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On non-planar ABJM anomalous dimensions from M2 branes in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

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Abstract: Planar parts of conformal dimensions of primary operators in $U_k(N) \times U_{-k}(N)$ ABJM theory are controlled by integrability. Strong coupling asymptotics of planar dimensions of operators with large spins can be found from the energy of semiclassical strings in $\mathrm{AdS}_4 \times \mathrm{CP}^3$ but computing non-planar corrections requires understanding higher genus string corrections. As was pointed out in arXiv:2408.10070, there is an alternative way to find the non-planar corrections by quantizing M2 branes in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ which are wrapped around the 11d circle of radius $1/k = \lambda/N$ and generalize spinning strings in $\mathrm{AdS}_4 \times \mathrm{CP}^3$. Computing the 1-loop correction to the energy of M2 brane that corresponds to the long folded string with large spin S in AdS_4 allowed to obtain a prediction for the large λ limit of non-planar corrections to the cusp anomalous dimension. Similar predictions were found for non-planar dimensions of operators dual to M2 branes that generalize the "short" and "long" circular strings with two equal spins $J_1 = J_2$ in CP^3. Here we consider two more non-trivial examples of 1-loop M2 brane computations that correspond to: (i) long folded string with large spin S in AdS_4 and orbital momentum J in CP^3 whose energy determines the generalized cusp anomalous dimension, and (ii) circular string with spin S in AdS_4 and spin J in CP^3. We find the leading terms of the expansion of the corresponding 1-loop M2 brane energies in $1/k$. We also discuss similar semiclassical 1-loop M2 brane computation in flat 11d background and comment on possible relation to higher genus corrections to energies in 10d string theory.

Keywords: $1/N$ Expansion, AdS-CFT Correspondence, M-Theory

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Contents

1 Introduction

An important problem in the study of superconformal quantum field theories admitting a large N expansion (such as $\mathcal{N} = 4$ SYM and ABJM [1]) is to compute the conformal dimensions Δ of primary operators as functions of the 't Hooft coupling λ at each order in $1/N$,

$$\Delta(\lambda, N) = \Delta_0(\lambda) + \frac{1}{N^2}\Delta_1(\lambda) + \dots . \tag{1.1}$$

The planar part $\Delta_0(\lambda)$ is determined in principle by integrability with its large λ expansion matched by the large tension expansion of string energies in the dual string theory [2]. This does not in general apply to non-planar Δ_n corrections. In particular, determining large λ behaviour of Δ_n is challenging on string theory even for the states with large quantum numbers as this requires computing higher genus corrections to semiclassical string energies.

While computing non-planar corrections at large λ is an open problem in the $\mathcal{N} = 4$ SYM theory it was recently realised [3] that it is possible to find the leading strong-coupling terms in non-planar corrections $\Delta_n(\lambda)$ in the $U(N)_k \times U(N)_{-k}$ ABJM theory by using its duality to M-theory, i.e. quantum M2 brane theory on $AdS_4 \times S^7/\mathbb{Z}_k$.[1]

[1] This was demonstrated earlier for other observables for which M2 brane predictions can be matched with localization results on the gauge theory side [4–6].

The idea is to consider an M2 brane counterpart of a semiclassical string solution with large quantum numbers in $AdS_4 \times CP^3$ and find the 1-loop correction to its energy as a function of the 11d radius $1/k$ or string coupling (see appendix A). The M2 brane action $S = S_B + S_F$ [7, 8] has the following bosonic part

$$S_B = S_V + S_{WZ}, \qquad S_V = -T_2 \int d^3\xi \sqrt{-g}, \qquad g_{ij} = \partial_i X^M \partial_j X^N G_{MN}(X), \tag{1.2}$$

$$S_{WZ} = -T_2 \int d^3\xi \ \tfrac{1}{3!}\epsilon^{ijk} C_{MNK}(X)\, \partial_i X^M \partial_j X^N \partial_k X^K, \qquad T_2 = \frac{1}{(2\pi)^2 \ell_P^3}. \tag{1.3}$$

The explicit form of the quadratic term in the fermionic part S_F that we use in this paper can be found, e.g., in [3].

The large N, fixed k expansion corresponds to the expansion in large effective dimensionless M2 brane tension (L is the 11d scale, see (A.1))

$$\mathrm{T}_2 = \mathrm{L}^3\, T_2 = \frac{1}{\pi}\sqrt{2Nk}, \qquad T_2 = \frac{1}{(2\pi)^2 \ell_P^3}. \tag{1.4}$$

In general, an observable (like the energy of a spinning membrane corresponding to the dimension of an operator with large quantum numbers) computed in the semiclassical expansion will then be

$$E = \mathrm{T}_2\, \mathrm{E}_0(k) + \mathrm{E}_1(k) + \mathrm{T}_2^{-1}\, \mathrm{E}_2(k) + \dots, \qquad \mathrm{T}_2 \gg 1. \tag{1.5}$$

Expanding further in large k with fixed $\lambda \equiv \frac{N}{k}$ corresponds to the 't Hooft expansion in the 3d gauge theory which is dual to perturbative expansion in type IIA string theory in $AdS_4 \times CP^3$ with string coupling g_s (A.8) and the effective dimensionless string tension given by (cf. (A.9), (A.12))

$$\mathrm{T} = \frac{\pi}{2k}\mathrm{T}_2 = \sqrt{\frac{\lambda}{2}} = \frac{\sqrt{\bar\lambda}}{2\pi}, \qquad \bar\lambda \equiv 2\pi^2\lambda. \tag{1.6}$$

Representing $\mathrm{E}_n(k)$ in (1.5) as a series in powers of

$$\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{g_s^2}{8\pi \mathrm{T}}, \tag{1.7}$$

and expressing it as an expansion in g_s^2 and then in T^{-1} determines the strong coupling (large λ) corrections at each order in $1/N^2$ to the corresponding anomalous dimension in gauge theory.

In particular, the 1-loop term $\mathrm{E}_1(k)$ encodes the leading inverse string tension corrections at each order in g_s^2 in type IIA theory on $AdS_4 \times CP^3$. Ref. [3] computed $\mathrm{E}_1(k)$ for the two types of M2 brane solutions generalizing (i) long folded string with spin S in AdS_4 related to cusp anomalous dimension, and (ii) "long" and "short" circular strings with two equal spins $J_1 = J_2$ in CP^3. We shall review the resulting expressions in section 1.1.

The aim of the present paper is to consider two more non-trivial examples in the (S, J) sector that generalize (i) long folded string with spin S in AdS_4 and orbital momentum J in CP^3 related to the generalized cusp anomalous dimension, and (ii) circular string with spin S in AdS_4 and spin J in CP^3. We shall summarize our conclusions in section 1.2.

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1.1 Review

Starting with a classical string solution in $AdS_4 \times CP^3$ with large quantum numbers $Q = \sqrt{\lambda}\,\mathcal{Q}$ (e.g. spins in both AdS_4 and CP^3) its AdS_4 energy computed in large tension expansion may be written as

$$E = \left[\sqrt{\lambda}\,\mathcal{E}_0(\mathcal{Q}) + \mathcal{E}_1(\mathcal{Q}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{Q}) + \mathcal{O}\Big(\frac{1}{(\sqrt{\lambda})^2}\Big)\right] + \mathcal{O}(g_s^2)\,. \tag{1.8}$$

1-loop string corrections $\mathcal{E}_1$ to the energies of spinning string solutions in $AdS_4 \times CP^3$ were computed, e.g., in [9–13]. Such solutions have direct M2 brane counterparts in $AdS_4 \times S^7/\mathbb{Z}_k$— the membrane is wrapped on 11d circle of radius $1/k$ and reduces to the string solution on the $AdS_4 \times CP^3$. The classical M2 brane energy, i.e. the first term in (1.5) then coincides with the first term in (1.8) with the tensions related as in (1.6), i.e. $\mathrm{E}_0 = \frac{\pi^2}{k}\mathcal{E}_0$. Also, $\mathrm{E}_1 = \mathcal{E}_1 + \mathcal{O}(\frac{1}{k^2})$, with subleading terms in $\frac{1}{k^2}$ determining g_s^2 corrections, etc.

Cusp anomalous dimension. The first example considered in [3] was the M2 brane analog of the long folded string with large spin $\mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1$. Its energy determines the null cusp anomalous dimension as in [14, 15], i.e. the leading coefficient $f(\lambda, N)$ in the large spin expansion of the conformal dimension of an operator like $O_S = \mathrm{tr}\big[D^S(Y^1 Y_4^\dagger)\big]$,[2]

$$E = \Delta(\lambda, S) \overset{S\gg 1}{=} S + f(\lambda, N)\,\log S + \dots\,, \qquad f(\lambda, N) = f_0(\lambda) + \frac{1}{N^2}\mathrm{f}_1(\lambda) + \frac{1}{N^4}\mathrm{f}_2(\lambda) + \dots\,, \tag{1.9}$$

$$f_0(\lambda) \overset{\lambda\gg 1}{=} \sqrt{2\lambda} + \mathrm{f}_0(\lambda), \qquad \mathrm{f}_r(\lambda) \overset{\lambda\gg 1}{=} \lambda^{2r}\big(a_{1r} + \tfrac{1}{\sqrt{\lambda}}a_{2r} + \dots\big), \qquad r = 0, 1, 2, \dots\,. \tag{1.10}$$

The leading coefficients in the planar part $f_0(\lambda)$ found by quantizing the long folded spinning string in $AdS_4 \times CP^3$ [9, 10, 16, 17] are

$$f_0(\lambda) = \sqrt{2\lambda} - \tfrac{5\log 2}{2\pi} - \big(\tfrac{K}{4\pi^2} + \tfrac{1}{24}\big)\tfrac{1}{\sqrt{2\lambda}} + \mathcal{O}\big(\tfrac{1}{(\sqrt{\lambda})^2}\big) = 2h(\lambda) - \tfrac{3\log 2}{2\pi} - \tfrac{K}{8\pi^2}h^{-1}(\lambda) + \mathcal{O}\big(h(\lambda)^{-2}\big), \tag{1.11}$$

where $h(\lambda)$ is a "renormalized tension" (containing a $\log 2$ correction [11] with the exact form suggested in [18])

$$h(\lambda) = \tfrac{1}{\sqrt{2}}\sqrt{\lambda - \tfrac{1}{24}} - \tfrac{\log 2}{2\pi} + \mathcal{O}(e^{-2\pi\sqrt{2\lambda}})\,. \tag{1.12}$$

The expression (1.11) is consistent with integrability and is related to the $AdS_5 \times S^5$ or $\mathcal{N} = 4$ SYM result by the replacement $\frac{\sqrt{\lambda_{\rm SYM}}}{2\pi} \to h(\lambda_{\rm ABJM})$ [19].[3]

The semiclassical quantization of the corresponding M2 brane solution leads to the following representation for f in (1.9) [3] (cf. (1.5))

$$f(k, \mathrm{T}_2) = \frac{\pi}{k}\mathrm{T}_2 + q_0(k) + \mathrm{T}_2^{-1}\,q_1(k) + \mathrm{T}_2^{-2}\,q_2(k) + \dots\,, \tag{1.13}$$

$$q_s(k) = k^s\left(p_s^{(0)} + \frac{1}{k^2}p_s^{(1)} + \frac{1}{k^4}p_r^{(2)} + \dots\right), \qquad s = 0, 1, 2, \dots \tag{1.14}$$

[2] Here Y^i are bi-fundamental scalars of ABJM theory and D is covariant derivative in a null direction.

[3] This is related to the fact that planar part of the cusp anomalous dimension is controlled just by the asymptotic Bethe ansatz. There is no reason to expect that a similar simple relation should hold at a non-planar level.

Comparing (1.9), (1.10) and (1.13) one finds that at any non-planar order $1/N^{2r}$ the leading large λ contribution to $\mathrm{f}_r(\lambda)$, i.e. the coefficient a_{1r}, is fixed in terms of the coefficients appearing in the large k expansion (1.14) of $q_0(k)$, i.e. $a_{1r} = p_0^{(r)}$, $r = 0, 1, 2, \ldots$. The expression for the 1-loop M2 brane correction $q_0(k)$ obtained in [3] is

$$\begin{aligned} q_0(k) &= \frac{1}{\pi}\left[-\frac{5}{2}\log 2 + \frac{4}{k^2}\zeta(2) + \frac{4}{k^4}\zeta(4) - \frac{1616}{15k^6}\zeta(6) + \ldots\right] \\ &= -\frac{5\log 2}{2\pi} + \frac{2\pi}{3}\frac{1}{k^2} + \frac{2\pi^3}{45}\frac{1}{k^4} - \frac{1616\pi^5}{14175}\frac{1}{k^6} + \ldots, \end{aligned} \tag{1.15}$$

where the expression for $\frac{1}{k^2}$ is given in (1.7). This determines the leading strong coupling asymptotics of the non-planar $\frac{1}{N^{2r}}$ terms in the cusp anomalous dimension.

