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Five-dimensional p-nilpotent restricted Lie algebras over algebraically closed fields of characteristic $p > 3$

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ABSTRACT

A classification of p-nilpotent 5-dimensional restricted Lie algebras over algebraically closed fields of characteristic $p > 3$ is provided. This is achieved by employing a natural restricted analogue of the known method by Skjelbred and Sund for classifying ordinary nilpotent Lie algebras as central extensions of Lie algebras of smaller dimension.

1. Introduction

The present paper is a contribution to the classification of restricted Lie algebras of low dimension. Similar classifications for ordinary Lie algebras represent a classical problem and have been carried out by several authors over the years. Up to dimension 5, the characterization of nilpotent Lie algebras over any field has been known for a long

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time. In 1958, a classification in dimension 6 over fields of characteristic zero was given by Morozov in [13], and other results in this framework appeared in [1,7,14,22]. However, these classifications differ and it was hard to compare them until de Graaf [5] provided a complete classification over arbitrary fields of characteristic not 2. Some years later, de Graaf's approach was revised and extended to characteristic 2 in [2]. Apparently, the classification in dimensions more than 6 seems to be still in progress (see e.g. [15,17]).

Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. We recall that $[p] : L \to L$, $x \mapsto x^{[p]}$, is called a p-map if satisfies the following:

(1) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, for all $x \in L$, $\lambda \in \mathbb{F}$,
(2) $\mathrm{ad} x^{[p]} = (\mathrm{ad} x)^p$, for all $x \in L$,
(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, for all $x, y \in L$,

where the terms $s_i(x, y)$ are determined by the relation

$$(\mathrm{ad}(x \otimes X + y \otimes 1))^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i s_i(x, y) \otimes X^{i-1}$$

in the Lie algebra $L \otimes_{\mathbb{F}} \mathbb{F}[X]$ over the polynomial ring $\mathbb{F}[X]$. A Lie algebra L with a fixed p-map is referred to as a restricted Lie algebra. Restricted Lie algebras play a predominant role in the theory of Lie algebras in positive characteristic, in connection with algebraic groups, homological algebra, representation theory and classification of simple Lie algebras (cf. [19–21]).

A restricted Lie algebra L is said to be p-nilpotent if there exists a positive integer n such that $x^{[p]^n} = 0$, for all $x \in L$. By Engel Theorem, every finite-dimensional p-nilpotent restricted Lie algebra is nilpotent. On the other hand, every finite-dimensional nilpotent restricted Lie algebra is a central extension of a p-nilpotent restricted Lie algebra by a torus (see [21, Chapter 2, Corollary 4.2]).

Now, a classification of p-nilpotent restricted Lie algebras of dimension at most 4 over perfect fields was obtained by Schneider and Usefi in [16]. Afterwords, a description of 5-dimensional p-nilpotent restricted Lie algebras over perfect fields of characteristic $p > 3$ was provided by Darijani and Usefi in [3]. Unfortunately, as explained in Section 2, the proof of the main result of [3] is partially incorrect and, actually, the list of isomorphism classes of restricted Lie algebras found by the authors is incomplete.

In this paper we provide a classification of p-nilpotent restricted Lie algebras of dimension 5 valid over algebraically closed fields of characteristic $p > 3$. This is achieved by means of a cohomological method that can be considered as the natural restricted analogue of the Skjelbred-Sund procedure for classifying ordinary nilpotent Lie algebras (cf. [18]). Similar ideas also appear in some recent work concerning the classification of other kinds of non-associative algebras: see e.g. [6,8,10–12].

Here we give a rough outline of the method that we use, for more details we refer to Section 2. It essentially consists of two steps. In the first step, we construct a possibly redundant list containing all n-dimensional p-nilpotent restricted Lie algebras. Secondly we remove the isomorphic copies from the list.

In the first step, p-nilpotent restricted Lie algebras are constructed as central extensions of p-nilpotent restricted Lie algebras of smaller dimension. Let L be a p-nilpotent restricted Lie algebra of dimension $n-1$ and V a vector space of dimension 1 regarded as a trivial left L-module. Let $H^2_*(L,V)$ denote the second restricted cohomology group of L with coefficient in V. Every $[\theta] \in H^2_*(L,V)$ defines a restricted Lie algebra structure on $L \oplus V$, which is called the central extension of L by $[\theta]$. By this construction we obtain all p-nilpotent restricted Lie algebras of dimension n (varying L and $[\theta]$). As different restricted 2-cocycles may yield isomorphic restricted Lie algebras, we will use the action of the restricted automorphism group $\mathrm{Aut}_p(L)$ of L on $H^2_*(L,V)$ in order to reduce their number. In fact, elements lying to the same $\mathrm{Aut}_p(L)$-orbit yield isomorphic restricted Lie algebras. However, the converse is not true and so we still need to eliminate the remaining redundancies from the list. As the conditions for isomorphism of two restricted Lie algebras are translated to polynomial equations, the assumption that the ground field is algebraically closed is used in a decisive way at this stage.

Note also that every 5-dimensional p-nilpotent restricted Lie algebra has nilpotency class at most 4. Therefore, as $p > 3$, all p-maps are p-semilinear, so the task of determining $H^2_*(L,V)$ and the action of $\mathrm{Aut}_p(L)$ is easier. In characteristic 2 and 3, the computations and the resulting list of restricted Lie algebras are somewhat different, which is the reason why these exceptional cases will be considered in a future paper.

The paper is organized as follows. In Section 2 we fix the notation, recall some basic facts about the second restricted cohomology space and describe our method for classifying p-nilpotent restricted Lie algebras. We also explain the main problems that the proof proposed in [3] present and how our approach is different. Sections 3, 4 and 5 are devoted to determine all the 1-dimensional central extensions of the 4-dimensional p-nilpotent restricted Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic $p > 3$. In Section 6 we eliminate redundancies, by detecting and removing isomorphic restricted Lie algebras from the list. Section 7 contains the statement of our main result, which provides the list of the p-nilpotent restricted Lie algebras of dimension 5 over $\mathbb{F}$. Finally, in Section 8 we briefly discuss our results and compare the classification in [3] to ours.

The main aim of this paper is to describe the methods used, and to present the main results. However, a detailed proof of these results involves many case distinctions, and is rather technical (Sections 3-6 contain a small part of it along with some guiding elaborated examples). Therefore, in order to avoid tedious repetition of the same arguments, most of the explicit computations have been omitted from the paper and left to the interested reader.

2. Preliminaries and summary of the method

Throughout this paper, $\mathbb{F}$ will denote an algebraically closed field of characteristic $p > 3$. Before explaining our method to classify p-nilpotent restricted Lie algebras of dimension 5 over $\mathbb{F}$, we recall some known facts about the second restricted cohomology space with coefficient in a trivial module and the central extensions of restricted Lie algebras (cf. [3,4,9]). Let L be a restricted Lie algebra over $\mathbb{F}$ and V a vector space regarded as a trivial L-module. Following [4], for $\phi \in \mathrm{Hom}_{\mathbb{F}}(\Lambda^2 L, V)$ and a map $\omega : L \to V$, we say that ω has $\star$-property with respect to ϕ if, for every $x, y \in L$ and $\lambda \in \mathbb{F}$, we have $\omega(\lambda x) = \lambda^p \omega(x)$ and

$$\omega(x+y) = \omega(x) + \omega(y) + \sum_{\substack{x_j = x \text{ or } y \\ x_1 = x, x_2 = y}} \frac{1}{\#x} \phi([x_1, x_2, \ldots, x_{p-1}], x_p),$$

where $\#x$ is the number of x_i equal to x. If L is nilpotent of class less than p, note that ω has $\star$-property precisely when it is p-semilinear. The set $C^2_*(L, V)$ consisting of all (ϕ, ω), where $\phi \in \mathrm{Hom}_{\mathbb{F}}(\Lambda^2 L, V)$ and ω has $\star$-property with respect to ϕ, is viewed as a vector space over $\mathbb{F}$.

Let $Z^2_*(L, V)$ denote the set consisting of all $(\phi, \omega) \in C^2_*(L, V)$ with the properties that

$$\phi([x_1, x_2], x_3) + \phi([x_3, x_1], x_2) + \phi([x_2, x_3], x_1) = 0, \quad \phi(x, y^{[p]}) = \phi([x, \underbrace{y, \ldots, y}_{p-1}], y), \tag{1}$$

for all $x, y \in L$. The elements of $Z^2_\star(L, V)$ are called restricted 2-cocycles. For a linear map $\psi : L \to V$, we define a map $\hat{\psi} : L \times L \to V$ as $\hat{\psi}(x, y) = \psi([x, y])$, and a map $\tilde{\psi} : L \to V$ as $\tilde{\psi}(x) = \psi(x^{[p]})$. The set $\{(\hat{\psi}, \tilde{\psi}) | \, \psi : L \to V \text{ is linear}\}$ is denoted by $B^2_*(L, V)$. It is routine to check that $B^2_*(L, V)$ is a subspace of $Z^2_*(L, V)$, and the elements of $B^2_*(L, V)$ are said to be restricted 2-coboundaries. The second restricted cohomology space of L with coefficient in V is defined as $H^2_*(L, V) = Z^2_*(L, V)/B^2_*(L, V)$.

We denote by $\mathrm{Aut}(L)$ and $\mathrm{Aut}_p(L)$, respectively, the automorphism group and the restricted automorphism group of L. The vector spaces defined in the previous paragraph can be viewed as $\mathrm{Aut}_p(L)$-modules. Indeed, for $A \in \mathrm{Aut}_p(L)$ and $\theta = (\phi, \omega) \in Z^2_*(L, V)$, we define $A\theta = (A\phi, A\omega) \in Z^2_*(L, V)$ by the conditions $(A\phi)(x, y) = \phi(A(x), A(y))$ and $(A\omega)(x) = \omega(A(x))$. This action makes $Z^2_*(L, V)$ an $\mathrm{Aut}_p(L)$-module and it is easy to see that $B^2_*(L, V)$ is an $\mathrm{Aut}_p(L)$-submodule. Hence the quotient $H^2_*(L, V)$ can also be viewed as an $\mathrm{Aut}_p(L)$-module.

Now, let L be a restricted Lie algebra, V a vector space over $\mathbb{F}$ and $\theta = (\phi, \omega) \in Z^2_*(L, V)$. Following [9], we say that a restricted ideal I of L is strongly abelian if I is abelian and $x^{[p]} = 0$ for every $x \in I$. We define a restricted Lie algebra L_θ as follows. The underlying space of L_θ is $L \oplus V$. For $x + v, y + w \in L_\theta$, Lie bracket and p-map on L_θ are defined by

$$[x+v, y+w] = [x,y]_L + \phi(x,y), \quad (x+v)^{[p]} = x^{[p]_L} + w(x),$$

where $[x,y]_L$ and $x^{[p]_L}$ denote the Lie bracket and the p-map of L, respectively. Then L_θ is a restricted Lie algebra and V is a strongly abelian ideal of L_θ contained in the center $Z(L_\theta)$. Moreover, $L_\theta/V \cong L$, therefore L_θ is a central extension of L. Further, if $\theta \in Z^2_*(L,V)$ and $\eta \in B^2_*(L,V)$, then $L_{\theta+\eta} \cong L_\theta$, so the isomorphism type of L_θ only depends on the element $[\theta] = \theta + B^2_*(L,V)$ of $H^2_*(L,V)$. Conversely, suppose that a restricted Lie algebra K has a strongly abelian restricted ideal V such that $0 \neq V \subseteq Z(K)$ and set $L = K/V$. Let $\pi : K \to L$ be the projection map and choose an injective linear map $\sigma : L \to K$ such that $\pi(\sigma(x)) = x$ for all $x \in L$. Define $\phi : L \times L \to V$ by $\phi(x,y) = [\sigma(x), \sigma(y)] - \sigma([x,y])$, and $\omega : L \to V$ by $\omega(x) = \sigma(x)^{[p]} - \sigma(x^{[p]})$. Then $\theta = (\phi, \omega) \in Z^2_*(L,V)$ and $K \cong L_\theta$. Note that θ depends on the choice of σ. However, the θ's corresponding to two different σ's differ by a coboundary. Therefore, $[\theta]$ is independent of σ and the central extension K of L determines a well-defined element of $H^2_*(L,V)$.

