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Solutions of the Yang–Baxter Equation and Strong Semilattices of Skew Braces

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Abstract. We prove that any set-theoretic solution of the Yang–Baxter equation associated with a dual weak brace is a strong semilattice of non-degenerate bijective solutions. This fact makes use of the description of any dual weak brace S we provide in terms of strong semilattice Y of skew braces B_α, with $\alpha \in Y$. Additionally, we describe the ideals of S and study its nilpotency by correlating it to that of each skew brace B_α.

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Introduction

The quantum Yang–Baxter equation takes its name from two independent works by Yang [35] and Baxter [3]. It is an important tool in several fields of research, among these are statistical mechanics, quantum field theory, and quantum group theory, whose study of solutions has been a major research area for the past 50 years. The challenging problem of determining all the set-theoretic solutions arose in 1992 in the paper by Drinfel'd [15] and is still open. Into the specific, given a set S, a map $r : S \times S \to S \times S$ is said to be a *set-theoretic solution of the Yang–Baxter equation*, or briefly a *solution*, if r satisfies the braid identity

$$(r \times \mathrm{id}_S)\,(\mathrm{id}_S \times r)\,(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)\,(r \times \mathrm{id}_S)\,(\mathrm{id}_S \times r)\,.$$

Writing $r\,(x, y) = (\lambda_x\,(y)\,, \rho_y\,(x))$, with λ_x, ρ_x maps from S into itself, then r is *left non-degenerate* if $\lambda_x \in \mathrm{Sym}_S$, *right non-degenerate* if $\rho_x \in \mathrm{Sym}_S$, for every $x \in S$, and *non-degenerate* if r is both left and right non-degenerate.

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Several techniques for constructing solutions starting from known solutions have been introduced over the years. For the purposes of this paper, we mention the *strong semilattice of solutions* [8], which is a method that allows for determining solutions starting from a semilattice Y, a family of disjoint sets $\{X_\alpha \mid \alpha \in Y\}$, and solutions r_α defined on these sets.
In 2007, Rump [31] innovatively showed how an involutive non-degenerate solution can be obtained starting from the special algebraic structure of *brace*. This type of approach had a large following in the last few years and other similar structures have been studied. Among these, we mention the *weak brace* [9] that is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

$$a \circ (b + c) = a \circ b - a + a \circ c \qquad \text{and} \qquad a \circ a^- = -a + a,$$

for all $a, b, c \in S$, where $-a$ and a^- denote the inverses of a with respect to $+$ and $\circ$, respectively. Clearly, the sets of the idempotents $\mathrm{E}(S, +)$ and $\mathrm{E}(S, \circ)$ coincide, so we will simply write $\mathrm{E}(S)$. In particular, if $|\,\mathrm{E}(S)| = 1$, then $(S, +)$ and $(S, \circ)$ are groups having the same identity, and so $(S, +, \circ)$ is a *skew brace* [17] which is a brace if the group $(S, +)$ is abelian. Necessarily, the additive structure is a Clifford semigroup, instead, in general, the multiplicative one is not. A class of weak braces having $(S, \circ)$ as a Clifford semigroup is obtained in [10, Theorem 16]. We call *dual* the weak braces for which $(S, \circ)$ is Clifford. Any weak brace $(S, +, \circ)$ gives rise to a solution $r : S \times S \to S \times S$ defined by

$$r\,(a, b) = \Big(-a + a \circ b,\ (-a + a \circ b)^- \circ a \circ b\Big),$$

for all $a, b \in S$ (see [9, Theorem 11]). Such a map r is close to being bijective, and, in the case of a dual one, r is also close to being non-degenerate (see [10, pp. 604–605]). Besides, a new family of solutions coming from dual weak braces has been investigated in [24]. For this kind of structure, a notion of *ideal* has been also introduced in [10, Definition 20]. Moreover, it turns out that the quotient structure is a new dual weak brace with semilattice of idempotents isomorphic to $\mathrm{E}\,(S)$.

In this paper, we entirely describe the structure of a dual weak brace $(S, +, \circ)$, by showing that it is a strong semilattice Y of skew braces $(B_\alpha, +, \circ)$, for every $\alpha \in Y$, where $(B_\alpha, +)$ and $(B_\alpha, \circ)$ are the groups fulfilling the structure of $(S, +)$ and $(S, \circ)$, respectively, as Clifford semigroups (see [18, Theorem 4.2.1]). As a consequence, we prove that the solution r associated with S is the strong semilattice Y of the non-degenerate bijective solutions r_α on each B_α. Any strong semilattice of skew braces is realized by combining skew brace homomorphisms $\phi_{\alpha,\beta}$ from B_α to B_β (whenever $\alpha \geq \beta$), thus this further motivates the study of such maps, a problem already emerged in literature (cf. [11, Problem 10.2], [34, Problem 2.18], [26,29,30]).
Despite the obtained description, the skew brace theory is not exhaustive for developing the theory of dual weak braces. In fact, for instance, although we show the ideals of any dual weak brace S are specific strong semilattices of ideals of every skew brace B_α, if we consider known ideals, such as its socle $\mathrm{Soc}(S)$, in general, it is not the strong semilattice Y of each $\mathrm{Soc}\,(B_\alpha)$. This

led us to deepen the theory of dual weak braces and not just reduce it to the study of every skew brace B_α. As a first step, we introduce the binary operation $\cdot$ on S given by $a \cdot b := -a + a \circ b - b$, for all $a, b \in S$, classically known in the context of radical Jacobson rings. We give some properties that are useful to characterize the ideals of S in terms of the operation $\cdot$. Furthermore, this has allowed us to investigate the right nilpotency and the nilpotency of S by relating them with those of each skew braces B_α. We highlight that nilpotency in skew braces has been intensively studied over the years by many authors (see, for instance, [1,2,4–6,12,20,32]) and it is still under investigation above all concerning multipermutation solutions [16].

1. Basics on Weak Braces

This section aims to give actual results on the structures of weak braces [9] paying particular attention to the behavior of the idempotents.

To make this paper self-contained and to set up the notation, throughout the paper where it will be needed, we will recall some notions contained in classical books on inverse semigroups, as [14,18,23,27]. A semigroup S is *inverse* if for each $a \in S$, there exists a unique $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. We call such an element a^{-1} the *inverse* of a. The class of inverse semigroups is very close to that of groups since $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$, for all $a, b \in S$. Note that aa^{-1} and $a^{-1}a$ are the idempotents of S, for every $a \in S$. An inverse semigroup S is called *Clifford* if its idempotents are central, or, equivalently, $aa^{-1} = a^{-1}a$, for every $a \in S$.

Definition 1. [9, Definition 5] Let S be a set endowed with two operations $+$ and $\circ$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups. Then, $(S, +, \circ)$ is a *weak brace* if

$$a \circ (b + c) = a \circ b - a + a \circ c \qquad \text{and} \qquad a \circ a^- = -a + a,$$

for all $a, b, c \in S$, where $-a$ and a^- denote the inverses of a with respect to $+$ and $\circ$, respectively.

Clearly, the sets of the idempotents $E(S, +)$ and $E(S, \circ)$ coincide, thus we will simply denote such a set by $E(S)$. Obviously, if $|\operatorname{E}(S)| = 1$, then $(S, +, \circ)$ is a skew brace [17].
In [9, Theorem 8], it is proved that the additive semigroup of any weak brace is necessarily Clifford. In general, the multiplicative one is not (see [9, Example 3]). Any Clifford semigroup $(S, \circ)$ determines two trivial weak braces having both Clifford structures, by setting $a + b := a \circ b$ or $a + b := b \circ a$, for all $a, b \in S$. A bigger class of weak braces having $(S, \circ)$ Cifford is studied in [10, Theorem 16].

Definition 2. [10, Definition 2] A weak brace $(S, +, \circ)$ is called *dual* if $(S, \circ)$ is Clifford.