Circular $J_1 = J_2$ solution. The second example discussed in [3] was the M2 brane generalization of the string solutions [12, 20] with two spins $J_1 = J_2 \equiv J$ in CP^3. The dual ABJM operator should have the structure $O_{J_1,J_2} = \mathrm{tr}[(Y^1 Y_2^\dagger)^{J_1}(Y^3 Y_4^\dagger)^{J_2}] + \ldots$. For the M2 brane solution which admits the "short" (or "slow" $\mathcal{J} \equiv \frac{J}{\sqrt{\bar\lambda}} \ll 1$) limit, the corresponding energy contains the following 1-loop M2 brane corrections [3]

$$\begin{aligned} E &= 2\sqrt{\sqrt{\bar\lambda}\,J} + \tfrac{1}{2} + \tfrac{1}{2}\bar\lambda^{-1/4}J^{1/2} - \tfrac{9}{4}\zeta(3)\bar\lambda^{-3/4}J^{3/2} + \mathcal{O}(\bar\lambda^{-1}J^2) \\ &\quad + \frac{1}{k^2}\Big[\zeta(2)\big(-4\bar\lambda^{3/4}J^{-3/2} + 8\bar\lambda^{1/4}J^{-1/2}\big) + \mathcal{O}(\bar\lambda^{-1/4}J^{1/2})\Big] + \mathcal{O}\big(\frac{1}{k^4}\big), \end{aligned} \tag{1.16}$$

where $\bar\lambda$ was defined in (1.6). Here the first line is the 1-loop string correction that represents the strong-coupling expansion of the conformal dimension $\Delta(J, \lambda)$ of the dual "short" operator in the planar limit. The second line is the large k expansion of the leading non-planar correction (cf. (1.13), (1.15)), i.e. the leading large tension asymptotics of the quantum string (torus) contribution. Equivalently, (1.16) may be written as

$$(E-\tfrac{1}{2})^2 = 4\sqrt{\bar\lambda}\,J\,\big(1+\tfrac{1}{2\sqrt{\bar\lambda}}+\ldots\big) - \tfrac{9}{\sqrt{\bar\lambda}}\,\zeta(3)J^2 + \ldots + \frac{1}{k^2}\,16\zeta(2)\big(-\bar\lambda\,J^{-1} + 2\sqrt{\bar\lambda} + \ldots\big) + \mathcal{O}\big(\frac{1}{k^4}\big). \tag{1.17}$$

For the M2 brane solution which admits the "long" (or "fast" $\mathcal{J} \gg 1$) limit, one finds [3][4]

$$\begin{aligned} E =&\, 2J + \tfrac{1}{4}\bar\lambda J^{-1}\big(1 - 2\log 2\,\bar\lambda^{-1/2} + \ldots\big) + \tfrac{1}{2}c_1\bar\lambda J^{-2} + \mathcal{O}(\bar\lambda^2 J^{-3}) \\ &+ \frac{1}{k^2}\zeta(2)\big(-8\bar\lambda^{-1/2}J - 2\bar\lambda^{1/2}J^{-1} + \tfrac{3}{16}\bar\lambda^{3/2}J^{-3} + \mathcal{O}(\bar\lambda^{5/2}J^{-5})\big) + \mathcal{O}\big(\frac{1}{k^4}\big). \end{aligned} \tag{1.18}$$

The first line is the sum of the classical energy and string 1-loop contribution expanded in the "long" limit [12] with $c_1 \simeq -0.336$. The second line represents the membrane contribution, or the leading non-planar correction, to the anomalous dimension of the dual ABJM operator with large spin J.

Let us recall that the string corrections in (1.18) that scale as odd powers of $1/J$, i.e. $E^{\rm odd}(J, \sqrt{\bar\lambda})$, do not receive wrapping corrections, i.e. they are controlled by the asymptotic

[4] Here $\zeta(2)$ is the standard Riemann zeta-function value, i.e. $\frac{\pi^2}{6}$. Like in (1.15), below we will keep the $\zeta(2n)$ factors in the $\frac{1}{k^{2n}}$ terms in their implicit form to emphasize their common origin from the summation over the M2 brane modes in the second circular direction (identified with the 11d direction).

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Bethe Ansatz [19] and are directly related to their counterparts $E^{\rm odd}_{\rm AdS_5}(J,\sqrt{\lambda})$ in the $\rm AdS_5\times S^5$ case as [11]

$$E^{\rm odd}_{\rm AdS_4}(J,\sqrt{\bar\lambda})=\tfrac12 E^{\rm odd}_{\rm AdS_5}(2J,2\bar h(\bar\lambda))\,, \qquad \bar h(\bar\lambda)\equiv 2\pi h(\lambda)=\sqrt{\bar\lambda}-\log 2+\dots\,, \tag{1.19}$$

where $h(\lambda)$ was given in (1.12). Thus the $\log 2$ term in the first line of (1.18) may be absorbed into the redefinition of the $\bar\lambda$ factor. This relation is also consistent with (1.9), (1.11).[5]

1.2 Summary

Generalized cusp anomalous dimension. Below we will first consider a generalization of the 1-loop M2 brane computation of non-planar corrections to the cusp anomaly in [3] to the case of the "generalized" cusp anomaly that includes also dependence on the angular momentum J in $\rm CP^3$. In the $\rm AdS_5\times S^5$ case the corresponding long folded string (S,J) solution and the associated cusp anomaly function was studied in [21–23]. For the string in $\rm AdS_4\times CP^3$, the 1-loop correction to the energy of such (S,J) solution was computed in [9, 10]. While for generic values of the spins the form of the folded string solution is complicated, it simplifies in the limit

$$\mathcal{S}\gg\mathcal{J}\gg 1, \qquad x\equiv\frac{\log\mathcal{S}}{\pi\mathcal{J}}=\text{fixed}, \qquad \mathcal{S}=\frac{S}{\sqrt{\bar\lambda}},\quad \mathcal{J}=\frac{J}{\sqrt{\bar\lambda}}\,. \tag{1.20}$$

In this case one finds a generalization of (1.9) where $f(\lambda,N)\to f(\lambda,N,x)$. The tree-level string (planar) term in the expansion of f in (1.9) is then the following generalization of (1.10)

$$f_0(\lambda,x)=\frac{\sqrt{\bar\lambda}}{\pi}\,\frac{\sqrt{1+x^2}}{x}+\mathrm{f}_0(\lambda,x)\,, \tag{1.21}$$

where $\mathrm{f}_0(\lambda,x)$ is the coefficient of $\log S$ in the 1-loop $\rm AdS_4\times CP^3$ string correction [9] that may be written as

$$E_1(J,S,\sqrt{\bar\lambda})=-\,\mathcal{J}\frac{x^2}{\sqrt{1+x^2}}\log 2+\tfrac12 E_1^{\rm AdS_5}(2J,2S,2\sqrt{\bar\lambda}), \tag{1.22}$$

where $E_1^{\rm AdS_5}(J,S,\sqrt{\lambda})$ is the corresponding expression in the $\rm AdS_5\times S^5$ string case [22]

$$E_1^{\rm AdS_5}(J,S,\sqrt{\lambda})=\mathcal{J}\frac{1}{\sqrt{1+x^2}}\Big[x(\sqrt{1+x^2}-x) \\ +2(1+x^2)\log(1+x^2)-(1+2x^2)\log\big[\sqrt{1+2x^2}(x+\sqrt{1+x^2})\big]\Big]. \tag{1.23}$$

Combining (1.22) with the classical string term in (1.21), i.e. $E_0=S+\sqrt{2\lambda}\sqrt{1+\frac{1}{x^2}}\log S+\dots$ one observes that the relation (1.22) is consistent with the direct generalization of (1.19) that includes also the spin S argument.[6]

---

[5] Note that the large $\mathcal{J}$ expansion of the classical string energy $E_0=\sqrt{\lambda}\mathcal{E}_0$ has the form $E_0(J,\sqrt{\lambda})=J+\sum_{n=0}^{\infty}c_n(\sqrt{\lambda})^{2n+2}J^{-2n-1}$, which implies that $E_0(J,\sqrt{\lambda})=\frac12 E_0(2J,2\sqrt{\lambda})$. This relation extends to odd terms in the string quantum (inverse tension) corrections with extra replacements of $\sqrt{\lambda}$ in the coefficients. The factor of 2 in front of $\bar h$ in (1.19) stems from a comparison of the BMN dispersion relation in the $\rm AdS_4\times CP^3$ and $\rm AdS_5\times S^5$ cases.

[6] Let us note also that the 2-loop corrections to the corresponding string energy or generalized cusp anomaly were computed in the $\rm AdS_5\times S^5$ case in [23] and in the $\rm AdS_4\times S^7/\mathbb{Z}_k$case in [17].

To find non-planar corrections to this generalized scaling function we shall consider the M2 brane generalization of the long folded (S, J) string in the limit (1.20). The resulting analog of (1.13) will have q_r now depending also on x with the 1-loop term $q_0(x, k)$ having the following large k expansion

$$
\begin{aligned}
q_0(x,k) = &-\frac{5\log 2}{2\pi}+\frac{\pi}{x}\left(\tfrac{2}{3}\sqrt{1+x^2}+\frac{1}{\sqrt{1+x^2}}\right)\frac{1}{k^2}\\
&+\frac{\pi^3}{x^3}\left[\tfrac{2}{45}\left(\sqrt{1+x^2}\right)^3+\tfrac{19}{45}\sqrt{1+x^2}-\tfrac{1}{60}\frac{1}{\sqrt{1+x^2}}\right]\frac{1}{k^4}+\dots\,.
\end{aligned}
\tag{1.24}
$$

This generalizes (1.15) to the non-zero J case ((1.24) reduces to (1.15) for $x \to \infty$). Explicitly, the large x and small x expansions of q_0 may be written as (cf. (1.6))

$$
q_0(x,k)\Big|_{x\gg 1} = -\frac{5\log 2}{2\pi}+\frac{2\pi}{3}\frac{\lambda^2}{N^2}\left(1+\frac{J^2}{\lambda\log^2 S}+\mathcal{O}\big(\lambda^{-2}J^4\log^{-4}S\big)\right)+\mathcal{O}\big(\frac{1}{N^4}\big)\,, \tag{1.25}
$$

$$
q_0(x,k)\Big|_{x\ll 1} = -\frac{5\log 2}{2\pi}+\frac{5\pi}{3\sqrt{2}}\frac{\lambda^2}{N^2}\left(\frac{J}{\sqrt{\lambda}\log S}+\mathcal{O}\big(\lambda^{1/2}J^{-1}\log S\big)\right)+\mathcal{O}\big(\frac{1}{N^4}\big)\,. \tag{1.26}
$$

These expressions represent the predictions for the non-planar corrections to the generalized ABJM cusp anomaly function in the strong coupling limit.

Circular (S, J) solution. Another 1-loop computation we will consider in this paper is for an M2 brane generalization of the circular (S, J) string solution, in which the string wraps a circle in AdS_4 and a circle in CP^3. The 1-loop correction to such solution in $AdS_5 \times S^5$ was computed in [24–26] while for its $AdS_4 \times CP^3$ analog this was done in [11]. We will consider string energy as a function of $J = \sqrt{\lambda}\,\mathcal{J}$ and[7]

$$
u \equiv \frac{\mathcal{S}}{\mathcal{J}} = \frac{S}{J}\,. \tag{1.27}
$$

In general, the M2 brane energy has the following structure

$$
E(S, J, k, \sqrt{\lambda}) = E_0 + E_1(u, \mathcal{J}) + E_1^{\rm M2}(u, \mathcal{J}, k) + \dots\,, \tag{1.28}
$$

where E_0 is the classical string (or membrane) energy, E_1 is the 1-loop string correction and $E_1^{\rm M2}(u, \mathcal{J}, k)$ in (1.28) stands for the genuine M2 brane 1-loop contribution, i.e. a series in $1/k^2$.

