Instanton contributions to the ABJM free energy from quantum M2 branes

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ABSTRACT: We present a quantum M2 brane computation of the instanton prefactor in the leading non-perturbative contribution to the ABJM 3-sphere free energy at large N and fixed level k. Using supersymmetric localization, such instanton contribution was found earlier to take the form $F^{\text{inst}}(N,k) = -(\sin^2 \frac{2\pi}{k})^{-1} \exp(-2\pi\sqrt{\frac{2N}{k}}) + \dots$. The exponent comes from the action of an M2 brane instanton wrapped on $S^3/\mathbb{Z}_k$, which represents the M-theory uplift of the $\mathbb{CP}^1$ instanton in type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$. The IIA string computation of the leading large k term in the instanton prefactor was recently performed in arXiv:2304.12340. Here we find that the exact value of the prefactor $(\sin^2 \frac{2\pi}{k})^{-1}$ is reproduced by the 1-loop term in the M2 brane partition function expanded near the $S^3/\mathbb{Z}_k$ instanton configuration. As in the Wilson loop example in arXiv:2303.15207, the quantum M2 brane computation is well defined and produces a finite result in exact agreement with localization.

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Contents

1 Introduction

The ABJM duality [1] between the supersymmetric Chern-Simons-matter theory and 11d M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, combined with exact localization results [2], provides a possibility to shed light on the structure of M-theory as a theory of quantum supermembranes.

A recent remarkable example was provided in [3], which considered the $\frac{1}{2}$-BPS circular Wilson loop expectation value $\langle W_{\frac{1}{2}} \rangle$ in the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ ABJM theory at large N and fixed level k. This has a dual description in terms of an M2 brane wrapped on $\mathrm{AdS}_2 \times S^1$ [4] in the M-theory background $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. The localization result [5] $\langle W_{\frac{1}{2}} \rangle = \frac{1}{2\sin(\frac{2\pi}{k})} e^{\pi\sqrt{\frac{2N}{k}}} + \dots$ has the exponential factor that comes from the classical value of the M2 brane action, while the k-dependent prefactor was exactly reproduced [3] by the one-loop term in the partition function of the quantum M2 brane. This consistent quantum M2 brane computation naturally suggests extensions to M-theory calculations for other observables that may be similarly compared to localization results.

Indeed, here we present an analogous quantum M2 brane computation of the instanton prefactor in the localization result for the leading large N non-perturbative contribution

to the ABJM free energy F on the 3-sphere, which has the form [6, 7] $F^{\text{inst}}(N, k) =$ $-\frac{1}{\sin^2(\frac{2\pi}{k})} e^{-2\pi\sqrt{\frac{2N}{k}}} + \ldots$. Here the exponent comes from the action of an M2 brane instanton with $S^3/\mathbb{Z}_k$ world-volume geometry. Such M2 instanton wraps the 11d circle S^1 and a $\mathbb{CP}^1$ in $\mathbb{CP}^3$, and it represents the M-theory uplift of the $\mathbb{CP}^1$ instanton in type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ [8]. The IIA superstring computation of the leading large k term in this instanton prefactor $\frac{1}{\sin^2(\frac{2\pi}{k})} \to \frac{k^2}{(4\pi)^2} = \frac{2T}{\pi g_{\text{s}}^2}$ (T is the string tension and g_{s} is the string coupling) was recently presented in a remarkable paper [9]. Here we show that the exact prefactor $\frac{1}{\sin^2(\frac{2\pi}{k})}$ is reproduced by the 1-loop term in the M2 brane partition function expanded near the $S^3/\mathbb{Z}_k$ instanton configuration. As in the Wilson loop example [3], the quantum M2 brane computation is well defined and produces a finite result which is in exact agreement with the localization prediction.

We shall start in section 2 with a review of the relevant localization results for the large N non-perturbative contributions to the ABJM free energy F on 3-sphere. We shall note a surprising similarity between the instanton prefactor in the non-perturbative part of F and in the leading perturbative term in the expectation value of the Wilson loop.

In section 3 we shall argue that the gauge theory free energy should be matched to the "first-quantized" M2 brane partition function. The leading large N perturbative terms should be captured by the 11d supergravity action plus higher derivative corrections (see [10] and refs. there), while non-perturbative contributions should come from M2 brane instanton contributions.

Section 4 will be devoted to the classical solution for the M2 brane wrapped on $S^3/\mathbb{Z}_k \subset S^7/\mathbb{Z}_k$ and the Lagrangian for the bosonic and fermionic quadratic fluctuations around it. We will use a static gauge, where the 11d circle is identified with one periodic M2 coordinate. Expanding in Fourier modes on this circle, we may represent the M2 brane 3d fluctuation action as an action for an infinite Kaluza-Klein tower (labeled by $n = 0, \pm 1, \pm 2, \ldots$) of 2d fields on $\mathbb{CP}^1$, in the presence of a background U(1) gauge field of a magnetic monopole (originating from the Hopf fibration representation of $S^3/\mathbb{Z}_k$). The lowest $n = 0$ level corresponds to the type IIA string fluctuations near the $\mathbb{CP}^1$ instanton already discussed in [8, 9].

The determinants of the resulting bosonic and fermionic operators on S^2 for charged massive 2d fields in the magnetic monopole background are computed in section 5.

In section 6 we perform the sum over the level n of all the 2d fluctuation contributions. We observe that the final result for the M2 brane 1-loop correction is UV finite provided one uses the standard analytic (Riemann ζ-function) regularization, in agreement with expectation of no 1-loop log UV divergences in a 3d theory. We also discover that almost all finite bosonic and fermionic terms mutually cancel and the 1-loop M2 brane partition function Z_1 appears to effectively "localize" to the contribution of just two bosonic 1d degrees of freedom on S^1, i.e. $\log \text{Z}_1 \to -\log \det'\left(-\frac{k^2}{4}\frac{d^2}{ds^2} - 1\right) = -2\sum_{n=1}^{\infty} \log\left(\frac{k^2}{4}n^2 - 1\right)$. Regularizing this 1d determinant in the standard way we get the $\frac{1}{\sin^2(\frac{2\pi}{k})}$ prefactor matching the localization result (up to an integer factor that requires consideration of string-level 0-modes as in [9]).

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We conclude in section 7 by making some remarks on several open problems. In particular, we comment on the role of the $\mathbb{RP}^3 \subset \mathbb{CP}^3$ M2 brane instanton (or the D2 brane instanton [6] in the type IIA string limit) with some details about this second type of instanton contributions to the free energy provided in appendix A, where we also comment on the cases of $k = 1, 2$ which need special consideration.

2 Free energy from localization

Let us start with a review of the localization result for the partition function $Z(N, k)$ of the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ ABJM theory on S^3. We shall assume that $k > 2$ (and in general finite). As a function of N the partition function can be represented as a sum of a perturbative part (given by a series in $\frac{1}{\sqrt{N}}$) and a non-perturbative part involving factors like $e^{-h(k)\sqrt{N}}$ that are exponentially suppressed at large N, i.e.

$$Z = Z^{\mathrm{p}}(N, k) + Z^{\mathrm{np}}(N, k). \tag{2.1}$$

In the Fermi gas approach [11] the localization expression for $Z(N, k)$ is expressed[1] in terms of the grand potential $J(\mu, k)$ of a non-trivial fermionic system as[2]

$$Z(N, k) \equiv e^{-F(N,k)} = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} \, e^{J(\mu,k) - N\mu}. \tag{2.2}$$

The grand potential $J(\mu, k)$ may be split into the sum of the perturbative (polynomial in μ) and non-perturbative (suppressed at large μ) parts

$$J(\mu, k) = J^{\mathrm{p}}(\mu, k) + J^{\mathrm{np}}(\mu, k) \tag{2.3}$$

where

$$J^{\mathrm{p}}(\mu, k) = \frac{1}{3} C(k) \, \mu^3 + B(k) \, \mu + A(k), \qquad C(k) = \frac{2}{\pi^2 k}, \qquad B(k) = \frac{k}{24} + \frac{1}{3k}. \tag{2.4}$$

Here $A(k)$ is the so-called constant map contribution first identified in [17] and admitting the following integral representation [17, 18]

$$A(k) = -\frac{\zeta(3)}{8\pi^2} \left(k^2 - \frac{16}{k} \right) + \frac{k^2}{\pi^2} \int_0^\infty dx \, \frac{x}{e^{kx} - 1} \, \log(1 - e^{-2x}). \tag{2.5}$$

If (2.2) is evaluated keeping only the perturbative part $J^{\mathrm{p}}(\mu, k)$ in (2.3) one finds the perturbative part of Z which is expressed in terms of the Airy function

$$Z^{\mathrm{p}}(N, k) = C(k)^{-\frac{1}{3}} \, e^{A(k)} \, \mathrm{Ai} \left[C(k)^{-\frac{1}{3}} \big(N - B(k) \big) \right]. \tag{2.6}$$

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[1]The localization matrix model representation for the partition function of the ABJM theory on S^3 was first derived in [12] and later mapped to a lens space matrix model solvable in planar limit [13]. Higher order $1/N$ corrections were computed in [6, 14] and resummed in [15] neglecting non-perturbative corrections.

[2]To be precise, the integration contour in (2.2) corresponds to the use of the so-called modified grand potential, see [16] for details.

As for the non-perturbative contributions to Z, the structure of $J^{\mathrm{np}}(\mu, k)$ implies that there are two basic types of exponential corrections.[3] Written as the exponentially suppressed contributions F^{np} to the free energy

$$F \equiv -\log Z = F^{\mathrm{p}} + F^{\mathrm{np}}, \tag{2.7}$$

where the large N expansion of the perturbative part follows from (2.6) (see [10] for details)

$$F^{\mathrm{p}} = -\log Z^{\mathrm{p}} = \tfrac{1}{3}\sqrt{2}\pi k^{1/2} N^{3/2} - \tfrac{\pi}{24\sqrt{2}}\left(k^2 + 8\right) k^{-1/2} N^{1/2} + \tfrac{1}{4}\log\tfrac{32N}{k} - A(k) + \mathsf{O}(N^{-1/2}), \tag{2.8}$$

they are given by the double sum [6, 7]

$$F^{\mathrm{np}} = \sum_{n_{\mathrm{I}}, n_{\mathrm{II}}=0}^{\infty} f_{n_{\mathrm{I}}, n_{\mathrm{II}}}(N, k) \exp\left[-2\pi\sqrt{N}\left(n_{\mathrm{I}}\sqrt{\tfrac{2}{k}} + n_{\mathrm{II}}\sqrt{\tfrac{k}{2}}\right)\right]. \tag{2.9}$$

In the type IIA string theory regime (i.e. in the limit of large N and k with $\lambda = \tfrac{N}{k}$=fixed) these may be interpreted as the contributions of the string world-sheet instantons (wrapping $\mathbb{CP}^1$ in $\mathbb{CP}^3$ [8]) and of the D2-brane instantons (wrapping a 3-cycle $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ in $\mathbb{CP}^3$) respectively [6]. In the M-theory regime (i.e. for large N with fixed k), the world-sheet instantons correspond to the M2 brane instantons wrapping the 11d circle and a $\mathbb{CP}^1$ in $\mathbb{CP}^3$, i.e. $S^3/\mathbb{Z}_k \subset S^7/\mathbb{Z}_k$, while the D2 instantons correspond to the M2 instantons wrapping the analog of $\mathbb{RP}^3 \subset \mathbb{CP}^3$ 3-cycle in $S^7/\mathbb{Z}_k$.

We denote the numbers of the two kinds of instanton respectively as n_{I} and n_{II}. The factors in the exponents in (2.9) correspond to the classical volumes of the two types of the membrane instantons. The terms with both n_{I} and n_{II} nonzero can be thought of as "bound state" contributions [7].

For $k > 2$ the dominant non-perturbative contribution to (2.9) comes from the $S^3/\mathbb{Z}_k$ instanton, i.e. from the $n_{\mathrm{I}} = 1$, $n_{\mathrm{II}} = 0$ term in the sum. Below we shall focus on this leading term and simply refer to it as the "instanton" contribution.