Given a weak brace $(S, +, \circ)$, let $\lambda : S \to \operatorname{End}(S, +)$, $a \mapsto \lambda_a$ and $\rho : S \to \operatorname{Map}(S)$, $b \mapsto \rho_b$ be the maps defined by

$$\lambda_a(b) = -a + a \circ b \qquad \text{and} \qquad \rho_b(a) = (-a + a \circ b)^- \circ a \circ b,$$

for all $a, b \in S$, respectively. One has that $\lambda_a(b) = a \circ (a^- + b)$, for all $a, b \in S$, and $\lambda_a(E(S)) \subseteq E(S)$. Besides, the map λ is a homomorphism of $(S, \circ)$ into the endomorphism semigroup of $(S, +)$ and the map ρ is an anti-homomorphism of $(S, \circ)$ into the monoid $\operatorname{Map}(S)$. Following [9, Theorem 11], the map $r : S \times S \to S \times S$ defined by $r(a, b) = (\lambda_a(b),\ \rho_b(a))$, for all $a, b \in S$, is a solution that has a behavior close to bijectivity: indeed, considered the solution r^{op} associated with the opposite weak brace $S^{op} := (S, +^{op}, \circ)$ of S they hold

$$r\, r^{op}\, r = r, \qquad r^{op}\, r\, r^{op} = r^{op}, \qquad \text{and} \qquad r r^{op} = r^{op} r.$$

In addition, if S is dual, r has also a behavior close to the non-degeneracy since

$$\lambda_a \lambda_{a^-} \lambda_a = \lambda_a, \qquad \lambda_{a^-} \lambda_a \lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \qquad \lambda_a \lambda_{a^-} = \lambda_{a^-} \lambda_a,$$
$$\rho_a \rho_{a^-} \rho_a = \rho_a, \qquad \rho_{a^-} \rho_a \rho_{a^-} = \rho_{a^-}, \qquad \text{and} \qquad \rho_a \rho_{a^-} = \rho_{a^-} \rho_a,$$

for every $a \in S$. Clearly, if S is a skew brace, r is non-degenerate and bijective with $r^{-1} = r^{op}$ [21]. Moreover, S is a brace if and only if r is involutive.

In the lemma below, we collect some useful properties of weak braces provided in [9,10].

Lemma 1. *Let $(S, +, \circ)$ be a weak brace. Then, the following hold:*

1. $\lambda_a(a^-) = -a$,
2. $a \circ (-b) = a - a \circ b + a$,
3. $a \circ b = a + \lambda_a(b)$,
4. $a + b = a \circ \lambda_{a^-}(b)$,

for all $a, b \in S$.

The following key lemma highlights the behavior of idempotents in any weak brace.

Lemma 2. *If $(S, +, \circ)$ is a weak brace and $e \in \mathrm{E}(S)$, then $\rho_e(a) = a \circ e$ and*

$$\lambda_e(a) = e \circ a = e + a, \tag{1}$$

for every $a \in S$. In particular, if S is a dual weak brace, then $e \circ a = e + a = a \circ e = a + e = \lambda_e(a) = \rho_e(a)$, for every $a \in S$..

Proof. Let $a \in S$. Then, by Lemma 1-3., $e \circ a = e + \lambda_e(a) = e - e + e \circ a = \lambda_e(a)$. As a consequence, we obtain $\lambda_e(a) = e \circ e \circ (e + a) = e \circ \lambda_e(a) = e + a$, where the last equality follows from Lemma 1-4.. In addition, by [9, Proposition 9-3.], $\rho_e(a)^- = e \circ a^- - e = \lambda_e(a^-) = e \circ a^-$, hence $\rho_e(a) = a \circ e$, thus the claim is satisfied. □

In light of Lemma 2, in any dual weak brace S the set of the idempotents $\mathrm{E}(S)$ gives rise to a structure of trivial weak sub-brace of S, the general definition of which is given below.

Definition 3. Let $(S, +, \circ)$ be a weak brace and $H \subseteq S$. Then, H is said to be a *weak sub-brace* of $(S, +, \circ)$ if H both is an inverse semigroup of $(S, +)$ and $(S, \circ)$ such that $\mathrm{E}(S) \subseteq H$.

Convention. From now on, we will only deal with dual weak braces and briefly write a^0 to denote the idempotent $a - a = a \circ a^-$, for every $a \in S$.
It follows by Lemma 2 that $a \circ b = a^0 + a \circ b = a \circ b + b^0$, for all $a, b \in S$.

In the last part of this section, we introduce a new operation on a dual weak brace $(S, +, \circ)$ that will allow characterizing its ideals. Such an operation is the usual multiplication in the context of rings and is already known for skew braces [12,22] and cancellative semi-braces [6]. Specifically, we define the operation $\cdot$ on S given by

$$a \cdot b := -a + a \circ b - b,$$

for all $a, b \in S$, that can be also written as $a \cdot b = \lambda_a(b) - b$ and, by Lemma 2, $a \cdot b = a^0 + a \cdot b = a \cdot b + b^0$. As it is usual in brace theory (cf. [31, Definition 2]), by (1), one has that

$$\lambda_a(b) = -a + a \circ b + b^0 = a \cdot b + b, \tag{2}$$

for all $a, b \in S$.

Lemma 3. *Let $(S, +, \circ)$ be a dual weak brace and $a, b \in S$. Then, the following are equivalent:*

(i) $a \cdot b \in \mathrm{E}(S)$,
(ii) $a \cdot b = a^0 + b^0$,
(iii) $a + b = a \circ b$.

Proof. If $a \cdot b \in \mathrm{E}(S)$, then we have that

$$a \cdot b = a \cdot b - a \cdot b = -a + a \circ b + b^0 - a \circ b + a = a^0 + b^0 + (a \circ b)^0 \underset{(1)}{=} a^0 + b^0.$$

Now, if $a \cdot b = a^0 + b^0$, then $a + b = a + a^0 + b^0 + b = a + a \cdot b + b \underset{(1)}{=} a \circ b$. Finally, if $a + b = a \circ b$, we obtain that $a \cdot b = -a + a \circ b - b = -a + a + b - b = a^0 + b^0 \in \mathrm{E}(S)$, which completes the proof. □

Using Lemma 3 and (1), we immediately obtain that, for all $a \in S$ and $e \in E(S)$,

$$a \cdot e = e \cdot a \in \mathrm{E}(S). \tag{3}$$

Proposition 1. *Let $(S, +, \circ)$ be a dual weak brace. Then, the following are satisfied:*

1. $a \circ b = a + a \cdot b + b$
2. $a \cdot (b + c) = a \cdot b + b + a \cdot c - b$,
3. $(a \circ b) \cdot c = a \cdot (b \cdot c) + b \cdot c + a \cdot c$,

for all $a, b, c \in S$.

Proof. Let $a, b, c \in S$. Firstly, by (1), we have $a \circ b = a^0 + a \circ b + b^0 = a + a \cdot b + b$. Moreover, $a \cdot (b + c) = -a + a \circ b + b^0 - a + a \circ c - c - b = a \cdot b + b + a \cdot c - b$.

Finally, using (1), we get

$$\begin{aligned} & a\cdot(b\cdot c)+b\cdot c+a\cdot c \\ & \quad = -a+a\circ(b\cdot c)+a\cdot c \\ & \quad = -a\circ b+a\circ(b\circ c-c)+a\cdot c && \text{by Lemma } 1-2. \\ & \quad = -a\circ b+a\circ b\circ c-a\circ c+a\circ c-c && \text{by Lemma } 1-2. \\ & \quad = -a\circ b+a\circ b\circ c\circ a^0\circ c^0-c \\ & \quad = -a\circ b+a\circ b\circ c-c \\ & \quad = (a\circ b)\cdot c, \end{aligned}$$

which completes our claim. □

Corollary 1. *Let $(S,+,\circ)$ be a dual weak brace. Then, they hold:*

1. $a\cdot(b+e)=a\cdot b+a\cdot e$,
2. $(e+a)\cdot b=e\cdot b+a\cdot b$,

for all $a,b\in S$ and $e\in E(S)$.