In the "short" string limit ($\mathcal{J} \ll 1$, u=fixed) one finds that

$$
E_0 = \sqrt{\lambda}\left[u+\sqrt{1+u^2}\,\mathcal{J}+\frac{u}{2(1+u^2)}\mathcal{J}^2+\dots\right]\,, \tag{1.29}
$$

$$
E_1(u, \mathcal{J}) = A_0(u) + A_1(u)\,\mathcal{J} + A_2(u)\,\mathcal{J}^2 + \dots\,, \tag{1.30}
$$

where A_i are given by convergent infinite sums (see appendix D), while (cf. (1.16), (1.18))

$$
\begin{aligned}
E_1^{\rm M2}(u,\mathcal{J},k) = &\frac{1}{k^2}\zeta(2)\left(-6u+\frac{10}{\sqrt{1+u^2}}\mathcal{J}+\frac{u(3+13u^2)}{(1+u^2)^2}\mathcal{J}^2+\dots\right)\\
&+\frac{1}{k^4}\zeta(4)\left(-\frac{3}{2}u-\frac{35}{\sqrt{1+u^2}}\mathcal{J}+\frac{162+165u^2-137u^4}{4u(1+u^2)^2}\mathcal{J}^2+\dots\right)+\mathcal{O}\big(\frac{1}{k^6}\big).
\end{aligned}
\tag{1.31}
$$

[7]In general, the classical solution depends on the two integer winding numbers n, m so that $\mathrm{n}\, S + \mathrm{m}\, J = 0$. We will fix $\mathrm{n} = 1$ (so that $u = -\mathrm{m}$) but the dependence on n can be always reinstated by an appropriate rescaling of J.

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In "long" string limit ($\mathcal{J} \gg 1$, u=fixed) the classical string energy is given by

$$E_0 = \sqrt{\lambda}\,\Big[(1+u)\,\mathcal{J} + \frac{1}{2\mathcal{J}}u(1+u) - \frac{1}{8\mathcal{J}^3}u(1+u)(1+3u+u^2) + \dots\Big]\,, \tag{1.32}$$

i.e. has the familiar "fast string" expansion ($E = S + J +$ terms with odd powers in $1/J$). String 1-loop correction E_1 expanded for $\mathcal{J} \gg 1$ contains terms with both odd and even powers of $1/J$. The former combine with the classical E_0 term to satisfy the analog of the relation (1.19) (including also the dependence on S) to the corresponding terms in the $AdS_5 \times S^5$ case [25, 27]

$$\begin{aligned} E^{\rm odd}_{\rm AdS_5}(J,S,\sqrt{\lambda}) = J + S + \frac{\lambda}{2J}\,u(1+u) - \frac{\lambda^2}{8J^3}\,u(1+u)(1+3u+u^2) \\ + \frac{\lambda^3}{16J^5}\Big[u(1+u)(1+7u+13u^2+7u^3+u^4) + \frac{1}{\sqrt{\lambda}}u^3(1+u)^3\Big] + \dots \end{aligned} \tag{1.33}$$

The even part of the AdS_4 string 1-loop term is

$$E^{\rm even}_1 = \Big[-\tfrac{3}{4}u^2(1+u)^2\zeta(2) + \tfrac{15}{8}u^3(1+u)^3\zeta(4) + \dots\Big]\frac{1}{\mathcal{J}^2} + \mathcal{O}\big(\frac{1}{\mathcal{J}^4}\big). \tag{1.34}$$

The coefficients of $1/\mathcal{J}^{2n}$ terms in (1.34) are given by infinite sums that converge for sufficiently small u [11, 27]. The membrane correction in (1.28) expanded at large $\mathcal{J}$ is found to have the following form (cf. (1.31))

$$\begin{aligned} E^{\rm M2}_1(u,\mathcal{J},k) = \frac{1}{k^2}\zeta(2)\Big[10\mathcal{J} - \frac{u(6+11u)}{\mathcal{J}} + \frac{u(12+68u+88u^2+27u^3)}{4\mathcal{J}^3} + \mathcal{O}(\mathcal{J}^{-5})\Big] \\ + \frac{1}{k^4}\zeta(4)\Big[\frac{81}{2u(1+u)}\mathcal{J}^3 + \frac{81+103u-221u^2}{4u(1+u)}\mathcal{J} + \mathcal{O}(\mathcal{J}^{-1})\Big] + \mathcal{O}\big(\frac{1}{k^6}\big)\,. \end{aligned} \tag{1.35}$$

Note that the leading $\frac{1}{k^2}\mathcal{J}$ term in (1.35) combined with the classical J term in (1.32) gives $E = (1 + \frac{10\zeta(2)}{k^2\lambda^{1/2}} + \dots)J + \dots$ which is similar to the structure of the linear in J term in the energy (1.18) for the $J_1 = J_2$ solution found in [3].

Flat space case. The matching of the leading strong-coupling asymptotics of non-planar corrections found from localization to the 1-loop M2 brane corrections in [4, 6] implies that the latter should be reproducing the leading in large string tension term in the quantum string higher genus contribution in $AdS_4 \times CP^3$(cf. (1.7)). In particular, the 1-loop energy of a semiclassical M2 brane state should be reproducing certain part of the quantum string 1-loop (torus) correction to the energy of the corresponding string state.

As was noted in [3] the same should then apply also in the flat space limit. Namely, starting with a semiclassical string state in 10d IIA string theory, considering its M-theory analog represented by an M2 brane wrapped on the 11d circle and computing the 1-loop correction to its energy should capture certain leading terms in the corresponding torus, etc., corrections [28–32] to the string energy or the mass shift in 10d string theory.

For example, starting with the $\mathbb{R}^{1,9}\times S^1$ counterpart of the ("short") circular $J_1 = J_2 = J$ string rotating in 2 orthogonal planes in $AdS_4 \times CP^3$and taking the flat-space limit [3] of the M2 brane 1-loop expression in (1.16) or (1.17) one finds[8]

$$E = 2\sqrt{\alpha'^{-1}J}\Big[1-\tfrac{1}{2}\zeta(2)\,g_s^2J^{-2}-\tfrac{19}{60}\zeta(6)\,g_s^6J^{-4}+\ldots\Big], \tag{1.36}$$

$$\alpha'E^2 = \alpha'E_0^2-4\zeta(2)\,g_s^2J^{-1}+\zeta^2(2)\,g_s^4J^{-3}-\tfrac{38}{15}\zeta(6)\,g_s^6J^{-3}+\ldots, \qquad \alpha'E_0^2=4J\,. \tag{1.37}$$

Below we will reproduce (1.36) by the 1-loop M2 brane computation directly in flat space and extend this expression to all orders in expansion in g_s.

Considering M2 brane in flat 11d background we have ($\mu=0,\ldots,9,\ \ \alpha'=\ell_s^2$)

$$ds_{11}^2 = dx^\mu dx_\mu + dx_{10}^2, \qquad x_{10}\equiv x_{10}+2\pi R_{11}\,, \tag{1.38}$$

$$T=\frac{1}{2\pi\ell_s^2}=2\pi R_{11}T_2\,, \qquad T_2=\frac{1}{(2\pi)^2\ell_P^3}=\frac{1}{(2\pi)^2\ell_s^3\,g_s}\,, \qquad R_{11}=g_s\,\ell_s\,, \qquad \ell_P=g_s^{1/3}\,\ell_s\,. \tag{1.39}$$

Here T is the tension of the 10d string related by the double dimensional reduction to the M2 brane action with tension T_2. As our aim is to compare the semiclassical M2 brane expansion to string perturbation theory, we express the M-theory parameters in terms of the string theory ones.

Specialising to the classical M2 brane solutions with topology $\Sigma\times S^1$, where S^1 wraps the M-theory circle of radius R_{11} and Σ is the world surface of the associated 10d string solution the corresponding quantum M2 brane corrections to its energy are organized as an expansion in $T_2^{-1} = (2\pi)^2\ell_s^3\,g_s$ as in (1.5),

$$E = E_0+\mathrm{E}_1+T_2^{-1}\mathrm{E}_2+T_2^{-2}\mathrm{E}_3+\ldots\,. \tag{1.40}$$

Here E_r depend on dimensionless parameters of the solution (spins) and on $R_{11}/\ell_s = g_s$. Note that in the flat space case there are no non-trivial tree-level (genus 0) α' corrections to string energies (i.e. E_0 is just the classical energy) so that E_n will represent the genuine M2 brane corrections depending on g_s. Expanding (1.40) in small g_s we get a series of g_s^n contributions that may be compared with higher genus string-theory corrections to the mass shift of the corresponding semiclassical string state.

In the above example of the $J_1 = J_2$ circular string solution we find the following structure of the 1-loop M2 brane correction to its energy (cf. (1.36))[9]

$$E=E_0+\mathrm{E}_1, \qquad \mathrm{E}_1=\frac{1}{\alpha'E_0}\bar{\mathrm{E}}_1, \qquad E^2=E_0^2+\Delta E^2, \qquad \alpha'\Delta E^2=2\bar{\mathrm{E}}_1+\frac{(\bar{\mathrm{E}}_1)^2}{\alpha'E_0^2}, \tag{1.41}$$

$$\bar{\mathrm{E}}_1=\sum_{n=1}^{\infty}c_n\zeta(4n-2)\frac{1}{\Lambda^{4n-2}}=-2\zeta(2)\,\frac{g_s^2}{J}-\tfrac{19}{15}\zeta(6)\,\frac{g_s^6}{J^3}+\ldots, \qquad \frac{1}{\Lambda^2}\equiv\frac{g_s^2}{2J}. \tag{1.42}$$

[8]Note that [33] discussed the 1-loop correction to the energy of a different $J_1 = J_2$ solution in flat 11d space: there the membrane was rotating in 2 planes with the "radii" being periodic functions of the two world-volume coordinates but was not wrapped on S^1.

[9]Note that J, $\bar{\mathrm{E}}_1$ and Λ are dimensionless while E has canonical dimension of inverse length.

Here c_n are rational coefficients that we will determine below for all n. We will also discuss the case of a folded string with spin J where $\bar{\rm E}_1$ should have a similar expansion in powers of $g_s^2 J^{-1}$.[10]

One may try to compare the g_s^2 term in (1.37) or (1.42) with the 1-loop mass shift for the corresponding string state $(\Delta E^2)_{\rm str} \equiv \Delta M^2$ that can be found from the torus 2-point amplitude

$$\alpha' \Delta E^2 = g_s^2 \big[{\rm R}(J) + i\,{\rm I}(J) \big] . \tag{1.43}$$

Here the real and imaginary parts are non-trivial functions of $J \sim E_0^2$. The presence of an imaginary part reflects the possibility of a decay of a massive states into lighter states.[11] The imaginary part has the form [30, 31, 34]

$$ {\rm I}(J) \sim J^\gamma , \tag{1.44}$$

with γ depending on a specific string state (for the folded string $\gamma = \frac{1}{2}$ [30, 34], while for the rotating circular string $\gamma = -2$ [30, 31]). The real part ${\rm R}(J)$ of the torus correction is given by a complicated modular integral and appears to be explicitly known only in an "averaged" (over states at a given mass level) form [35], e.g.,

$$ {\rm R}(J) \sim J^{-3/4} . \tag{1.45}$$

We conjecture that the leading M2 brane result $\sim g_s^2 J^{-1}$ in (1.37), (1.42) may be reproducing the large J asymptotics of the function ${\rm R}(J)$ corresponding to the spinning circular string state. The terms subleading in large J may be captured by higher-loop M2 brane corrections in (1.40). A resummation of higher loop terms in (1.40) may also produce an imaginary part in the resulting ΔE^2 that should be present in the string theory result but absent in the M2 brane 1-loop correction in (1.37), (1.42).

The rest of this paper is organized as follows. In section 2 we review the 1-loop correction to energy of long folded (S, J) string in $AdS_4 \times CP^3$. Then in section 3 we compute the leading $1/k^2$ M2 brane corrections to its energy or generalized cusp anomaly. In section 4 we present a similar discussion of the circular (S, J) string in $AdS_4 \times CP^3$ while in section 5 we generalize the computation of the 1-loop correction to its energy to the M2 brane case. The flat space case is discussed in section 6. There are also several appendices with some details of computations in the main part.

2 Long folded (S, J) string in $AdS_4 \times CP^3$

In this section we review the structure of the 1-loop correction [9] to the energy of the folded string with one spin S in AdS_4 and one angular momentum J in CP^3. Using the coordinates defined in (A.2), (A.5), the folded string solution in the long string limit (1.20) [22, 36] takes a simple form

$$ t = \kappa\tau, \quad \rho = \mu\sigma, \quad \alpha = 0, \quad \beta = \kappa\tau, \qquad \eta = \nu\tau, \quad \gamma = \tfrac{\pi}{4}, \quad \theta_{1,2} = \tfrac{\pi}{2}, \quad \phi_{1,2} = 0, \tag{2.1}$$

$$ \kappa^2 = \mu^2 + \nu^2, \qquad x \equiv \frac{\mu}{\nu} = \text{fixed}, \qquad \kappa, \nu, \mu \gg 1 . \tag{2.2}$$

[10] Note that since $J = 2\pi T \mathcal{J} = \mathcal{J}/\alpha'$ where $\mathcal{J}$ is a parameter of a spinning string solution, the expansion in $\frac{1}{\Lambda^2} = \frac{g_s^2}{4\pi T \mathcal{J}}$ is analogous to the expansion in $\frac{1}{k^2} = \frac{g_s^2}{2\pi {\rm T}}$ in (1.7) discussed above (cf. (1.14)).