The previous argument applies, in particular, when K is a n-dimensional p-nilpotent restricted Lie algebra. Indeed, by Engel's Theorem, K is nilpotent of class at most $n-1$, so we can find a central element x such that $x^{[p]} = 0$ and $V = \mathbb{F}x$ is a 1-dimensional strongly abelian restricted ideal contained in $Z(K)$. We conclude that all p-nilpotent restricted Lie algebras of dimension n can be obtained as 1-dimensional central extensions of p-nilpotent restricted Lie algebras L of dimension $n-1$ (via restricted 2-cocycles). The number of isomorphic restricted Lie algebras obtained in this way can be then reduced by using the action of $\mathrm{Aut}_p(L)$ on $H^2_*(L,V)$. In fact, it turns out that if $[\theta_1]$ and $[\theta_2]$ belong to the same $\mathrm{Aut}_p(L)$-orbit, then $L_{\theta_1} \cong L_{\theta_2}$ by [3, Theorem 2.13]. However the converse is not true and, moreover, one can obtain isomorphic restricted Lie algebras as central extensions of non-isomorphic restricted Lie algebras. Thus we have to eliminate all redundancies from the list.

Our procedure to classify 5-dimensional p-nilpotent restricted Lie algebras can be summarized as follows.

1. Take a 4-dimensional p-nilpotent restricted Lie algebra $(L, [p])$ listed in Theorem 2.2, determine $\mathrm{Aut}_p(L)$ and $H^2_*(L,\mathbb{F})$.
2. Find a (possibly redundant) list of representatives of the orbits of $\mathrm{Aut}_p(L)$ acting on $H^2_*(L,\mathbb{F})$.
3. For each $[\theta]$ found in 2, construct L_θ.
4. Detect and remove isomorphic restricted Lie algebras from the list obtained by varying L and $[\theta]$.

Clearly, $\mathrm{Aut}_p(L)$ is determined as the subgroup of $\mathrm{Aut}(L)$ consisting of all $A \in \mathrm{Aut}(L)$ such that $A(x^{[p]}) = A(x)^{[p]}$ for all $x \in L$. The space $H^2_*(L,\mathbb{F})$ can be calculated in the following straightforward way. Let x_1, x_2, x_3, x_4 be a basis of L and $(\phi, \omega) \in C^2_*(L,\mathbb{F})$. Then we have $\phi = \sum_{1 \le i < j \le 4} c_{ij}\Delta_{ij}$, where Δ_{ij} denotes the skew-symmetric matrix with (i,j)-entry equal to 1, (j,i)-entry equal to -1 and all other entries equal to 0. Moreover, as L is nilpotent of class at most 3 and $p > 3$, the map ω has $\star$-property with respect to ϕ if and only if $\omega(\lambda x) = \lambda^p \omega(x)$ and $\omega(x+y) = \omega(x) + \omega(y)$ for all $\lambda \in \mathbb{F}$ and $x, y \in L$, that is, ω is p-semilinear. Since ω is determined by its evaluation on x_1, x_2, x_3, x_4, we will write $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, where $f_i(x_j) = \delta_{i,j}$, and $\alpha, \beta, \gamma, \delta \in \mathbb{F}$. The set consisting of the elements $(\Delta_{ij}, 0)$ and $(0, f_i)$ is a basis of $C^2_*(L, \mathbb{F})$. We have that $(\phi, \omega) \in Z^2_*(L, \mathbb{F})$ if and only if the conditions (1) hold. As L has nilpotency class at most 3 and $p > 3$, note that the second condition in (1) reduces to $\phi(x, y^{[p]}) = 0$, for all $x, y \in L$. Also, the restricted 2-cocycles $(\phi, \omega) \in B^2_*(L, \mathbb{F})$ are found by requiring that $(\phi, \omega) = (\hat{\psi}, \tilde{\psi})$ for some linear map $\psi : L \to \mathbb{F}$, and $H^2_*(L, \mathbb{F})$ is the space of cosets of $B^2_*(L, \mathbb{F})$ in $Z^2_*(L, \mathbb{F})$.

For Step 2 there is no general method and it has to be handled by a direct case-by-case analysis. As this process is quite tedious and involves a big amount of routine computations, we will include the details only in one case. For the other restricted Lie algebras listed in Theorem 2.2, a description of $\mathrm{Aut}_p(L)$, a basis of $H^2_*(L, \mathbb{F})$, and the 1-dimensional central extensions of L are given without including the computations. We now provide an explicit description of the action of $\mathrm{Aut}_p(L)$ on $H^2_*(L, \mathbb{F})$. An element $A \in \mathrm{Aut}_p(L)$ is represented by an invertible matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Let $[\theta] = [(\phi, \omega)] \in H^2_*(L, \mathbb{F})$. If $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$ and $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$, by a direct computation we have $A\phi = a'\Delta_{12} + b'\Delta_{13} + c'\Delta_{14} + d'\Delta_{23} + e'\Delta_{24} + f'\Delta_{34}$, where

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{pmatrix} = \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{42} - a_{41}a_{12} & a_{21}a_{32} - a_{31}a_{22} & a_{21}a_{42} - a_{41}a_{22} & a_{31}a_{42} - a_{41}a_{32} \\ a_{11}a_{23} - a_{21}a_{13} & a_{11}a_{33} - a_{31}a_{13} & a_{11}a_{43} - a_{13}a_{41} & a_{21}a_{33} - a_{31}a_{23} & a_{21}a_{43} - a_{23}a_{41} & a_{31}a_{43} - a_{33}a_{41} \\ a_{11}a_{24} - a_{14}a_{21} & a_{11}a_{34} - a_{14}a_{31} & a_{11}a_{44} - a_{14}a_{41} & a_{21}a_{34} - a_{24}a_{31} & a_{21}a_{44} - a_{24}a_{41} & a_{31}a_{44} - a_{34}a_{41} \\ a_{12}a_{23} - a_{22}a_{13} & a_{12}a_{33} - a_{32}a_{13} & a_{12}a_{43} - a_{13}a_{42} & a_{22}a_{33} - a_{32}a_{23} & a_{22}a_{43} - a_{23}a_{42} & a_{32}a_{43} - a_{33}a_{42} \\ a_{12}a_{24} - a_{14}a_{22} & a_{12}a_{34} - a_{14}a_{32} & a_{12}a_{44} - a_{14}a_{42} & a_{22}a_{34} - a_{24}a_{32} & a_{22}a_{44} - a_{24}a_{42} & a_{32}a_{44} - a_{34}a_{42} \\ a_{13}a_{24} - a_{14}a_{23} & a_{13}a_{34} - a_{14}a_{33} & a_{13}a_{44} - a_{14}a_{43} & a_{23}a_{34} - a_{24}a_{33} & a_{23}a_{44} - a_{24}a_{43} & a_{33}a_{44} - a_{34}a_{43} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \tag{2}$$

and $A\omega = \alpha' f_1 + \beta' f_2 + \gamma' f_3 + \delta' f_4$, where

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a_{11}^p & a_{21}^p & a_{31}^p & a_{41}^p \\ a_{12}^p & a_{22}^p & a_{32}^p & a_{42}^p \\ a_{13}^p & a_{23}^p & a_{33}^p & a_{43}^p \\ a_{14}^p & a_{24}^p & a_{34}^p & a_{44}^p \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{3}$$

The first 3 steps of the procedure described above provide a complete (but redundant) list of isomorphism classes of p-nilpotent restricted Lie algebras of dimension 5. Therefore, it remains the problem to detect and remove redundancies from this list, so that the remaining algebras are pairwise non-isomorphic. As we will see in Section 6, deciding whether two restricted Lie algebras are isomorphic is equivalent to the existence of a solution over $\mathbb{F}$ of a set of polynomial equations.

We include here the list of nilpotent ordinary Lie algebras of dimension 5 (cf. [5]) and the list of p-nilpotent restricted Lie algebras of dimension 4 over algebraically closed fields of characteristic $p > 3$ (cf. [16]). For nilpotent ordinary Lie algebras we keep the same notation $L_{5,j}$ used in [5]. As usual, unspecified elements $[x_i, x_j]$ or $x_i^{[p]}$ are intended to be zero.

Theorem 2.1 *([5], Section 9). The isomorphism classes of all nilpotent Lie algebras of dimension 5 over an arbitrary field are the following:*

- $L_{5,1} =$ *abelian;*
- $L_{5,2} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3 \rangle$;
- $L_{5,3} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$;
- $L_{5,4} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_5, [x_3, x_4] = x_5 \rangle$;
- $L_{5,5} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle$;
- $L_{5,6} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5 \rangle$;
- $L_{5,7} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5 \rangle$;
- $L_{5,8} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_4, [x_1, x_3] = x_5 \rangle$;
- $L_{5,9} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 \rangle$.

Theorem 2.2 *([16], Theorem 2.1). Let L be a nilpotent Lie algebra of dimension 4 over an algebraically closed field $\mathbb{F}$ of characteristic $p > 3$. Then the equivalence classes of the $[p]$-maps on L are as follows:*

- *If $L = \langle x_1, x_2, x_3, x_4 \rangle$, then*
  (1) *Trivial p-map;*
  (2) $x_1^{[p]} = x_2$.
  (3) $x_1^{[p]} = x_2$, $x_3^{[p]} = x_4$.
  (4) $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$.
  (5) $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$, $x_3^{[p]} = x_4$.
- *If $L = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3 \rangle$, then*
  (1) *Trivial p-map;*
  (2) $x_1^{[p]} = x_3$.
  (3) $x_1^{[p]} = x_4$.
  (4) $x_1^{[p]} = x_3$, $x_2^{[p]} = x_4$.

(5) $x_3^{[p]} = x_4$.
(6) $x_3^{[p]} = x_4$, $x_2^{[p]} = x_3$.
(7) $x_4^{[p]} = x_3$.
(8) $x_4^{[p]} = x_3$, $x_2^{[p]} = x_4$.

- *If* $L = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$, *then*
  (1) *Trivial p-map;*
  (2) $x_1^{[p]} = x_4$.
  (3) $x_3^{[p]} = x_4$.
  (4) $x_2^{[p]} = x_4$.