Proof. If $a,b\in S$ and $e\in E(S)$, we obtain

$$\begin{aligned} a\cdot(b+e) &= a\cdot b+b+a\cdot e-b && \text{by Proposition } 1-2. \\ &= a\cdot b+b^0+a\cdot e && \text{by } (3) \\ &= a\cdot b+a\cdot e && \text{by } (1) \end{aligned}$$

and

$$\begin{aligned} (e+a)\cdot b &= (e\circ a)\cdot b && \text{by } (1) \\ &= e\cdot(a\cdot b)+a\cdot b+e\cdot b && \text{by Proposition } 1-3. \\ &= e\circ(a\cdot b)^0+a\cdot b+e\cdot b && \text{by } (1) \\ &= e\cdot b+a\cdot b && \text{by } (3)-(1) \end{aligned}$$

Therefore, the claim follows. □

2. A Description of Dual Weak Braces and Their Solutions

In this section, we provide a description theorem for dual weak braces by showing that they are strong semilattices of specific skew braces. This description is consistent with the fact that Clifford semigroups are strong semilattices of groups (see [27, Theorem II.2.6]).

Let Y be a (lower) semilattice and $\{G_\alpha \mid \alpha\in Y\}$ a family of disjoint groups. For all $\alpha,\beta\in Y$ such that $\alpha\geq\beta$, let $\phi_{\alpha,\beta}: G_\alpha\to G_\beta$ be a homomorphism of groups such that

1. $\phi_{\alpha,\alpha}$ is the identical automorphism of G_α, for every $\alpha\in Y$,
2. $\phi_{\beta,\gamma}\phi_{\alpha,\beta}=\phi_{\alpha,\gamma}$, for all $\alpha,\beta,\gamma\in Y$ such that $\alpha\geq\beta\geq\gamma$.

Then, $S=\bigcup\limits_{\alpha\in Y} G_\alpha$ endowed with the operation defined by $a\,b:=\phi_{\alpha,\alpha\beta}(a)\,\phi_{\beta,\alpha\beta}(b)$, for all $a\in G_\alpha$ and $b\in G_\beta$, is a *Clifford semigroup*, also called *strong semilattice Y of groups G_α*, usually written as $S=[Y;G_\alpha;\phi_{\alpha,\beta}]$. Conversely, any Clifford semigroup is of this form.

Theorem 1. *Let Y be a (lower) semilattice, $\{B_\alpha \mid \alpha \in Y\}$ a family of disjoint skew braces. For each pair α, β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ be a skew brace homomorphism such that*

1. *$\phi_{\alpha,\alpha}$ is the identical automorphism of B_α, for every $\alpha \in Y$,*
2. *$\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$.*

Then, $S = \bigcup\{B_\alpha \mid \alpha \in Y\}$ endowed with the addition and the multiplication, respectively, defined by

$$a + b := \phi_{\alpha,\alpha\beta}(a) + \phi_{\beta,\alpha\beta}(b) \quad \text{and} \quad a \circ b := \phi_{\alpha,\alpha\beta}(a) \circ \phi_{\beta,\alpha\beta}(b),$$

for all $a \in B_\alpha$ and $b \in B_\beta$, is a dual weak brace with $\mathrm{E}(S)$ isomorphic to Y, called strong semilattice S of skew braces B_α *and denoted by $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$. Conversely, any dual weak brace is a strong semilattice of skew braces.*

Proof. The proof of the sufficient condition is contained [10, Corollary 6]. Conversely, let $(S, +, \circ)$ be a dual weak brace and let $[Y; B_\alpha; \phi_{\alpha,\beta}]$ and $[Z; H_i; \psi_{i,j}]$ the two Clifford semigroups $(S, +)$ and $(S, \circ)$, respectively. Since $\mathrm{E}(S, +)$ and $\mathrm{E}(S, \circ)$ coincide and they are isomorphic to Y and Z, respectively, there exists a semilattice isomorphism $f : Y \to Z$, that, formally, can be written as $f(\alpha) = i_\alpha$, for every $\alpha \in Y$. Keeping in mind this fact, it is not restrictive to consider $Y = Z$. Now, let $\alpha \in Y$ and show that $B_\alpha = H_\alpha$. Thus, if $a \in B_\alpha$, then there exists $\beta \in Y$ such that $a \in H_\beta$. Denoted by e_α and e_β the identities of the groups $(B_\alpha, +)$ and $(H_\beta, \circ)$, respectively, note that

$$e_\alpha = a - a = a \circ a^- = e_\beta,$$

hence $a \in B_\alpha \cap H_\alpha \subseteq H_\alpha$. Reversing the role of $+$ and $\circ$, one similarly obtains that $H_\alpha \subseteq B_\alpha$. Finally, if $\alpha, \beta \in Y$ and $\alpha \geq \beta$, observe that

$$\phi_{\alpha,\beta}(a) = a + e_\beta \underset{(1)}{=} a \circ e_\beta = \psi_{\alpha,\beta}(a),$$

for every $a \in B_\alpha$. Therefore, $\{B_\alpha \mid \alpha \in Y\}$ is a family of disjoint skew braces and the claim follows. □

The following result is a consequence of [27, Proposition II.2.8] and Theorem 1.

Proposition 2. *Two dual weak braces $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ and $T = [Z; C_i; \psi_{i,j}]$ are isomorphic if and only if there exists a semilattice homomorphism $\eta : Y \to Z$ and a family of skew brace isomorphisms $\{\Theta_\alpha : B_\alpha \to C_{\eta(\alpha)} \mid \alpha \in Y\}$ such that $\Theta_\beta\phi_{\alpha,\beta} = \psi_{\eta(\alpha),\eta(\beta)}\Theta_\alpha$, for every $\alpha \geq \beta$.*

The following example shows that Theorem 1 allows for obtaining new dual weak braces even if we start from trivial skew braces.

Example 1. Let us consider $Y = \{\alpha, \beta\}$, with $\alpha > \beta$, B_α and B_β the trivial skew braces on the cyclic group C_3 and on the symmetric group Sym_3 of order 3, respectively, and $\phi_{\alpha,\beta} : C_3 \to \mathrm{Sym}_3$ the homomorphism given by $\phi_{\alpha,\beta}(0) = \mathrm{id}_3$, $\phi_{\alpha,\beta}(1) = (123)$, $\phi_{\alpha,\beta}(2) = (132)$. Then, $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ is a dual weak brace.

The following natural question arises intending to concretely construct dual weak braces.

Question *Determining all skew brace homomorphisms.*

Note that the problem of studying homomorphisms between skew braces has emerged yet in the literature, such as [11, Problem 10.2], [34, Problem 2.18], [26,29], and [30], and also in a recent conference talk held by Civino [13].

Now, by Theorem 1, we illustrate that the solution r associated with any dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ is the *strong semilattice of the solutions* r_α on each skew brace B_α. In this regard, we recall the more general result contained in [8, Theorem 4.1].

Theorem 2. *Let Y be a (lower) semilattice, let $\{r_\alpha \mid \alpha \in Y\}$ be a family of disjoint solutions on X_α indexed by Y such that for each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, there is a map $\phi_{\alpha,\beta} : X_\alpha \to X_\beta$. Let X be the union*

$$X = \bigcup \{X_\alpha \mid \alpha \in Y\}$$

and let $r : X \times X \longrightarrow X \times X$ be the map defined as

$$r\left(x, y\right) := r_{\alpha\beta}\left(\phi_{\alpha,\alpha\beta}\left(x\right), \phi_{\beta,\alpha\beta}\left(y\right)\right),$$

for all $x \in X_\alpha$ and $y \in X_\beta$. Then, if the following conditions are satisfied

1. *$\phi_{\alpha,\alpha}$ is the identity map of X_α, for every $\alpha \in Y$,*
2. *$\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$,*
3. *$\left(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta}\right) r_\alpha = r_\beta \left(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta}\right)$, for all $\alpha, \beta \in Y$ such that $\alpha \geq \beta$.*

then r is a solution on X called a strong semilattice of solutions r_α indexed by Y.