[11] By optical theorem, the imaginary part is related to the decay rate as $\Gamma = \frac{1}{2E_0} {\rm Im}(\Delta M^2)$.

Here $\rho(\sigma)$ periodic in $\sigma \in (0, 2\pi)$ is actually a combination of 4 segments: it is given by $\mu\,\sigma$ for $\sigma \in (0, \frac{\pi}{2})$, by $\mu\,(\pi - \sigma)$ for $\sigma \in (\frac{\pi}{2}, \pi)$, etc. The corresponding energy and the two spins are, in general, defined as[12]

$$(E_0, S, J) = \sqrt{\lambda}\,(\mathcal{E}_0, \mathcal{S}, \mathcal{J})\,, \qquad \mathcal{E}_0 = \frac{\kappa}{2\pi}\int_0^{2\pi} d\sigma \cosh^2\rho = \mathcal{S} + \kappa, \tag{2.3}$$

$$\mathcal{S} = \frac{\kappa}{2\pi}\int_0^{2\pi} d\sigma \sinh^2\rho, \qquad \mathcal{J} = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, \nu = \nu. \tag{2.4}$$

In the $\mu \gg 1$ limit

$$\mathcal{S} = \frac{\kappa}{4\pi\mu} e^{\pi\mu} + \ldots, \qquad \mu = \frac{1}{\pi}\log\mathcal{S} + \ldots, \tag{2.5}$$

and thus from (2.2) we get

$$\mathcal{E}_0 = \mathcal{S} + \mathcal{J}\sqrt{1+x^2} + \ldots = \mathcal{S} + \frac{1}{\pi}\frac{\sqrt{1+x^2}}{x}\log\mathcal{S} + \ldots\,, \tag{2.6}$$

where the coefficient of $\log\mathcal{S}$ corresponds to the leading term in the scaling function in (1.21).

To compute the 1-loop string correction to the energy one needs to start with the quadratic fluctuation action and find the fluctuation frequencies p_0 [9, 22]. Using the static gauge and denoting by $p_1 \in \mathbb{Z}$ the mode number corresponding to the periodic σ-direction we find for the bosonic and fermionic fluctuation frequencies[13]

$$\text{B}: \quad (p_0)_1 = \sqrt{p_1^2 + 2\kappa^2 - \nu^2}, \qquad (p_0)_{2,3} = (p_0)_{4,5} = \pm\tfrac{1}{2}\nu + \sqrt{p_1^2 + \tfrac{1}{4}\nu^2}, \tag{2.7}$$

$$(p_0)_6 = \sqrt{p_1^2 + \nu^2}, \qquad (p_0)_{7,8} = \sqrt{p_1^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + p_1^2\nu^2}}\,, \tag{2.8}$$

$$\text{F}: \quad (p_0)_{1,2} = \sqrt{p_1^2 + \kappa^2}, \qquad (p_0)_{3,4} = \sqrt{p_1^2 + \kappa^2} \pm \nu, \tag{2.9}$$

$$(p_0)_{5,6} = (p_0)_{7,8} = \tfrac{1}{\sqrt{2}}\sqrt{2p_1^2 + \kappa^2 \pm \sqrt{\kappa^4 + 4p_1^2\nu^2}}. \tag{2.10}$$

Then the 1-loop correction to the energy is given by

$$E_1 = \frac{1}{2\kappa}\sum_{p_1=-\infty}^{\infty}\sum_{\{p_0\}}(-1)^{\text{F}}\, p_0\,. \tag{2.11}$$

In the limit $\kappa \gg 1$ the sum over the spatial mode number p_1 may be converted into an integral, and then one obtains the expression (1.22) quoted in the Introduction. Let us note that the

[12]The string tension in (1.6) is given by $\frac{\sqrt{\lambda}}{2\pi}$. Note that while the radii of AdS_4 and CP^3 in (A.7) differ by 2, the $d\eta$ term in (A.5) that determines the angular momentum J has the prefactor $\cos^2\gamma\sin^2\gamma = \frac{1}{4}$ that compensates for this difference which explains why (2.3), (2.4) have the same form as in the similar $AdS_5 \times S^5$ case.

[13]Some frequencies have constant shifts compared to the ones in [9] as we use a different parametrization of the fluctuation fields adapted to our choice of coordinates. These shifts cancel in the contribution to the energy.

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small/large x expansions of E_1 in the AdS_4 and the AdS_5 cases read (cf. (1.22))

$$x \ll 1: \quad E_1^{\mathrm{AdS}_4} = \mathcal{J}\,(-\log 2\; x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{2}\log 2\, x^4 + \tfrac{2}{5}x^5 + \dots), \qquad E_1^{\mathrm{AdS}_5} = \mathcal{J}\,(-\tfrac{4}{3}x^3 + \tfrac{4}{5}x^5 + \dots), \tag{2.12}$$

$$x \gg 1: \quad E_1^{\mathrm{AdS}_4} = \mathcal{J}\,(-\tfrac{5}{2}\log 2\; x + \tfrac{3+2\log 2+4\log x}{4x} + \dots), \qquad E_1^{\mathrm{AdS}_5} = \mathcal{J}\,(-3\log 2\; x + \tfrac{3+4\log x}{2x} + \dots). \tag{2.13}$$

3 1-loop energy of long folded (S, J) M2 brane in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

The above folded string solution has a straightforward uplift to the M2 brane solution in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with the second spatial direction of the membrane wrapped on the 11d circle $\varphi \in (0, 2\pi)$ in (A.3). The values of the classical membrane action in (1.2) (given just by the volume term) and of the conserved charges are then the same as in (2.3), (2.4) (the M2 brane (1.4) and the string tensions are related by (1.6)).

To find the 1-loop M2 brane correction to the energy (2.11) we will use as in [3] the static gauge relating the world-volume coordinates $\xi^i = (\tau, \sigma, \sigma')$ to the target space coordinates t, ρ and φ. The classical value of the induced 3d metric corresponding to the solution (2.1) and $\varphi = \sigma'$ is then (see (A.1), (1.2))

$$\bar g_{ij} = \frac{\mathrm{L}^2}{4}\begin{pmatrix} -\mu^2 & 0 & 0 \\ 0 & \mu^2 & 0 \\ 0 & 0 & \frac{4}{k^2} \end{pmatrix}, \qquad \sqrt{-\bar g} = \frac{\mathrm{L}^3}{4k}\,\mu^2. \tag{3.1}$$

Bosonic fluctuations. The fluctuations of the 8 "transverse" bosonic coordinates (denoted by a hat) will be defined as (cf. (2.1))

$$\alpha = \frac{1}{\sinh(\mu\,\sigma)}\;\hat\alpha(\xi), \qquad \beta = \kappa\tau + \frac{1}{\sinh^2(\mu\,\sigma)}\hat\beta(\xi)\,, \tag{3.2}$$

$$\eta = \nu\tau + \hat\eta(\xi), \qquad \gamma = \tfrac{\pi}{4} + \hat\gamma(\xi), \qquad \theta_{1,2} = \tfrac{\pi}{2} + \hat\theta_{1,2}(\xi), \qquad \phi_{1,2} = \hat\phi_{1,2}(\xi). \tag{3.3}$$

Expanding the volume part of the M2 brane action (1.2) to quadratic order in fluctuations we get

$$S^{(2)}_{\mathrm{V}} = \int d^3\xi \sqrt{-\bar g}\; L^{(2)}_{\mathrm{V}}, \qquad L^{(2)}_{\mathrm{V}} = \tfrac{1}{2}\hat\Phi_p\big[\mathrm{K}_{\mathrm{V}}(\partial_i)\big]_{pq}\hat\Phi_q, \tag{3.4}$$

where M_{V} is a 2nd order differential operator with constant coefficients in the $\mu \gg 1$ limit (cf. in (2.2)). Similarly, expanding the WZ term in (1.3) gives (in the long string limit $\coth(\mu\sigma) \to 1$)

$$\begin{aligned} S^{(2)}_{\mathrm{WZ}} &= -T_2(-\tfrac{3}{8}\mathrm{L}^3)\int \cosh(\mu\sigma)\sinh^2(\mu\sigma)\frac{1}{\sinh(\mu\sigma)}\hat\alpha\, d(\kappa\tau) \wedge (\mu d\sigma) \wedge \frac{1}{\sinh^2(\mu\sigma)}\partial_{\sigma'}\hat\beta\, d\sigma' \\ &\to \tfrac{3}{8}\mathrm{T}_2\kappa\mu\int d\tau d\sigma d\sigma'\;\hat\alpha\partial_{\sigma'}\hat\beta = \int d^3\xi\sqrt{-\bar g}\, L^{(2)}_{\mathrm{WZ}}, \qquad L^{(2)}_{\mathrm{WZ}} = \frac{3k\kappa}{2\mu}\,\hat\alpha\partial_{\sigma'}\hat\beta. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) we get the total kinetic operator $\mathrm{K}(\partial_i) = \mathrm{K_V} + \mathrm{K_{WZ}}$ so that the equation for fluctuations reads

$$\left[\mathrm{K}(\partial_i)+\mathrm{K}^{\mathrm{T}}(-\partial_i)\right]_{pq}\hat{\Phi}_q = 0\,, \qquad \hat{\Phi}_q = \sum_{p_1,p_2\in\mathbb{Z}}\int\frac{dp_0}{2\pi}\,\tilde{\Phi}_q(p_0,p_1,p_2)\,e^{i(p_0\tau+p_1\sigma+p_2\sigma')}\,. \tag{3.6}$$

The characteristic frequencies $p_0 = p_0(p_1,p_2)$ can be found from the equation

$$\mathcal{D}_{\mathrm{B}}(p_0,p_1,p_2) \equiv \det\left[\mathrm{K}(ip_i) + \mathrm{K}^{\mathrm{T}}(-ip_i)\right] = 0\,, \tag{3.7}$$

$$\mathcal{D}_{\mathrm{B}}(p_0,p_1,p_2) = P_8(p_0,p_1,p_2)\prod_{s_1,s_2\in\{-1,1\}}\left[p_0^2 - p_1^2 + s_1 p_0\sqrt{\kappa^2-\mu^2} - \tfrac{1}{4}\mu^2 k p_2(kp_2+2s_2)\right]. \tag{3.8}$$

Here $P_8(p_0,p_1,p_2)$ is a complicated polynomial of degree 8 in p_0 that we do not give explicitly here. In the string theory limit, i.e. for $p_2 = 0$, one recovers the expressions for the bosonic fluctuation frequencies given in (2.7), (2.8).

Fermionic fluctuations. To find the fermionic fluctuation frequencies we will follow the approach described in appendix A of [3]. One defines an orthonormal basis $\mathrm{e}_i = \mathrm{e}_i^M\partial_M$ on the membrane 3d world volume satisfying $\langle \mathrm{e}_i,\mathrm{e}_j\rangle = G_{MN}\,\mathrm{e}_i^M\mathrm{e}_j^N = \eta_{ij}$ (cf. (1.2)). Using the fact that the induced metric (3.1) is diagonal with constant coefficients, it is enough to take e_i^M to be proportional to $\partial_i X^M$, i.e.

$$\mathrm{e}_0 = \frac{2}{\mu}(\kappa\partial_t + \kappa\partial_\beta + \nu\partial_\eta), \qquad \mathrm{e}_1 = \frac{2}{\mu}\partial_\rho, \qquad \mathrm{e}_2 = k\partial_\varphi. \tag{3.9}$$

An orthonormal basis in the normal bundle can be chosen as

$$\begin{gathered}\mathrm{n}_1 = \frac{2}{\sinh\rho}\partial_\alpha, \qquad \mathrm{n}_2 = 2(\tanh\rho\;\partial_\rho + \coth\rho\;\partial_\beta), \qquad \mathrm{n}_3 = \frac{2}{\mu}(\nu\partial_\rho + \nu\partial\beta + \kappa\partial_\eta),\\ \mathrm{n}_4 = \partial_\gamma, \qquad \mathrm{n}_5 = 2\sqrt{2}\partial_{\theta_1}, \qquad \mathrm{n}_6 = 2\sqrt{2}\partial_{\theta_2}, \qquad \mathrm{n}_7 = 2\sqrt{2}\partial_{\phi_1}, \qquad \mathrm{n}_8 = 2\sqrt{2}\partial_{\phi_2}.\end{gathered} \tag{3.10}$$