As mentioned in the introduction, by using a cohomological method different from ours, a classification of 5-dimensional p-nilpotent restricted Lie algebras over perfect fields of characteristic $p \geq 5$ was proposed by Darijani and Usefi in [3]. In their approach, the authors start with a 5-dimensional nilpotent Lie algebra H, a central element $z \in H$ and then aim to find all possible p-maps on H such that $z^{[p]} = 0$. For this purpose, they consider the Lie algebra $L = H/\langle z \rangle$ and all possible p-maps on L and then try to construct all 1-dimensional central extensions of L that lead to H by choosing $\theta = (\phi, \omega) \in Z_*^2(L, \mathbb{F})$ such that L_θ is isomorphic to H as a Lie algebra. Unfortunately, some crucial arguments used in that paper are not correct. We briefly explain the main problems. Indeed, in [3], the following lemma is proved:

Lemma 2.3 *([3], Lemma 4.1). Let* $\mathbb{F}$ *be a perfect field of characteristic* $p \geq 5$. *Let* $K = L_{5,2}$ *and* $[p] : K \to K$ *a p-map on* K *such that* $x_3^{[p]} = 0$. *Let* $L = \frac{K}{M}$, *where* $M = \langle x_3 \rangle_{\mathbb{F}}$. *Then* $K \cong L_\theta$, *where* $\theta = (\Delta_{12}, \omega) \in Z_*^2(L, \mathbb{F})$.

Now, let S be the 4-dimensional abelian restricted Lie algebra with basis $\{x_1, x_2, x_3, x_4\}$ and p-map defined by $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$. Let $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$. In Section 4.1 of [3], by using Lemma 2.3, it is claimed that since $a = \phi(x_1, x_2) = 0$, then $L_{5,2}$ cannot be obtained from S. This conclusion is not true. For instance, for $\theta = (\Delta_{34}, 0) \in Z^2(L, \mathbb{F})$, the central extension S_θ is clearly isomorphic to $L_{5,2}$ as an ordinary Lie algebra. Similar invalid arguments also occur in other parts of [3] (see for instance the applications of Lemma 5.1 in Section 5 and Lemma 10.1 in Section 10). As a consequence, the classification of Darijani and Usefi lacks of several restricted Lie algebras.

3. L abelian

In this section we focus on the central extensions of an abelian 4-dimensional restricted Lie algebra $L := \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{F}}$ by a 1-dimensional vector space $V = \mathbb{F}x_5$. Note that the automorphism group $\mathrm{Aut}(L)$ of L as an ordinary Lie algebra consists of all the invertible matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

According to Theorem 2.2, up to isomorphisms, the possible p-maps on L are the following:

1. Trivial p-map;
2. $x_1^{[p]} = x_2$;
3. $x_1^{[p]} = x_2$, $x_3^{[p]} = x_4$;
4. $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$;
5. $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$, $x_3^{[p]} = x_4$.

In the sequel we will freely use the following property

Remark 3.1. *Let G be a group acting on the sets A and B. Consider the action of G on $A \times B$ defined by $g \cdot (a, b) = (ga, gb)$, for all $g \in G$, $a \in A$, $b \in B$. Let S be an orbit representative system of the action of G on A. For every $\alpha \in S$, let J_α be an orbit representative system of the action of the stabilizer G_α of α on B. Then $\{(\alpha, \beta)|\ \alpha \in S, \beta \in J_\alpha\}$ is an orbit representative system of the action of G on $A \times B$.*

3.1. Strongly Abelian case

We deal with the central extensions of a 4-dimensional strongly abelian restricted Lie algebra L over an algebraically closed field $\mathbb{F}$ of characteristic $p > 3$ by a 1-dimensional vector space V. This will also serve as a guiding example showing how the computations can be performed in the remaining cases.

Let L be the abelian 4-dimensional restricted Lie algebra with trivial p-map. Then the restricted automorphism group $\mathrm{Aut}_p(L)$ of L is clearly given by

$$\mathrm{Aut}_p(L) = \mathrm{GL}(4, \mathbb{F}).$$

Thus, $\mathrm{Aut}_p(L)$ consists of all matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

with $\det A \neq 0$. Moreover, a basis for $H_*^2(L, \mathbb{F})$ is given by the following elements (see Section 2 for the definition of the Δ_{ij}):

$$[(\Delta_{12}, 0)], [(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(\Delta_{34}, 0)], [(0, f_1)], [(0, f_2)],$$
$$[(0, f_3)], [(0, f_4)].$$

Let $[(\phi,\omega)] \in H^2_*(L,\mathbb{F})$. By Remark 3.1, in order to determine the orbits of $\mathrm{Aut}_p(L)$ on $H^2_*(L,\mathbb{F})$, in a first stage we will only concern with the action of $\mathrm{Aut}_p(L)$ on the ϕ's, regardless the ω's. This allows to focus on elements of the form $[(\phi_i,\omega)]$, where the ϕ_i are of a particularly convenient special form. In a second stage, for every ϕ_i, we determine the orbit representatives of the action on the ω's by the subgroup of $\mathrm{Aut}_p(L)$ consisting of the restricted automorphisms A such that $A\phi_i = \phi_i$.

By symmetry, without loss of generality we need only to consider the following cases:

C.1 $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, where $a,b,c \in \mathbb{F}^\times = \mathbb{F}\setminus\{0\}$,
C.2 $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, where $a,b,d \in \mathbb{F}^\times$,
C.3 $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, where $a,b,e \in \mathbb{F}^\times$,
C.4 $\phi = a\Delta_{12} + f\Delta_{34}$, $a,f \in \mathbb{F}^\times$,
C.5 $\phi = a\Delta_{12} + b\Delta_{13}$, $a,b \in \mathbb{F}^\times$,
C.6 $\phi = a\Delta_{12}$, $a \in \mathbb{F}^\times$,
C.7 $\phi = 0$.

Indeed, for every $\phi \in Z^2(L,\mathbb{F})$ one can easily reduce to one of the previous cases by means of a suitable restricted automorphism. For instance, if $\phi = a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$, where $a,d,e \in \mathbb{F}^\times$, by considering

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}_p(L),$$

one has $A\phi = -a\Delta_{12} + d\Delta_{13} + e\Delta_{14} + b\Delta_{23} + c\Delta_{24} + f\Delta_{34}$, which is of the form C.1 in the previous list.

We will then consider separately each of these cases.

3.1.1. Case C.1

Let $\lambda_1 = a^{-1}b^{-1}d - b^{-1}c^{-1}f + a^{-1}c^{-1}e \in \mathbb{F}$. If $\lambda_1 \neq 0$, take

$$A = \begin{pmatrix} 1 & 1+b^{-1}c^{-1}f & a^{-1}b^{-1}d & \lambda^{-1}(a^{-1}b^{-1}d + b^{-1}c^{-1}f) \\ 0 & 0 & -a^{-1} & -a^{-1}\lambda^{-1} \\ 0 & b^{-1} & b^{-1} & 0 \\ 0 & 0 & 0 & c^{-1}\lambda^{-1} \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Then by (2) we have

$$A\phi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

On the other hand, if $\lambda_1 = 0$, by considering

$$A = \begin{pmatrix} 1 & 1+b^{-1}c^{-1}f & a^{-1}b^{-1}d & (a^{-1}b^{-1}d+b^{-1}c^{-1}f) \\ 0 & 0 & -a^{-1} & -a^{-1} \\ 0 & b^{-1} & b^{-1} & 0 \\ 0 & 0 & 0 & c^{-1} \end{pmatrix} \in \mathrm{Aut}_p(L),$$

by (2) one has

$$A\phi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, in this case $[(\phi, \omega)]$ belongs to an orbit represented by $[(\phi', \omega')]$, where ϕ' is one of the following:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Cases C.2 and C.3 can be managed in a similar way and yield the same orbits.

3.1.2. Case C.4

Consider $A = \mathrm{diag}(a^{-1}, 1, 1, f^{-1}) \in \mathrm{Aut}_p(L)$. Then the action (2) gives

$$A\phi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

3.1.3. Case C.5

Consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^{-1} & -a^{-1}b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Then the action (2) gives

$$A\phi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Cases C.6 and C.7 are obvious, so we conclude that the representatives for the $\mathrm{Aut}_p(L)$-action on $H^2_*(L,\mathbb{F})$ are of the form $[(\phi,\omega)]$, where ϕ is one of the following

$$\phi_1 = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix},\ \phi_2 = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},\ \phi_3 = \begin{pmatrix}1\\0\\0\\0\\0\\1\end{pmatrix}.$$

Case $\phi_1 = (0,0,0,0,0,0)^T$. In this case, we obviously have that $A\phi = \phi$ for every $A \in \mathrm{Aut}_p(L)$. Let $\omega = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$. Without loss of generality, it is enough to consider the following subcases:

1. $\alpha = \beta = \gamma = \delta = 0$.
2. $\alpha \in \mathbb{F}^\times$, $\beta = \gamma = \delta = 0$.
3. $\alpha, \beta \in \mathbb{F}^\times$, $\gamma = \delta = 0$.
4. $\alpha, \beta, \gamma \in \mathbb{F}^\times$, $\delta = 0$.
5. $\alpha, \beta, \gamma, \delta \in \mathbb{F}^\times$.

In fact, any other case can be reduced to one of the previous cases via a suitable restricted automorphism by using the action (3). Clearly, Subcase 1 leads to the class represented by $(0,0,0,0)^T$. Suppose that Subcase 2 holds. Consider the restricted automorphism

$$A = \mathrm{diag}(\alpha^{-\frac{1}{p}},1,1,1) \in \mathrm{Aut}_p(L).$$

Then from (3) we have

$$A\omega = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

If Subcase 3 holds, then from (3) we have

$$A\omega = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix}\alpha^{-\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0\\ 0 & \alpha^{\frac{1}{p}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \in \mathrm{Aut}_p(L).$$

Next, suppose that Subcase 4 holds. Then

$$A\omega = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix}\alpha^{-\frac{1}{p}} & -\alpha^{-\frac{1}{p}} & -\alpha^{-\frac{1}{p}} & 0\\ 0 & \beta^{-\frac{1}{p}} & 0 & 0\\ 0 & 0 & \gamma^{-\frac{1}{p}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \in \mathrm{Aut}_p(L).$$

If Subcase 5 holds, then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & -\alpha^{-\frac{1}{p}} & -\alpha^{-\frac{1}{p}} & -\alpha^{-\frac{1}{p}} \\ 0 & \beta^{-\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & 0 \\ 0 & 0 & 0 & \delta^{-\frac{1}{p}} \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Therefore, the set of ω's such that $[(0,\omega)]$ is an orbit representative of the $\mathrm{Aut}_p(L)$-action on $H^2_*(L,\mathbb{F})$ is as follows:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Case $\phi_2 = (1,0,0,0,0,0)^T$. For every $A \in \mathrm{Aut}_p(L)$, in order to have $A\phi = \phi$ the following conditions must be satisfied:

$$a_{11}a_{22} - a_{12}a_{21} = 1;$$
$$a_{11}a_{23} - a_{21}a_{13} = 0;$$
$$a_{11}a_{24} - a_{21}a_{14} = 0;$$
$$a_{12}a_{23} - a_{22}a_{13} = 0;$$
$$a_{12}a_{24} - a_{22}a_{14} = 0;$$
$$a_{13}a_{24} - a_{23}a_{14} = 0.$$

Without loss of generality, we need only to consider the following cases:

1. $\alpha = \beta = \gamma = \delta = 0$;
2. $\alpha \in \mathbb{F}^\times$, $\beta = \gamma = \delta = 0$;
3. $\gamma \in \mathbb{F}^\times$, $\alpha = \beta = \delta = 0$;
4. $\alpha, \beta \in \mathbb{F}^\times$, $\gamma = \delta = 0$;
5. $\gamma, \delta \in \mathbb{F}^\times$, $\alpha = \beta = 0$;
6. $\alpha, \gamma \in \mathbb{F}^\times$, $\beta = \delta = 0$;
7. $\alpha, \beta, \gamma \in \mathbb{F}^\times$, $\delta = 0$;
8. $\alpha, \gamma, \delta \in \mathbb{F}^\times$, $\beta = 0$;
9. $\alpha, \beta, \gamma, \delta \in \mathbb{F}^\times$.