Given a dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$, clearly the conditions of Theorem 2 are satisfied by each solution r_α on each skew brace B_α. In particular, the condition 3. follows by the fact that the map $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ is a skew brace homomorphism, for every $\alpha \geq \beta$. Moreover, it is easy to see that also the converse is true. Thus, we can sum it all up in the following result.

Proposition 3. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace. Then, the solution r associated with S is the strong semilattice of the solutions r_α associated with each skew brace B_α.*

To be thorough, we know from [33, Theorem 4.13] that in the finite case the order of each solution r_α, or equivalently the period $p\left(r_\alpha\right)$ of r_α, for every $\alpha \in Y$, is even and, by [8, Theorem 5.2], we can establish the order of the overall solution r. In the next result, for a finite dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ we mean that Y and B_α are finite, for every $\alpha \in Y$.

Corollary 2. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a finite dual weak brace and r the solution associated with S. Then, $r^{2k+1} = r$ with $2k = \operatorname{lcm}\{p\left(r_\alpha\right) \mid \alpha \in Y\}$, where r_α is the solution on each skew brace B_α, for every $\alpha \in Y$.*

Corollary 3. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a strong semilattice of braces such that Y is finite. Then, the solution r associated with S is cubic, namely, $r^3 = r$.*

3. Ideals of Dual Weak Braces

In this section, we entirely describe the structure of any ideal of a dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ through ideals of each skew brace B_α.

Let us start by recalling the notion of ideal contained in [10, Definition 20]. To this end, by a *normal subsemigroup* I of a Clifford semigroup S we will mean a subset I of S such that

1. $\mathrm{E}(S) \subseteq I$ (i.e., I is *full* in S),
2. $\forall a, b \in I \quad ab \in I \quad$ and $\quad a^{-1} \in I$,
3. $\forall a \in S \quad a^{-1} I a \subseteq I$.

Note that such a definition is equivalent to that of *normal subset* contained in [28, Definition VI.1.2].

Definition 4. (cf. [10, Definition 20]) Let $(S, +, \circ)$ be a dual weak brace and I a subset of S. Then, I is an *ideal* of $(S, +, \circ)$ if they hold

1. I is a normal subsemigroup of $(S, +)$,
2. $\lambda_a(I) \subseteq I$, for every $a \in S$,
3. I is a normal subsemigroup of $(S, \circ)$.

It follows from the definition of every ideal I that $\mathrm{E}(S) \subseteq I$. In particular, I is a dual weak sub-brace of S, thus S and $\mathrm{E}(S)$ are trivial ideals of S. According to [10, Proposition 23], another known ideal of S is its *socle* (or *left annihilator*), i.e., the set

$$\mathrm{Soc}\,(S) = \{a \,|\, a \in S,\ \forall\ b \in S \quad a + b = a \circ b \quad \text{and} \quad a + b = b + a\}.$$

In addition, note that if I and J are ideals of S, then, as proved in [10, Proposition 24], $I + J$ and $I \circ J$ also are, and one can easily see from Lemma 1 that they coincide.

In light of Theorem 1, we can give the following structure theorem for every ideal of a dual weak brace.

Theorem 3. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace, I_α an ideal of each skew brace B_α, and set $\psi_{\alpha,\beta} := {\phi_{\alpha,\beta}}_{|_{I_\alpha}}$, for all $\alpha \geq \beta$. If $\phi_{\alpha,\beta}(I_\alpha) \subseteq I_\beta$, for any $\alpha > \beta$, then $I = [Y; I_\alpha; \psi_{\alpha,\beta}]$ is an ideal of S and, conversely, every ideal of S is of this form.*

Proof. To show that $I = [Y; I_\alpha; \psi_{\alpha,\beta}]$ is an ideal of S, by [27, Exercises III.1.9(ii)], it is enough to prove the λ-invariance of I. If $a \in S$ and $x \in I$, there exist $\alpha, \gamma \in Y$ such that $a \in B_\alpha$ and $x \in I_\gamma$, and so, since by the assumption $\phi_{\gamma,\alpha\gamma}\,(x) \in I_{\alpha\gamma}$, we have that

$$\lambda_a\,(x) = \phi_{\alpha,\alpha\gamma}\,(-a) + \phi_{\alpha,\alpha\gamma}\,(a) \circ \phi_{\gamma,\alpha\gamma}\,(x) = \lambda_{\phi_{\alpha,\alpha\gamma}(a)}\phi_{\gamma,\alpha\gamma}\,(x) \in I_{\alpha\gamma},$$

hence $\lambda_a\,(x) \in I$.

Vice versa, if I is an ideal of $(S, +, \circ)$, since I is a dual weak brace too, it follows by Theorem 1 and [27, Exercises III.1.9(ii)] that $I = [Y; I_\alpha; \psi_{\alpha,\beta}]$, with $I_\alpha = B_\alpha \cap I$, for every $\alpha \in Y$, $\psi_{\alpha,\beta} := {\phi_{\alpha,\beta}}_{|_{I_\alpha}}$, for all $\alpha \geq \beta$, such that $\phi_{\alpha,\beta}(I_\alpha) \subseteq I_\beta$, for any $\alpha > \beta$. To get the claim, we prove the λ-invariance of each I_α, i.e., I_α is an ideal of the skew brace B_α, for every $\alpha \in Y$. Indeed, we have that $\lambda_a(x) \in B_\alpha \cap I = I_\alpha$, for all $a \in B_\alpha$ and $x \in I_\alpha$. Therefore, the claim follows. $\square$

By Theorem 3, the ideals of any dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ are strong semilattices Y of certain ideals I_α of each skew brace B_α. However, the skew brace theory is not exhaustive for developing the theory of dual weak braces. In fact, for instance, if we consider the ideal $\mathrm{Soc}(S)$, in general, it is not the strong semilattice Y of the socles $\mathrm{Soc}\,(B_\alpha)$ of B_α (see Example 1).

Lemma 4. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace. Then, denoting by $\phi^-_{\alpha,\beta}$ the inverse image of $\phi_{\alpha,\beta}$, we have*

$$\mathrm{Soc}(S) = \big[Y; \bigcap_{\alpha\geq\beta} \phi^-_{\alpha,\beta}\left(\mathrm{Soc}(B_\beta)\right); \psi_{\alpha,\beta}\big],$$

where $\psi_{\alpha,\beta} := \phi_{\alpha,\beta}|_{\mathrm{Soc}(B_\alpha)}$, for all $\alpha \geq \beta$.

Proof. By the proof of Theorem 3, to prove the claim it is enough to show that $\mathrm{Soc}(S)\cap B_\alpha = \bigcap_{\alpha\geq\beta} \phi^-_{\alpha,\beta}\left(\mathrm{Soc}(B_\beta)\right)$, for every $\alpha \in Y$. Let $x \in \mathrm{Soc}(S)\cap B_\alpha$ and $\beta \in Y$ such that $\alpha \geq \beta$. Then, if $b \in B_\beta$,

$$\phi_{\alpha,\beta}\left(x\right) + b = x + b = x \circ b = \phi_{\alpha,\beta}\left(x\right) \circ b$$
$$\phi_{\alpha,\beta}\left(x\right) + b = x + b = b + x = b + \phi_{\alpha,\beta}\left(x\right),$$

hence $x \in \bigcap_{\alpha\geq\beta} \phi^-_{\alpha,\beta}\left(\mathrm{Soc}(B_\beta)\right)$. Conversely, if $x \in \bigcap_{\alpha\geq\beta} \phi^-_{\alpha,\beta}\left(\mathrm{Soc}(B_\beta)\right)$ and $y \in B_\gamma$, with $\gamma \in Y$, then $x \in B_\alpha$ and, since $\phi_{\alpha,\alpha\gamma}\left(x\right) \in \mathrm{Soc}\left(B_{\alpha\gamma}\right)$, we get

$$x + y = \phi_{\alpha,\alpha\gamma}\left(x\right) + \phi_{\gamma,\alpha\gamma}\left(y\right) = \phi_{\alpha,\alpha\gamma}\left(x\right) \circ \phi_{\gamma,\alpha\gamma}\left(y\right) = x \circ y$$
$$x + y = \phi_{\alpha,\alpha\gamma}\left(x\right) + \phi_{\gamma,\alpha\gamma}\left(y\right) = \phi_{\gamma,\alpha\gamma}\left(y\right) + \phi_{\alpha,\alpha\gamma}\left(x\right) = y + x,$$

therefore $x \in \mathrm{Soc}(S) \cap B_\alpha$. □

As a direct consequence of Theorem 3 and Lemma 4, we have the following.