The $n_{\mathrm{II}} = 0$ part of (2.9) originates from the following contribution to $J^{\mathrm{np}}(\mu, k)$ [16]

$$J^{\mathrm{np}}(\mu, k) = \sum_{n_{\mathrm{I}}=1}^{\infty} d_{n_{\mathrm{I}}}(k)\, e^{-\frac{4n_{\mathrm{I}}}{k}\mu} + \dots, \tag{2.10}$$

where the function $d_{n_{\mathrm{I}}}(k)$ may be determined using that the ABJM matrix integral is dual to the partition function of a topological string theory on $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, the $n_{\mathrm{I}} = 1$ instanton term has the coefficient [16]

$$d_1(k) = \frac{1}{\sin^2\left(\tfrac{2\pi}{k}\right)}. \tag{2.11}$$

From (2.2) we find that the contribution of the 1-instanton term to the non-perturbative part of the partition function $Z^{\mathrm{np}}(N, k)$ can be expressed in terms of the perturbative part

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[3]Note that in the saddle point evaluation of (2.2), the large μ and the large N limits are correlated due to the form of the perturbative grand potential.

Z^{p} in (2.6) as

$$Z^{\mathrm{inst}}(N,k) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i}\, e^{J^{\mathrm{p}}(\mu,k)-N\mu}\, d_1(k)\, e^{-\frac{4}{k}\mu} = d_1(k)\, Z^{\mathrm{p}}(N+\tfrac{4}{k},k)\,. \qquad (2.12)$$

The corresponding 1-instanton term in the non-perturbative part of free energy (2.7) is then (cf. (2.6), (2.9))

$$F^{\mathrm{np}}(N,k) = F^{\mathrm{inst}}(N,k) + \cdots\,, \qquad (2.13)$$

$$F^{\mathrm{inst}}(N,k) = -d_1(k)\frac{\mathrm{Ai}\left[C(k)^{-\frac{1}{3}}(N-B(k)+\frac{4}{k})\right]}{\mathrm{Ai}\left[C(k)^{-\frac{1}{3}}(N-B(k))\right]}$$

$$= F_1^{\mathrm{inst}}(N,k)\left[1 + \tfrac{\pi}{\sqrt{2k}}\tfrac{k^2-40}{12k}\tfrac{1}{\sqrt{N}} + \ldots\right]\,, \qquad (2.14)$$

$$F_1^{\mathrm{inst}}(N,k) = -d_1(k)\, e^{-2\pi\sqrt{N}\sqrt{\frac{2}{k}}} = -\frac{1}{\sin^2(\frac{2\pi}{k})}\, e^{-2\pi\sqrt{\frac{2N}{k}}}\,. \qquad (2.15)$$

Here F_1^{inst} is the leading large N term in the 1-instanton contribution.

It is interesting to notice a close resemblance of the expression for $F^{\mathrm{inst}}(N,k)$ in (2.14) and the one for the perturbative part of the expectation value of the $\frac{1}{2}$-BPS circular Wilson loop in the ABJM theory derived using localization in [5] (see also [11, 14, 15, 19])[4]

$$\langle W_{\frac{1}{2}}\rangle = \frac{1}{2\sin(\frac{2\pi}{k})}\frac{\mathrm{Ai}\left[C(k)^{-\frac{1}{3}}(N-B(k)-\frac{2}{k})\right]}{\mathrm{Ai}\left[C(k)^{-\frac{1}{3}}(N-B(k))\right]}$$

$$= \frac{1}{2\sin(\frac{2\pi}{k})}e^{\pi\sqrt{\frac{2N}{k}}}\left[1 - \tfrac{\pi}{\sqrt{2k}}\tfrac{k^2+32}{24k}\tfrac{1}{\sqrt{N}} + \ldots\right]\,. \qquad (2.16)$$

Compared to (2.14) here the prefactor is $\frac{1}{2\sin(\frac{2\pi}{k})}$ instead of $-\frac{1}{\sin^2(\frac{2\pi}{k})}$ in (2.15) and the argument of the Airy function in the numerator is $N-B(k)-\frac{2}{k} = N - \frac{k}{24} - \frac{7}{3k}$ instead of $N-B(k)+\frac{4}{k} = N - \frac{k}{24} + \frac{11}{3k}$. These different shifts explain why the coefficients in the exponentials in (2.15) and (2.16) differ by factor of -2.[5]

This factor of -2 has a string theory (or wrapped M2 brane) interpretation: the regularized area of the AdS_2 minimal surface in the case of the Wilson loop is -2π (minus area of a disk), while the area of $\mathbb{CP}^1$ is $+4\pi$.[6] Also, the fact that the prefactor in the instanton contribution to the free energy in (2.15) is proportional to the square of the prefactor in the Wilson loop in (2.16) may be attributed to the a heuristic expectation that the partition function on a 2-sphere is related to a square of the partition function on a disk.

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[4]We use the same normalization of $\langle W_{\frac{1}{2}}\rangle$ as in [3, 20].

[5]Indeed, the large x asymptotics of $\mathrm{Ai}(x) \sim x^{-1/4}e^{-2/3x^{3/2}}$ implies that for the ratio with the arguments $N-B(k)+\frac{a}{k}$ and $N-B(k)$ the asymptotics is $e^{-a\pi\sqrt{\frac{N}{2k}}}$.

[6]In more detail, for the WL we have an AdS_2 inside AdS_4 of radius $R/2$ so the area is $-\frac{1}{2}\pi R^2$. For the M2 brane instanton, the $\mathbb{CP}^3$ factor in the metric has radius R but the $\mathbb{CP}^1$ we wrap around is a sphere of radius $R/2$, so the resulting area is πR^2. Thus the relative factor is still -2 as above.

Our aim below will be to reproduce (2.15) on the dual M-theory side by a quantum M2 brane computation, in full analogy to how that was done in [3] for the leading term in the Wilson loop expression in (2.16).

3 Free energy from M2 brane partition function

Motivated by the expected duality between the ABJM theory and M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ it is natural to expect that the perturbative part of the free energy (2.8) should be reproduced by some higher derivative extension of the 11d supergravity action evaluated on the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ background (for a recent discussion and references to related work see [10, 21]). Indeed, it was found in [14] that the leading $N^{3/2}$ term in (2.8) is matched by the on-shell value of the Euclidean 11d supergravity action.[7]

More generally, we shall conjecture that the gauge theory free energy should be reproduced by some properly defined supermembrane partition function,

$$F \sim Z_{\mathrm{M2}}, \qquad Z_{\mathrm{M2}} = \int [dx\, d\theta]\, e^{-S_{\mathrm{M2}}[x,\theta]}, \tag{3.1}$$

where S_{M2} is the M2 brane action on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with the dimensionless coefficient of the effective tension (R is the radius of S^7 or twice the radius of AdS_4, see (4.1) below)

$$\mathrm{T}_2 \equiv R^3 T_2 = \frac{1}{(2\pi)^2} \frac{R^3}{\ell_P^3} = \frac{\sqrt{2k}}{\pi}\sqrt{N}, \qquad \frac{R}{\ell_P} = (32\pi^2 N k)^{1/6}. \tag{3.2}$$

Then for fixed k (or fixed radius of the 11d circle) the semiclassical large T_2 expansion of Z_{M2} should be equivalent to the large N expansion on the gauge theory side.

One may further conjecture that the perturbative part of Z_{M2} in the large $\mathrm{T}_2 \sim \sqrt{N}$ limit may be captured by an expansion near "point-like" M2 branes or, more precisely, degenerate 3-surfaces with a topology of S^1 times a point which have zero 3-volume. At the same time, the non-perturbative $e^{-a\mathrm{T}_2} = e^{-a\frac{\sqrt{2k}}{\pi}\sqrt{N}}$ contributions may come from saddle points with non-vanishing 3-volumes, e.g. from M2 branes wrapping the M-theory circle and a $\mathbb{CP}^1 \subset \mathbb{CP}^3$, or a 3-cycle in $\mathbb{CP}^3$ (and their superpositions). Symbolically, we may write

$$Z_{\mathrm{M2}} = Z_{\mathrm{M2}}^{(0)} + Z_{\mathrm{M2}}^{\mathrm{inst}} + \dots. \tag{3.3}$$

Here the first term (coming from contributions of "degenerate" M2 brane surfaces) when expanded at large k should represent the sum of all perturbative tree level plus higher loop type IIA string corrections to the on-shell value of the partition function.

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[7]More precisely, if one directly evaluates the 11d action on the $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ solution one gets the leading term in (2.8) with an extra factor of $-\frac{1}{2}$ (see discussion in appendix B of [10]). There is a subtlety here: as $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ is an "electric" 11d solution, one may use a prescription that the sign of the flux F_{mnkl}^2 term is to be reversed when evaluating the on-shell value of the 11d action (this prescription is equivalent to adding a particular boundary term; a related observation is that adding a total derivative 4d term $\int d^4 x\, \epsilon^{mnkl} F_{mnkl}$ changes the value of the 4d cosmological constant [22]). This then gives the same value as found from the effective 4d action reconstructed to have the same AdS_4 space as its solution (see [23, 24] and refs. there for related discussions). It is the latter 4d action that was used as a starting point in [14] following [25].

A motivation for this suggestion comes from considering the perturbative type IIA string limit ($k \sim N \gg 1$) in which the radius of the 11d circle is small and thus the membrane partition function should effectively reduce to the type IIA string partition function (cf. [26–30]). The latter, expanded in the inverse string tension, should be closely related to the low-energy string effective action [31–35] and thus also to the on-shell value of the latter.[8] The contribution of the type IIA string instanton saddle point [8, 9] will then naturally supplement the perturbative part of the string partition function.

One should add of course a reservation that, as the M2 brane action is highly non-linear (even its bosonic part does not become quadratic in any gauge), it is not clear how to define its expansion near a degenerate "point-like" membrane configuration; this apparently requires a non-perturbative approach to the corresponding quantum 3d world-volume theory.[9]

In contrast, the semiclassical expansion of the M2 brane path integral near a classical solution with a non-zero 3d volume is well defined [36–40] as in this case one is able to fix a static gauge and thus develop the standard gaussian perturbation theory. A remarkable recent example is the computation [3] of the Wilson loop prefactor in (2.16) from the quantum M2 fluctuation determinants near the corresponding $AdS_2 \times S^1$ minimal 3-surface.

Below we will perform a similar semiclassical computation in the case of the M2 brane instanton wrapping $S^3/\mathbb{Z}_k$, reproducing the $\frac{1}{\sin^2 \frac{2\pi}{k}}$ prefactor in (2.15) from the corresponding 1-loop fluctuation determinants.

To be able to make a precise comparison to the localization result for the free energy one is to decide about the proportionality coefficient in (3.1). We will assume (as was also done in the string-theory limit in [9]) that

$$F = -Z_{\text{M2}}^{(0)} - Z_{\text{M2}}^{\text{inst}} + \ldots = (S_{\text{sugra}} + \ldots) - Z_{\text{M2}}^{\text{inst}} + \ldots. \tag{3.4}$$

This assumption is based on the expectation that in the string theory limit the (on-shell value of) string effective action should be given by minus string partition function as was originally suggested in [32].[10]

This may be motivated [32] by analogy with the first-quantized point-particle representation of the standard quantum field theory effective action. For example, the expression for the torus part of the string partition function should effectively reduce, in

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[8]To see directly how a non-zero on-shell value of the string effective action is reproduced in the case of AdS space factors and RR backgrounds requires understanding how boundary terms are captured by the string path integral, which is currently an open problem.

[9]In particular, going off shell, it is not even clear how one would compute the leading terms of the 11d supergravity action by starting with the M2 brane path integral.