Clearly, the first case leads to $(0,0,0,0)^T$. If Subcase 2 holds, then by (3) we have

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \mathrm{diag}(\alpha^{-\frac{1}{p}}, \alpha^{\frac{1}{p}}, 1, 1) \in \mathrm{Aut}_p(L).$$

In Subcase 3 we have

$$A\omega = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \operatorname{diag}(1, 1, \gamma^{-\frac{1}{p}}, 1) \in \operatorname{Aut}_p(L).$$

Assume now that Subcase 4 holds. Then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \operatorname{Aut}_p(L).$$

Consider now Subcase 5. We have

$$A\omega = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & -\delta^{\frac{1}{p}} \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \operatorname{Aut}_p(L).$$

If Subcase 6 holds, then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \operatorname{diag}(\alpha^{-\frac{1}{p}}, \alpha^{\frac{1}{p}}, \gamma^{-\frac{1}{p}}, \gamma^{\frac{1}{p}}) \in \operatorname{Aut}_p(L).$$

Next, suppose that Subcase 7 holds. Then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \operatorname{Aut}_p(L).$$

If Subcase 8 holds, then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \alpha^{-\frac{1}{p}} & 0 & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & -\delta^{\frac{1}{p}} \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \operatorname{Aut}_p(L).$$

Finally, suppose that Subcase 9 holds. Then (3) yields

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{\frac{1}{p}} & -\delta^{\frac{1}{p}} \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \operatorname{Aut}_p(L).$$

Now, by (3), we have

$$\hat{A}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \text{where}\quad \hat{A}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\-1&0&1&0\\0&0&0&1\end{pmatrix}\in \mathrm{Aut}_p(L).$$

We conclude that the $\omega's$ such that $[(\Delta_{12},\omega)]$ are representatives of the orbits of the $\mathrm{Aut}_p(L)$-action on $H^2_*(L,\mathbb{F})$ are the following

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

Case $\phi_3=(1,0,0,0,0,1)^T$. One has that $A\in \mathrm{Aut}_p(L)$ satisfies $A\phi=\phi$ if and only if the following hold:

$$a_{11}a_{22}-a_{12}a_{21}+a_{31}a_{42}-a_{41}a_{32}=1;$$
$$a_{11}a_{23}-a_{21}a_{13}+a_{31}a_{43}-a_{41}a_{33}=0;$$
$$a_{11}a_{24}-a_{21}a_{14}+a_{31}a_{44}-a_{41}a_{34}=0;$$
$$a_{12}a_{23}-a_{22}a_{13}+a_{32}a_{43}-a_{42}a_{33}=0;$$
$$a_{12}a_{24}-a_{22}a_{14}+a_{32}a_{44}-a_{42}a_{34}=0;$$
$$a_{13}a_{24}-a_{23}a_{14}+a_{33}a_{44}-a_{34}a_{43}=1.$$

Without loss of generality, we need only to consider the following cases:

1. $\alpha=\beta=\gamma=\delta=0$;
2. $\alpha\in\mathbb{F}^\times,\beta=\gamma=\delta=0$;
3. $\gamma\in\mathbb{F}^\times,\alpha=\beta=\delta=0$;
4. $\alpha,\beta\in\mathbb{F}^\times,\gamma=\delta=0$;
5. $\gamma,\delta\in\mathbb{F}^\times,\alpha=\beta=0$;
6. $\alpha,\gamma\in\mathbb{F}^\times,\beta=\delta=0$;
7. $\alpha,\beta,\gamma\in\mathbb{F}^\times,\delta=0$;
8. $\alpha,\gamma,\delta\in\mathbb{F}^\times,\beta=0$;
9. $\alpha,\beta,\gamma,\delta\in\mathbb{F}^\times$.

Clearly, the first case leads to $(0,0,0,0)^T$. If Subcase 2 holds, then we deduce from (3) that

$$A\omega=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \text{where}\quad A=\mathrm{diag}(\alpha^{-\frac{1}{p}},\alpha^{\frac{1}{p}},1,1)\in \mathrm{Aut}_p(L).$$

Suppose that Subcase 3 holds. Then

$$A\omega = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \mathrm{diag}(1, 1, \gamma^{-\frac{1}{p}}, \gamma^{\frac{1}{p}}) \in \mathrm{Aut}_p(L).$$

Next consider Subcase 4. We have

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Assume now that Subcase 5 holds. Then

$$A\omega = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & -\delta^{\frac{1}{p}} \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Let $\bar{\omega} = (0, 0, 1, 0)^T$. Then by (3) we have

$$\bar{A}\bar{\omega} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad \bar{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Let Subcase 6 holds. Then we have

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \mathrm{diag}(\alpha^{-\frac{1}{p}}, \alpha^{\frac{1}{p}}, \gamma^{-\frac{1}{p}}, \gamma^{\frac{1}{p}}) \in \mathrm{Aut}_p(L).$$

If Subcase 7 holds, then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & 0 \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Now assume that Subcase 8 holds. Consider the restricted automorphism

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & 0 & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & -\delta^{\frac{1}{p}} \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \mathrm{Aut}_p(L).$$

Finally, assume that Subcase 9 holds. Then

$$A\omega = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \alpha^{-\frac{1}{p}} & -\beta^{\frac{1}{p}} & 0 & 0 \\ 0 & \alpha^{\frac{1}{p}} & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{p}} & -\delta^{\frac{1}{p}} \\ 0 & 0 & 0 & \gamma^{\frac{1}{p}} \end{pmatrix} \in \mathrm{Aut}_p(L).$$

We conclude that the set of ω's such that $[(\Delta_{12} + \Delta_{34}, \omega)]$ is a representative of the $\mathrm{Aut}_p(L)$−action on $H^2_*(L, \mathbb{F})$ is as follows:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

By summarizing, the central extensions obtained from a 4-dimensional strongly abelian restricted Lie algebra L by a 1-dimensional vector space V are the following:

- $L_1 = \langle x_1, x_2, x_3, x_4, x_5 \rangle$;
- $L_2 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^{[p]} = x_5 \rangle$;
- $L_3 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_5 \rangle$;
- $L_4 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_5, x_1^{[p]} = x_5 \rangle$;
- $L_5 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_5, x_3^{[p]} = x_5 \rangle$;
- $L_6 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_5 = [x_3, x_4] \rangle$;
- $L_7 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_5 = [x_3, x_4], x_1^{[p]} = x_5 \rangle$;
- $L_8 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_5 = [x_3, x_4], x_1^{[p]} = x_5, x_3^{[p]} = x_5 \rangle$.

3.2. p-map $x_1^{[p]} = x_2$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11}^p & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}.$$

Basis of $H^2_*(L, \mathbb{F})$:

$$[(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{34}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_9 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^{[p]} = x_2 \rangle$;
- $L_{10} = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^{[p]} = x_2,\ x_3^{[p]} = x_5 \rangle$;
- $L_{11} = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^{[p]} = x_2,\ x_2^{[p]} = x_5 \rangle$;

- $L_{12} = \langle x_1, x_2, x_3, x_4, x_5 | [x_3, x_4] = x_5,\ x_1^{[p]} = x_2 \rangle$;
- $L_{13} = \langle x_1, x_2, x_3, x_4, x_5 | [x_3, x_4] = x_5,\ x_1^{[p]} = x_2,\ x_3^{[p]} = x_5 \rangle$;
- $L_{14} = \langle x_1, x_2, x_3, x_4, x_5 | [x_3, x_4] = x_5,\ x_1^{[p]} = x_2,\ x_2^{[p]} = x_5 \rangle$;
- $L_{15} = \langle x_1, x_2, x_3, x_4, x_5 | [x_3, x_4] = x_5,\ x_1^{[p]} = x_2,\ x_2^{[p]} = x_5,\ x_3^{[p]} = x_5 \rangle$;
- $L_{16} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_4] = x_5,\ x_1^{[p]} = x_2 \rangle$;
- $L_{17} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_4] = x_5,\ x_1^{[p]} = x_2,\ x_3^{[p]} = x_5 \rangle$;
- $L_{18} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_4] = x_5,\ x_1^{[p]} = x_2,\ x_2^{[p]} = x_5 \rangle$;
- $L_{19} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_4] = x_5,\ x_1^{[p]} = x_2,\ x_4^{[p]} = x_5 \rangle$.

3.3. p-map $x_1^{[p]} = x_2,\ x_3^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & a_{11}^p & a_{23} & a_{13}^p \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{31}^p & a_{43} & a_{33}^p \end{pmatrix}.$$

Basis of $H_*^2(L, \mathbb{F})$:

$$[(\Delta_{13}, 0)], [(0, f_2)], [(0, f_4)].$$

Central extensions:

- $L_{20} = \langle x_1, x_2, x_3, x_4, x_5 | x_1^{[p]} = x_2,\ x_3^{[p]} = x_4 \rangle$;
- $L_{21} = \langle x_1, x_2, x_3, x_4, x_5 | x_1^{[p]} = x_2,\ x_2^{[p]} = x_5,\ x_3^{[p]} = x_4 \rangle$;
- $L_{22} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_3] = x_5,\ x_1^{[p]} = x_2,\ x_3^{[p]} = x_4 \rangle$;
- $L_{23} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_3] = x_5,\ x_1^{[p]} = x_2,\ x_2^{[p]} = x_5,\ x_3^{[p]} = x_4 \rangle$.

3.4. p-map $x_1^{[p]} = x_2,\ x_2^{[p]} = x_3$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11}^p & 0 & 0 \\ a_{31} & a_{21}^p & a_{11}^{p^2} & a_{34} \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}.$$

Basis of $H_*^2(L, \mathbb{F})$:

$$[(\Delta_{14}, 0)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{24} = \langle x_1, x_2, x_3, x_4, x_5 | \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3 \rangle$;
- $L_{25} = \langle x_1, x_2, x_3, x_4, x_5 | \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_5 \rangle$;
- $L_{26} = \langle x_1, x_2, x_3, x_4, x_5 | \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, \, x_4^{[p]} = x_5 \rangle$;
- $L_{27} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3 \rangle$;
- $L_{28} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, x_3^{[p]} = x_5 \rangle$;
- $L_{29} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, x_4^{[p]} = x_5 \rangle$.

