

$$\frac{1}{\rho_{K_{n_h}(i)}(s_{i,n_h})} + \left(\sum_{j \in M_{n_h}(K_{n_h}(i))} \frac{1}{\rho_j(s_{i,n_h})^{1/(p-1)}} \right) > \frac{1}{\delta_1}$$

and, for every $i \in I_2$, there exists $t_{i,n_h} \in [0, 1)$ such that

$$\left(\sum_{j \in M_{n_h}(i)} \frac{1}{\rho_j(t_{i,n_h})^{1/(p-1)}} \right) > \frac{1}{\delta_2}.$$

Then, for every $i \in I_1$

$$\frac{1}{\rho_{K_{n_h}(i)}(s_{i,n_h})} > \frac{1}{2\delta_1} \quad \text{or} \quad \left(\sum_{j \in M_{n_h}(K_{n_h}(i))} \frac{1}{\rho_j(s_{i,n_h})^{1/(p-1)}} \right) > \frac{1}{2\delta_1},$$

hence we can write $I_1 = J_1 \cup J_2$ where

$$J_1 := \left\{ i \in I_1 : \frac{1}{\rho_{K_{n_h}(i)}(s_{i,n_h})} > \frac{1}{2\delta_1} \right\},$$

$$J_2 := \left\{ i \in I_1 : \left(\sum_{j \in M_{n_h}(K_{n_h}(i))} \frac{1}{\rho_j(s_{i,n_h})^{1/(p-1)}} \right) > \frac{1}{2\delta_1} \right\}.$$

By Lemma 2.1, for each $i \in J_2 \setminus J_1$ there exists a family of numbers $(u_j^i)_{j \in M_{n_h}(K_{n_h}(i))}$, $u_j^i \geq 0$ such that

$$\sum_{j \in M_{n_h}(K_{n_h}(i))} u_j^i = 1 \quad \sum_{j \in M_{n_h}(K_{n_h}(i))} |u_j^i|^p \rho_j(s_{i,n_h}) < (2\delta_1)^{p-1},$$

and for every $i \in I_2$ there exists $(\nu_j^i)_{j \in M_{n_h}(i)}$, $\nu_j^i \geq 0$ such that

$$\sum_{j \in M_{n_h}(i)} \nu_j^i = 1, \quad \sum_{j \in M_{n_h}(i)} |\nu_j^i|^p \rho_j(t_{i,n_h}) < \delta_2^{p-1}.$$

Now, define the function $g \in L_\rho^p(L(G))$ such that

$$g_i(s) := \begin{cases} (f_1)_i(s) & \text{if } i \in J_1; \\ (f_1)_i(s)(1 - u_i^i) & \text{if } i \in J_2 \setminus J_1; \\ -(f_1)_l(s)u_i^l & \text{if } i \in M_{n_h}(K_{n_h}(l)) \text{ for some } l \in J_2 \setminus J_1 \text{ and } i \notin I_1; \\ (f_2)_l(s)\nu_i^l & \text{if } i \in M_{n_h}(l) \text{ for some } l \in I_2; \\ 0 & \text{otherwise.} \end{cases}$$

If $l \in J_1$ and $j \in M_{n_h}(K_{n_h}(l))$, then, by (3.14) and (1.1), $j \notin M_{n_h}(I_2)$ and $j \notin M_{n_h}(K_{n_h}(J_2 \setminus J_1))$, so

$$(T_{n_h}g)_{K_{n_h}(l)}(s) = \sum_{j \in M_{n_h}(K_{n_h}(l))} g_j(s) = (f_1)_l(s). \quad (3.15)$$

If $l \in J_2 \setminus J_1$, then for every $j \in M_{n_h}(K_{n_h}(l))$ it holds that $j \notin M_{n_h}(I_2)$ and

$$(T_{n_h}g)_{K_{n_h}(l)}(s) = \sum_{j \in M_{n_h}(K_{n_h}(l))} g_j(s) = (f_1)_l(s) - \sum_{j \in M_{n_h}(K_{n_h}(l))} (f_1)_l(s)u_j^l = 0. \quad (3.16)$$

If $l \in M_{n_h}(I_2)$ then

$$(T_{n_h}g)_{K_{n_h}(l)}(s) = \sum_{j \in M_{n_h}(K_{n_h}(l))} (f_2)_{K_{n_h}(l)}(s)\nu_j^{K_{n_h}(l)} = (f_2)_{K_{n_h}(l)}(s). \quad (3.17)$$

Finally, if $i \in I \setminus (K_{n_h}(J_1) \cup K_{n_h}(J_2 \setminus J_1) \cup K_{n_h}(M_{n_h}(I_2))) = I \setminus K_{n_h}(I_1 \cup M_{n_h}(I_2))$, then $M_{n_h}(i)$ has empty intersection with the support of g , hence $(T_{n_h}g)_i(s) = 0$.