One can introduce a dual basis satisfying $\mathrm{e}^i(\mathrm{e}_j) = \delta^i_j$, $\mathrm{n}^p(\mathrm{n}_q) = \delta^p_q$, $\mathrm{n}^p(\mathrm{e}_i) = \mathrm{e}^i(\mathrm{n}_p) = 0$. We define the set of 11d gamma matrices with respect to the orthonormal frame (3.9), (3.10), i.e. we introduce $\Gamma_A = (\rho_i, \gamma_p)$,[14]

$$\{\rho_i,\rho_j\} = 2\eta_{ij}\mathrm{I}_{32}\,, \qquad i = 0,1,2; \qquad \{\gamma_p,\gamma_q\} = 2\delta_{pq}\mathrm{I}_{32}\,, \qquad p = 1,\ldots,8. \tag{3.11}$$

Assuming the κ-symmetry gauge $(1+\Gamma)\theta = 0$ the quadratic term in the fermionic part of the M2 brane action may be written as ($\rho^i = \eta^{ij}\rho_j$)

$$S_{\mathrm{F}} = T_2\int d^3\xi\sqrt{-g}\,\bar{\theta}(1-\Gamma)\rho^i D_{e_i}\theta, \qquad \Gamma = \tfrac{1}{3!}\epsilon^{ijk}\rho_i\rho_j\rho_k\,. \tag{3.12}$$

The corresponding Dirac operator is

$$\not{D} = \rho^i D_{e_i} = \rho^i\Big[\nabla_{e_i} + \tfrac{1}{12}\big(\rho_i\mathrm{F}_4 - 3\mathrm{F}_{4,i}\big)\Big], \qquad \nabla_{e_i} = \partial_{e_i} + \tfrac{1}{4}\Omega_i^{AB}\Gamma_{AB}, \tag{3.13}$$

$$\mathrm{F}_4 = \tfrac{1}{4!}F_{ABCD}\,\Gamma^{ABCD}, \qquad \mathrm{F}_{4,i} = \tfrac{1}{3!}F_{iBCD}\,\Gamma^{BCD}, \tag{3.14}$$

[14] Here A is the 11d tangent space index; for the explicit form of the matrices see appendix B.

where $\Omega^{AB}(e_i)$ is the spin connection on $AdS_4 \times S^7/\mathbb{Z}_k$. Using the frame (3.9), (3.10), we get

$$\not{D} = \rho^i \nabla^{\perp}_{e_i} + \frac{3\kappa}{2\mu}\rho^1\rho^2\gamma^1\gamma^2, \qquad \nabla^{\perp}_{e_i} = \partial_{e_i} + \tfrac{1}{4}\Omega^{pq}_i \gamma_{pq}, \tag{3.15}$$

with the normal bundle connection $\Omega^{pq}_i = \langle \mathrm{n}^p, \nabla_{e_i}\mathrm{n}^q\rangle$ given in the appendix B.[15]

The resulting determinant of the fermionic operator computed for a single Fourier mode as in (3.7) has the following factorized structure

$$\mathcal{D}_{\rm F}(p_0,p_1,p_2) = P_4(p_0,p_1,p_2)\ P_4(-p_0,p_1,p_2)\ \tilde{P}_4(p_0,p_1,p_2)\ \tilde{P}_4(p_0,p_1,-p_2), \tag{3.16}$$

$$\begin{aligned} P_4(p_0,p_1,p_2) = {}& p_0^4 - 2\nu p_0^3 + p_0^2(-2\kappa^2+\nu^2-2p_1^2-\tfrac{1}{2}k^2\mu^2p_2^2) + \tfrac{1}{2}\nu p_0(4\kappa^2+4p_1^2+k^2\mu^2p_2^2) \\ & + p_1^4 + \tfrac{1}{2}p_1^2(4\kappa^2-2\nu^2+k^2\mu^2p_2^2) + \tfrac{1}{16}\mu^2(16\kappa^2-8k^2\kappa^2p_2^2+k^4\mu^2p_2^4), \end{aligned} \tag{3.17}$$

$$\tilde{P}_4(p_0,p_1,p_2) = p_0^4 - p_0^2(2p_1^2+\kappa^2+\tfrac{1}{2}\mu^2kp_2(kp_2-2)) + (p_1^2+\tfrac{1}{4}\mu^2k^2p_2^2)(p_1^2+\tfrac{1}{4}\mu^2(kp_2-2)^2).$$

The characteristic frequencies $p_0(p_1,p_2)$ are found by solving $\mathcal{D}_{\rm F}(p_0,p_1,p_2)=0$. In the string theory limit $p_2=0$ they reduce to the ones given in (2.9), (2.10).

1-loop correction to energy. The 1-loop correction to the energy generalizing (2.11) to the M2 brane case can be represented in terms of the determinants in (3.7) and (3.15) as (see, e.g., [3, 37])

$$\mathrm{E}_1 = \frac{1}{2\kappa}\sum_{p_1,p_2\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{dw}{2\pi} \log\frac{\mathcal{D}_{\rm B}(-iw,p_1,p_2)}{\mathcal{D}_{\rm F}(-iw,p_1,p_2)} = E_1 + E_1^{\rm M2}\,. \tag{3.18}$$

Here the $p_2=0$ term in the sum represents the string mode contribution E_1 in (2.11), (1.22), while the terms with $p_2 \neq 0$ give the additional M2 brane mode contribution denoted as $E_1^{\rm M2}$ as in (1.28). We can evaluate this part in the large k limit by converting the sum over p_1 into an integral in the long string limit, by rescaling w and p_1 by k and then expanding in large k. This yields

$$\begin{aligned} E_1^{\rm M2}(x,\mathcal{J},k) &= \frac{k^2}{2\pi\kappa}\sum_{p_2=1}^{\infty}\int_{-\infty}^{\infty} dw' \int_{-\infty}^{\infty} dp_1' \log\frac{\mathcal{D}_{\rm B}(-ikw',kp_1',p_2)}{\mathcal{D}_{\rm F}(-ikw',kp_1',p_2)} \\ &= \sum_{p_2=1}^{\infty}\left[\mathcal{C}_2\frac{1}{(kp_2)^2} + \mathcal{C}_4\frac{1}{(kp_2)^4} + \dots\right] = \mathcal{C}_2\frac{\zeta(2)}{k^2} + \mathcal{C}_4\frac{\zeta(4)}{k^4} + \dots, \end{aligned} \tag{3.19}$$

where

$$\mathcal{C}_2 = \frac{2(2\kappa^2+3\nu^2)}{\kappa}, \qquad \mathcal{C}_4 = \frac{8\kappa^2+76\kappa^2\nu^2-3\nu^4}{2\kappa(\kappa^2-\nu^2)}. \tag{3.20}$$

For some higher order $\mathcal{C}_{2n}$ terms see appendix C. Expressing these coefficients in terms of $\mathcal{J}=\nu$ and x, we get

$$\mathcal{C}_{2n} = \mathcal{J}\, C_{2n}(x), \qquad C_2(x) = \frac{2(5+2x^2)}{\sqrt{1+x^2}} = 4\sqrt{1+x^2} + \frac{6}{\sqrt{1+x^2}}, \tag{3.21}$$

$$C_4(x) = \frac{81+92x^2+8x^4}{2x^2\sqrt{1+x^2}} = 4\frac{(1+x^2)^{3/2}}{x^2} + 38\frac{\sqrt{1+x^2}}{x^2} - \frac{3}{2}\frac{1}{x^2\sqrt{1+x^2}}. \tag{3.22}$$

[15] As in the cases discussed in appendix A of [3], the membrane metric (3.1) is flat, implying $\Omega^{ij}_k = 0$. Also, since the classical solution represents a minimal surface (WZ term does not contribute at the classical level) the term involving $\rho^i\rho^j\Omega_{jp,i}$ vanishes.

Since $\mathcal{J}x = \frac{\log \mathcal{S}}{\pi}$ (see (1.20)) the expression (3.19) gives the $1/k$ corrections to the generalized cusp anomaly $q_0(x,k)$ in (1.13) (1.24) already quoted in (1.24)

$$q_0(x,k) = -\frac{5\log 2}{2\pi} + \frac{C_2(x)}{6x}\frac{\pi}{k^2} + \frac{C_4(x)}{90x}\frac{\pi^3}{k^4} + \dots . \tag{3.23}$$

The small/large x expansions of (3.21), (3.22) are

$$x \ll 1: \quad C_2(x) = 10 - x^2 + \tfrac{7}{4}x^4 + \dots, \qquad C_4(x) = \tfrac{81}{2x^2} + \tfrac{103}{4} - \tfrac{61}{16}x^2 + \dots, \tag{3.24}$$

$$x \gg 1: \quad C_2(x) = 4x + \tfrac{8}{x} + \dots, \qquad C_4(x) = 4x + \tfrac{44}{x} + \dots . \tag{3.25}$$

4 Circular $(\boldsymbol{S}, \boldsymbol{J})$ string in $\mathbf{AdS_4} \times \mathbf{CP^3}$

Let us now consider the circular string solution with spins S and J in $AdS_4 \times CP^3$ which was discussed in [11] (for the analogous solution in $AdS_5 \times S^5$ see [24, 25]). Using the coordinates in (A.2), (A.5) it is described by

$$t = \kappa\,\tau, \qquad \rho = \rho_* = \text{const}, \qquad \alpha = 0, \qquad \beta = w\,\tau + \mathrm{n}\,\sigma, \qquad \eta = \omega\,\tau + \mathrm{m}\,\sigma, \tag{4.1}$$

with other angles being trivial and n and m being integer winding numbers. The conformal gauge (Virasoro) constraints read ($r_0 = \cosh\rho_*$, $r_1 = \sinh\rho_*$, $r_0 = \sqrt{r_1^2+1}$)

$$w^2 - (\kappa^2 + \mathrm{n}^2) = 0, \qquad r_1^2 w\mathrm{n} + \omega\mathrm{m} = 0, \qquad -r_0^2\kappa^2 + r_1^2(w^2+\mathrm{n}^2) + \omega^2 + \mathrm{m}^2 = 0. \tag{4.2}$$

The three conserved charges are as in (2.3) given by

$$(E_0, S, J) = \sqrt{\bar\lambda}\,(\mathcal{E}_0, \mathcal{S}, \mathcal{J}), \qquad \mathcal{E}_0 = r_0^2\,\kappa, \qquad \mathcal{S} = r_1^2\,w, \qquad \mathcal{J} = \omega. \tag{4.3}$$

The second constraint in (4.2) implies that

$$\mathrm{n}\,S + \mathrm{m}\,J = 0. \tag{4.4}$$

In the following we will set

$$\mathrm{n} = 1, \qquad u \equiv \frac{\mathcal{S}}{\mathcal{J}} = -\mathrm{m}\,. \tag{4.5}$$

The dependence on n can be restored by rescaling of $\mathcal{J}$ and we will formally treat u as a continuous parameter. The solution with minimal energy for given values of spins corresponds to $u = -\mathrm{m} = 1$. The conditions (4.2) can be solved in terms of independent parameters u, r_1 as

$$\omega = r_1^2\frac{\sqrt{1+2r_1^2+u^2}}{\sqrt{u^2-r_1^4}}, \qquad w = \frac{u\omega}{r_1^2} = u\frac{\sqrt{1+2r_1^2+u^2}}{\sqrt{u^2-r_1^4}}, \qquad \kappa = \frac{r_1^2+u^2}{\sqrt{u^2-r_1^4}}, \tag{4.6}$$

so that

$$\mathcal{E}_0 = \frac{(1+r_1^2)(r_1^2+u^2)}{\sqrt{u^2-r_1^4}}. \tag{4.7}$$

In the case of $S = J$ or $u = 1$ one finds explicitly $r_1^2 = \frac{1}{4}\mathcal{J}(\sqrt{8+\mathcal{J}^2} - \mathcal{J})$ and thus

$$u = 1: \qquad \mathcal{E}_0(\mathcal{J}) = \sqrt{1 + \tfrac{5}{2}\mathcal{J}^2 - \tfrac{1}{8}\mathcal{J}^4 + \tfrac{1}{8}\mathcal{J}\sqrt{(8+\mathcal{J}^2)^3}}\,. \tag{4.8}$$

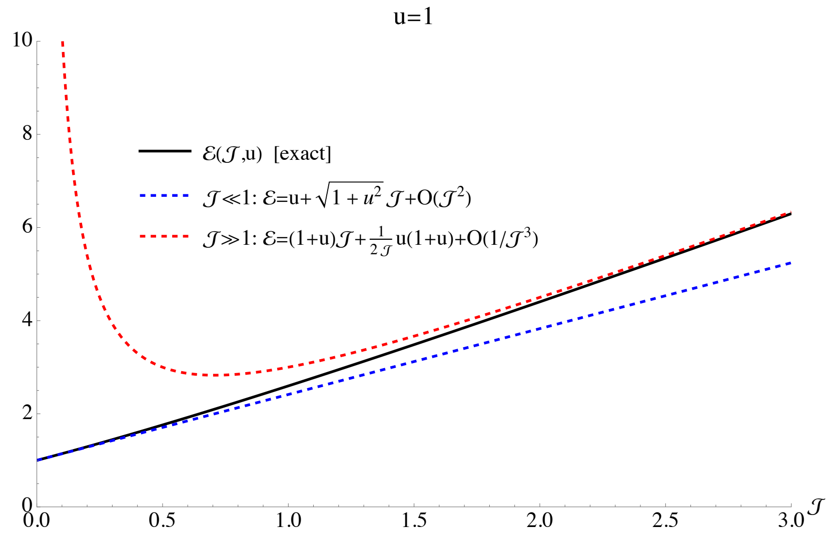

Figure 1. Comparison of the exact expression $\mathcal{E}_0(\mathcal{J}, u = 1)$ with its small and large $\mathcal{J}$ expansions.