Proposition 4. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace, $\psi_{\alpha,\beta} := \phi_{\alpha,\beta}|_{\mathrm{Soc}(B_\alpha)}$, for all $\alpha \geq \beta$, and assume that $I := [Y; \mathrm{Soc}\,(B_\alpha)\,; \psi_{\alpha,\beta}]$ is an ideal of S. Then, $I = \mathrm{Soc}(S)$.*

According to [10, Theorem 21], ideals allow for obtaining quotient structures. Into the specific, if I is an ideal of a dual weak brace $(S, +, \circ)$, then the relation $\sim_I$ on S given by

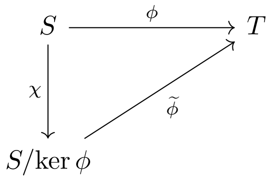
$$\forall\, a, b \in S \qquad a \sim_I b \iff a^0 = b^0 \;\text{ and }\; -a + b \in I$$

is an idempotent separating congruence of $(S, +, \circ)$. Furthermore, $S/I := S/\sim_I$ is a dual weak brace with semilattice of idempotents isomorphic to $\mathrm{E}\,(S)$.
As it is natural to expect, we can consider the canonical epimorphism $\chi : S \to S/I,\ a \mapsto a + I$ and give the usual homomorphism theorems.

Theorem 4. *Let $\phi : S \to T$ be a homomorphism between two dual weak braces $(S, +, \circ)$ and $(T, +, \circ)$. Then, the following statements hold:*

1. *$\mathrm{Im}\,\phi$ is a dual weak sub-brace of $(T, +, \circ)$;*
2. *$\ker\phi = \{a \mid a \in S \quad \exists\, e \in \mathrm{E}(S) \quad \phi(a) = \phi(e)\}$ is an ideal of $(S, +, \circ)$;*

3. *there exists a monomorphism* $\widetilde{\phi}: S/\ker\phi \to T$ *of dual weak braces such that* $\operatorname{Im}\widetilde{\phi} = \operatorname{Im}\phi$ *and the diagram*



commutes.

Remark 1. Let us observe that any ideal I of a dual weak brace S is the kernel of the canonical epimorphism χ. Indeed, if $a \in I$, then $a + I = a^0 + I$, hence $a \in \ker\chi$. Conversely, if $a \in \ker\chi$, then there exists $e \in E(S)$ such that $\chi(a) = \chi(e)$, hence $a = a + a^0 \in a + I = e + I \subseteq I$.

Corollary 4. *Let* $(S, +, \circ)$ *be a dual weak brace,* I *an ideal of* S*, and* H *a dual weak sub-brace of* S*. Then,*

1. $I + H$ *is a dual weak sub-brace of* S*;*
2. I *is an ideal of* $I + H$*;*
3. $I \cap H$ *is an ideal of* H*;*
4. $H/(I \cap H)$ *is isomorphic to* $(I + H)/I$*.*

Corollary 5. *Let* $(S, +, \circ)$ *be a dual weak brace and* I, J *ideals of* S *such that* $J \subset I$*. Then,* S/I *is isomorphic to* $(S/J)/(I/J)$*.*

Corollary 6. *Let* I *be an ideal of a dual weak brace* $(S, +, \circ)$*. There is a one-to-one correspondence between the set of ideals of* S *containing* I *and the set of ideals of* S/I*. Moreover, ideals of* S *containing* I *correspond to ideals of* S/I*.*

4. Left Ideals of Dual Weak Braces

In this section, we provide a characterization of ideals of any dual weak brace which makes use of the concept of left ideals and the operation $\cdot$.

Let us start by introducing the notions of the left ideal and strong left ideal of a dual weak brace, consistently with [12] and [19, Definition 2.3].

Definition 5. Let $(S, +, \circ)$ be a dual weak brace. Then, a subset I of S is a *left ideal* of $(S, +, \circ)$ if

1. I is a full inverse subsemigroup of $(S, +)$,
2. $\lambda_a(I) \subseteq I$, for every $a \in S$.

A left ideal I is a *strong left ideal* if I is a normal subsemigroup of $(S, +)$.

Note that any left ideal I of a dual weak brace S is a full inverse subsemigroup of $(S, \circ)$. Indeed, by Lemma 1, if $a, b \in I$, then $a \circ b = a + \lambda_a(b) \in I$ and $a^- = \lambda_{a^-}(-a) \in I$.

Similar to Theorem 3, we can describe the structure of any strong left ideal, as follows.

Proposition 5. *Let* $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ *be a dual weak brace,* I_α *a strong left ideal of each skew brace* B_α*, and set* $\psi_{\alpha,\beta} := \phi_{\alpha,\beta}|_{I_\alpha}$*, for all* $\alpha \geq \beta$*. If* $\phi_{\alpha,\beta}(I_\alpha) \subseteq I_\beta$*, for any* $\alpha > \beta$*, then* $I = [Y; I_\alpha; \psi_{\alpha,\beta}]$ *is a strong left ideal of* S *and, conversely, every strong left ideal of* S *is of this form.*

Example 2. Let $(S, +, \circ)$ be a dual weak brace. Every full endomorphism-invariant subsemigroup I of $(S, +)$, i.e., $\varphi(I) \subseteq I$, for every $\varphi \in \mathrm{End}(S, +)$, is a strong left ideal (cf. [19, cf. Example 2.4]).

In skew braces theory [12], a known left ideal of a skew brace $(B, +, \circ)$ is $\mathrm{Fix}(B)$, that is the set of the elements of B that are fixed by the map λ_a, for every $a \in B$. For a dual weak brace S, it can be defined similarly, as we show below.

Proposition 6. *Let* $(S, +, \circ)$ *be a dual weak brace. Then, the following set*

$$\begin{aligned}\mathrm{Fix}(S) :&= \{b \mid b \in S,\ \forall\ a \in S \quad a + b = a \circ b\}\\ &= \{b \mid b \in S,\ \forall\ a \in S \quad \lambda_a(b) = a^0 + b\}\end{aligned}$$

is a left ideal of S*.*

Proof. Initially, by (1), $E(S) \subseteq \mathrm{Fix}(S)$. Moreover, $(\mathrm{Fix}(S), +)$ is trivially closed with respect to $+$ and, by Lemma 1-2., $a \circ (-b) = a - a \circ b + a = a - b + a^0 = a - b$, for all $a \in S$ and $b \in \mathrm{Fix}(S)$. Finally, if $a, c \in S$ and $b \in \mathrm{Fix}(S)$, it follows that

$$\begin{aligned}c + \lambda_a(b) &= c + c^0 + a^0 + b && b \in \mathrm{Fix}(S)\\ &= c + c^0 \circ a^0 + b && \text{by (1)}\\ &= c - c \circ a + c \circ a + b\\ &= c - c \circ a + c \circ a \circ b && b \in \mathrm{Fix}(S)\\ &= c + \lambda_{c\circ a}(b)\\ &= c \circ \lambda_a(b) && \text{by Lemma 1} - 3.\end{aligned}$$

Therefore, $\mathrm{Fix}(S)$ is a left ideal of S. □

Definition 6. If $(S, +, \circ)$ is a dual weak brace, then we call the set $Zl(S) := \mathrm{Fix}(S) \cap \zeta(S, +)$ the *left center* (or *right annihilator*) of S, where $\zeta(S, +)$ denotes the center of $(S, +)$.

It is easy to check that $Zl(S)$ is a strong left ideal of any dual weak brace S. In general, $Zl(S)$ is not an ideal as shown in the context of skew braces in [2, Example 5.7].