[10]This is true, e.g., in the open string theory where the string partition function on the disk in conformal gauge is negative-definite as the regularized value of the $SL(2,\mathbb{R})$ Möbius volume is negative, $\text{vol}(SL(2,\mathbb{R})) = \text{vol}(AdS_2)\,\text{vol}(S^1) = -4\pi^2$ (cf. [41–46]). In the closed bosonic string sigma model on S^2 near a conformal point one has $Z_{\text{str}} = \frac{1}{\Omega}Z$, where $Z = \int d^D x \sqrt{G}\, e^{-2\phi} \exp(C\,\chi \log \Lambda + \ldots)$, $C = \frac{1}{6}D + \ldots$, $\chi(S^2) = 2$ and Ω is the regularized volume of the $SL(2,\mathbb{C})$ Möbius group, $\Omega = \text{vol}(SL(2,\mathbb{C})) = \text{vol}(\frac{SO(1,3)}{SO(3)})\,\text{vol}(SO(3)) = -4\pi^3 \log \Lambda$, where $\text{vol}(\frac{SO(1,3)}{SO(3)}) = \text{vol}(AdS_3) = -2\pi \log \Lambda$. Then $S_{\text{str}} = -Z_{\text{str}} = \frac{1}{2\pi^3}\int d^D x \sqrt{G}\, e^{-2\phi} C$. In general, one is to replace $\frac{1}{\log \Lambda}$ by $\frac{d}{d\log \Lambda}$ (and renomalize the fields) [33, 34, 47] getting the effective action $S_{\text{str}} = \frac{1}{2\pi^3}\int d^D x \sqrt{G}\, e^{-2\phi} \tilde{\beta}^\phi$, $\tilde{\beta}^\phi = C + \ldots = \frac{1}{6}(D-26) - \frac{1}{4}\alpha' R^{(D)} + \ldots$, which thus has the right sign for a Euclidean action.

the point-like limit, to the familiar relation between the 1-loop quantum effective action and minus the particle path integral on a circle, $\Gamma_1 = \frac{1}{2} \log \det(-\partial^2 + \ldots) = -Z_{\text{part.}}$, with $Z_{\text{part.}} = \int_0^\infty \frac{dt}{2t} \int [dx(\tau)] \, e^{-\int_0^t d\tau(\frac{1}{2}\dot{x}^2 + \ldots)}$.[11]

Before proceeding, let us comment on the interpretation of (2.15) expanded at large k from the point of view of type IIA string theory. Let us recall the relations for the type IIA string theory coupling and tension [1]

$$g_{\text{s}} = \sqrt{\pi} \left(\frac{2}{k}\right)^{5/4} N^{1/4} = \frac{\sqrt{\pi}\,(2\lambda)^{5/4}}{N} \, , \qquad \lambda = \frac{N}{k} \, , \tag{3.5}$$

$$T = \frac{1}{8\pi} \frac{R_{10}^2}{\alpha'} = g_{\text{s}}^{2/3} \frac{R^2}{8\pi\alpha'} = \frac{\sqrt{\lambda}}{\sqrt{2}} \, , \qquad \frac{g_{\text{s}}^2}{8\pi\,T} = \frac{\lambda^2}{N^2} = \frac{1}{k^2} \, , \tag{3.6}$$

where $R/2$ and $R_{10}/2$ are the AdS curvature radii in the 11d and 10d metrics. Just like the $\frac{1}{2\sin\frac{2\pi}{k}}$ prefactor in the Wilson loop expectation value (2.16) sums up all the leading large T terms in the higher genus string corrections to the prefactor $\frac{1}{\sqrt{2\pi}} \frac{\sqrt{T}}{g_{\text{s}}} = \frac{k}{4\pi}$ from the disk diagram [20], similarly the prefactor in F_1^{inst} in (2.15) may be written as

$$\frac{1}{\sin^2\left(\frac{2\pi}{k}\right)} = \frac{1}{\sin^2\left(\sqrt{\frac{\pi}{2}} \frac{g_{\text{s}}}{\sqrt{T}}\right)} = \frac{2}{\pi} \frac{T}{g_{\text{s}}^2} + \frac{1}{3} + \frac{\pi}{30} \frac{g_{\text{s}}^2}{T} + \frac{\pi^2}{378} \frac{g_{\text{s}}^4}{T^2} + \ldots \, , \tag{3.7}$$

where the leading term is the contribution of the string tree level 2-sphere diagram, and corrections are coming from higher genus contributions. To leading order in large tension T at each order in g_{s}^2 we expect that the exponential with the classical instanton action in (2.15) will remain the same (the area of handles attached to 2-sphere will be effectively negligible) but the prefactor will get corrected order by order in g_{s} as in (3.7). This picture is corroborated by the M2-brane derivation of (2.15), (3.7) below.

The leading 2-sphere term in (3.7) was recently discussed in [9], where its overall factor was fixed using relative normalization to other available data. Like in the WL computation in [3], here we will not have to rely on indirect arguments to fix the tension dependence of the prefactor (all UV divergences will cancel automatically) but will still need to address the issue of the 0-modes (appearing only from the string-level fluctuations) arguing for the related overall factor of 2 following [9].

4 M2 brane wrapped on $S^3/\mathbb{Z}_k$

4.1 Classical M2 brane action in $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ background

Let us start with a review of some basic relations for the $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ background and the classical M2 brane action (see, e.g., [3, 4]).

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[11]The integral over the Schwinger parameter t can be understood as arising from the path integral over the einbein upon gauge fixing the worldline diffeomorphism invariance (see for instance [48]).

The 11d metric is $(n, m = 1, 2, 3; \ y \equiv y + 2\pi)$

$$ds^2 = \frac{R^2}{4} ds^2_{AdS_4} + R^2 ds^2_{S^7/\mathbb{Z}_k}, \qquad ds^2_{S^7/\mathbb{Z}_k} = ds^2_{\mathbb{CP}^3} + \frac{1}{k^2}(dy + kA)^2, \qquad (4.1)$$

$$ds^2_{\mathbb{CP}^3} = \frac{(1 + |w|^2)\, dw^n\, d\bar{w}^n - w^n \bar{w}^m dw^m d\bar{w}^n}{(1 + |w|^2)^2}, \qquad (4.2)$$

$$A = \frac{i}{2}(\bar{\partial} - \partial) \log(1 + |w|^2) = \frac{i}{2} \frac{1}{1 + |w|^2}(w^n d\bar{w}^n - \bar{w}^n dw^n). \qquad (4.3)$$

We shall assume that AdS_4 has Euclidean signature with boundary S^3. This 11d background is then supported by $F_4 = dC_3 = -i\frac{3}{8}R^3\text{vol}(AdS_4)$.

The action for a (Euclidean) M2 brane in this background is given by [37, 49–52]

$$S = T_2 \left(\int d^3\xi \sqrt{\det G} + i \int C_3 \ + \ \text{fermionic terms} \right), \qquad T_2 = \frac{1}{(2\pi)^2} \frac{1}{\ell_P^3}. \qquad (4.4)$$

Here we are interested in the M2 brane configuration with $S^3/\mathbb{Z}_k$ world-volume, that is wrapped on the 11d circle y of radius R/k and on $\mathbb{CP}^1 \subset \mathbb{CP}^3$. This is the M2 uplift of the IIA string $\mathbb{CP}^1$ instanton of [8]. The $\mathbb{CP}^1$ will be chosen as the $w^2 = w^3 = 0$ surface in $\mathbb{CP}^3$.[12] We fix the world-volume reparametrization invariance using the static gauge: we identify $(w_1, \bar{w}_1, y)$ with the 3 real world-volume coordinates $\xi^i = (u, v, s)$ according to

$$w^1 \equiv z = u + iv, \qquad \bar{w}^1 = \bar{z} = u - iv, \qquad y = s, \qquad s \in (0, 2\pi]. \qquad (4.5)$$

As the C_3 potential has only the AdS_4 components the bosonic part of the corresponding Euclidean M2 brane action is given by

$$S_{\text{cl}} = T_2 \int d^3\xi \sqrt{g}, \qquad (4.6)$$

where g_{ij} is the induced world-volume metric

$$ds^2_3 = g_{ij}d\xi^i d\xi^j = R^2 \frac{dz\, d\bar{z}}{(1 + |z|^2)^2} + \frac{R^2}{k^2}\big[ds + kA(z, \bar{z})\big]^2. \qquad (4.7)$$

This is the metric of $S^3/\mathbb{Z}_k$ (for $k = 1$ this is the standard Hopf metric of S^3 with radius R). The explicit form of the metric and the 1-form $A = A_u du + A_v dv$ in the real basis (u, v, s) is

$$g_{ij} = R^2 \begin{pmatrix} \kappa^2(1 + v^2) & -\kappa^2 uv & \frac{1}{k}A_u \\ -\kappa^2 uv & \kappa^2(1 + u^2) & \frac{1}{k}A_v \\ \frac{1}{k}A_u & \frac{1}{k}A_v & \frac{1}{k^2} \end{pmatrix}, \quad A = \kappa(-vdu + udv), \quad \kappa \equiv (1 + u^2 + v^2)^{-1},$$

$$(4.8)$$

$$g^{ij} = \frac{1}{\kappa^2 R^2} \begin{pmatrix} 1 & 0 & -kA_u \\ 0 & 1 & -kA_v \\ -kA_u & -kA_v & k^2\kappa \end{pmatrix}, \qquad \sqrt{g} = \frac{R^3}{k}\kappa^2. \qquad (4.9)$$

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[12]This corresponds to a particular $\frac{1}{2}$-BPS M2 brane solution discussed in [53].

The resulting classical value of the action (4.6) is

$$S_{\text{cl}} = T_2 R^3 \, \text{vol}(S^3/\mathbb{Z}_k) = \frac{1}{k} T_2 R^3 \, \text{vol}(S^3) = \frac{2\pi^2}{k} \text{T}_2 \,. \tag{4.10}$$

Here the effective dimensionless tension is

$$\text{T}_2 \equiv R^3 T_2 = \frac{1}{(2\pi)^2} \frac{R^3}{\ell_P^3} = \frac{1}{\pi} \sqrt{2Nk} \,, \tag{4.11}$$

so that

$$S_{\text{cl}} = 2\pi \sqrt{\frac{2N}{k}}. \tag{4.12}$$

This is also the same as the value of the classical action of the string world sheet wrapped on $\mathbb{CP}^1$ in $\text{AdS}_4 \times \mathbb{CP}^3$ [8], i.e. $S_{\text{cl}} = 2\pi \sqrt{2\lambda}$ (cf. (3.6)).

4.2 Quadratic fluctuation Lagrangian

Our aim will be to compute the 1-loop prefactor Z_1 in the corresponding 1-instanton contribution to the M2 brane partition function in (3.3)

$$Z_{\text{M2}}^{\text{inst}} = \text{Z}_1 \, e^{-S_{\text{cl}}} + \dots \,. \tag{4.13}$$

The factor Z_1 will be expressed in terms of the determinants of operators of the bosonic and fermionic fluctuations which will be functions of the 3d coordinates (u, v, s) in the static gauge defined in (4.5).

In this static gauge we will have 8 real bosonic fluctuations: 4 in the AdS_4 directions and 4 in the 2 complex transverse $\mathbb{CP}^3$ directions w_2, w_3. Fixing a κ-symmetry gauge (like in [37]), we will also have 8 fermionic fluctuations.

Expanding the action (4.4), one finds that the fluctuations of w_2 and w_3 decouple and their contributions are the same. Considering, e.g., w_2 and setting (cf. (4.8))[13]

$$w_2(u, v, s) = \kappa^{-1/2} \, \phi(u, v, s) \,, \tag{4.14}$$

we get for the corresponding quadratic fluctuation Lagrangian

$$\mathcal{L}_2(\phi) = \frac{R^2}{2} \sum_{i,j=1}^{3} g^{ij} D_i \bar{\phi} \, D_j \phi - \bar{\phi}\phi - \frac{i}{2} k (\bar{\phi} \partial_s \phi - \partial_s \bar{\phi} \, \phi) \,, \tag{4.15}$$

$$D_i \phi = (\partial_i - i \, A_i) \, \phi, \qquad D_i \bar{\phi} = (\partial_i + i \, A_i) \, \bar{\phi} \,. \tag{4.16}$$

Here $\partial_i = (\partial_u, \partial_v, \partial_s)$ and $A_i = (A_u, A_v, 0)$ is the 3d gauge potential with A_u, A_v defined in (4.8).