For general u one can expand $\mathcal{E}_0$ at large or small $\mathcal{J}$, corresponding to $r_1 \to \sqrt{u}$ or $r_1 \to 0$:

$$\mathcal{J}\gg 1: \quad r_1^2 = u - \frac{1}{2}u(1+u)^2\frac{1}{\mathcal{J}^2} + \frac{1}{8}u(1+u)^2(3+u)(1+3u)\frac{1}{\mathcal{J}^4} + \ldots, \tag{4.9}$$

$$\mathcal{E}_0(\mathcal{J},u) = (1+u)\mathcal{J} + \frac{1}{2\mathcal{J}}u(1+u) - \frac{1}{8\mathcal{J}^3}u(1+u)(1+3u+u^2) + \ldots, \tag{4.10}$$

$$\mathcal{J}\ll 1: \quad r_1^2 = \frac{u}{\sqrt{1+u^2}}\mathcal{J} - \frac{u^2}{(1+u^2)^2}\mathcal{J}^2 - \frac{u(1-3u^2+u^4)}{2(1+u^2)^{7/2}}\mathcal{J}^3 + \frac{2u^2(-1+u^2)^2}{(1+u^2)^5}\mathcal{J}^4 + \ldots, \tag{4.11}$$

$$\mathcal{E}_0(\mathcal{J},u) = u + \sqrt{1+u^2}\,\mathcal{J} + \frac{u}{2(1+u^2)}\mathcal{J}^2 - \frac{u^2}{2(1+u^2)^{5/2}}\mathcal{J}^3 + \ldots. \tag{4.12}$$

The expression in (4.10) starts with the familiar "fast-string" term $\mathcal{E}_0 = (1+u)\mathcal{J} + \cdots = \mathcal{S} + \mathcal{J} + \ldots$, while the first term in (4.12) $\mathcal{E}_0(\mathcal{J}, u) = u + \ldots = -\mathrm{m} + \ldots$ is the contribution to the energy due to the string winding the circle. A comparison of (4.10) and (4.12) for $u = 1$ with the exact expression for $\mathcal{E}_0(\mathcal{J})$ in (4.8) is illustrated in figure 1. To compute the 1-loop correction to the energy one needs to combine the fluctuation frequencies $p_0 = p_0(p_1)$ as in (2.11). Fixing the static gauge on t and η one finds as in [11] that[16]

$$\mathrm{CP}^3: \quad (p_0)_{1,2,3,4} = \sqrt{p_1^2 + \tfrac{1}{4}(\mathcal{J}^2 - u^2)}, \qquad (p_0)_5 = \sqrt{p_1^2 + \mathcal{J}^2 - u^2}, \tag{4.13}$$

$$\mathrm{AdS}_4: \quad p_0 = \sqrt{p_1^2 + \kappa^2}, \qquad (p_0^2 - p_1^2)^2 + 4r_1^2\kappa^2 p_0^2 - 4(1+r_1^2)\left(p_0\sqrt{\kappa^2+1} - p_1\right)^2 = 0, \tag{4.14}$$

where two out of the total three AdS_4 frequencies are given by the positive solutions of the quartic equation in (4.14).

[16] As in the folded string case discussed above, we note that different parametrizations of fluctuations may introduce constant shifts of p_0 but these do not affect the 1-loop correction to the energy.

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The fermionic spectrum contains four different frequencies, each being doubly-degenerate. Two are given by [11]

$$\mathrm{F}: \quad p_0 = \pm\frac{r_0^2\kappa u}{2(u^2+r_1^2)} + \sqrt{(p_1\pm b)^2+\mathcal{J}^2+r_1^2}, \qquad b \equiv -\frac{\kappa u}{w}\frac{w^2-\mathcal{J}^2}{2(u^2+r_1^2)}, \tag{4.15}$$

and two are the solutions of the equation which is very similar to the one in (4.14)

$$\mathrm{F}: \quad (p_0^2-p_1^2)^2 + r_1^2\kappa^2p_0^2 - (1+r_1^2)\left(p_0\sqrt{\kappa^2+1}-p_1\right)^2 = 0\,. \tag{4.16}$$

The resulting 1-loop correction E_1 is given by the sum (2.11). It can be computed explicitly in the large $\mathcal{J}$ expansion (cf. (1.33), (1.34)), where the "odd" term part is related (1.19) to the corresponding $\mathrm{AdS}_5\times S^5$ expression (1.33). The small $\mathcal{J}$ expansion of E_1 was not studied previously and we present it in appendix D.

5 1-loop energy of circular (S,J) M2 brane in $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$

Let us now consider the direct uplift of the circular string solution of the previous section to the M2 brane solution in $\mathrm{AdS}_4\times S^7/\mathbb{Z}_k$ where in addition to (4.1) we set $\varphi=\sigma'$ as in section 3. Using (4.6) the resulting induced metric may be written as (cf. (3.1)).

$$\bar{g}_{ij} = \frac{\mathrm{L}^2}{4}\begin{pmatrix} -(r_1^2+u^2) & 0 & 0\\ 0 & r_1^2+u^2 & 0\\ 0 & 0 & \frac{4}{k^2}\end{pmatrix}, \qquad \sqrt{-\bar{g}} = \frac{\mathrm{L}^3}{4k}(r_1^2+u^2). \tag{5.1}$$

The classical conserved charges are the same as in (4.3) (WZ term in (1.3) does not contribute at the classical level). Choosing the static gauge where t, β and φ do not fluctuate we have ($\mathrm{n}=1$)

$$t=\kappa\tau, \qquad \rho=\rho_*+\hat\rho(\xi), \qquad \alpha=\hat\alpha(\xi), \qquad \beta=w\,\tau+\sigma, \tag{5.2}$$

$$\eta=\omega\tau+\mathrm{m}\,\sigma+\hat\eta(\xi), \quad \gamma=\tfrac{\pi}{4}+\hat\gamma(\xi), \quad \theta_{1,2}=\tfrac{\pi}{2}+\hat\theta_{1,2}(\xi), \quad \phi_{1,2}=\hat\phi_{1,2}(\xi), \qquad \varphi=\sigma'\,. \tag{5.3}$$

Bosonic fluctuations. The expansion of the volume part of the M2 brane action gives the quadratic fluctuation Lagrangian of the same form as in (3.4) while the analog of (3.5) is

$$S^{(2)}_{\mathrm{WZ}} = -T_2(-\tfrac{3}{8}\mathrm{L}^3)\int\cosh\rho_*\sinh^2\rho_*\,\hat\alpha\,d(\kappa\tau)\wedge d\hat\rho\wedge d(w\,\tau+\sigma) = \tfrac{3}{8}\mathrm{T}_2r_1^2r_0\kappa\int d\tau d\sigma d\sigma'\;\hat\alpha\partial_{\sigma'}\hat\rho,$$

$$L^{(2)}_{\mathrm{WZ}} = \frac{3}{2}\frac{kr_1^2r_0\kappa}{r_1^2+u^2}\,\hat\alpha\partial_{\sigma'}\hat\rho. \tag{5.4}$$

The bosonic fluctuation frequencies are determined by solving the analog of eq. (3.7). In the string theory limit ($p_2=0$) one finds for the characteristic polynomial

$$\begin{aligned}\mathcal{D}_{\mathrm{B}}(p_0,p_1,0) = {}&(p_0^2-p_1^2-\kappa^2)(p_0^2-p_1^2-\omega^2+u^2)\\ &\times\left[\left(p_0+\tfrac{1}{2}\omega\right)^2-\left(p_1-\tfrac{1}{2}u\right)^2-\tfrac{1}{4}\left(\omega^2-u^2\right)\right]^2\left[\left(p_0-\tfrac{1}{2}\omega\right)^2-\left(p_1+\tfrac{1}{2}u\right)^2-\tfrac{1}{4}\left(\omega^2-u^2\right)\right]^2\\ &\times\left[(p_0^2-p_1^2)^2+4r_1^2\kappa^2p_0^2-4(1+r_1^2)\left(\sqrt{1+\kappa^2}p_0-p_1\right)^2\right].\end{aligned} \tag{5.5}$$

Then the solutions of $\mathcal{D}_{\rm B}(p_0, p_1, 0) = 0$ are equivalent to the frequencies in (2.7), (2.8) (up to constant shifts related to the choice of parametrization of fluctuations). For generic p_2 we get

$$\mathcal{D}_{\rm B}(p_0, p_1, p_2) = P_4(p_0, p_1, p_2)\, P_4(-p_0, -p_1, +p_2)\, P_8(p_0, p_1, p_2), \tag{5.6}$$

where P_r stands for a polynomial of degree r in p_0. From the zeroes of the two P_4 factors we get

$$p_0 = \tfrac{1}{2}\left[s_1\omega \pm \sqrt{4p_1^2 + 4s_2 u p_1 + \omega^2 + (r_1^2 + u^2)kp_2(kp_2 + 2s_3)}\ \right], \tag{5.7}$$

with $s_{1,2,3} = \pm 1$ (8 frequencies in total). The zeros of P_8 cannot be easily written in a closed form but can be found in an expansion in large $\mathcal{J}$ (see appendix E) or small $\mathcal{J}$.

Fermionic fluctuations. The determination of fermionic frequencies is analogous to the case of the long folded M2 solution in section 3. The orthonormal basis in the tangent and normal bundles may be chosen as (cf. (3.9))

$$\mathrm{e}_0 = \frac{2}{\mathrm{v}}(\kappa\partial_t + w\partial_\beta + \omega\partial_\eta), \qquad \mathrm{e}_1 = \frac{2}{\mathrm{v}}(\partial_\beta + \mathrm{m}\partial_\eta), \qquad \mathrm{e}_2 = k\partial_\varphi, \qquad \mathrm{v} \equiv \sqrt{r_1^2 + u^2}, \tag{5.8}$$

$$\mathrm{n}_1 = 2\partial_\rho, \qquad \mathrm{n}_2 = \frac{2\omega}{r_0 r_1}\partial_t + \frac{2\kappa u r_0}{\mathrm{v}^2 r_1}\partial_\beta + \frac{2\kappa r_1 r_0}{\mathrm{v}^2}\partial_\eta, \qquad \mathrm{n}_3 = \frac{2}{r_1}\partial_\alpha, \qquad \mathrm{n}_4 = \partial_\gamma,$$

$$\mathrm{n}_5 = 2\sqrt{2}\partial_{\theta_1}, \qquad \mathrm{n}_6 = 2\sqrt{2}\partial_{\theta_2}, \qquad \mathrm{n}_7 = 2\sqrt{2}\partial_{\phi_1}, \qquad \mathrm{n}_8 = 2\sqrt{2}\partial_{\phi_2}. \tag{5.9}$$

The analog of the Dirac operator in the κ-fixed quadratic fermionic action reads (cf. (3.13), (3.15))

$$\not{D} = \rho^i \nabla^\perp_{e_i} + \frac{3}{2}\frac{\kappa r_0 r_1}{\mathrm{v}^2}\rho^0\rho^1\gamma^1\gamma^3, \qquad \nabla^\perp_{e_i} = \partial_{e_i} + \frac{1}{4}\Omega_i^{pq}\gamma_{pq}, \tag{5.10}$$

with the spin connection presented in appendix B (see (B.3)). The zeros of the corresponding determinant for one Fourier mode $\mathcal{D}_{\rm F}(p_0, p_1, p_2)$ give $p_0 = p_0(p_1, p_2)$. In the string limit $p_2 = 0$ we get ($s_{1,2} = \pm 1$)

$$p_0 = s_1 \frac{r_0^2 \kappa u}{2(u^2 + r_1^2)} + \tfrac{1}{2}s_2\omega + \sqrt{\left(p_1 + s_1 b + \tfrac{1}{2}s_2 u\right)^2 + (\omega^2 + r_1^2)}, \qquad b = -\frac{\kappa u}{w}\frac{w^2 - \omega^2}{2(u^2 + r_1^2)}, \tag{5.11}$$

which are the shifted versions of those in (4.15), and also find the fermionic frequencies in (4.16) (with no shifts).