3.5. p-map $x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11}^p & 0 & 0 \\ a_{31} & a_{21}^p & a_{11}^{p^2} & 0 \\ a_{41} & a_{31}^p & a_{21}^{p^2} & a_{11}^{p^3} \end{pmatrix}.$$

Basis of $H_*^2(L, \mathbb{F})$:

$$[(0, f_4)].$$

Central extensions:

- $L_{30} = \langle x_1, x_2, x_3, x_4, x_5 | \, x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_4 \rangle$;
- $L_{31} = \langle x_1, x_2, x_3, x_4, x_5 | x_1^{[p]} = x_2, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_4, \, x_4^{[p]} = x_5 \rangle$.

4. Case $L = \langle x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3 \rangle$

In this section we focus on the 1-dimensional central extensions of

$$L := L_{4,2} = \langle x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3 \rangle_{\mathbb{F}}.$$

The automorphism group $\mathrm{Aut}(L)$ of L as an ordinary Lie algebra consists of the invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21}$.

According to Theorem 2.2, up to isomorphisms, the possible p-maps on L are the following:

(1) $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.
(2) $x_1^{[p]} = x_3$, $x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.
(3) $x_1^{[p]} = x_4$, $x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.
(4) $x_1^{[p]} = x_3$, $x_2^{[p]} = x_4$, $x_3^{[p]} = x_4^{[p]} = 0$.
(5) $x_3^{[p]} = x_4$, $x_1^{[p]} = x_2^{[p]} = x_4^{[p]} = 0$.
(6) $x_3^{[p]} = x_4$, $x_2^{[p]} = x_3$, $x_1^{[p]} = x_4^{[p]} = 0$.
(7) $x_4^{[p]} = x_3$, $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = 0$.
(8) $x_4^{[p]} = x_3$, $x_2^{[p]} = x_4$, $x_1^{[p]} = x_3^{[p]} = 0$.

We are going to consider these cases, separately, in the next subsections.

4.1. p−map $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$

$\mathrm{Aut}_p(L) = \mathrm{Aut}(L)$.
Basis of $H^2_*(L, \mathbb{F})$:

$$[(\Delta_{13}, 0)], [(\Delta_{14}, 0)], [(\Delta_{23}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{32} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3 \rangle$;
- $L_{33} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ x_4^{[p]} = x_5 \rangle$;
- $L_{34} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ x_3^{[p]} = x_5 \rangle$;
- $L_{35} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ x_1^{[p]} = x_5 \rangle$;
- $L_{36} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_2, x_3] = x_5 \rangle$;
- $L_{37} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_2, x_3] = x_5,\ x_3^{[p]} = x_5,\ x_4^{[p]} = x_5 \rangle$;
- $L_{38}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_2, x_3] = x_5,\ x_1^{[p]} = \alpha x_5,\ x_3^{[p]} = x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{39} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_2, x_3] = x_5,\ x_4^{[p]} = x_5 \rangle$;
- $L_{40} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_2, x_3] = x_5,\ x_2^{[p]} = x_5 \rangle$;
- $L_{41} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_2, x_3] = x_5,\ x_1^{[p]} = x_5 \rangle$;
- $L_{42} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5 \rangle$;
- $L_{43} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ x_4^{[p]} = x_5 \rangle$;
- $L_{44} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ x_3^{[p]} = x_5 \rangle$;
- $L_{45} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ x_1^{[p]} = x_5 \rangle$;
- $L_{46} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ x_2^{[p]} = x_5,\ x_4^{[p]} = x_5 \rangle$;
- $L_{47} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ x_2^{[p]} = x_5 \rangle$;
- $L_{48} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ [x_2, x_3] = x_5 \rangle$;
- $L_{49} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3,\ [x_1, x_4] = x_5,\ [x_2, x_3] = x_5,\ x_4^{[p]} = x_5 \rangle$;

- $L_{50} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, [x_2, x_3] = x_5, \, x_3^{[p]} = x_5 \rangle$;
- $L_{51} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, [x_2, x_3] = x_5, \, x_2^{[p]} = x_5 \rangle$;
- $L_{52} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, [x_2, x_3] = x_5, \, x_1^{[p]} = x_5 \rangle$.

4.2. p-map $x_1^{[p]} = x_3$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11}^{p-1} & 0 & 0 \\ a_{31} & a_{32} & a_{11}^{p} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix}.$$

Basis of $H^2_*(L, \mathbb{F})$:

$$[(\Delta_{14}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{53} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_3 \rangle$;
- $L_{54} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_3, \, x_4^{[p]} = x_5 \rangle$;
- $L_{55} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_3, \, x_3^{[p]} = x_5 \rangle$;
- $L_{56} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, x_1^{[p]} = x_3, \, x_2^{[p]} = x_5 \rangle$;
- $L_{57} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_3 + x_5 \rangle$;
- $L_{58} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_2, x_4] = x_5, \, x_1^{[p]} = x_3 \rangle$;
- $L_{59}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_2, x_4] = x_5, \, x_1^{[p]} = x_3 + \alpha x_5, \, x_4^{[p]} = x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{60} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_2, x_4] = x_5, \, x_1^{[p]} = x_3, \, x_3^{[p]} = x_5 \rangle$;
- $L_{61}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_2, x_4] = x_5, \, x_1^{[p]} = x_3 + x_5, \, x_2^{[p]} = \alpha x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{62} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_2, x_4] = x_5, \, x_1^{[p]} = x_3, \, x_2^{[p]} = x_5 \rangle$;
- $L_{63} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_3 \rangle$;
- $L_{64}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_3, \, x_2^{[p]} = \alpha x_5, \, x_4^{[p]} = x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{65} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_3, \, x_3^{[p]} = x_5 \rangle$;
- $L_{66} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_3, \, x_2^{[p]} = x_5 \rangle$;
- $L_{67} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_3 + x_5 \rangle$.

4.3. p-map $x_1^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & 0 & a_{11}^p \end{pmatrix}.$$

Basis of $H^2_*(L, \mathbb{F})$:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{68} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, x_1^{[p]} = x_4 \rangle$;
- $L_{69} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, x_1^{[p]} = x_4,\, x_4^{[p]} = x_5 \rangle$;
- $L_{70} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, x_1^{[p]} = x_4,\, x_3^{[p]} = x_5 \rangle$;
- $L_{71} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, x_1^{[p]} = x_4,\, x_3^{[p]} = x_5,\, x_4^{[p]} = x_5 \rangle$;
- $L_{72} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, x_1^{[p]} = x_4,\, x_2^{[p]} = x_5 \rangle$;
- $L_{73} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_2, x_3] = x_5,\, x_1^{[p]} = x_4 \rangle$;
- $L_{74} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_2, x_3] = x_5,\, x_1^{[p]} = x_4,\, x_4^{[p]} = x_5 \rangle$;
- $L_{75}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_2, x_3] = x_5,\, x_1^{[p]} = x_4,\, x_3^{[p]} = x_5,\, x_4^{[p]} = \alpha x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{76} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_2, x_3] = x_5,\, x_1^{[p]} = x_4,\, x_2^{[p]} = x_5 \rangle$;
- $L_{77} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_1, x_3] = x_5,\, x_1^{[p]} = x_4 \rangle$;
- $L_{78}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_1, x_3] = x_5,\, x_1^{[p]} = x_4,\, x_3^{[p]} = \alpha x_5,\, x_4^{[p]} = x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{79}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_1, x_3] = x_5,\, x_1^{[p]} = x_4,\, x_2^{[p]} = \alpha x_5,\, x_3^{[p]} = x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{80} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, [x_1, x_3] = x_5,\, x_1^{[p]} = x_4,\, x_2^{[p]} = x_5 \rangle$.

4.4. p-map $x_1^{[p]} = x_3,\, x_2^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{11}^{p-1} & 0 & 0 \\ a_{31} & a_{32} & a_{11}^p & a_{12}^p \\ a_{41} & a_{42} & 0 & a_{11}^{p^2-p} \end{pmatrix}.$$

Basis of $H^2_*(L, \mathbb{F})$:

$$[(0, f_1)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{81} = \langle x_1, x_2, x_3, x_4, x_5 |\, [x_1, x_2] = x_3,\, x_1^{[p]} = x_3,\, x_2^{[p]} = x_4 \rangle$;

- $L_{82} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_1^{[p]} = x_3 + x_5, x_2^{[p]} = x_4 \rangle$;
- $L_{83} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_3^{[p]} = x_5 \rangle$;
- $L_{84} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_1^{[p]} = x_3, x_2^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$.

4.5. p-map $x_3^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & r & 0 \\ a_{41} & a_{42} & 0 & r^p \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21}$.
Basis of $H_*^2(L, \mathbb{F})$:

$$[(\Delta_{13}, 0)], [(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_4)].$$

Central extensions:

- $L_{85} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_3^{[p]} = x_4 \rangle$;
- $L_{86} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle$;
- $L_{87} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle$;
- $L_{88} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$;
- $L_{89} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, x_3^{[p]} = x_4 \rangle$;
- $L_{90} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5\ x_1^{[p]} = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$;
- $L_{91} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5\ x_1^{[p]} = x_5, x_3^{[p]} = x_4 \rangle$;
- $L_{92} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5\ x_2^{[p]} = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$;
- $L_{93} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, x_2^{[p]} = x_5, x_3^{[p]} = x_4 \rangle$;
- $L_{94} = \langle x_1, x_2, x_3, x_4, x_5 | [x_1, x_2] = x_3, [x_1, x_3] = x_5, x_3^{[p]} = x_4, x_4^{[p]} = x_5 \rangle$.

4.6. p-map $x_2^{[p]} = x_3,\ x_3^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{22}^{p-1} & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{22}^p & 0 \\ a_{41} & a_{42} & 0 & a_{22}^{p^2} \end{pmatrix}.$$

Basis of $H_*^2(L, \mathbb{F})$:

$$[(0, f_1)], [(0, f_2)], [(0, f_4)].$$

Central extensions:

- $L_{95} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_4 \rangle$;
- $L_{96} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_5, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_4 \rangle$;
- $L_{97} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_2^{[p]} = x_3 + x_5, \, x_3^{[p]} = x_4 \rangle$;
- $L_{98} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_2^{[p]} = x_3, \, x_3^{[p]} = x_4, \, x_4^{[p]} = x_5 \rangle$.

4.7. p-map $x_4^{[p]} = x_3$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & r & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{12}a_{21} = a_{44}^p$.
Basis of $H^2_*(L, \mathbb{F})$:

$$[(\Delta_{14}, 0)], [(\Delta_{24}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{99} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_4^{[p]} = x_3 \rangle$;
- $L_{100} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_5, \, x_4^{[p]} = x_3 \rangle$;
- $L_{101} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_3^{[p]} = x_5, \, x_4^{[p]} = x_3 \rangle$;
- $L_{102} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_5, \, x_4^{[p]} = x_3 + x_5 \rangle$;
- $L_{103} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_4^{[p]} = x_3 + x_5 \rangle$;
- $L_{104} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_4^{[p]} = x_3 \rangle$;
- $L_{105}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = \alpha x_5, \, x_4^{[p]} = x_3 + x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{106}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_2^{[p]} = \alpha x_5, \, x_4^{[p]} = x_3 + x_5 \rangle$, where $\alpha \in \mathbb{F}^\times$;
- $L_{107} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_2^{[p]} = x_5, \, x_4^{[p]} = x_3 \rangle$;
- $L_{108} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5, \, x_1^{[p]} = x_5, \, x_4^{[p]} = x_3 \rangle$;
- $L_{109} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_4] = x_5 \, x_3^{[p]} = x_5, \, x_4^{[p]} = x_3 \rangle$.