As s is a periodic coordinate we may interpret the corresponding 3d action $\int d^3\xi \sqrt{g} \, \mathcal{L}_2$ as a 2d action for an infinite tower of the Fourier modes of ϕ by setting $\phi(u, v, s) = \sum_n \phi_n(u, v) \, e^{ins}$. This 2d action will be defined on $\mathbb{CP}^1$ with the metric g_{ab} of a 2-sphere of radius $R/2$ (the first term in (4.7)). The corresponding Lagrangian for a tower of 2d

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[13]Here $\kappa(u, v)$ is defined in (4.8). We also rescale all fluctuation fields by $\text{T}_2^{-1/2}$.

charged massive complex scalars ϕ_n on the 2-sphere coupled to the background 2d abelian gauge field potential A_a is then (here $a, b = 1, 2$ label the u, v directions)

$$\mathcal{L}_2(\phi_n) = \frac{R^2}{2}\Big[\sum_{a,b=1}^{2} g^{ab} D_a \bar{\phi}_n \, D_b \phi_n + M^2 \, \bar{\phi}_n \phi_n \Big], \tag{4.17}$$

$$D_a \phi_n = (\partial_a - i\,q A_a)\phi_n, \qquad D_a \bar{\phi}_n = (\partial_a + i\,q A_a)\phi_n, \tag{4.18}$$

$$q = 1 + nk, \qquad\qquad R^2 M^2 = -2 + 2nk + n^2 k^2. \tag{4.19}$$

The fluctuations in the AdS$_4$ directions may be represented by 4 real 3d massless scalars η^r ($r = 1, 2, 3, 4$) with the 3d Lagrangian

$$\mathcal{L}_2(\eta) = \frac{R^2}{2} \sum_{i,j=1}^{3} g^{ij} \partial_i \eta^r \, \partial_j \eta^r. \tag{4.20}$$

The corresponding 2d Lagrangian for the tower of the Fourier modes of $\eta^r(u, v, s) = \sum_n \eta_n^r(u, v)\, e^{\mathrm{ins}}$ (with $\eta_{-n}^r = \bar{\eta}_n^r$) then has a similar form to (4.17)

$$\mathcal{L}_2(\eta_n) = \frac{R^2}{2}\left[\sum_{a,b=1}^{2} g^{ab} D_a \eta_{-n}^r \, D_b \eta_n^r + M^2 \, \eta_{-n}^r \eta_n^r \right], \tag{4.21}$$

$$D_a \eta_n^r = (\partial_a - iq A_a)\, \eta_n^r, \qquad q = nk, \qquad R^2 M^2 = n^2 k^2. \tag{4.22}$$

Let us comment on the explicit form of the background metric and gauge field in the 2d actions corresponding to (4.17), (4.21). The metric g_{ab} obtained by the restriction of the induced metric to the u, v subspace is the metric of the 2-sphere with radius $L = \frac{R}{2}$:

$$g_{ab} d\xi^a d\xi^b = R^2 \frac{du^2 + dv^2}{(1 + u^2 + v^2)^2} = L^2(d\theta^2 + \sin^2\theta \, d\varphi^2), \qquad u + iv = \tan\frac{\theta}{2}\, e^{i\varphi}, \qquad L = \frac{R}{2}. \tag{4.23}$$

The gauge potential A in (4.8) written in these angular coordinates θ, φ reads

$$A = \frac{1}{2}(1 - \cos\theta)\, d\varphi, \quad F = dA = \frac{1}{2}\sin\theta \, d\theta \wedge d\varphi, \quad \frac{1}{2\pi}\int_{S^2} F = 1. \tag{4.24}$$

It may be interpreted as a field of a unit-charge 3d monopole placed at the center of a unit-radius S^2.

The fluctuation operators in (4.17) (4.21) are thus the standard 2nd order operators on S^2 of radius L in the magnetic monopole background

$$\Delta = -D^2 + M^2, \qquad D_a = \partial_a - iq A_a. \tag{4.25}$$

Measuring the masses in terms of the radius $L = R/2$ of S^2 we thus get the following bosonic spectrum: 2 towers of complex ϕ_n modes and 4 towers of $\eta_n = \bar{\eta}_{-n}$ modes with

$$\phi_n: \quad m^2 \equiv L^2 M^2 = -\frac{3}{4} + \frac{1}{4}(1 + nk)^2, \quad q = 1 + nk; \qquad \eta_n: \quad m^2 = \frac{1}{4}(nk)^2, \quad q = nk. \tag{4.26}$$

The M2 brane action in $AdS_4 \times S^7/\mathbb{Z}_k$ background is related to the type IIA string in the corresponding $AdS_4 \times \mathbb{CP}^3$ background by the double dimensional reduction [26]. Indeed, the $n = 0$ parts of the 2d fluctuation Lagrangians (4.17), (4.21) are equivalent to the ones in the type IIA string case in [9].

This relation is even more direct in the fermionic sector as the fermionic fields in the M2 brane and the type IIA GS string actions are in direct correspondence (their components are essentially the same, the only difference is due to the M2 brane fields depending on the extra coordinate s).

It is thus straightforward to reconstruct the quadratic part of the M2 brane fermionic action from its lowest KK level $n = 0$ term, i.e. by starting with the fermionic part of the type IIA superstring action used in [9]. The detailed structure of the quadratic fermionic Lagrangian in the string case[14] shows that it is equivalent to the sum of 2d fermionic terms $\bar{\psi}\mathcal{D}\psi$ where $\mathcal{D}$ is the standard 2d Dirac operator on the 2-sphere of radius $L = R/2$ (4.23) in the monopole background (4.24) with a particular mass term (cf. (4.25))

$$\mathcal{D} = i\,\slashed{D} + M_1\sigma_3 + M_2, \qquad \slashed{D} = \sigma^{\mathrm{a}}e^a_{\mathrm{a}}\left(\partial_a + \frac{i}{2}\omega_a\sigma_3 - iqA_a\right). \qquad (4.27)$$

Here e^a_{a} is the inverse zweibein on the sphere ($\mathrm{a} = 1, 2$), ω_a is the 2d spin connection, and $\sigma_i = (\sigma_{\mathrm{a}}, \sigma_3)$ are the Pauli matrices. The explicit values of the dimensionless mass parameters are

$$m_1 \equiv LM_1 = -\frac{1}{4}(u - u'), \qquad m_2 \equiv LM_2 = -\frac{1}{4} - \frac{3}{4}uu', \qquad u, u' \in \{1, -1\}, \quad (4.28)$$

where u, u' represent 4 independent sign factors arising from 10-d Gamma matrices in a suitable representation. Thus one finds 8 fermionic modes organized as 2d fermionic fields with 4 choices of mass parameters in (4.27) $m_1 = (-\frac{1}{2}, \frac{1}{2}, 0, 0)$, $m_2 = (\frac{1}{2}, \frac{1}{2}, -1, -1)$. In addition, the values of the charges are $q = (1, -1, 0, 0)$ [9].

To generalize this to the M2 brane case, again expanding the fermions in the Fourier modes $\theta(u, v, s) = \sum_n \theta_n(u, v)\, e^{ins}$ in the 3rd direction s (with s identified with the 11-th direction y in the static gauge (4.5) we use), i.e. we need to account for the contribution of the corresponding term

$$\bar{\theta}(\Gamma^A E^y_A \partial_s + \ldots)\theta = \bar{\theta}(\Gamma^{11} E^y_{11}\partial_s + \Gamma^a E^y_a \partial_s + \ldots)\theta\,, \qquad (4.29)$$

in the supermembrane action [37, 49].[15] As Γ_{11} gets expressed in terms of σ_3 (times a unit matrix) we learn that the M_1 term in (4.28) gets a shift (due to $\partial_s\theta \to in\theta_n$) while M_2 stays the same, i.e.[16] $\Delta m_1 = -\frac{1}{2}nk$, $\Delta m_2 = 0$. In addition, the presence of the off-diagonal Ady term in the 11d metric (4.1), and thus in the corresponding vielbein, implies also that the

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[14]We thank the authors of [9] for kindly sharing their unpublished notes on the derivation of the fermionic fluctuation part of the type IIA superstring action in the background of the $\mathbb{CP}^1$ instanton.

[15]The quadratic fermionic term in the M2 brane action is built using the gravitino covariant derivative in 11d supergravity and is straightforward to analyze, given that here $y = x^{11}$ is an isometric direction.

[16]Here $n = 0, \pm 1, \pm 2, \ldots$ so the sign of the shift is not important.

∂_s term in (4.29) leads to a shift of the coefficient of the A-term in the covariant derivative in (4.27), i.e. to the shift of the charge $\Delta q = nk$.[17]

Thus we find that the Lagrangian for the tower of the 2d fermionic modes originating from the quadratic fermionic part of the M2 brane action can be represented by a collection of 4 2d fermionic fields with the Dirac-like operators (4.27) where the parameters depend on k as

$$m_1 = -\frac{1}{4}(u - u') - \frac{1}{2}nk, \qquad m_2 = -\frac{1}{4} - \frac{3}{4}uu', \qquad q = -2m_1. \qquad (4.30)$$

5 Determinants of operators on S^2 in monopole background

5.1 Spectra of operators and formal spectral sums

For a massless scalar field of charge q on S^2 in the field of a monopole normalized as in (4.24) the spectrum of the corresponding Laplace operator (4.25) was found in [54]. Its eigenvalues (normalized to the radius L of the sphere, i.e. multiplied by L^2) and degeneracies are given by

$$\lambda_\ell = \ell(\ell+1) - \frac{q^2}{4}, \qquad \ell - \frac{|q|}{2} = 0, 1, 2, \dots, \qquad \deg\lambda_\ell = 2\ell + 1. \qquad (5.1)$$

Inclusion of mass term in the operator can be done by the obvious shift $\lambda_\ell \to \lambda_\ell + m^2, \ \ m \equiv LM$. Then the formal (unregularized) expression for the corresponding determinant may be written as

$$\log\det\left[L^2(-D^2 + M^2)\right] = \sum_{\ell=\frac{|q|}{2}}^{\infty} (2\ell+1)\log\left[\ell(\ell+1) - \frac{q^2}{4} + m^2\right] = \sum_{\ell=\frac{|q|+1}{2}}^{\infty} 2\ell\log\left[\ell^2 - \frac{1}{4} - \frac{q^2}{4} + m^2\right]. \qquad (5.2)$$

To obtain the last relation we redefined the summation index ℓ by $\frac{1}{2}$; such a shift is allowed assuming one adopts a spectral regularization like spectral zeta function (as we will do below). Introducing the notation

$$s_p(\mu) \equiv \sum_{\ell=p, \ \ell^2 \neq \mu}^{\infty} 2\ell\log(\ell^2 - \mu), \qquad (5.3)$$

where p and μ are some parameters and $\ell - p$ takes non-negative integer values, eq. (5.2) may be written as

$$\log\det\left[L^2(-D^2 + M^2)\right] = s_p(\mu), \qquad p = \frac{|q|+1}{2}, \qquad \mu = \frac{1}{4} + \frac{q^2}{4} - m^2. \qquad (5.4)$$

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[17]Let us note that the reason why the mass m_1 gets a shift $\frac{1}{2}nk$ while the charge gets a shift of nk (that seems to contradict the usual KK intuition) has to do with the fact that as follows from the structure of the 11d metric the relevant $S^3/\mathbb{Z}_k$ subspace has the radius of the 2-sphere being $L = R/2$ and that translates into the factor 2 in the relative nk shift of charge compared to mass. The same is seen also in the bosonic spectrum in (4.26).

Turning to the 2d spin $\frac{1}{2}$ charged field case, the corresponding massless Dirac operator $i\not{D}$ defined in (4.27) has eigenspinors with the following eigenvalues (normalized again to the radius L of the sphere) and degeneracies [55, 56]

$$\lambda_\ell = \pm\sqrt{\ell^2 - \frac{q^2}{4}}, \qquad \ell - \frac{|q|}{2} = 0, 1, 2, \ldots, \qquad \deg\lambda_\ell = 2\ell. \tag{5.5}$$

For the minimal value $\ell = \frac{|q|}{2}$ (assuming $|q| \geqslant 1$), we get $|q|$ zero modes with definite chirality (or the eigenvalue of the σ_3 matrix) equal to the sign of q, consistently with the Atiyah-Singer index theorem on the 2-sphere [57].