1-loop correction to energy. The 1-loop correction to the M2 brane energy is given again by (3.18), (3.19). The expressions for the few leading $1/k^2$ coefficients in (3.19) are

$$\mathcal{C}_2 = \frac{2(5r_1^2 + 8r_1^4 - 3u^2)}{\sqrt{u^2 - r_1^4}}, \qquad \mathcal{C}_4 = \frac{232r_1^6 + 148r_1^8 - 70r_1^2u^2 - 3u^4 + r_1^4(81 - 64u^2)}{2(u^2 - r_1^4)^{3/2}}, \tag{5.12}$$

$$\mathcal{C}_6 = \frac{2}{15(u^2 - r_1^4)^{5/2}}\Big[-175u^6 + u^4r_1^2(1192 + 1717r_1^2) - u^2r_1^4(3751 + 9886r_1^2 + 6660r_1^4)$$
$$+ r_1^6(1926 + 9529r_1^2 + 14472r_1^4 + 7044r_1^6)\Big]. \tag{5.13}$$

Expanding in large/small $\mathcal{J}$ for fixed u we find (making use of (4.9), (4.11))

$$\begin{aligned}
\mathcal{J}\gg 1:\quad & \mathcal{C}_2=10\mathcal{J}-\frac{u(6+11u)}{\mathcal{J}}+\frac{u(12+68u+88u^2+27u^3)}{4\mathcal{J}^3}+\dots,\\
&\mathcal{C}_4=\frac{81}{2u(1+u)}\mathcal{J}^3+\frac{81+103u-221u^2}{4u(1+u)}\mathcal{J}-\frac{81+729u+1077u^2+173u^3-499u^4}{16u(1+u)}\frac{1}{\mathcal{J}}+\dots,\\
&\mathcal{C}_6=\frac{1284}{5u^2(1+u)^2}\mathcal{J}^5-\frac{2(-1926-2027u+4714u^2)}{15u^2(1+u)^2}\mathcal{J}^3+\dots,
\end{aligned}\tag{5.14}$$

$$\begin{aligned}
\mathcal{J}\ll 1:\quad & \mathcal{C}_2=-6u+\frac{10}{\sqrt{1+u^2}}\mathcal{J}+\frac{u(3+13u^2)}{(1+u^2)^2}\mathcal{J}^2+\dots,\\
&\mathcal{C}_4=-\frac{3}{2}u-\frac{35}{\sqrt{1+u^2}}\mathcal{J}+\frac{162+165u^2-137u^4}{4u(1+u^2)^2}\mathcal{J}^2+\dots,\\
&\mathcal{C}_6=-\frac{70}{3}u+\frac{2384}{15\sqrt{1+u^2}}\mathcal{J}+\frac{-7502-7327u^2+2559u^4}{15u(1+u^2)^2}\mathcal{J}^2+\dots.
\end{aligned}\tag{5.15}$$

In appendix F we present the generating function for the coefficients of the leading $\sim u\mathcal{J}^0$ terms in the small $\mathcal{J}$ expansion generalizing those given in (5.15).

In the special case of $\mathcal{S}=\mathcal{J}$, i.e. $u=1$ we get explicitly

$$\begin{aligned}
u=1:\qquad E_1^{\rm M2} &= \Big(-6+5\sqrt{2}\,\mathcal{J}+4\mathcal{J}^2+\dots\Big)\frac{\zeta(2)}{k^2}+\Big(-\tfrac{3}{2}-\tfrac{35}{\sqrt{2}}\mathcal{J}+\tfrac{95}{8}\mathcal{J}^2+\dots\Big)\frac{\zeta(4)}{k^4}+\dots\\
&=\left[-\left(\frac{\pi}{k}\right)^2-\tfrac{1}{60}\left(\frac{\pi}{k}\right)^4+\dots\right]+\left[\tfrac{5}{3\sqrt{2}}\left(\frac{\pi}{k}\right)^2-\tfrac{7}{18\sqrt{2}}\left(\frac{\pi}{k}\right)^4+\dots\right]\mathcal{J}\\
&\quad+\left[\tfrac{2}{3}\left(\frac{\pi}{k}\right)^2+\tfrac{19}{244}\left(\frac{\pi}{k}\right)^4+\dots\right]\mathcal{J}^2+\mathcal{O}(\mathcal{J}^3).
\end{aligned}\tag{5.16}$$

6 1-loop correction to semiclassical M2 brane energy in flat space

As discussed in the Introduction, similar semiclassical 1-loop computations can be done for M2 branes in flat 11d background.

Expanding M2 brane action (1.2) in flat background near a classical solution, the action for quadratic fluctuations for 8 "transverse" bosonic and fermionic fluctuations may be written as (see, e.g., appendix A in [3])[17]

$$S_{\rm B}=\frac{1}{2}\int d^3\xi\sqrt{-\bar{g}}\,\bar{g}^{ij}\Big(\langle\nabla_i^{\perp}{\rm X},\nabla_j^{\perp}{\rm X}\rangle-\langle\mathcal{K}^{ij},{\rm X}\rangle\langle\mathcal{K}_{ij},{\rm X}\rangle\Big),\tag{6.1}$$

$$S_{\rm F}=\int d^3\xi\sqrt{-\bar{g}}\,\bar{\theta}(1-\Gamma)\nabla\!\!\!\!/\,\theta,\qquad \Gamma=\tfrac{1}{3!}\epsilon^{ijk}\rho_i\rho_j\rho_k,\quad \nabla\!\!\!\!/=\rho^i\nabla_{{\rm e}_i},\quad \nabla_{{\rm e}_i}=\partial_{{\rm e}_i}+\tfrac{1}{4}\Omega_i^{AB}\Gamma_{AB}.\tag{6.2}$$

Here $\bar{g}_{ij}$ is the classical induced metric, $\nabla^{\perp}$ is the connection on the normal bundle, ${\rm e}_i$ denotes a local orthonormal frame tangent to the M2 brane surface and $\mathcal{K}_{ij}$ is the extrinsic curvature. The latter is given by $\mathcal{K}_{ij}=({\rm I}-{\rm P})(\partial_i\partial_j X)$ where P is the projector on the tangent bundle of the classical M2 brane surface.

[17] We denote by X the classical solution and by X the corresponding 3d fluctuations.

For example, the M2 brane solution generalizing the circular string rotating in two planes and wrapped on the 11d circle is given by ($\xi^i = (\tau, \sigma, \sigma')$; cf. (1.38))

$$X^0 = \kappa\,\tau, \qquad X^1 + iX^2 = \tfrac{1}{2}\kappa\, e^{i(\tau+\sigma)}, \qquad X^3 + iX^4 = \tfrac{1}{2}\kappa\, e^{i(\tau-\sigma)}, \qquad X^{10} = R_{11}\sigma', \tag{6.3}$$

$$\bar{g}_{ij} = \mathrm{diag}\big(- \tfrac{1}{2}\kappa^2,\ \tfrac{1}{2}\kappa^2,\ R_{11}^2 \big)\,, \tag{6.4}$$

where $\sigma' \in (0, 2\pi)$ and $\bar{g}_{ij}$ is the induced 3d metric. The M2 brane energy and spins are given by

$$E_0 = \frac{\kappa R_{11}}{\ell_P^3} = \alpha'^{-1}\kappa, \qquad J_1 = J_2 \equiv J = \frac{\kappa^2 R_{11}}{4\,\ell_P^3} = \tfrac{1}{4}\alpha'^{-1}\kappa^2, \qquad E_0 = 2\sqrt{\alpha'^{-1}J}\,, \tag{6.5}$$

where we used (1.39). Considering quadratic fluctuations near this solution (for details see appendix G) one finds that the 1-loop correction to the energy can be expressed in terms of the determinants of the bosonic and fermionic fluctuation operators as in (3.18) where (p_1, p_2 are again the integer mode numbers)

$$\begin{aligned}
\mathcal{D}_\mathrm{B}(-iw,p_1,p_2) = {}& \big(\Lambda^2p_2^2+p_1^2+w^2\big)^5\Big[\Lambda^6p_2^6+3\Lambda^4p_2^4p_1^2-16\Lambda^2p_2^2+3\Lambda^2p_2^2p_1^4-8\Lambda^2p_2^2p_1^2\\
&+w^4\big(3\Lambda^2p_2^2+3p_1^2+8\big)-w^2\big(-3\Lambda^4p_2^4-8\Lambda^2p_2^2-6\Lambda^2p_2^2p_1^2-3p_1^4-16\big)\\
&+p_1^6-8p_1^4+16p_1^2+w^6\Big],\\
\mathcal{D}_\mathrm{F}(-iw,p_1,p_2) = {}& \Big[\Lambda^4p_2^4+2p_1^2\big(\Lambda^2p_2^2+w^2-1\big)+2w^2\big(\Lambda^2p_2^2+1\big)+p_1^4+w^4+1\Big]^4,
\end{aligned} \tag{6.6}$$

$$\Lambda^2 \equiv \frac{\kappa^2}{2R_{11}^2} = \frac{2J}{g_s^2}. \tag{6.7}$$

One finds that the string-mode ($p_2 = 0$) contribution vanishes. Using the integral approximation for the sum over p_1 for $p_2 \neq 0$ we then get for the M2 brane mode contribution (cf. (1.41); $\kappa = \alpha' E_0$)

$$\mathrm{E}_1 = \frac{1}{\kappa}\bar{\mathrm{E}}_1\,, \qquad \bar{\mathrm{E}}_1 = \sum_{p_2=1}^{\infty}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty}\frac{dw}{2\pi}\log\frac{\mathcal{D}_\mathrm{B}(-iw,p_1,p_2)}{\mathcal{D}_\mathrm{F}(-iw,p_1,p_2)}\,. \tag{6.8}$$

This can be expanded in the parameter $\Lambda \gg 1$ after rescaling $w \to \Lambda w$, $p_1 \to \Lambda p_1$. As a result,

$$\bar{\mathrm{E}}_1(\Lambda) = \sum_{p_2=1}^{\infty} F(p_2\Lambda) = \sum_{p_2=1}^{\infty}\Big(-\frac{4}{p_2^2\Lambda^2} - \frac{152}{15p_2^6\Lambda^6} + \dots\Big) = -\frac{4\,\zeta(2)}{\Lambda^2} - \frac{152\,\zeta(6)}{15\Lambda^6} + \dots\,, \tag{6.9}$$

where the function $F(x)$ has the following integral representation

$$\begin{aligned}
F(x) = \tfrac{1}{2}x^2\int_0^\infty dt\,\Bigg[&-3 + \frac{6}{1+t} + \frac{4x^4(t^2-1)-4}{\sqrt{x^8(1+t)^4+2x^4(1+2t-t^2)+1}}\\
&+\frac{x^4(1-t)(1+t)^3-16(1+2t-t^2)}{\sqrt{x^8(1+t)^8-32x^4(1+t)^4(1+t^2)+256(1-t^2)^2}}\Bigg].
\end{aligned} \tag{6.10}$$

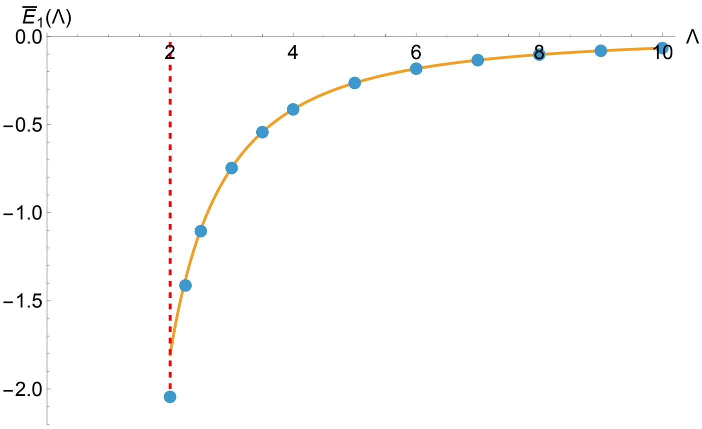

Figure 2. The values of $\bar{\mathrm{E}}_1(\Lambda)$ for $\Lambda \geqslant 2$ found by numerical evaluation of the sum in (6.9) (blue circles). The orange line represents the sum of the first two terms of the large Λ expansion of $\bar{\mathrm{E}}_1(\Lambda)$.