By (3.15), (3.16) and (3.17) we have:

$$\begin{aligned} \|T_{n_h}g - f_2\|_{p,\rho}^p &= \int_0^1 \sum_{l \in J_1} |(f_1)_l(s)|^p \rho_{K_{n_h}(l)}(s) ds \leq \int_0^1 \sum_{l \in J_1} |(f_1)_l(s)|^p \frac{C_2}{C_1} \rho_{K_{n_h}(l)}(s_{l,n_h}) ds \\ &\leq \frac{C_2}{C_1} 2\delta_1 \int_0^1 \sum_{l \in J_1} |(f_1)_l(s)|^p ds < \frac{C_2}{C_1^2} 2\delta_1 \int_0^1 \sum_{l \in J_1} |(f_1)_l(s)|^p \rho_l(s) ds = \frac{C_2}{C_1^2} 2\delta_1 \|f_1\|_{p,\rho}^p. \end{aligned}$$

Moreover,

$$\begin{aligned} \|f_1 - g\|_{p,\rho}^p &= \sum_{i \in I} \int_0^1 |(f_1)_i(s) - g_i(s)|^p \rho_i(s) ds \\ &= \sum_{l \in I_2} \sum_{j \in M_{n_h}(l)} \int_0^1 |(f_2)_l(s)\nu_j^l|^p \rho_j(s) ds \\ &\quad + \sum_{l \in J_2 \setminus J_1} \sum_{j \in M_{n_h}(K_{n_h}(l))} \int_0^1 |(f_1)_l(s)u_j^l|^p \rho_j(s) ds \\ &= \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p \sum_{j \in M_{n_h}(l)} |\nu_j^l|^p \rho_j(s) ds + \\ &\quad \sum_{l \in J_2 \setminus J_1} \int_0^1 |(f_1)_l(s)|^p \sum_{j \in M_{n_h}(K_{n_h}(l))} |u_j^l|^p \rho_j(s) ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_2}{C_1} \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p \left(\sum_{j \in M_{n_h}(l)} |\nu_j^l|^p \rho_j(t_{l,n_h}) \right) ds + \\
&\quad \frac{C_2}{C_1} \sum_{l \in J_2 \setminus J_1} \int_0^1 |(f_1)_l(s)|^p \left(\sum_{j \in M_{n_h}(K_{n_h}(l))} |u_j^l|^p \rho_j(s_{l,n_h}) \right) ds \\
&\leq \frac{C_2}{C_1^2} \delta_2^{p-1} \sum_{l \in I_2} \int_0^1 |(f_2)_l(s)|^p \rho_l(s) ds + \frac{C_2}{C_1^2} (2\delta_1)^{p-1} \sum_{l \in J_2 \setminus J_1} \int_0^1 |(f_1)_l(s)|^p \rho_l(s) ds \leq \\
&\leq \frac{C_2}{C_1^2} \delta_2^{p-1} \|f_2\|_{p,\rho}^p + (2\delta_1)^{p-1} \frac{C_2}{C_1^2} \|f_1\|_{p,\rho}^p.
\end{aligned}$$

Now, choosing

$$\delta_1 = \left(\frac{\varepsilon^p C_1^2}{4C_2 \|f_1\|_{p,\rho}^p} \right)^{1/(p-1)} \quad \text{and} \quad \delta_2 = \left(\frac{\varepsilon^p C_1^2}{2C_2 \|f_2\|_{p,\rho}^p} \right)^{1/(p-1)}$$

we get the assertion.

Following a similar argument, it can be proved that if condition (iii) holds, then given $\epsilon > 0$ and $f_1, f_2, f_3, f_4 \in F_\rho^p$ there exists some $k_0 \in \mathbb{N}$ and $g_1, g_2 \in L_\rho^p(L(G))$ such that

$$\|f_1 - g_1\|_{p,\rho} < \epsilon, \quad \|f_2 - g_2\|_{p,\rho} < \epsilon$$

and also

$$\|T_{n_{k_0}} g_1 - f_3\|_{p,\rho} < \epsilon, \quad \|T_{n_{k_0}} g_2 - f_4\|_{p,\rho} < \epsilon,$$

hence $\{T_t\}_{t \geq 0}$ is weakly mixing. $\square$

Remark 3.4. As in the rooted case, one can show that either on $L_\rho^1(L(G))$ or on $L_\rho^p(L(G))$, the translation semigroup is mixing if and only if condition (iii) of Theorem 3.3 holds for the full sequence $(n)_{n \in \mathbb{N}}$.

Remark 3.5. In the case considered in Remark 1.5, the conditions in Theorems 3.1 and 3.3 are exactly those ensuring that the left translation semigroup is hypercyclic when acting on $[0, +\infty[$ or on $\mathbb{R}$ (see e.g. [10, Example 7.10] and [3]).

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