In the case of the massive operator $i\not{D} + M_1\sigma_3 + M_2$ in (4.27) this spectrum leads to the following expression for the determinant (here $m_a = LM_a$)

$$\log\det\left[L(i\not{D} + M_1\sigma_3 + M_2)\right] = |q| \log\left|\mathrm{sign}(q)\, m_1 + m_2\right| + \sum_{\ell=\frac{|q|}{2}+1}^{\infty} 2\ell \log\left(\ell^2 - \frac{q^2}{4} + m_1^2 - m_2^2\right). \tag{5.6}$$

Here the first term represents the contribution of the 0-mode of $\not{D}$ which has a definite chirality equal to $\mathrm{sign}(q)$, as explained above.[18] That the eigenvalues of all other modes contain the effective mass-squared parameter $m_1^2 - m_2^2$ (cf. (5.2)) follows from the direct evaluation of the determinant in (5.6) or can be seen from "squaring" the first-order operator as in (5.16).

The fermionic counterpart $\tilde{s}_p(\mu; \mathrm{w})$ of the function $s_p(\mu)$ in (5.3) may be defined as

$$\tilde{s}_p(\mu; \mathrm{w}) \equiv 2p \log \mathrm{w} + s_{p+1}(\mu), \tag{5.7}$$

so that (cf. (5.3), (5.4))

$$\log\det\left[L(i\not{D} + M_1\sigma_3 + M_2)\right] = \tilde{s}_p(\mu; \mathrm{w}), \tag{5.8}$$

$$p = \frac{|q|}{2}, \qquad \mu = \frac{q^2}{4} - m_1^2 + m_2^2, \qquad \mathrm{w} = \left|\mathrm{sign}(q)\, m_1 + m_2\right|. \tag{5.9}$$

5.2 Computing determinants using spectral ζ-function

The determinant of an elliptic 2nd order operator Δ can be expressed in terms of the spectral ζ-function $\zeta_\Delta(z) = \sum_\ell \lambda_\ell^{-z}$, $\lambda_\ell \neq 0$, as

$$\log\det\Delta = -\zeta_\Delta(0) \log(\Lambda^2 L^2) + (\log\det\Delta)_{\mathrm{fin}}, \qquad (\log\det\Delta)_{\mathrm{fin}} = -\zeta_\Delta'(0), \tag{5.10}$$

where Λ is a 2d UV cutoff. In particular, for the bosonic operator in (5.4) we get

$$\zeta_\Delta(z) = \sum_{\ell=p}^{\infty} 2\ell\,(\ell^2 - \mu)^{-z}. \tag{5.11}$$

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[18]More details on the explicit eigenspinors of the Dirac operator on S^2 are discussed in [58]. The fermionic spectral problem on S^2 in the monopole background is treated also in [59–61].

This can be computed by expanding in series of μ

$$\zeta_\Delta(z) = \sum_{\ell=p}^{\infty} 2\ell^{-2z+1} \left(1 - \mu\ell^{-2}\right)^{-z} = 2 \sum_{k=0}^{\infty} \binom{-z}{k} (-\mu)^k \sum_{\ell=p}^{\infty} \ell^{-2z-2k+1}$$

$$= 2 \sum_{k=0}^{\infty} \binom{-z}{k} (-\mu)^k \zeta(2z + 2k - 1, p), \tag{5.12}$$

where $\zeta(x, a)$ is the Hurwitz ζ-function. As a result,

$$\zeta_\Delta(0) = -\frac{1}{6} + \mu + p(1 - p). \tag{5.13}$$

In particular, for the values of p and μ corresponding to the bosonic operator in (5.4) we get

$$\zeta_\Delta(0) = \frac{1}{3} - m^2, \qquad m^2 = L^2 M^2. \tag{5.14}$$

This matches the value of the second Seeley coefficient B_2 that controls the log UV divergent part of $\log\det$ of the operator $-D^2 + X$ on S^2 in the heat kernel regularization:[19] $B_2 = \frac{1}{4\pi} \int d^2\xi \sqrt{g} \left(\frac{1}{6}\mathcal{R} - X\right) = \frac{1}{3} - L^2 M^2$ where $X = M^2$ and $\mathcal{R} = \frac{2}{L^2}$ is the curvature of S^2 with area $4\pi L^2$.

In the case of the fermionic operator in (5.6), (5.8) we need to account for the fact that for the lowest value of $\ell = p$ there is just one and not two eigenvalues with multiplicity p (cf. (5.7)). Then instead of (5.13) we get for the ζ-function corresponding to the operator in (5.6)[20]

$$\zeta_\Delta(0) = -\frac{1}{6} + \mu + p(1 - p) - p = -\frac{1}{6} - m^2, \qquad m^2 = L^2(M_1^2 - M_2^2). \tag{5.15}$$

This again is in correspondence with the value of the B_2 coefficient. To see this note that $\log\det\mathcal{D} = \frac{1}{2}\det\Delta$ where Δ is the corresponding "squared" Dirac operator (here $\tilde{\mathcal{D}} = S\mathcal{D}S^{-1}$ where S is an appropriate spinor rotation matrix; we suppress spinor indices)

$$\hat{\Delta} = \tilde{\mathcal{D}}\mathcal{D} = -D^2 + \frac{1}{4}\mathcal{R} + \frac{1}{2}i\, q\, \epsilon^{ab} F_{ab}\sigma_3 + M_1^2 - M_2^2. \tag{5.16}$$

As this operator is of the general form $-D^2 + X$ we have $B_2 = \frac{1}{4\pi} \int d^2\xi \sqrt{g}\ \mathrm{tr}(\frac{1}{6}\mathcal{R} - X)$ and thus $B_2 = 2\frac{1}{4\pi} \int d^2\xi \sqrt{g} \left[\frac{1}{6}\mathcal{R} - (\frac{1}{4}\mathcal{R} + M_1^2 - M_2^2)\right] = 2[-\frac{1}{6} - L^2(M_1^2 - M_2^2)]$ which is consistent with (5.15) corresponding to the 1st order operator in (5.6).

Let us note that in general $\zeta_\Delta(0)$ represents the regularized total number of all non-zero modes. In the above discussion we were assuming that the values of parameters are such that there are no zero modes. In the case when there are N_0 zero modes one should find that $B_2 = \zeta_\Delta(0) + N_0$. For a particular system of operators discussed in the next section we will have equal numbers of the bosonic and fermionic zero modes and thus the sum of $\sum_i (-1)^{F_i} B_2(\Delta_i)$ and $\sum_i (-1)^{F_i} \zeta_{\Delta_i}(0)$ will still be equal.

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[19]Note that B_2 does not depend on 2d gauge field A and indeed the dependence on q cancels in (5.13).

[20]Equivalently, we are to add p to the case with $p' = p + 1$, i.e. $p'(1 - p') + p = p(1 - p) - p = -p^2$.

	# real scalars	m^2	q	μ	p
AdS_4	4	$\frac{1}{4}n^2k^2$	nk	$\frac{1}{4}$	$\frac{1+|n|k}{2}$
$\mathbb{CP}^3$	$2+2$	$\frac{1}{4}(-2+2nk+n^2k^2)$	$\pm(1+nk)$	1	$\frac{1+|1+nk|}{2}$

Table 1. Bosonic spectrum. The parameters μ and p are defined in eq. (5.4).

Let us now compute the finite part $(\log \det \Delta)_{\text{fin}}$ in (5.10). In the bosonic case we find from (5.12) (here $\psi(x) = (\log \Gamma(x))'$)

$$\frac{d}{d\mu}(\log \det \Delta)_{\text{fin}} = \sum_{k=0}^{\infty} \frac{2}{(2k)!}\psi^{(2k)}(p)\,\mu^k = \psi(p + \sqrt{\mu}) + \psi(p - \sqrt{\mu}). \qquad (5.17)$$

The integration constant is fixed by the value of $\zeta'_\Delta(0)$ at $\mu = 0$ which is also obtained from (5.12)

$$\zeta_\Delta(z)\big|_{\mu=0} = 2\zeta(2z-1,p) \qquad \rightarrow \qquad \zeta'_\Delta(0)\big|_{\mu=0} = 4\zeta'(-1,p). \qquad (5.18)$$

Then from (5.17) we find (see, e.g., [62] and [9])

$$(\log \det \Delta)_{\text{fin}} = -4\,\zeta'(-1,p) + \int_0^\mu dx\,\left[\psi(p + \sqrt{x}) + \psi(p - \sqrt{x})\right] \equiv s_p(\mu), \qquad (5.19)$$

where in what follows we will define $s_p(\mu)$ in (5.3), (5.4) by this regularized expression.

Similarly, in the case of the fermionic operator in (5.6) we obtain the finite part of $(\log \det[L(i\slashed{D} + M_1\sigma_3 + M_2)])_{\text{fin}} = \widetilde{s}_p(\mu; \text{w})$ by using (5.8) and (5.7) where now $\widetilde{s}_p(\mu; \text{w})$ is expressed in terms of $s_{p+1}(\mu)$ in its regularized form (5.19), i.e.

$$\widetilde{s}_p(\mu; \text{w}) = s_{p+1}(\mu) + 2p\log\text{w} = s_p(\mu) - 2p\log(p^2 - \mu) + 2p\log\text{w}\,. \qquad (5.20)$$

Here we used that according to the definition in (5.3) one finds that $s_p(\mu) = 2p\log(p^2 - \mu) + s_{p+1}(\mu)$ (the same relation follows also from (5.19)). Thus the bosonic and fermionic log det contributions with the same p and μ have the same non-trivial part (5.19), i.e. differ only by the two logarithmic terms in (5.20).

6 One-loop instanton prefactor in the M2 brane partition function

6.1 Summary of the spectral data

We are now ready to combine the above results to compute the prefactor Z_1 in (4.27). Let us start with summarizing in tables 1 and 2 the data about the bosonic and fermionic fluctuation spectra in (4.26) and (4.30) and the corresponding parameters p and μ in the operators in (5.4) and (5.9). The type IIA string ($n = 0$) part of the spectral data in tables 1 and 2 was found earlier in [9].

u	u'	m_1	m_2	$m^2 = m_1^2 - m_2^2$	q	p	μ	$\mathrm{w}_{n>0}$	$\mathrm{w}_{n<0}$	$\mathrm{w}_{n=0}$
$+1$	-1	$-\frac{nk+1}{2}$	$\frac{1}{2}$	$\frac{nk(nk+2)}{4}$	$1+nk$	$\frac{|nk+1|}{2}$	$\frac{1}{4}$	$\frac{nk}{2}$	$-\frac{nk}{2}-1$	0
-1	$+1$	$-\frac{nk-1}{2}$	$\frac{1}{2}$	$\frac{nk(nk-2)}{4}$	$-1+nk$	$\frac{|nk-1|}{2}$	$\frac{1}{4}$	$\frac{nk}{2}-1$	$-\frac{nk}{2}$	0
$+1$	$+1$	$-\frac{nk}{2}$	-1	$\frac{k^2n^2}{4}-1$	nk	$\frac{|n|k}{2}$	1	$\frac{nk}{2}+1$	$-\frac{nk}{2}+1$	1
-1	-1	$-\frac{nk}{2}$	-1	$\frac{k^2n^2}{4}-1$	nk	$\frac{|n|k}{2}$	1	$\frac{nk}{2}+1$	$-\frac{nk}{2}+1$	1

Table 2. Fermionic spectrum. The parameters μ, p and w are defined in eq. (5.9).

The 4 rows in table 2 correspond to the parameters of the complex 2-component 2d fermions (one complex 2d fermion represents the same number of degrees of freedom as one complex scalar). The values of the parameter μ in (5.9) and the charge q in the fermionic spectrum are the same as in the bosonic spectrum (were μ is defined in (5.4)). This should be an indication of an underlying 2d supersymmetry which should be related to the fact that the M2 brane instanton background preserves half of the target space supersymmetry.