In the following, if X and Y are subsets of a dual weak brace $(S, +, \circ)$, we denote by $X \cdot Y$ the additive inverse subsemigroup of S generated by the elements of the form $x \cdot y$, with $x \in X$ and $y \in Y$ (cf. [27, Definition II.1.11]).

Proposition 7. *Let* $(S, +, \circ)$ *be a dual weak brace. Then, the following hold:*

1. *a full inverse subsemigroup* I *of* $(S, +)$ *is a left ideal of* S *if and only if* $S \cdot I \subseteq I$*;*
2. *if* I *is an ideal of* S*, then* $S \cdot I \subseteq I$ *and* $I \cdot S \subseteq I$*;*

3. a normal subsemigroup I of $(S,+)$ is an ideal of S if and only if $I\cdot S\subseteq I$ and $\lambda_a(I)\subseteq I$, for every $a\in S$.

Proof. 1. If I is a left ideal of S, $a\in S$ and $x\in I$, then $a\cdot x=\lambda_a(x)-x\in I$. Conversely, if $S\cdot I\subseteq I$, $a\in S$ and $x\in I$, by (2), we obtain that $\lambda_a(x)=a\cdot x+x\in I$.
2. By 1., it is enough to see that if $a\in S$ and $x\in I$, then

$$x\cdot a=-x+a-a+a\circ a^-\circ x\circ a-a=-x+\underbrace{a+\lambda_a\left(a^-\circ x\circ a\right)-a}_{\in I}\in I.$$

3. It is enough to show that if $a\in S$ and $x\in I$, then by Lemma 1

$$\begin{aligned}a\circ x\circ a^- &= a+\lambda_a\left(x+\lambda_x\left(a^-\right)\right)=a+\lambda_a(x)+\lambda_{a\circ x}\left(a^-\right)+a-a\\ &= a+\lambda_a\left(x+\lambda_x\left(a^-\right)-a^-\right)-a=a+\lambda_a\underbrace{\left(x+x\cdot a^-\right)}_{\in I}-a\in I,\end{aligned}$$

which completes the proof. □

As a consequence of Proposition 7, one can characterize the ideals of a dual weak brace as is usual in ring theory.

Corollary 7. *Let $(S,+,\circ)$ be a dual weak brace. Then, a normal subsemigroup I of $(S,+)$ is an ideal of S if and only if $S\cdot I\subseteq I$ and $I\cdot S\subseteq I$.*

To conclude this section, similarly to [4, Proposition 1.3], we show how to obtain instances of left ideals starting from ideals. If X,Y are ideals of a dual weak brace S, we denote by $[X,\,Y]_+=\left\langle\,[x,\,y]_+\mid x\in X,\ y\in Y\,\right\rangle$.

Proposition 8. *Let $(S,+,\circ)$ be a dual weak brace and I,J ideals of S. Then, $I\cdot J$, $I\cdot J+J\cdot I$, and $[I,\,J]_+$ are left ideals of S.*

Proof. Initially, note that $I\cdot J$ and $I\cdot J+J\cdot I$ trivially contain $\mathrm{E}(S)$. Moreover, $I\cdot J$ clearly is an inverse subsemigroups of $(S,+)$ and if $s\in S$, $x\in I$, and $h\in J$, then, by Lemma 1-2. and (3),

$$\begin{aligned}s\cdot(x\cdot h) &= \lambda_{s\circ x}(h)-\lambda_s(h)-x\cdot h=\lambda_{s\circ x\circ s^-}\lambda_s(h)-\lambda_s(h)-x\cdot h\\ &= \left(s\circ x\circ s^-\right)\cdot\lambda_s(h)-x\cdot h\in I\cdot J,\end{aligned}$$

hence, by Proposition 7-1., $I\cdot J$ is a left ideal of S.
To prove that $I\cdot J+J\cdot I$ is an inverse subsemigroup of $(S,+)$, let us check that $J\cdot I+I\cdot J\subseteq I\cdot J+J\cdot I$. If $x,y\in I$ and $h,k\in J$, since $(y\cdot(h\cdot x))^0=y^0+y^0\circ(h\cdot x)^0+(h\cdot x)^0=(h\cdot x)^0+y^0$ and $y\cdot k=y^0+y\cdot k$, it follows that

$$\begin{aligned}h\cdot x+y\cdot k &= (y\cdot(h\cdot x))^0+h\cdot x+y\cdot k+(h\cdot x)^0\\ &= -y\cdot(h\cdot x)+y\cdot(h\cdot x+k)+h\cdot x\in I\cdot J+J\cdot I,\end{aligned}$$

where in the last equality we use Proposition 1-2.. Furthermore, by the first part of the proof, we trivially get $\lambda_a(I\cdot J+J\cdot I)\subseteq I\cdot J+J\cdot I$, for every $a\in S$.
Finally, $[I,\,J]_+$ trivially is a left ideal of S. Therefore, the claim is proved. □

5. Right Nilpotent Dual Weak Braces

Following [12,31], if $(S,+,\circ)$ is a dual weak brace, set $S^{(1)} := S$, we inductively define

$$S^{(n+1)} = S^{(n)} \cdot S,$$

for every $n \geq 1$, and obtain that it is an ideal, whose proof can be proved similarly to [12, Proposition 2.1]. However, unlike what happens for the socle, here we can use the characterization of ideals in Theorem 3.

Proposition 9. *If $(S,+,\circ)$ is a dual weak brace, then $S^{(n)}$ is an ideal of S, for every $n \geq 1$.*

Proof. Initially, by Theorem 1, we have that $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ and one clearly obtains $S^{(n)} \subseteq \dot{\bigcup}_{\alpha\in Y} B_\alpha^{(n)}$. The claim follows by Theorem 3, by showing that $\phi_{\alpha,\beta}\left(B_\alpha^{(n)}\right) \subseteq B_\beta^{(n)}$, for any $\alpha > \beta$, by proceeding by an easy induction on n. □

Definition 7. A dual weak brace $(S,+,\circ)$ is said to be *right nilpotent* if $S^{(n)} = E(S)$ for some $n \geq 1$. The smallest positive integer m such that $S^{(m+1)} = \mathrm{E}(S)$ is called *right nilpotency index* of S.

It is easy to check that if S is a right nilpotent dual weak brace, then any ideal I and quotient S/I are right nilpotent as well. At the present state of our knowledge, we do not know whether the vice versa is true in general. Surely, it is true when I is the socle of any dual weak brace, as we show in the following result. At first, note that $\operatorname{Soc}(S)$ is right nilpotent since, using Lemma 3, it can be rewritten in terms of the $\cdot$ operation, as

$$\operatorname{Soc}(S) = \{a \mid a \in S,\ \forall\, b \in S \quad a \cdot b \in \mathrm{E}(S) \quad \text{and} \quad a + b = b + a\}.$$

Proposition 10. *Let $(S,+,\circ)$ be a dual weak brace such that $S/\operatorname{Soc}(S)$ is right nilpotent. Then, S is right nilpotent.*

Proof. Since there exists $m \in \mathbb{N}$ such that $(S/\operatorname{Soc}(S))^{(m)} = E(S/\operatorname{Soc}(S))$, it follows that $S^{(m)} \subseteq \operatorname{Soc}(S)$. Hence, $S^{(m+1)} = S^{(m)} \cdot S \subseteq \operatorname{Soc}(S) \cdot S \subseteq E(S)$. Thus, the claim is proved. □

Definition 8. Let $(S,+,\circ)$ be a dual weak brace. Set $\operatorname{Soc}_0(S) := E(S)$, we define $\operatorname{Soc}_n(S)$ to be the ideal of S containing $\operatorname{Soc}_{n-1}(S)$ such that

$$\operatorname{Soc}_n(S) / \operatorname{Soc}_{n-1}(S) = \operatorname{Soc}(S/\operatorname{Soc}_{n-1}(S)),$$

for every positive integer n.