4.8. p-map $x_2^{[p]} = x_4, \, x_4^{[p]} = x_3$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{22}^{p^2-1} & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{22}^{p^2} & a_{42}^{p} \\ 0 & a_{42} & 0 & a_{22}^{p} \end{pmatrix}.$$

Basis of $H^2_*(L, \mathbb{F})$:

$$[(0, f_1)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{110} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_2^{[p]} = x_4, \, x_4^{[p]} = x_3 \rangle$;
- $L_{111} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_2^{[p]} = x_4, \, x_3^{[p]} = x_5, \, x_4^{[p]} = x_3 \rangle$;
- $L_{112}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = \alpha x_5, \, x_2^{[p]} = x_4, \, x_4^{[p]} = x_3 + x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{113} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, x_1^{[p]} = x_5, \, x_2^{[p]} = x_4, \, x_4^{[p]} = x_3 \rangle$.

5. Case $L = \langle x_1, x_2, x_3, x_4 | \, [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$

In this section we deal with the 1-dimensional central extensions of $L := L_{4,3} = \langle x_1, x_2, x_3, x_4 | [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle_{\mathbb{F}}$. The automorphism group $\mathrm{Aut}(L)$ of L consists of the invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2 a_{22} \end{pmatrix}.$$

According to Theorem 2.2, up to isomorphisms, the possible p-maps on L are the following:

(1) $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.
(2) $x_1^{[p]} = x_4, \, x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.
(3) $x_3^{[p]} = x_4, \, x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.
(4) $x_2^{[p]} = x_4, \, x_1^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$.

5.1. *p-map* $x_1^{[p]} = x_2^{[p]} = x_3^{[p]} = x_4^{[p]} = 0$

$\mathrm{Aut}_p(L)$: Invertible matrices A of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^2 a_{22} \end{pmatrix}.$$

Basis of $H^2_*(L,\mathbb{F})$:

$$[(\Delta_{14},0)],[(\Delta_{23},0)],[(0,f_1)],[(0,f_2)],[(0,f_3)],[(0,f_4)].$$

Central extensions:

- $L_{114}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4\rangle$;
- $L_{115}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,x_4^{[p]}=x_5\rangle$;
- $L_{116}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,x_3^{[p]}=x_5\rangle$;
- $L_{117}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,x_2^{[p]}=x_5\rangle$;
- $L_{118}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,x_1^{[p]}=x_5\rangle$;
- $L_{119}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_2,x_3]=x_5\rangle$;
- $L_{120}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_2,x_3]=x_5,\,x_4^{[p]}=x_5\rangle$;
- $L_{121}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_2,x_3]=x_5,\,x_3^{[p]}=x_5\rangle$;
- $L_{122}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_2,x_3]=x_5,\,x_2^{[p]}=x_5\rangle$;
- $L_{123}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_2,x_3]=x_5,\,x_1^{[p]}=x_5\rangle$;
- $L_{124}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5\rangle$;
- $L_{125}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,x_4^{[p]}=x_5\rangle$;
- $L_{126}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,x_3^{[p]}=x_5\rangle$;
- $L_{127}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,x_2^{[p]}=x_5\rangle$;
- $L_{128}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,x_1^{[p]}=x_5\rangle$;
- $L_{129}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,[x_2,x_3]=x_5\rangle$;
- $L_{130}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,[x_2,x_3]=x_5,\,x_4^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$;
- $L_{131}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,[x_2,x_3]=x_5,\,x_3^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$;
- $L_{132}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,[x_2,x_3]=x_5,\,x_2^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$;
- $L_{133}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\,[x_1,x_3]=x_4,\,[x_1,x_4]=x_5,\,[x_2,x_3]=x_5,\,x_1^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$.

5.2. *p-map* $x_1^{[p]}=x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A=\begin{pmatrix} a_{11} & 0 & 0 & 0\\ a_{21} & a_{11}^{p-2} & 0 & 0\\ a_{31} & a_{32} & a_{11}^{p-1} & 0\\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^{p}\end{pmatrix}.$$

Basis of $H^2_*(L,\mathbb{F})$:

$$[(\Delta_{23},0)],[(0,f_1)],[(0,f_2)],[(0,f_3)],[(0,f_4)].$$

Central extensions:

- $L_{134}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_1^{[p]}=x_4\rangle$;
- $L_{135}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_1^{[p]}=x_4,\ x_4^{[p]}=x_5\rangle$;
- $L_{136}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_1^{[p]}=x_4,\ x_3^{[p]}=x_5\rangle$;
- $L_{137}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_1^{[p]}=x_4,\ x_2^{[p]}=x_5\rangle$;
- $L_{138}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_1^{[p]}=x_4+x_5\rangle$;
- $L_{139}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ [x_2,x_3]=x_5,\ x_1^{[p]}=x_4\rangle$;
- $L_{140}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ [x_2,x_3]=x_5,\ x_1^{[p]}=x_4,\ x_4^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$;
- $L_{141}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ [x_2,x_3]=x_5,\ x_1^{[p]}=x_4,\ x_3^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$;
- $L_{142}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ [x_2,x_3]=x_5,\ x_1^{[p]}=x_4,\ x_2^{[p]}=\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$;
- $L_{143}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ [x_2,x_3]=x_5,\ x_1^{[p]}=x_4+\alpha x_5\rangle$, where $\alpha\in\mathbb{F}^\times$.

5.3. p-map $x_3^{[p]}=x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A=\begin{pmatrix} a_{11} & 0 & 0 & 0\\ a_{21} & a_{11}^{\frac{2-p}{p-1}} & 0 & 0\\ 0 & 0 & a_{11}^{\frac{1}{p-1}} & 0\\ a_{41} & a_{42} & 0 & a_{11}^{\frac{p}{p-1}}\end{pmatrix}.$$

Basis of $H^2_*(L,\mathbb{F})$:

$$[(\Delta_{23},0)],[(0,f_1)],[(0,f_2)],[(0,f_3)],[(0,f_4)].$$

Central extensions:

- $L_{144}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_3^{[p]}=x_4\rangle$;
- $L_{145}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_3^{[p]}=x_4+\alpha x_5,\ x_4^{[p]}=x_5\rangle$, where $\alpha\in\mathbb{F}$;
- $L_{146}=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_3^{[p]}=x_4+x_5\rangle$;
- $L_{147}(\alpha)=\langle x_1,x_2,x_3,x_4,x_5|\,[x_1,x_2]=x_3,\ [x_1,x_3]=x_4,\ x_1^{[p]}=x_5,\ x_3^{[p]}=x_4+\alpha x_5\rangle$, where $\alpha\in\mathbb{F}$;

- $L_{148}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_2^{[p]} = x_5, \, x_3^{[p]} = x_4 + \alpha x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{149} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_2^{[p]} = x_5, \, x_3^{[p]} = x_4 \rangle$;
- $L_{150} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_1^{[p]} = x_5, \, x_3^{[p]} = x_4 \rangle$;
- $L_{151} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_3^{[p]} = x_4 \rangle$;
- $L_{152}(\alpha, \beta) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_3^{[p]} = x_4 + \alpha x_5, \, x_4^{[p]} = \beta x_5 \rangle$, where $(\alpha, \beta) \in \mathbb{F}^2 \setminus (0, 0)$;
- $L_{153}(\alpha, \beta) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_1^{[p]} = \beta x_5, \, x_3^{[p]} = x_4 + \alpha x_5 \rangle$, where $(\alpha, \beta) \in \mathbb{F}^2 \setminus (0, 0)$;
- $L_{154}(\alpha, \beta) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_2^{[p]} = \beta x_5, \, x_3^{[p]} = x_4 + \alpha x_5 \rangle$, where $(\alpha, \beta) \in \mathbb{F}^2 \setminus (0, 0)$;
- $L_{155}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_1^{[p]} = \alpha x_5, \, x_3^{[p]} = x_4 \rangle$, where $\alpha, \in \mathbb{F}^\times$.

5.4. p-map $x_2^{[p]} = x_4$

$\mathrm{Aut}_p(L)$: Invertible matrices of the form

$$A = \begin{pmatrix} a_{22}^{\frac{p-1}{2}} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{22}^{\frac{p+1}{2}} & 0 \\ a_{41} & a_{42} & a_{22}^{\frac{p-1}{2}} a_{32} & a_{22}^p \end{pmatrix}.$$

Basis of $H_*^2(L, \mathbb{F})$:

$$[(\Delta_{23}, 0)], [(0, f_1)], [(0, f_2)], [(0, f_3)], [(0, f_4)].$$

Central extensions:

- $L_{156} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_2^{[p]} = x_4 \rangle$;
- $L_{157} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_2^{[p]} = x_4, \, x_4^{[p]} = x_5 \rangle$;
- $L_{158} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_2^{[p]} = x_4, \, x_3^{[p]} = x_5 \rangle$;
- $L_{159}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_1^{[p]} = \alpha x_5, \, x_2^{[p]} = x_4 + x_5 \rangle$, where $\alpha \in \mathbb{F}$;
- $L_{160} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, x_1^{[p]} = x_5, \, x_2^{[p]} = x_4 \rangle$;
- $L_{161} = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_2^{[p]} = x_4 \rangle$;
- $L_{162}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_2^{[p]} = x_4, \, x_4^{[p]} = \alpha x_5 \rangle$, where $\alpha \in \mathbb{F}^\times$;
- $L_{163}(\alpha) = \langle x_1, x_2, x_3, x_4, x_5 | \, [x_1, x_2] = x_3, \, [x_1, x_3] = x_4, \, [x_2, x_3] = x_5, \, x_2^{[p]} = x_4, \, x_3^{[p]} = \alpha x_5 \rangle$, where $\alpha \in \mathbb{F}^\times$;

- $L_{164}(\alpha,\beta) = \langle x_1,x_2,x_3,x_4,x_5 \,|\, [x_1,x_2] = x_3,\ [x_1,x_3] = x_4,\ [x_2,x_3] = x_5,\ x_1^{[p]} = \beta x_5,\ x_2^{[p]} = x_4+\alpha x_5\rangle$, where $(\alpha,\beta)\in\mathbb{F}^2\setminus(0,0)$.