Before proceeding, let us comment on the zero modes in the spectrum that are not included in the determinants computed via spectral ζ-function and thus should be treated separately.

In the bosonic sector, zero modes may appear when $\ell^2 = \mu$ (cf. (5.3), (5.4)). From the data in table 1 where $\mu = 1$ or $\mu = \frac{1}{4}$ and given that ℓ starts with p we conclude (assuming as always that $k > 2$) that this is possible only for $n = 0$ modes with $\ell = \frac{1}{2}$ for four AdS$_4$ fluctuations and $\ell = 1$ for four $\mathbb{CP}^3$ fluctuations. Taking into account the degeneracy 2ℓ of each mode we get in total $4 \times (1 + 2) = 12$ real bosonic zero modes.

The same number of 12 real zero modes is found also in the fermionic sector: again zero modes appear only for $n = 0$ and thus their count is equivalent to the one in [8, 9]. For the fermionic determinant in (5.6), (5.9) a zero mode is found if $\mathrm{w} = 0$ or $\ell^2 = \mu = \frac{q^2}{4} - m^2$ for $\ell \geqslant p = \frac{|q|}{2}$. We thus get $2 + 2$ modes from the first two lines in table 2 and another $4 + 4$ from the last two lines.

The four bosonic AdS$_4$ 0-modes correspond to the point-like position of the instanton in AdS$_4$. As for the $\mathbb{CP}^3$ modes, their count is related to earlier discussions of the instanton 0-modes in the standard $\mathbb{CP}^N$ sigma model on 2-sphere [63–66] as follows.

Considering a $\mathbb{CP}^1$ instanton in the standard $\mathbb{CP}^3$ sigma model one finds 2 "longitudinal" fluctuations and 4 "transverse" fluctuations. If we were to quantize the AdS$_4 \times \mathbb{CP}^3$ string in the conformal instead of the static gauge then the contribution of the longitudinal fluctuations would cancel against that of the conformal gauge ghost operator on 2-sphere. The latter has 6 real $SL(2, C)$ Möbius zero modes (corresponding to the conformal Killing vectors on S^2). The same six 0-modes are found also for the longitudinal fluctuation operator whose contribution cancels against the ghost determinant one. This cancellation extends also to the 0-mode factors, leaving the static gauge result where only the contributions of

the transverse fluctuations are present.[21] Thus the total number of the 0-modes in the $\mathbb{CP}^3$ string sigma model in the conformal gauge is $6 + 8 = 14$. This agrees with the number $4N + 2$ of 0-modes in the $\mathbb{CP}^N$ sigma model expanded near $\mathbb{CP}^1$ instanton [67]: $(4N + 2)\big|_{N=3} = 14$.

6.2 The total contribution of the fluctuation determinants

Let us now sum up the log det contributions of all 2d fluctuation fields with the spectral data in tables 1 and 2. The corresponding 1-loop correction is (cf. (5.10))

$$\Gamma = \Gamma_B - \Gamma_F = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_i (-1)^{F_i} \log \det{}' \Delta_i = -\zeta_{\text{tot}}(0) \log(\Lambda L) - \frac{1}{2} \zeta'_{\text{tot}}(0) \,, \qquad (6.1)$$

where each term is counted as coming from a real 2d field. We are not including here the contribution of the bosonic and fermionic modes that appear only for $n = 0$; their contribution produces just a constant that can be found as suggested in [9] (see also below).

Let us first compute the value of the total coefficient $\zeta_{\text{tot}}(0) = \sum_{n=-\infty}^{\infty} \zeta_{\text{tot},\,n}(0)$ of the logarithmic divergence in (6.1). Applying the expressions in (5.14) and (5.15) we find that for a fixed level n the result is $\zeta_{\text{tot},\,n}(0) = 2 - 2nk$. Here the $n = 0$ is the non-vanishing type II GS string theory coefficient which is, in general, given by the Euler number of the 2-surface [20, 68, 69]. This is 2 for S^2 in the present case while it was 1 in the Wilson loop case in [3, 20] corresponding to disk topology. Summing over n using an analytic regularization (Riemann ζ-function) we then find that

$$\zeta_{\text{tot}}(0) = \sum_{n=-\infty}^{\infty} (2 - 2nk) = \sum_{n=-\infty}^{\infty} 2 = 2 + 4\zeta_R(0) = 0 \,. \qquad (6.2)$$

Thus, as in the 1-loop correction in the case of the M2 brane wrapped on $AdS_2 \times S^1$ in [3], the UV divergences coming from the tower of $n \neq 0$ modes effectively cancel the non-zero contribution of the $n = 0$ string modes so that the full M2 brane 1-loop correction is UV finite. As discussed in [3], this cancellation can be understood as being a consequence of the fact that the 2d model with an infinite set of modes considered here is equivalent to the original 3d model where there are no logarithmic UV divergences.[22]

The vanishing of $\zeta_{\text{tot}}(0)$ implies also that Γ in (6.1) will not depend on the radius L of the 2-sphere so that the result will only be a function of the dimensionless parameter k in the spectrum.

Let us first consider the part of the sum in (6.1) which comes from the $n = 0$ (string-level) modes. The contribution of the $n = 0$ modes in tables 1 and 2 can be computed using the expressions for the corresponding determinants in (5.4), (5.8), (5.7), (5.19) and is found to vanish

$$\Gamma_0 = 2\left[s_{\frac{1}{2}}(\tfrac{1}{4}) + s_1(1)\right] - 2\left[\widetilde{s}_{\frac{1}{2}}(\tfrac{1}{4}, 0) + \widetilde{s}_0(1; 1)\right] = 0 \,. \qquad (6.3)$$

---

[21]Note that in the static gauge in string theory there is no factor of the Möbius volume as it cancels against the 0-mode factor in the determinant of the longitudinal modes (cf. also a discussion in [20]).

[22]To be precise, the relation between 2d and 3d UV divergences requires the use of an analytic regularization that disposes of all power divergences and is effectively consistent with symmetries of the 3d theory that are restored in the UV limit.

To show the vanishing we used the relation (5.20) between s_p and $\widetilde{s}_p$ and dropped all 0-mode contributions (corresponding to the lowest values of ℓ in the respective sums) that should be treated separately. That all the non-trivial finite contributions of the $n = 0$ string-theory modes mutually cancel was observed already in [9].

Let us now determine the contributions of the rest of the M2 brane fluctuation modes with $n \neq 0$. Using the data in table 1 and the expression in (5.19) we find that the bosonic mode contribution to Γ in (6.1) is

$$\Gamma_B = 2 \sum_{n \neq 0} \left[s_{\frac{1+|1+nk|}{2}}(1) + s_{\frac{1+|n|k}{2}}\left(\tfrac{1}{4}\right) \right] = 2 \sum_{n=1}^{\infty} \left[s_{\frac{nk}{2}+1}(1) + 2 s_{\frac{nk}{2}+\frac{1}{2}}\left(\tfrac{1}{4}\right) + s_{\frac{nk}{2}}(1) \right]. \quad (6.4)$$

Similarly, in the fermionic case we are to use the expression in (5.8), (5.7) where $\widetilde{s}_p(\mu)$ is defined in terms of the finite expression for $s_p(\mu)$ in (5.19). Separating the $n > 0$ and $n < 0$ contributions[23] we obtain

$$\begin{aligned}
\Gamma_F &= \sum_{n=1}^{\infty} \left[\widetilde{s}_{\frac{nk+1}{2}}\left(\tfrac{1}{4}; \tfrac{nk}{2}\right) + \widetilde{s}_{\frac{nk-1}{2}}\left(\tfrac{1}{4}; \tfrac{nk}{2} - 1\right) + 2 \widetilde{s}_{\frac{nk}{2}}\left(1; \tfrac{nk}{2} + 1\right) \right] \\
&\quad + \sum_{n=1}^{\infty} \left[\widetilde{s}_{\frac{nk-1}{2}}\left(\tfrac{1}{4}; \tfrac{|n|k}{2} - 1\right) + \widetilde{s}_{\frac{nk+1}{2}}\left(\tfrac{1}{4}; \tfrac{nk}{2}\right) + 2 \widetilde{s}_{\frac{nk}{2}}\left(1; \tfrac{nk}{2} + 1\right) \right] \\
&= 2 \sum_{n=1}^{\infty} \left[\widetilde{s}_{\frac{nk+1}{2}}\left(\tfrac{1}{4}; \tfrac{nk}{2}\right) + \widetilde{s}_{\frac{nk-1}{2}}\left(\tfrac{1}{4}; \tfrac{nk}{2} - 1\right) + 2 \widetilde{s}_{\frac{nk}{2}}\left(1; \tfrac{nk}{2} + 1\right) \right]. \quad (6.5)
\end{aligned}$$

If we now use the definition of $\widetilde{s}$ in (5.8), i.e. $\widetilde{s}_p(\mu; \mathrm{w}) \equiv 2p \log \mathrm{w} + s_{p+1}(\mu)$, we find

$$\begin{aligned}
\Gamma_F = 2 \sum_{n=1}^{\infty} \Big[&(nk + 1) \log \tfrac{nk}{2} + s_{\frac{nk+3}{2}}\left(\tfrac{1}{4}\right) + (nk - 1) \log\left(\tfrac{nk}{2} - 1\right) + s_{\frac{nk+1}{2}}\left(\tfrac{1}{4}\right) \\
&+ 2\,nk \log\left(\tfrac{nk}{2} + 1\right) + 2\, s_{\frac{nk}{2}+1}(1) \Big]. \quad (6.6)
\end{aligned}$$

Hence, combining (6.4) and (6.6) we get

$$\begin{aligned}
\Gamma = \Gamma_B - \Gamma_F = 2 \sum_{n=1}^{\infty} \Big[&s_{\frac{nk}{2}}(1) - s_{\frac{nk}{2}+1}(1) + s_{\frac{nk}{2}+\frac{1}{2}}\left(\tfrac{1}{4}\right) - s_{\frac{nk}{2}+\frac{3}{2}}\left(\tfrac{1}{4}\right) \\
&- (nk+1) \log \tfrac{nk}{2} - (nk-1) \log\left(\tfrac{nk}{2} - 1\right) - 2\,nk \log\left(\tfrac{nk}{2} + 1\right) \Big]. \quad (6.7)
\end{aligned}$$

Here the first line can be simplified by using the relation for s_{p+1} used in (5.20) (cf (5.3)), i.e. $s_{p+1}(\mu) - s_p(\mu) = -2p \log(p^2 - \mu)$. Then we observe a remarkable cancellation (which should be attributed again to an underlying supersymmetry) of all the non-trivial functions $s_p(\mu)$ (cf. (5.19)) with only few logarithms remaining.

Thus, notwithstanding the complexity of the intermediate expressions, the final result for Γ is very simple

$$\Gamma = 2 \sum_{n=1}^{\infty} \log\left(\tfrac{n^2 k^2}{4} - 1 \right). \quad (6.8)$$

---

[23]Note that the values of w in table 2 depend on the sign of n.

Surprisingly, this is the same expression (up to overall factor of 2) as was found in [3] for the 1-loop correction in the case of the $AdS_2 \times S^1$ M2 brane solution, despite the fact that the two fluctuation spectra are very different. This is, however, in line with the fact that the prefactors in the localization results in (2.15) and (2.16) happen to be closely related.