In particular, $\operatorname{Soc}_n(S) := \{a \mid a \in S,\ \forall\, b \in S \quad a \cdot b \in \operatorname{Soc}_{n-1}(S),\ [a,b]_+ \in \operatorname{Soc}_{n-1}(S)\}$, for every positive integer n (cf. [6, Remark 28]).

Definition 9. Let $(S,+,\circ)$ be a dual weak brace. An *s-series* (or *left annihilator series*) of S is a sequence of ideals of S

$$\mathrm{E}(S) = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S$$

such that $I_j/I_{j-1} \subseteq \operatorname{Soc}(S/I_{j-1})$, for each $j \in \{1,\ldots,m\}$.

One can check that S admits an s-series if and only if there exists a positive integer n such that $S = \mathrm{Soc}_n(S)$, as in the case of the skew braces (cf. [12, Lemma 2.15]). In light of this fact, we give the following definitions.

Definition 10. A dual weak brace $(S, +, \circ)$ is called *left annihilator nilpotent* if S admits an s-series. Consequently, if S is left annihilator nilpotent, we call *socle series* (or *upper left annihilator series*) the series introduced in Definition 8. The smallest non-negative integer n such that $S = \mathrm{Soc}_n(S)$ is called *left annihilator nilpotency index* of S.

Proposition 11. *Let $(S, +, \circ)$ be a left annihilator nilpotent dual weak brace. Then, S is right nilpotent.*

Proof. Let $\mathrm{E}(S) = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S$ an s-series of S. Let $k \in \{1, \ldots, m\}$, we prove that $S^{(k)} \subseteq I_{m-k+1}$ by proceeding by induction on k. If $k = 1$, then $S = I_m$. Now, let $k \in \mathbb{N}$ and assume that $S^{(k)} \subseteq I_{m-k+1}$. Then, considered the canonical epimorphism $\chi : S \to S/I_{m-k}$, it holds that $\chi(I_{m-k+1}) \cdot \chi(S) \subseteq E(S/I_{m-k})$, and so $I_{m-k+1} \cdot S \subseteq I_{m-k}$. Hence,

$$S^{(k+1)} = S^{(k)} \cdot S \subseteq I_{m-k+1} \cdot S \subseteq I_{m-k}$$

and the claim follows by induction. Therefore, $S^{(m+1)} = E(S)$ and so S is right nilpotent of right nilpotency index less or equal to m. □

The converse of Proposition 11 is true in the particular case in which $(S, +)$ is a nilpotent Clifford semigroup. The notion of nilpotent Clifford semigroup which we adopt in this work is consistent with the one given in [25].

Proposition 12. *Let $(S, +, \circ)$ be a dual weak brace such that $(S, +)$ is nilpotent. Then, S is right nilpotent if and only if S is left annihilator nilpotent.*

Proof. The necessary condition follows by Proposition 11. The sufficient one is similar to the proof of [12, Lemma 2.16] by using lower central series of $(S, +)$ in [25, Definition 3.4]. □

Below, we relate the left annihilator nilpotency of a dual weak brace $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ with those of each skew brace B_α. It will be a direct consequence of the following lemma.

Lemma 5. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace. Then, $B_\alpha \cap \mathrm{Soc}_k(S) \subseteq \mathrm{Soc}_k(B_\alpha)$, for all $\alpha \in Y$ and $k \in \mathbb{N}$.*

Proof. Let us check that $B_\alpha \cap \mathrm{Soc}_k(S) \subseteq \mathrm{Soc}_k(B_\alpha)$, by proceeding by induction on k. If $k = 1$ the claim follows by Theorem 3. Now, suppose that the statement holds for $k \in \mathbb{N}$ and let $a \in B_\alpha \cap \mathrm{Soc}_{k+1}(S)$. Then, for any $b \in B_\alpha$, $a \cdot b \in \mathrm{Soc}_k(S)$, $[a, b]_+ \in \mathrm{Soc}_k(S)$, and $a \cdot b \in B_\alpha$, and so, by the inductive hypothesis, $a \in \mathrm{Soc}_{k+1}(B_\alpha)$. □

Proposition 13. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a left annihilator nilpotent dual weak brace of index n. Then, for every $\alpha \in Y$ each skew brace B_α is left annihilator nilpotent of index less or equal to n.*
Vice versa, if each skew brace B_α is left annihilator nilpotent of index k_α, then S is left annihilator nilpotent of index equal to the maximum between k_α, for every $\alpha \in Y$.

6. Annihilator Nilpotency of Dual Weak Braces

Let us start by introducing a further ideal of a dual weak brace $(S, +, \circ)$, that is the *annihilator* of S. Denoted by $\zeta(S, \circ)$ the center of $(S, \circ)$, it is the set

$$\operatorname{Ann}(S) := \operatorname{Soc}(S) \cap \zeta(S, \circ),$$

namely, $\operatorname{Ann}(S) = \{a \mid a \in S \quad \forall\, b \in B \quad a + b = b + a = a \circ b = b \circ a\}$. Note that the definition is consistent with that given in [7, Definition 7] for skew braces.

Proposition 14. *Let $(S, +, \circ)$ be a dual weak brace. Then, $\operatorname{Ann}(S)$ is an ideal of S.*

Proof. Clearly, $E(S)$ is contained in $\operatorname{Ann}(S)$. Moreover, if $x \in \operatorname{Ann}(S)$ and $a \in S$,

$$\begin{aligned}
b \circ (-a + x + a) &= b - b \circ a + b^0 + b \circ x - b + b \circ a && \text{by Lemma 1} - 2.\\
&= b - b \circ a + x \circ b - b + b \circ a && x \in \zeta(S, \circ)\\
&= b - b \circ a + x + b^0 + b \circ a && x \in \operatorname{Soc}(S)\\
&= b + x + (b \circ a)^0 && x \in \operatorname{Soc}(S)\\
&= b + x + a^0 + b^0 &&\\
&= (-a + x + a) + b && x \in \operatorname{Soc}(S)\\
&= (-a + x + a) \circ b. && -a + x + a \in \operatorname{Soc}(S)
\end{aligned}$$

for every $b \in S$. Besides, one has that $\lambda_a(x) = -a + x \circ a = -a + x + a \in \operatorname{Ann}(S)$. Finally, $a^- \circ x \circ a = a^0 \circ x = \lambda_a(x) \in \operatorname{Ann}(S)$, which completes the proof. □

As it happens for the socle, in general, the annihilator of a dual weak brace S does not coincide with the union of the annihilators of each skew brace B_α (see, for instance, Example 1). Similar to Proposition 4, we obtain the following result.

Proposition 15. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be a dual weak brace, $\psi_{\alpha,\beta} := {\phi_{\alpha,\beta}}_{|_{\operatorname{Ann}(B_\alpha)}}$, for all $\alpha \geq \beta$, and assume that $I := [Y; \operatorname{Ann}(B_\alpha); \psi_{\alpha,\beta}]$ is an ideal of S. Then, $I = \operatorname{Ann}(S)$.*

Remark 2. As a consequence of Lemma 3, if $(S, +, \circ)$ is a dual weak brace, it is also easy to see that

$$\operatorname{Ann}(S) = \{a \,|\, a \in S, \ \forall\, b \in S \quad a \cdot b = b \cdot a = [a, b]_+ = a^0 + b^0\}.$$

Similar to [20], as is usual in ring theory, we define the kth annihilator of a dual weak brace $(S, +, \circ)$.

Definition 11. Let $(S, +, \circ)$ be a dual weak brace. Set $\operatorname{Ann}_0(S) := E(S)$, we define $\operatorname{Ann}_k(S)$ to be the ideal of S containing $\operatorname{Ann}_{k-1}(S)$ such that

$$\operatorname{Ann}_k(S) \,/\, \operatorname{Ann}_{k-1}(S) = \operatorname{Ann}(S / \operatorname{Ann}_{k-1}(S)),$$

for every positive integer k.

Note that $\text{Ann}_k(S) = \{a \mid a \in S,\ \forall\, b \in S \quad a \cdot b,\ b \cdot a,\ [a,b]_+ \in \text{Ann}_{k-1}(S)\}$, for every positive integer k.