6. Detecting isomorphisms

In the previous sections we determined the 1-dimensional central extensions of the 4-dimensional p-nilpotent restricted Lie algebras. This gives a complete (but redundant) list of isomorphism classes of p-nilpotent restricted Lie algebras of dimension 5. In order to eliminate redundancies, we proceeded in the following way. We first separated the classes according with the 5-tuple of invariants ($\dim L'$, $\dim Z(L)$, $\dim L^{[p]}$, $\dim L^{[p]^2}$, $\dim L^{[p]^3}$), where $L^{[p]^i}$ denotes the restricted subalgebra generated by all the elements $x^{[p]^i}$. Now, if two restricted Lie algebras K_1 and K_2 have different 5-tuple of invariants, then it is clear that $K_1 \ncong K_2$. On the other hand, if K_1 and K_2 have the same invariants, we first established whether they are isomorphic as ordinary Lie algebras. If so, by means of a suitable change of basis, we reduced to one of the 9 Lie algebras listed in Theorem 2.1 and, finally, we looked for possible restricted isomorphisms $f : K_1 \to K_2$. This gives rise to the study of a set of polynomial equations. Of course, the given restricted Lie algebras are isomorphic if and only if such a system admits solutions. Here the assumption that the ground field is algebraically closed has been used in a crucial way. As it would be impractical to include here all the straightforward computations needed to detect and remove isomorphic p-nilpotent restricted Lie algebras, this has been omitted from this work. However, we illustrate by an example how the problem has been solved. Consider

- $L_{140}(\alpha) = \langle x_1,x_2,x_3,x_4,x_5 \,|\, [x_1,x_2] = x_3,\ [x_1,x_3] = x_4,\ [x_2,x_3] = x_5,\ x_1^{[p]} = x_4,\ x_4^{[p]} = \alpha x_5\rangle$, $\alpha\in\mathbb{F}^\times$;
- $L_{152}(\alpha,\beta) = \langle x_1,x_2,x_3,x_4,x_5 \,|\, [x_1,x_2] = x_3,\ [x_1,x_3] = x_4,\ [x_2,x_3] = x_5,\ x_3^{[p]} = x_4+\alpha x_5,\ x_4^{[p]} = \beta x_5\rangle$, $\alpha\in\mathbb{F}$, $\beta\in\mathbb{F}^\times$;
- $L_{162}(\alpha) = \langle x_1,x_2,x_3,x_4,x_5 \,|\, [x_1,x_2] = x_3,\ [x_1,x_3] = x_4,\ [x_2,x_3] = x_5,\ x_2^{[p]} = x_4,\ x_4^{[p]} = \alpha x_5\rangle$, $\alpha\in\mathbb{F}^\times$.

If L is any of the previous restricted Lie algebras, then we have $\dim L' = 3$, $\dim Z(L) = 2$, $\dim L^{[p]} = 2$, $\dim L^{[p]^2} = 1$, $\dim L^{[p]^3} = 0$. Clearly, all these restricted Lie algebras are isomorphic to $L_{5,9}$ as ordinary Lie algebras. Therefore, $\mathrm{Aut}(L)$ is the set of the invertible matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & r & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32}-a_{31}a_{12} & a_{11}r & a_{12}r \\ a_{51} & a_{52} & a_{21}a_{32}-a_{31}a_{22} & a_{21}r & a_{22}r \end{pmatrix},$$

where $r = a_{11}a_{22} - a_{21}a_{12}$.

We first show that $L_{140}(\alpha) \cong L_{140}(1)$. Indeed, let $f_1 : L_{140}(\alpha) \to L_{140}(1)$ be the element in $\mathrm{Aut}(L)$ defined by the following conditions: $f_1(x_1) = \alpha^{\frac{1}{p^2-2p+3}} x_1$, $f_1(x_2) =$

$\alpha^{\frac{p-2}{p^2-2p+3}}x_2$, $f_1(x_3) = \alpha^{\frac{p-1}{p^2-2p+3}}x_3$, $f_1(x_4) = \alpha^{\frac{p}{p^2-2p+3}}x_4$ and $f_1(x_5) = \alpha^{\frac{2p-3}{p^2-2p+3}}x_5$. Then it is easy to verify that f_1 is a restricted isomorphism. Similarly, let $f_2 : L_{162}(\alpha) \to L_{162}(1)$ be the element in $\mathrm{Aut}(L)$ defined by the following conditions: $f_2(x_1) = \alpha^{\frac{p-1}{2p^2-p-3}}x_1$, $f_2(x_2) = \alpha^{\frac{2}{2p^2-p-3}}x_2$, $f_2(x_3) = \alpha^{\frac{p+1}{2p^2-p-3}}x_3$, $f_2(x_4) = \alpha^{\frac{2p}{2p^2-p-3}}x_4$, $f_2(x_5) = \alpha^{\frac{p+3}{2p^2-p-3}}x_5$. Then one can check that f_2 is a restricted isomorphism, so $L_{162}(\alpha) \cong L_{162}(1)$.

Finally, we consider $f_3 : L_{152}(\alpha, \beta) \to L_{152}(0, 1) \in \mathrm{Aut}(L)$ defined by $f_3(x_1) = \beta^{\frac{p-1}{p^2+p-3}}x_1 - \alpha\beta^{\frac{2-p}{p^2+p-3}}x_2$, $f_3(x_2) = \beta^{\frac{2-p}{p^2+p-3}}x_2$, $f_3(x_3) = \beta^{\frac{1}{p^2+p-3}}x_3$, $f_3(x_4) = \beta^{\frac{p}{p^2+p-3}}x_4 - \alpha\beta^{\frac{3-p}{p^2+p-3}}x_5$, $f_3(x_5) = \beta^{\frac{3-p}{p^2+p-3}}x_5$. Then f_3 is a restricted isomorphism and so $L_{152}(\alpha, \beta) \cong L_{152}(0, 1)$.

We now show that the restricted Lie algebras $L_{140}(1)$, $L_{152}(0, 1)$ and $L_{162}(1)$ are pairwise non-isomorphic. Denote by $[p]_1$, $[p]_2$, $[p]_3$, respectively, the p-maps of $L_{140}(1)$, $L_{152}(0, 1)$, $L_{162}(1)$. Let $f : L_{152}(0, 1) \to L_{162}(1) \in \mathrm{Aut}(L)$ and suppose by contradiction that f is restricted. Then we must have $0 = f(x_1^{[p]_2}) = f(x_1)^{[p]_3} = a_{21}^p x_4 + a_{41}^p x_5$ and $0 = f(x_2^{[p]_2}) = f(x_2)^{[p]_3} = a_{22}^p x_4 + a_{42}^p x_5$. Consequently, we have $a_{21} = a_{22} = 0$, thus f is not bijective, a contradiction. Hence $L_{152}(0, 1) \ncong L_{162}^p(1)$.

Let $h : L_{152}(0, 1) \to L_{140}(1) \in \mathrm{Aut}(L)$ and suppose by contradiction that h is restricted. Then we must have $0 = h(x_1^{[p]_2}) = h(x_1)^{[p]_1} = a_{11}^p x_4 + a_{41}^p x_5$ and $0 = h(x_2^{[p]_2}) = h(x_2)^{[p]_1} = a_{12}^p x_4 + a_{42}^p x_5$. It follows that $a_{11} = a_{12} = 0$, hence h is not bijective, a contradiction. Thus, $L_{152}(0, 1) \ncong L_{162}(1)$.
Finally, let $g : L_{162}(1) \to L_{140}(1) \in \mathrm{Aut}(L)$ and suppose, if possible, that g is restricted. We must have $0 = g(x_1^{[p]_3}) = g(x_1)^{[p]_1} = a_{11}^p x_4 + a_{41}^p x_5$ and $0 = g(x_5^{[p]_3}) = g(x_5)^{[p]_1} = a_{12}^p r^p x_5$. Then, as $a_{11} = a_{12}r = 0$, g is not bijective, a contradiction. Therefore we have that $L_{162}(1) \ncong L_{140}(1)$.

We now re-list the central extensions taking into account of the isomorphisms found by the just described method. The title of each subsection will refer to the underlying Lie algebra $L_{5,i}$ to which the central extensions are isomorphic as ordinary Lie algebras (cf. Theorem 2.1).

6.1. *Underlying Lie algebra $L_{5,1}$*

$$L_1; \quad L_2 \cong L_9; \quad L_{10} \cong L_{20}; \quad L_{11} \cong L_{24}; \quad L_{21} \cong L_{26}; \quad L_{25} \cong L_{30}; \quad L_{31}.$$

6.2. *Underlying Lie algebra $L_{5,2}$*

$L_3 \cong L_{32}$; $L_4 \cong L_{53}$; $L_5 \cong L_{99}$; $L_{12} \cong L_{33} \cong L_{103}$; $L_{13} \cong L_{54} \cong L_{102}$;

$L_{14} \cong L_{15}$; $L_{16} \cong L_{35} \cong L_{57} \cong L_{68}$; $L_{17} \cong L_{100}$; $L_{18} \cong L_{110}$; $L_{19} \cong L_{56} \cong L_{81}$;

$L_{22} \cong L_{72} \cong L_{82}$; $L_{23} \cong L_{113}$; $L_{27} \cong L_{69} \cong L_{112}(0)$; L_{28};

$L_{29} \cong L_{84} \cong L_{112}(\alpha)$, $\alpha \neq 0$; $L_{34} \cong L_{85}$; $L_{55} \cong L_{95}$;

$L_{70} \cong L_{86} \cong L_{87} \cong L_{97}$; L_{71}; $L_{83} \cong L_{96}$; L_{88}; L_{98}; L_{101}; L_{111}.

6.3. Underlying Lie algebra $L_{5,3}$

$$L_{36} \cong L_{114}; \quad L_{37}; \quad L_{38}(\alpha) \cong L_{144}; \quad L_{39}; \quad L_{40} \cong L_{134}; \quad L_{41} \cong L_{156};$$
$$L_{73} \cong L_{117} \cong L_{159}(0); \quad L_{74}; \quad L_{75}(\alpha) \cong L_{75}(1) \cong L_{78}(\alpha'),\ \alpha, \alpha' \neq 0;$$
$$L_{75}(0) \cong L_{148}(0) \cong L_{149}; \quad L_{76} \cong L_{137} \cong L_{159}(\alpha),\ \alpha \neq 0;$$
$$L_{77} \cong L_{118} \cong L_{138}; \quad L_{78}(0); \quad L_{79}(\alpha) \cong L_{147}(0) \cong L_{150}; \quad L_{80} \cong L_{160};$$
$$L_{89} \cong L_{116} \cong L_{146}; \quad L_{90} \cong L_{92} \cong L_{94}; \quad L_{91} \cong L_{136} \cong L_{147}(\alpha),\ \alpha \neq 0;$$
$$L_{93} \cong L_{158} \cong L_{148}(\alpha),\ \alpha \neq 0; \quad L_{115}; \quad L_{135}; \quad L_{145}(\alpha) \cong L_{145}(1),\ \alpha \neq 0;$$
$$L_{145}(0); \quad L_{157}.$$

6.4. Underlying Lie algebra $L_{5,4}$

$$L_6; \quad L_7 \cong L_8.$$

6.5. Underlying Lie algebra $L_{5,5}$

$$L_{48}; \quad L_{49}; \quad L_{50}; \quad L_{51}; \quad L_{52}.$$

6.6. Underlying Lie algebra $L_{5,6}$

$$L_{129}; \quad L_{130}(\alpha) \cong L_{130}(1); \quad L_{131}(\alpha) \cong L_{131}(1);$$
$$L_{132}(\alpha) \cong L_{132}(1); \quad L_{133}(\alpha) \cong L_{133}(1).$$