The sum in (6.8) may thus be computed in the same way as in [3] (i.e. using again the Riemann ζ-function regularization, namely, $\zeta_R(0) = -\frac{1}{2}$, $\zeta'_R(0) = -\frac{1}{2}\log(2\pi)$):

$$
\begin{aligned}
\Gamma &= 4 \sum_{n=1}^{\infty} \log \frac{nk}{2} + 2 \sum_{n=1}^{\infty} \log \left(1 - \frac{4}{n^2 k^2}\right) \\
&= 4 \log \frac{k}{2} \zeta_R(0) - 4\zeta'_R(0) + 2 \log \left(\frac{k}{2\pi} \sin \frac{2\pi}{k}\right) = 2 \log \left(2 \sin \frac{2\pi}{k}\right).
\end{aligned}
\tag{6.9}
$$

It is interesting to note that (6.8) may be interpreted as twice a 1d "massive" determinant on S^1 or the contribution of a loop of a point particle in the (inverted) harmonic oscillator potential on a circle $(s \equiv s + 2\pi)$[24]

$$
\Gamma = 2 \times \frac{1}{2} \log \det' \left(-\frac{k^2}{4} \frac{d^2}{ds^2} - 1\right) = 2 \sum_{n=1}^{\infty} \log \left(\frac{k^2}{4} n^2 - 1\right),
\tag{6.10}
$$

where prime means we project out the negative mode corresponding to $n = 0$ (there is no zero modes if $k > 2$). Thus the simple form of the final result (6.8) suggests that what happens is that out of all the fluctuation modes of the M2 brane in the instanton background only two particular bosonic 1d modes survive after the fermion-boson cancellations, reminiscent of some kind of localization. The same reduction to just one 1d bosonic mode happened in the case of the $AdS_2 \times S^1$ membrane computation in [3].

We conclude that the 1-loop instanton prefactor in (4.13) is

$$
Z_1 = \gamma \, e^{-\Gamma} = \gamma \, \frac{1}{4 \sin^2\left(\frac{2\pi}{k}\right)},
\tag{6.11}
$$

where we introduced a numerical (k-independent) factor γ to account for the contribution of the 0-modes that we omitted above and also of possible degeneracy of the instanton saddle contributions. In the large k limit this reduces to (cf. (3.7)) $Z_1 = \gamma \frac{k^2}{16\pi^2} + \ldots = \frac{2\gamma}{\pi} \frac{T}{g_s^2} + \ldots$, i.e. to the expression found in [9]. Here we get it directly as a limit of the UV finite M2 brane contribution, without need to fix the form of the overall factor by some indirect considerations as was done in [9] using an analogy with the string Wilson loop case computation in [20].

To determine the value of γ let us note first that we are to add a factor of 2 due to the equal instanton and anti-instanton contributions (these become distinguished if there is a constant C_3 background). We also need a further factor of 2 that was argued in [9] to represent the contribution of the 0-modes of the string fluctuations.[25] It would be very interesting to derive this result systematically by introducing the collective coordinates

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[24]Note that the Riemann ζ-function regularization used in (6.9) is the standard prescription of how to define similar 1d determinants appearing in various path integral representations.

[25]The contributions of the equal number of 12+12 bosonic and fermionic modes can be regularized and shown to cancel after introducing a squashing parameter in the $\mathbb{CP}^3$ metric. Then there are two possible string $\mathbb{CP}^1$ saddles contributing equally and thus giving an extra factor of 2 [9].

for the bosonic and fermionic 0-modes and computing the volume of the corresponding supercoset (that may turn out to be finite like the volume of the superMöbius group on the disk [44]).[26]

To conclude, using that $\gamma = 4$ and accounting for the minus sign in (3.4), we precisely match the 1-instanton prefactor in the localization result in (2.15).

7 Concluding remarks and open problems

In this paper we presented a new remarkable test of the $\text{AdS}_4/\text{CFT}_3$ duality between ABJM theory with large rank of the gauge group N and finite level k and M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$. We reproduced the leading (at large N and fixed $k > 2$) instanton prefactor in the localization result for the non-perturbative part of the ABJM free energy on S^3 in (2.15) from a quantum 1-loop correction to the classical action factor of the M2 brane $S^3/\mathbb{Z}_k$ instanton. This generalizes to finite k the analysis of the string $\mathbb{CP}^1 \subset \mathbb{CP}^3$ instanton contribution in type IIA string theory [8, 9].

Like in the earlier $\text{AdS}_2 \times S^1$ M2 brane example in [3], the quantum 1-loop M2 brane computation described above was fully consistent (UV finite) and gave an unambiguous prediction for the resulting function of k in agreement with (2.15). One subtle issue that would be interesting to clarify further is the factor 2 associated with the zero mode contribution that originates from the string-level fluctuations; here we fixed it following [9] but there may exist a more direct derivation.

There are several possible extensions of our work. One may consider the leading perturbative $\frac{1}{\sqrt{N}}$ correction to the prefactor in (2.15), (2.14)

$$F^{\text{inst}}(N, k) = -\frac{1}{\sin^2\left(\frac{2\pi}{k}\right)} \left[1 + \frac{1}{\sqrt{N}} h_1(k) + \dots \right] e^{-2\pi\sqrt{\frac{2N}{k}}} + \dots, \qquad h_1(k) = \frac{\pi}{\sqrt{2k}} \frac{k^2 - 40}{12k}, \tag{7.1}$$

and try to reproduce the coefficient $h_1(k)$ from the 2-loop M2 brane correction, which should come with a factor of the inverse of the effective M2 brane tension in (3.2), i.e. $\text{T}_2^{-1} = \frac{\pi}{\sqrt{2k}} \frac{1}{\sqrt{N}}$.[27] Such a 2-loop calculation would require the use of the quartic bosonic and fermionic terms in the corresponding supermembrane action [50–52]. It would be important to check if the 2-loop M2 brane contribution is, in fact, UV finite, despite the apparent non-renormalizability of the membrane action.[28]

As was mentioned in section 2, the subleading (at large N and $k > 2$) non-perturbative contributions to the ABJM free energy involve, in addition to the contributions of M2 brane

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[26]Let us also recall that the equal number of the bosonic and fermionic 0-modes in the $\mathbb{CP}^1$ instanton background are found also in the (2,2) supersymmetric 2d $\mathbb{CP}^1$ sigma model [70] where their contributions to the instanton measure (and thus e.g. to the β-function) effectively cancel each other.

[27]As was suggested in [3], a similar 2-loop computation in the case of the $\text{AdS}_2 \times S^1$ M2 brane surface should reproduce the coefficient of the $\frac{1}{\sqrt{N}}$ correction to the prefactor of the Wilson loop expectation value in (2.16). Note that the string theory values of the coefficients of these $\frac{1}{\sqrt{N}}$ corrections in (7.1) and (2.16) are sensitive to the precise form of the relation between the string theory parameters in (3.2) and gauge theory parameters N, k, i.e. to the shift $N \to N - \frac{1}{24}(k - k^{-1})$ suggested in [71].

[28]Such 2-loop computation is, however, going to be hard as the background 3d metric in the static gauge (4.7) is that of $S^3/\mathbb{Z}_k$, i.e. is not flat.

instantons wrapping $S^3/\mathbb{Z}_k \subset S^7/\mathbb{Z}_k$, also the contributions of M2 brane instanton wrapping $\mathbb{RP}^3$ 3-cycle in $\mathbb{CP}^3 \subset S^7/\mathbb{Z}_k$ (not involving 11d circle), corresponding to the D2-brane instanton in the type IIA string limit. One may wonder if the analog of the semiclassical M2 brane computation that we performed in this paper for the first ($S^3/\mathbb{Z}_k$) instanton can be also repeated for the second ($\mathbb{RP}^3 = S^3/\mathbb{Z}_2$) one, thus determining a k-dependent prefactor of the corresponding exponential $e^{-\pi\sqrt{2Nk}}$ in (2.9).

As we discuss in appendix A, in this second case the M2 brane solution is not wrapping the 11d circle in (4.1) and thus the corresponding induced 3-metric and the quantum M2 brane fluctuation determinants do not have a non-trivial dependence on k (cf. (4.8), (4.15)). At the same time, the localization result for the prefactor of $e^{-\pi\sqrt{2Nk}}$ in (2.9) vanishes for odd k has singular dependence on even k (see (A.5)). We point out that for even k the contribution of the $\mathbb{RP}^3$ instanton in fact "mixes" with that of $\frac{k}{2}$-instanton of the $S^3/\mathbb{Z}_k$ type (both have the same classical action) and thus should not be considered in isolation. In general, one is to combine together all of the contributions in (2.9) that have the same exponential factor and then the prefactor will have a regular value for any given integer k [72].

In particular, one may study the special cases of $k = 1, 2$. While for $k \geqslant 2$ the leading non-perturbative correction to F in (2.9) is given by the single $(n_\mathrm{I}, n_\mathrm{II}) = (1,0)$ instanton contribution, for $k = 1, 2$ the two terms in the exponent in (2.9) become of the same order suggesting that the two types of instantons should be treated on an equal footing. We comment on this further in appendix A.

In the 1-loop quantum M2 brane computation near the $S^3/\mathbb{Z}_k$ instanton described in this paper we assumed $k > 2$ but it is straightforward to extend it to $k = 1, 2$ by determining the fluctuation spectra directly in these special cases (like that was done in the $\mathrm{AdS}_2 \times S^1$ case in [3]). In particular, for $k = 1$ we find that there are 4 negative modes (corresponding to $n = -1$ for each of the 4 $\mathbb{CP}^3$ fluctuations in table 1) reflecting instability of the S^3 instanton in S^7 in line with $\pi_3(S^7)$ being trivial. In addition, there are extra 0-mode contributions that appear in these cases with enhanced supersymmetry.

Another interesting open problem is to try to generalize both the computation in [3] and in the present paper in order to determine the leading instanton correction to the expectation value of the circular BPS Wilson loop from the quantum M2 brane theory to match the corresponding localization result on the ABJM gauge theory side [73] (cf. [74]).[29] The corresponding minimal 3-surface should be a superposition of $\mathrm{AdS}_2 \times S^1$ in [3, 4] and $S^3/\mathbb{Z}_k$ discussed above. Here a natural static gauge choice may be along the $\mathrm{AdS}_2 \times S^1$ subspace as in [3], implying a more involved structure of fluctuations in all of $\mathbb{CP}^3$ directions.[30]

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[29]The leading instanton corrections to the $\frac{1}{2}$-BPS Wilson loop in the fundamental representation is given by [73]

$$\langle W \rangle = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i}\, e^{-N\mu}\, \frac{e^{\frac{2\mu}{k}}}{2\sin\frac{2\pi}{k}}\, \mathcal{Q}(\mu, k), \quad \mathcal{Q} = 1 + 2Q + 3Q^2 + 10Q^3 + \left(49 - 32\sin^2\frac{2\pi}{k}\right)Q^4 + \cdots, \quad Q \equiv -e^{-\frac{4\mu}{k}}.$$

Here the corrections due to 1,2,3 instantons do not introduce a new k-dependence. $\mathcal{Q}$ has a smooth $k \to \infty$ limit which is known in exact form from the instanton corrections to the disk amplitude in topological string model [75].

[30]In particular, one would need to include explicitly the contribution of the "longitudinal" fluctuations along the $\mathbb{CP}^1 \subset \mathbb{CP}^3$ directions.