Definition 12. A dual weak brace $(S, +, \circ)$ is said to be *annihilator nilpotent* if there exists $n \in \mathbb{N}$ such that $\text{Ann}_n(S) = S$.

Clearly, any ideal I and quotient S/I of an annihilator nilpotent dual weak brace S are annihilator nilpotent. The next example shows that, unlike what happens in ring theory, the converse is not true.

Example 3. Let $B := (\mathbb{Z}_6, +, \circ)$ be the brace having as an additive group the cyclic group of 6 elements and multiplication defined by $a \circ b := a + (-1)^a\, b$, for all $a, b \in B$. Then, $I := \{0, 2, 4\}$ is an ideal of B such that $\text{Ann}(I) = I$, so I is annihilator nilpotent. Clearly, $\text{Ann}(B/I) = B/I$, so B/I also is annihilator nilpotent. However, B is not annihilator nilpotent since $\text{Ann}(B) = \{0\}$.

Following [4, Definition 2.3], given a dual weak brace $(S, +, \circ)$ and an ideal I of S, set $\Gamma_0(I) := I$, we can inductively define

$$\Gamma_k(I) := \left\langle\, \Gamma_{k-1}(I) \cdot S,\ S \cdot \Gamma_{k-1}(I),\ [\Gamma_{k-1}(I), S]_+ \,\right\rangle_+.$$

Observe that, set $\Gamma(I) := \Gamma_1(I)$, then $\Gamma_k(I) = \Gamma(\Gamma_{k-1}(I))$, for every $k \in \mathbb{N}$.

Proposition 16. *Let $(S, +, \circ)$ be a dual weak brace and I an ideal of S. Then, $\Gamma_k(I)$ is an ideal of S, for every $k \in \mathbb{N}$.*

Proof. Assuming $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$, by an easy induction, one can show that $\phi_{\alpha,\beta}(\Gamma_k(B_\alpha)) \subseteq \Gamma_k(B_\beta)$, for any $\alpha > \beta$. Hence, the proof follows by Theorem 3. □

One can show that if $(B, +, \circ)$ is a skew brace and I an ideal of B, then $\Gamma(I) \subseteq [I, B]$, where $[I, B]$ denotes the *commutator* introduced in [5], which is the smallest ideal of B containing $[I, B]_+, [I, B]_\circ$, and the set $\{i \circ b - (i + b) \mid i \in I,\ b \in B\}$. We can not establish whether $\Gamma(I) = [I, B]$.

We give the following lemmas, which is useful to prove the main result of this section and can be compared with [1, Proposition 15] regarding commutators in the context of skew braces.

Lemma 6. *Let M, N ideals of a dual weak brace $(S, +, \circ)$ with $M \subseteq N$. The following statements are equivalent:*

(i) $N/M \subseteq \text{Ann}(S/M)$,
(ii) $\Gamma(N) \subseteq M$.

Proof. Initially, assume that $N/M \subseteq \text{Ann}(S/M)$. Hence, $M + (n \circ s) = M + (n + s)$, for all $n \in N$ and $s \in S$. Then,

$$n \cdot s = -n + \underbrace{n^0 + (n \circ s)}_{M + (n \circ s)} - s = \underbrace{(-n + m + n)}_{M} + s - s \in M,$$

for some $m \in M$. Similarly, $s \cdot n \in M$. Furthermore, since $M + (n + s) = (s + n) + M$,

$$[n, s]_+ = -n - s + \underbrace{n^0 + n + s}_{M + (n + s)} = \underbrace{-n - s + s + n}_{M} + m \in M,$$

for some $m \in M$. Therefore, $\Gamma(N) \subseteq M$. The rest of the claim directly follows by Remark 2. □

Lemma 7. *Let $(S, +, \circ)$ be a dual weak brace and M, N ideals of S. Then, it holds that $\Gamma(M + N) = \Gamma(M) + \Gamma(N)$.*

Proof. It is a direct consequence of the definition of $\Gamma(I)$, for every ideal I of S, and Proposition 1- 2 and 3. □

Definition 13. Let $(S, +, \circ)$ be a dual weak brace. An ascending series

$$\mathrm{E}(S) = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k = S$$

of ideals of S is called an *annihilator series* of S if $I_{j+1}/I_j \subseteq \mathrm{Ann}\,(S/I_j)$, for $0 \leq j \leq k$.

Theorem 5. *Let $(S, +, \circ)$ be a dual weak brace. If $\mathrm{E}(S) = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k = S$ is an annihilator series of S, then*

$$\Gamma_{k-j}(S) \subseteq I_j \subseteq \mathrm{Ann}_j(S),$$

for $0 \leq j \leq k$.

Proof. First, we set $i := k - j$ and show that $\Gamma_i(S) \subseteq I_{k-i}$, by proceeding by induction on i. If $i = 0$, then $\Gamma_0(S) = S = I_k$. Now, let $i \in \mathbb{N}$ and assume that $\Gamma_i(S) \subseteq I_{k-i}$. Thus, by Lemma 6,

$$\Gamma_{i+1}(S) = \Gamma(\Gamma_i(S)) \subseteq \Gamma(I_{k-i}) \subseteq I_{k-(i+1)},$$

which completes the first part of the proof.
Now, we prove that $I_j \subseteq \mathrm{Ann}_j(S)$ by induction on j. If $j = 0$, then $I_0 = \mathrm{E}(S) = \mathrm{Ann}_0(S)$. Now, let $j \in \mathbb{N}$ and assume that $I_j \subseteq \mathrm{Ann}_j(S)$. By Lemma 7 and Lemma 6, we have that

$$\Gamma\left(I_j + \mathrm{Ann}_{j-1}(S)\right) = \Gamma(I_j) + \Gamma\left(\mathrm{Ann}_{j-1}(S)\right) \subseteq I_{j-1} + \mathrm{Ann}_{j-1}(S) \subseteq \mathrm{Ann}_{j-1}(S),$$

where the last inclusion follows by the inductive hypothesis. Thus, since by Lemma 6

$$\left(I_j + \mathrm{Ann}_{j-1}(S)\right) / \mathrm{Ann}_{j-1}(S) \subseteq \mathrm{Ann}\left(S/\mathrm{Ann}_{j-1}(S)\right) = \mathrm{Ann}_j(S)/\mathrm{Ann}_{j-1}(S),$$

we obtain that $I_j + \mathrm{Ann}_{j-1}(S) \subseteq \mathrm{Ann}_j(S)$. Consequently, $I_j \subseteq \mathrm{Ann}_j(S)$ and the claim follows. □

In light of Theorem 5 we can give the following definitions.

Definition 14. Let $(S, +, \circ)$ be a dual weak brace. The smallest non-negative integer c such that $\Gamma_{c+1}(S) = \mathrm{E}(S)$ and $\mathrm{Ann}_c(S) = S$ is called *nilpotency index* (or *annihilator nilpotency index*) of S. The series

$$\mathrm{E}(S) = \mathrm{Ann}_0(S) \subseteq \mathrm{Ann}_1(S) \subseteq \cdots \subseteq \mathrm{Ann}_c(S) = S$$

is called *upper annihilator series* of S. The series

$$\mathrm{E}(S) = \Gamma_{c+1}(S) \subseteq \Gamma_c(S) \subseteq \cdots \subseteq \Gamma_0(S) = S$$

is named *lower annihilator series* of S.

Similarly to Lemma 5 and Proposition 13, one can prove the following result.

Proposition 17. *Let $S = [Y; B_\alpha; \phi_{\alpha,\beta}]$ be an annihilator nilpotent dual weak brace of nilpotency index n. Then, for every $\alpha \in Y$ each skew brace B_α is annihilator nilpotent of index less or equal to n.*
Vice versa, if each skew brace B_α is annihilator nilpotent of index k_α, then S is annihilator nilpotent of index equal to the maximum between k_α, for every $\alpha \in Y$.

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Declarations

Conflict of interest The authors declare no competing interests.

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