6.7. Underlying Lie algebra $L_{5,7}$

$$L_{124}; \quad L_{125}; \quad L_{126}; \quad L_{127}; \quad L_{128}.$$

6.8. Underlying Lie algebra $L_{5,8}$

$$L_{42}; \quad L_{43} \cong L_{46} \cong L_{58} \cong L_{61}(0) \cong L_{105}(0); \quad L_{44}; \quad L_{45} \cong L_{63} \cong L_{67};$$
$$L_{47} \cong L_{104}; \quad L_{59}(\alpha) \cong L_{106}(\alpha) \cong L_{107}; \quad L_{60};$$
$$L_{61}(\alpha) \cong L_{62} \cong L_{64}(\alpha') \cong L_{105}(\alpha''),\ \alpha, \alpha'' \neq 0; \quad L_{65}; \quad L_{66} \cong L_{108}; \quad L_{109}.$$

6.9. Underlying Lie algebra $L_{5,9}$

$$L_{119}; \quad L_{120}; \quad L_{121} \cong L_{151} \cong L_{152}(\alpha, 0) \cong L_{153}(\alpha, 0) \cong L_{154}(\alpha, 0);$$
$$L_{122} \cong L_{139} \cong L_{143}(\alpha) \cong L_{164}(\alpha, 0); \quad L_{123} \cong L_{161}; \quad L_{140}(\alpha) \cong L_{140}(1);$$
$$L_{141}(\alpha) \cong L_{141}(1); \quad L_{142}(\alpha) \cong L_{142}(1) \cong L_{164}(\alpha, \beta),\ \beta \neq 0;$$

$L_{152}(\alpha,\beta) \cong L_{152}(0,1),\ \beta \neq 0;$

$L_{153}(\alpha,\beta) \cong L_{153}(0,1) \cong L_{154}(\alpha',\beta') \cong L_{155}(\alpha'') \cong L_{163}(\alpha'''),\ \beta,\beta' \neq 0;$

$L_{162}(\alpha) \cong L_{162}(1).$

7. The classification theorem

In the previous sections we found the 1-dimensional central extensions of the 4-dimensional p-nilpotent restricted Lie algebras and removed isomorphic copies from the list. Thus we are now in position to state the main result of this paper. Let $[p]_1$ and $[p]_2$ be two p-maps on a Lie algebra L defined over a field of characteristic $p > 0$. Following [16], we will say that $[p]_1$ and $[p]_2$ are *equivalent* if the restricted Lie algebras $(L, [p]_1)$ and $(L, [p]_2)$ are isomorphic. We have

Theorem 7.1. *Let L be a p-nilpotent restricted Lie algebra of dimension 5 over an algebraically closed field $\mathbb{F}$ of characteristic $p > 3$. Then the equivalence classes of the p-maps on L are as follows:*

- *If $L = L_{5,1} = \langle x_1, \dots, x_5 \rangle$ is abelian, then*

  (a) *Trivial p-map;*
  (b) $x_1^{[p]} = x_2$;
  (c) $x_1^{[p]} = x_2$, $x_3^{[p]} = x_4$;
  (d) $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$;
  (e) $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$, $x_4^{[p]} = x_5$;
  (f) $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$, $x_3^{[p]} = x_4$;
  (g) $x_1^{[p]} = x_2$, $x_2^{[p]} = x_3$, $x_3^{[p]} = x_4$, $x_4^{[p]} = x_5$.

- *If $L = L_{5,2} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3 \rangle$, then*

  (a) *Trivial p-map;*
  (b) $x_1^{[p]} = x_3$;
  (c) $x_4^{[p]} = x_3$;
  (d) $x_4^{[p]} = x_5$;
  (e) $x_1^{[p]} = x_3$, $x_4^{[p]} = x_5$;
  (f) $x_4^{[p]} = x_5$, $x_5^{[p]} = x_3$;
  (g) $x_1^{[p]} = x_4$;
  (h) $x_1^{[p]} = x_5$, $x_4^{[p]} = x_3$;
  (i) $x_2^{[p]} = x_4$, $x_4^{[p]} = x_3$;
  (j) $x_1^{[p]} = x_3$, $x_2^{[p]} = x_4$;
  (k) $x_1^{[p]} = x_4$, $x_2^{[p]} = x_5$;
  (l) $x_1^{[p]} = x_5$, $x_2^{[p]} = x_4$, $x_4^{[p]} = x_3$;
  (m) $x_1^{[p]} = x_4$, $x_4^{[p]} = x_5$;
  (n) $x_1^{[p]} = x_4$, $x_4^{[p]} = x_5$, $x_5^{[p]} = x_3$;
  (o) $x_1^{[p]} = x_3$, $x_2^{[p]} = x_4$, $x_4^{[p]} = x_5$;
  (p) $x_3^{[p]} = x_4$;
  (q) $x_2^{[p]} = x_3$, $x_3^{[p]} = x_4$;
  (r) $x_1^{[p]} = x_4$, $x_3^{[p]} = x_5$;
  (s) $x_1^{[p]} = x_4$, $x_3^{[p]} = x_5$, $x_4^{[p]} = x_5$;
  (t) $x_1^{[p]} = x_3$, $x_2^{[p]} = x_4$, $x_3^{[p]} = x_5$;
  (u) $x_3^{[p]} = x_4$, $x_4^{[p]} = x_5$;
  (v) $x_2^{[p]} = x_3$, $x_3^{[p]} = x_4$, $x_4^{[p]} = x_5$;
  (w) $x_3^{[p]} = x_5$, $x_4^{[p]} = x_3$.
  (x) $x_2^{[p]} = x_4$, $x_3^{[p]} = x_5$, $x_4^{[p]} = x_3$.

- *If $L = L_{5,3} = \langle x_1, \dots, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4 \rangle$, then*

(a) *Trivial p-map;*
(b) $x_3^{[p]} = x_4,\ x_5^{[p]} = x_4;$
(c) $x_3^{[p]} = x_4;$
(d) $x_5^{[p]} = x_4;$
(e) $x_1^{[p]} = x_4;$
(f) $x_2^{[p]} = x_4;$
(g) $x_2^{[p]} = x_5;$
(h) $x_2^{[p]} = x_5,\ x_5^{[p]} = x_4;$
(i) $x_1^{[p]} = x_5,\ x_3^{[p]} = x_4,\ x_5^{[p]} = x_4;$
(j) $x_2^{[p]} = x_5,\ x_3^{[p]} = x_4;$
(k) $x_1^{[p]} = x_4,\ x_2^{[p]} = x_5;$
(l) $x_1^{[p]} = x_5;$
(m) $x_1^{[p]} = x_5,\ x_5^{[p]} = x_4;$
(n) $x_1^{[p]} = x_5,\ x_3^{[p]} = x_4;$
(o) $x_1^{[p]} = x_5,\ x_2^{[p]} = x_4;$
(p) $x_3^{[p]} = x_5;$
(q) $x_3^{[p]} = x_5,\ x_5^{[p]} = x_4;$
(r) $x_1^{[p]} = x_4,\ x_3^{[p]} = x_5;$
(s) $x_2^{[p]} = x_4,\ x_3^{[p]} = x_5;$
(t) $x_4^{[p]} = x_5;$
(u) $x_1^{[p]} = x_4,\ x_4^{[p]} = x_5;$
(v) $x_3^{[p]} = x_4 + x_5,\ x_4^{[p]} = x_5.$
(w) $x_3^{[p]} = x_4,\ x_4^{[p]} = x_5;$
(x) $x_2^{[p]} = x_4,\ x_4^{[p]} = x_5.$

- *If* $L = L_{5,4} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_5,\ [x_3, x_4] = x_5\rangle$, *then*

(a) *Trivial p-map;*
(b) $x_1^{[p]} = x_5.$

- *If* $L = L_{5,5} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_5,\ [x_2, x_4] = x_5\rangle$, *then*

(a) *Trivial p-map;*
(b) $x_4^{[p]} = x_5;$
(c) $x_3^{[p]} = x_5;$
(d) $x_1^{[p]} = x_5;$
(e) $x_2^{[p]} = x_5.$

- *If* $L = L_{5,6} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_3] = x_5\rangle$, *then*

(a) *Trivial p-map;*
(b) $x_4^{[p]} = x_5;$
(c) $x_3^{[p]} = x_5;$
(d) $x_2^{[p]} = x_5;$
(e) $x_1^{[p]} = x_5.$

- *If* $L = L_{5,7} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5\rangle$, *then*

(a) *Trivial p-map;*
(b) $x_4^{[p]} = x_5;$
(c) $x_3^{[p]} = x_5;$
(d) $x_2^{[p]} = x_5;$
(e) $x_1^{[p]} = x_5.$

- *If* $L = L_{5,8} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_4,\ [x_1, x_3] = x_5\rangle$, *then*

(a) *Trivial p-map;*
(b) $x_3^{[p]} = x_5;$
(c) $x_4^{[p]} = x_5;$
(d) $x_1^{[p]} = x_5;$
(e) $x_2^{[p]} = x_5;$
(f) $x_2^{[p]} = x_5,\ x_3^{[p]} = x_4;$

(g) $x_2^{[p]} = x_4$, $x_4^{[p]} = x_5$;
(h) $x_1^{[p]} = x_4$, $x_3^{[p]} = x_5$;
(i) $x_1^{[p]} = x_4$, $x_4^{[p]} = x_5$;
(j) $x_1^{[p]} = x_4$, $x_2^{[p]} = x_5$;
(k) $x_3^{[p]} = x_4$, $x_4^{[p]} = x_5$.

- *If* $L = L_{5,9} = \langle x_1, \ldots, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5 \rangle$, *then*

(a) *Trivial p-map;*
(b) $x_4^{[p]} = x_5$;
(c) $x_3^{[p]} = x_5$;
(d) $x_2^{[p]} = x_5$;
(e) $x_1^{[p]} = x_5$;
(f) $x_1^{[p]} = x_4$, $x_4^{[p]} = x_5$;
(g) $x_1^{[p]} = x_4$, $x_3^{[p]} = x_5$;
(h) $x_1^{[p]} = x_4$, $x_2^{[p]} = x_5$.
(i) $x_3^{[p]} = x_4$, $x_4^{[p]} = x_5$;
(j) $x_1^{[p]} = x_5$, $x_3^{[p]} = x_4$;
(k) $x_2^{[p]} = x_4$, $x_4^{[p]} = x_5$.

8. Comments

Let L be a p-nilpotent restricted Lie algebra of dimension 5 over any field $\mathbb{F}$ of characteristic $p > 3$ and denote by $\bar{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$. Since p-nilpotency is preserved under extensions of the ground field, $L \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ is clearly isomorphic to one of the restricted Lie algebras in Theorem 7.1. In particular, this allows to compare the classification by Darijani and Usefi in [3] with ours. According to the conditions given in the main results of [3], it is easily seen that all the variables appearing in the parametrized families of their classification can be settled to 1 over algebraically closed fields.

We first point out that, in the notation of [3], the abelian restricted Lie algebras $L_{5,1}^5$ and $L_{5,1}^6$ are isomorphic over any field. We also note that the restricted structures (f), (n) and (t) on $L_{5,2}$ are missing in [3]. Moreover, the p-maps (h), (o) and (v) on $L_{5,3}$ do not appear in [3]. Finally, the classification in [3] lacks of the p-map (f) on $L_{5,8}$. It is also worth noticing that the restricted Lie algebras $L_{5,9}^6$ and $L_{5,9}^9$ in [3] are isomorphic over algebraically closed fields.

Data availability

No data was used for the research described in the article.

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