A Comments on M2 brane instantons wrapping $\mathbb{RP}^3 \subset \mathbb{CP}^3$ of $S^7/\mathbb{Z}_k$

To recall, the non-perturbative corrections to the free energy in (2.9) are labeled by a pair of integers $(n_{\text{I}}, n_{\text{II}})$ with $(n_{\text{I}}, 0)$ and $(0, n_{\text{II}})$ representing the two types ($S^3/\mathbb{Z}_k$ and $\mathbb{RP}^3$) of (multi)instantons. The corresponding terms in the non-perturbative part of the grand potential (2.3) are (cf. (2.10))

$$J^{\text{np}}(\mu, k) = \sum_{n_{\text{I}}, n_{\text{II}} = 0}^{\infty} J^{\text{np}}_{(n_{\text{I}}, n_{\text{II}})}(\mu, k), \tag{A.1}$$

$$J^{\text{np}}_{(n_{\text{I}}, 0)} = d_{n_{\text{I}}}(k) e^{-\frac{4 n_{\text{I}}}{k} \mu}, \qquad J^{\text{np}}_{(0, n_{\text{II}})} = \left[a_{n_{\text{II}}}(k) \mu^2 + b_{n_{\text{II}}}(k) \mu + c_{\text{II}}(k) \right] e^{-2 n_{\text{II}} \mu}. \tag{A.2}$$

These translate into the coefficients appearing in the prefactor of the non-perturbative part of the free energy (2.9). In particular, the leading $(n_{\text{I}}, n_{\text{II}}) = (0, 1)$ contribution to F^{inst} in (2.13) is [16] (cf. the $(1, 0)$ contribution in (2.14))

$$\begin{aligned} F^{\text{inst}}_{(0,1)}(k, N) = & -\left[C(k)^{-1} \left(N + 2 - B(k) \right) a_1(k) - c_1(k) \right] \frac{\text{Ai}[C(k)^{-\frac{1}{3}}(N + 2 - B(k))]}{\text{Ai}[C(k)^{-\frac{1}{3}}(N - B(k))]} \\ & - C(k)^{-\frac{1}{3}} b_1(k) \frac{\text{Ai}'[C(k)^{-\frac{1}{3}}(N + 2 - B(k))]}{\text{Ai}[C(k)^{-\frac{1}{3}}(N - B(k))]}. \end{aligned} \tag{A.3}$$

The coefficients $d_{n_{\text{I}}}(k)$ corresponding to multi-instantons of the first type may be computed using the topological string representation [16] (cf. (2.11))

$$d_1(k) = \frac{1}{\sin^2 \frac{2\pi}{k}}, \qquad d_2(k) = -\frac{1}{2 \sin^2 \frac{4\pi}{k}} - \frac{1}{\sin^2 \frac{2\pi}{k}}, \qquad d_3(k) = \frac{1}{3 \sin^2 \frac{6\pi}{k}} + \frac{3}{\sin^2 \frac{2\pi}{k}}, \quad \cdots. \tag{A.4}$$

The coefficients a_1, b_1, c_1 corresponding to the single instanton of the second kind in (A.2) have been conjectured in [16, 72, 76] and obtained systematically from the refined topological string representation in [77]

$$\begin{aligned} a_1(k) &= -\frac{4}{\pi^2 k} \cos \frac{\pi k}{2}, \qquad & b_1(k) &= \frac{2}{\pi \tan \frac{\pi k}{2}} \cos \frac{\pi k}{2}, \\ c_1(k) &= \left(-\frac{2}{3k} + \frac{5k}{12} + \frac{k}{2 \sin^2 \frac{\pi k}{2}} + \frac{1}{\pi \tan \frac{\pi k}{2}} \right) \cos \frac{\pi k}{2}. \end{aligned} \tag{A.5}$$

Note that these vanish for odd k and two of them are singular for even k.

In the dual M-theory setting, the M2 brane wrapped on $\mathbb{RP}^3$ 3-cycle in $\mathbb{CP}^3$ which is part of $S^7/\mathbb{Z}_k$ with the metric (4.1) can be explicitly described as follows [6]. Using the angular parametrization of $\mathbb{CP}^3$ (here $\alpha, \theta_1, \theta_2 \in [0, \pi]$; $\chi, \varphi_1, \varphi_2 \in [0, 2\pi]$)[31]

$$ds^2_{\mathbb{CP}^3} = \frac{1}{4}d\alpha^2 + \frac{1}{4}\cos^2\frac{\alpha}{2}(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + \frac{1}{4}\sin^2\frac{\alpha}{2}(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2)$$
$$+ \frac{1}{4}\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(2d\chi + \cos\theta_1 d\varphi_1 - \cos\theta_2 d\varphi_2)^2, \tag{A.6}$$

the relevant classical M2 brane solution is wrapped on $\mathbb{RP}^3 \subset \mathbb{CP}^3$ parametrised by $\theta_1 = \theta_2$, $\varphi_1 = -\varphi_2$, χ and $\alpha = 0$ [6, 53]. The static gauge adapted to this solution is thus (here we label the world-volume coordinates as $\xi^i = (s_1, s_2, s_3)$, cf. (4.5))

$$\theta_1 = \theta_2 = s_1 \in [0, \pi], \qquad \varphi_1 = -\varphi_2 = s_2 \in [0, 2\pi], \qquad \chi = s_3 \in [0, 2\pi]. \tag{A.7}$$

The induced world-volume metric is that of $\mathbb{RP}^3$, i.e. $ds^2 = \frac{R^2}{4}[ds_1^2 + \sin^2 s_1 ds_2^2 + (ds_3 + \cos s_1 ds_2)^2]$ which is the standard metric of S^3 of radius R in Hopf parametrization but with the angle s_3 having period 2π instead of 4π:

$$g_{ij} = \frac{R^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos s_1 \\ 0 & \cos s_1 & 1 \end{pmatrix}, \qquad \sqrt{g} = \frac{R^3}{8}\sin s_1. \tag{A.8}$$

The classical value of the M2 brane action is then given by (cf. (4.10), (4.12))

$$S_{\rm cl} = T_2 \frac{R^3}{8}(2\pi)^2 \int_0^\pi ds_1 \sin s_1 = \pi\sqrt{2kN}. \tag{A.9}$$

This is of course the same as the action of the corresponding D2 brane wrapped on $\mathbb{RP}^3 \subset \mathbb{CP}^3$ in type IIA theory and is also the value appearing as a factor of $n_{\rm II}$ in the exponent in (2.9).

Expanding the M2 brane action (4.4) near this solution we observe that, as the 11d angle y in (4.1) has a trivial classical value, the quadratic fluctuation Lagrangian will depend on k only via the term $\frac{1}{k^2}(\partial\tilde{y})^2$.[32] Thus, in contrast to what we found for the $S^3/\mathbb{Z}_k$ instanton, here the corresponding 1-loop M2 brane partition function will not have a non-trivial dependence on k.[33]

At the same time, the localization coefficients (A.5) have a peculiar dependence on k. They do not appear to admit a regular large k limit, vanish for odd values of k and singular for even k. For generic even k the non-perturbative contributions in (2.9) coming from $(n_{\rm I}, n_{\rm II}) = (\frac{k}{2}, 0)$ and $(0, 1)$ have the same exponential weight and thus should be treated on

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[31]In this parametrization A in (4.1) is given by $A = \frac{1}{2}\left(\cos\alpha\, d\chi + \cos^2\frac{\alpha}{2}\cos\theta_1 d\varphi_1 + \sin^2\frac{\alpha}{2}\cos\theta_2 d\varphi_2\right)$.

[32]The classical value of the 1-form A in (4.1) is $A = \frac{1}{2}(ds_3 + \cos s_1 ds_2)$ but there are similar off-diagonal terms in (A.8) so the term $\frac{1}{k}\partial\tilde{y}A$ will also not lead to a non-trivial contribution. Note also that the leading quadratic part of the fermionic Lagrangian can not depend on k as the bosonic coordinates have classical values only along the $\mathbb{RP}^3 \subset \mathbb{CP}^3$.

[33]Rescaling of $\tilde{y}$ by k will not lead to a factor of k in the partition function as the analog of $\zeta(0)$ vanishes in 3d.

an equal footing. Indeed, the corresponding $\frac{k}{2}$-instanton of the $S^3/\mathbb{Z}_k$ type has the same action as the 1-instanton of the $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ type. This suggests that the $\mathbb{RP}^3$ instanton should not be considered in isolation, resolving the above puzzle about k-independence of the corresponding 1-loop M2 contribution.

In the main part of this paper we have been assuming $k > 2$ in which case the leading non-perturbative correction to F in (2.9) is given by the single $(n_\mathrm{I}, n_\mathrm{II}) = (1, 0)$ instanton contribution. It is, however, necessary to include the contributions of the second type of M2 brane instanton in order to compute non-perturbative corrections to the free energy for the lowest values of $k = 1, 2$. These values are special since the corresponding ABJM model (and also its dual M-theory counterpart) should have the enhanced supersymmetry: from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ [1, 78–80].

Indeed, for $k = 1, 2$ the two terms in the exponent in (2.9) become of the same order and should be treated on an equal footing. The relevant combinations of the two instanton numbers are[34]

$$k = 1: \quad (1, 0) + (0, 2); \qquad k = 2: \quad (1, 0) + (0, 1). \tag{A.10}$$

For $k = 1$ we thus have the S^3 one-instanton and the $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ two-instanton which are indeed essentially the same. Similarly, for $k = 2$ both the $(1, 0)$ and $(0, 1)$ instanton correspond to $S^3/\mathbb{Z}_2$.

Note that for these values of $(n_\mathrm{I}, n_\mathrm{II})$ the corresponding coefficients in (A.2), (A.4), (A.5) are singular but their sums are finite and give the following expressions for J^np in (A.1) [76][35]

$$J^\mathrm{np}(\mu, 1) = \frac{1}{4\pi^2}(16\mu^2 + 4\mu + 1)e^{-4\mu} + \cdots, \qquad J^\mathrm{np}(\mu, 2) = \frac{1}{\pi^2}(4\mu^2 + 2\mu + 1)e^{-2\mu} + \cdots. \tag{A.11}$$

These expressions for the grand potential plugged into (2.2) give the following leading non-perturbative contributions to F^inst in (2.13) for $k = 1$ and $k = 2$ [16]

$$F^\mathrm{inst}_\mathrm{leading}(1, N) = -\left(2N + \frac{C(1)}{8} + \frac{29}{4}\right) \frac{\mathrm{Ai}\left[C(1)^{-\frac{1}{3}}\left(N + \frac{29}{8}\right)\right]}{\mathrm{Ai}\left[C(1)^{-\frac{1}{3}}\left(N - \frac{3}{8}\right)\right]} + \frac{C(1)^{\frac{2}{3}}}{2} \frac{\mathrm{Ai}'\left[C(1)^{-\frac{1}{3}}\left(N + \frac{29}{8}\right)\right]}{\mathrm{Ai}\left[C(1)^{-\frac{1}{3}}\left(N - \frac{3}{8}\right)\right]},$$

$$F^\mathrm{inst}_\mathrm{leading}(2, N) = -(4N + C(2) + 7) \frac{\mathrm{Ai}\left[C(2)^{-\frac{1}{3}}\left(N + \frac{7}{4}\right)\right]}{\mathrm{Ai}\left[C(2)^{-\frac{1}{3}}\left(N - \frac{1}{4}\right)\right]} + 2C(2)^{\frac{2}{3}} \frac{\mathrm{Ai}'\left[C(2)^{-\frac{1}{3}}\left(N + \frac{7}{4}\right)\right]}{\mathrm{Ai}\left[C(2)^{-\frac{1}{3}}\left(N - \frac{1}{4}\right)\right]}. \tag{A.12}$$

Expanded to leading order in large N, these expressions give, respectively, for $k = 1$ [81] and $k = 2$

$$F^\mathrm{inst}_\mathrm{leading}(1, N) = -2N\, e^{-2\pi\sqrt{2N}} + \cdots, \qquad F^\mathrm{inst}_\mathrm{leading}(2, N) = -4N\, e^{-2\pi\sqrt{N}} + \cdots. \tag{A.13}$$

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[34]For $k = 1$ the dominant contribution would naively come from the $(0, 1)$ contribution but this term is actually absent [81]; it is conjectured to vanish for all odd k, and that was checked for $k = 5, 7$ in [76]. Note that this is consistent with the expressions in (A.5), which vanish for odd k.

[35]Let us note also that for generic even k the sum of the $(n_\mathrm{I}, n_\mathrm{II}) = (\frac{k}{2}, 0)$ and $(0, 1)$ contributions is again finite after cancellations between the two separately divergent terms [72].

Here the N factor comes from the μ^2 term with the coefficient $a_{\mathrm{II}}(k)$ in (A.2) (cf. the first term in (A.3) in the $k = 2$ case). It would be very interesting to reproduce these contributions to F^{np} in (2.9) by a quantum M2 brane computation.[36] Note that the fact that $F^{\mathrm{inst}}_{\mathrm{leading}}(1, N)$ is real does not contradict the instability of the S^3 instanton in S^7 as there are 4 negative bosonic modes contributing $i^4 = 1$.

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[36] It is in principle straightforward to repeat the 1-loop analysis described in sections 3–6 above for the special cases of $k = 1, 2$. Let us mention only that the masses in the bosonic and fermionic charged sectors in tables 1 and 2 depend on $2nk + n^2 k^2$ and thus take the same values for n and $-n - 2/k$. Hence, there is an extra degeneracy in the spectrum when $k = 1, 2$. This should be related to the world-volume analog of the enhanced target space supersymmetry in these cases.

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