Its asymptotic expansion for $x \to \infty$ is given by (cf. (1.42))

$$F(x) = \sum_{n=1}^{\infty} \frac{c_n}{x^{2n-4}} = -\frac{4}{x^2} - \frac{152}{15}\frac{1}{x^6} - \frac{2496}{35}\frac{1}{x^{10}} - \frac{9439168}{15015}\frac{1}{x^{14}} - \frac{4871031808}{765765}\frac{1}{x^{18}} + \ldots , \tag{6.11}$$

with the $\bar{\mathrm{E}}_1$ in (6.9) determining all terms in (1.36), (1.42) (with $\sum_{p_2=1}^{\infty} p_2^{-2n+4} = \zeta(2n-4)$). We shall present the exact formula for the coefficients c_n in (6.11) in appendix G.

One can evaluate the sum in (6.9) numerically getting the values of the function $\bar{\mathrm{E}}_1(\Lambda) = \sum_{p_2=1}^{\infty} F(p_2\Lambda)$ that turn out to be real for $\Lambda \geqslant 2$. The results for several values of Λ are represented by blue circles in figure 2, by summing up to $p_{2,\max} = 2000$. They are in good agreement with the plot of just the first two terms in (6.9) (orange curve). More precisely, the relative error is below 0.1 percent already at $\Lambda = 2.5$ and fully negligible at $\Lambda = 3.0$. A more careful evaluation at the convergence radius $\Lambda = 2$ (see appendix G) is beyond our aims and would require a larger maximum value of p_2.

One may repeat a similar analysis of the 1-loop M2 brane correction in the case of the solution generalizing the folded string rotating in one plane in flat space (cf. (6.3), (6.4), (6.5))

$$X^0 = \kappa\,\tau\,, \qquad X^1 = \kappa\,\sin\sigma\,\cos\tau\,, \qquad X^2 = \kappa\,\sin\sigma\,\sin\tau\,, \qquad X^{10} = R_{11}\sigma'\,, \tag{6.12}$$

$$\bar{g}_{ij} = \mathrm{diag}\big(-\kappa^2\cos^2\sigma,\ \kappa^2\cos^2\sigma,\ R_{11}^2\big)\,, \tag{6.13}$$

$$E_0 = \frac{\kappa R_{11}}{\ell_P^3} = \alpha'^{-1}\kappa\,, \qquad J = \frac{\kappa^2 R_{11}}{2\ell_P^3} = \tfrac{1}{2}\alpha'^{-1}\kappa^2\,, \qquad E_0 = \sqrt{2\alpha'^{-1}J}\,. \tag{6.14}$$

In contrast to the circular solution in (6.3) here the induced metric is no longer flat which complicates the derivation of the fluctuation spectrum. After the Fourier expansion in extra σ' direction one finds that the 7+1 bosonic and 8 fermionic fluctuation operators are given by (see appendix G for details)

$$\Delta = -\frac{1}{\sqrt{-\bar g}}\partial_i\left(\sqrt{-\bar g}\,\bar g^{ij}\partial_j\right) \;\to\; \partial_\tau^2-\partial_\sigma^2+\Lambda^2 p_2^2\cos^2\sigma\,, \qquad \Lambda^2 \equiv \frac{\kappa^2}{R_{11}^2} = \frac{2J}{g_s^2}\,, \tag{6.15}$$

$$\Delta + R^{(2)} \;\to\; \partial_\tau^2-\partial_\sigma^2+\Lambda^2 p_2^2\cos^2\sigma + \frac{2}{\cos^2\sigma}\,, \tag{6.16}$$

$$\nabla\!\!\!/ = \rho^\alpha\partial_\alpha + i\Lambda\, p_2\cos\sigma\,. \tag{6.17}$$

While finding their eigenvalues for general p_2 appears to be non-trivial, the fact that they depend on J and g_s only through the parameter Λ implies that the M2 brane correction to the energy should have a similar structure (1.42), (6.9) as in the above circular case.

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A Definitions and notation

The $U(N)_k \times U(N)_{-k}$ ABJM theory in the large N limit with fixed k is dual to the M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with the following metric and 3-form background

$$ds_{11}^2 = \mathrm{L}^2(\tfrac14 ds^2_{AdS_4} + ds^2_{S^7/\mathbb{Z}_k}), \qquad \mathrm{L} = (2^5\pi^2 Nk)^{1/6}\ell_P, \tag{A.1}$$

$$ds^2_{AdS_4} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\alpha^2+\cos^2\alpha\, d\beta^2), \tag{A.2}$$

$$ds^2_{S^7/\mathbb{Z}_k} = ds^2_{CP^3} + \frac{1}{k^2}(d\varphi + k\,\mathrm{A})^2, \qquad \varphi \equiv \varphi + 2\pi, \tag{A.3}$$

$$C_3 = -\tfrac38\,\mathrm{L}^3\cosh\rho\,\sinh^2\rho\,\sin\alpha\, dt\wedge d\rho\wedge d\beta. \tag{A.4}$$

We shall use the following parametrization of $ds^2_{CP^3}$ and A

$$\begin{aligned} ds^2_{CP^3} &= d\gamma^2 + \cos^2\gamma\sin^2\gamma\big(d\eta + \tfrac12\cos\theta_1 d\phi_1 - \tfrac12\cos\theta_2 d\phi_2\big)^2 \\ &\quad + \tfrac14\cos^2\gamma\left(d\theta_1^2+\sin^2\theta_1 d\phi_1^2\right) + \tfrac14\sin^2\gamma\left(d\theta_2^2+\sin^2\theta_2 d\phi_2^2\right), \end{aligned} \tag{A.5}$$

$$\mathrm{A} = \tfrac12\big[\cos(2\gamma)d\eta + \cos^2\gamma\cos\theta_1 d\phi_1 + \sin^2\gamma\cos\theta_2\, d\phi_2\big]. \tag{A.6}$$

The limit of large k with fixed $\lambda \equiv \frac{N}{k}$ corresponds to the 't Hooft expansion in the 3d gauge theory which is dual to the perturbative type IIA string theory on $AdS_4 \times CP^3$ background with the metric

$$ds^2_{10} = L^2(\tfrac14 ds^2_{AdS_4} + ds^2_{CP^3}), \qquad L = g_s^{1/3}\,\mathrm{L}\,. \tag{A.7}$$

The string coupling g_s and the effective dimensionless string tension T (defined with respect to the radius $\frac{1}{2}L$ of the AdS_4) are given by (we set $\ell_P = \ell_s = \sqrt{\alpha'}$)

$$g_s = \left(\frac{L}{k\,\ell_P}\right)^{3/2} = \frac{\sqrt{\pi}(2\lambda)^{5/4}}{N}\,, \tag{A.8}$$

$$T = \frac{\frac{1}{4}L^2}{2\pi\alpha'} = \sqrt{\frac{\lambda}{2}} = \frac{\sqrt{\bar\lambda}}{2\pi}\,, \qquad \bar\lambda \equiv 2\pi^2\lambda\,, \tag{A.9}$$

$$\frac{1}{k^2} = \frac{\lambda^2}{N^2} = \frac{g_s^2}{8\pi T}\,. \tag{A.10}$$

The bosonic part of the M2 brane action is given by (1.3). The M-theory expansion corresponding to the large N with fixed k expansion on the gauge theory side is for $L \gg \ell_P$. This is the expansion in the large effective dimensionless M2 brane tension

$$T_2 \equiv L^3\, T_2 = \frac{1}{\pi}\sqrt{2Nk}\,. \tag{A.11}$$

It is related to the effective dimensionless type IIA string tension in (A.8) as

$$T = \frac{\pi}{2k}\, T_2\,, \tag{A.12}$$

which follows also upon double dimensional reduction [38] of the membrane action in $AdS_4 \times S^7/\mathbb{Z}_k$ to the type IIA GS string action in $AdS_4 \times CP^3$.

B Gamma matrices and spin connection

The explicit choice for the gamma matrices (ρ^i, γ^p) used in this paper is

$$\begin{aligned}
&\rho^0 = (i\sigma^2)\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2, && \rho^1 = \sigma^1\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2, && \rho^2 = \sigma^3\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2\otimes\sigma^2,\\
&\gamma^1 = \sigma^0\otimes\sigma^2\otimes\sigma^0\otimes\sigma^1\otimes\sigma^2, && \gamma^2 = \sigma^0\otimes\sigma^3\otimes\sigma^0\otimes\sigma^0\otimes\sigma^0, && \gamma^3 = \sigma^0\otimes\sigma^2\otimes\sigma^1\otimes\sigma^2\otimes\sigma^0,\\
&\gamma^4 = \sigma^0\otimes\sigma^2\otimes\sigma^3\otimes\sigma^2\otimes\sigma^0, && \gamma^5 = \sigma^0\otimes\sigma^2\otimes\sigma^2\otimes\sigma^0\otimes\sigma^1, && \gamma^6 = \sigma^0\otimes\sigma^2\otimes\sigma^0\otimes\sigma^3\otimes\sigma^2,\\
&\gamma^7 = \sigma^0\otimes\sigma^2\otimes\sigma^2\otimes\sigma^0\otimes\sigma^3, && \gamma^8 = \sigma^0\otimes\sigma^1\otimes\sigma^0\otimes\sigma^0\otimes\sigma^0, &&
\end{aligned} \tag{B.1}$$

where σ^i are Pauli matrices and σ^0 is a unit 2×2 matrix.

The spin connection in the normal bundle in the basis (3.9), (3.10) corresponding to the long folded M2 brane solution can be written as

$$\Omega_{pq}(e_i) = \langle n_p, \nabla_{e_i} n_q\rangle, \qquad \Omega = E_{2,3}\frac{2\nu}{\mu}e^1 + E_{5,7}\left(\frac{\nu}{\mu}e^0 + e^2\right) + E_{6,8}\big(-\frac{\nu}{\mu}e^0 + e^2\big) - E_{3,4}\frac{\kappa}{\mu}e^2, \tag{B.2}$$

where $E_{p,q}$ is a matrix that has $+1$ in (p,q) place and -1 in (q,p) place with other entries being 0.

Similarly, the connection in the normal bundle in the basis (5.8), (5.9) for the circular (S,J) M2 brane solution is given by

$$\begin{aligned}
\Omega = {}& E_{1,2}\left(-\frac{2\omega v}{\kappa r_1^2}e^0 - \frac{2\kappa u r_0^2}{v^3}e^1\right) + E_{5,7}\left(\frac{\omega}{v}e^0 - \frac{u}{v}e^1 + e^2\right)\\
&+ E_{6,8}\left(-\frac{\omega}{v}e^0 + \frac{u}{v}e^1 + e^2\right) + E_{2,4}\left(-\frac{\kappa r_0 r_1}{v^2}e^2\right), \qquad v \equiv \sqrt{r_1^2 + u^2}\,.
\end{aligned} \tag{B.3}$$

C Higher order $1/k^2$ corrections in (3.19)

The explicit expressions for the next two coefficients in the large k expansion of the folded membrane 1-loop energy in (3.19) are found to be

$$\begin{aligned}
\mathcal{C}_6 &= -\mathcal{J}\,\frac{2(808\kappa^6-1892\kappa^4\nu^2-667\kappa^2\nu^4-175\nu^6)}{15\kappa(\kappa^2-\nu^2)^2} = -\mathcal{J}\,\frac{2(-1926-2027x^2+532x^4+808x^6)}{15x^4\sqrt{1+x^2}},\\
\mathcal{C}_8 &= -\mathcal{J}\,\frac{467328\kappa^8-250560\kappa^6\nu^2-729664\kappa^4\nu^4-231824\kappa^2\nu^6+13685\nu^8}{420\kappa(\kappa^2-\nu^2)^3}\\
&= -\mathcal{J}\,\frac{-731035-573520x^2+1322624x^4+1618752x^6+467328x^8}{420x^6\sqrt{1+x^2}}.
\end{aligned} \tag{C.1}$$

Their small/large x expansions are (cf. (3.24), (3.25))

$$x\ll 1: \qquad \begin{aligned}\mathcal{C}_6 &= \mathcal{J}\big(\tfrac{1284}{5}\tfrac{1}{x^4}+\tfrac{2128}{15}\tfrac{1}{x^2}-\tfrac{3293}{30}+\dots\big),\\ \mathcal{C}_8 &= \mathcal{J}\big(\tfrac{146207}{84}\tfrac{1}{x^6}+\tfrac{83201}{168}\tfrac{1}{x^4}-\tfrac{10681967}{3360}\tfrac{1}{x^2}-\tfrac{3106619}{1344}+\dots\big),\end{aligned} \tag{C.2}$$

$$x\gg 1: \qquad \mathcal{C}_6 = \mathcal{J}\big(-\tfrac{1616}{15}x-\tfrac{256}{15}\tfrac{1}{x}+\dots\big), \qquad \mathcal{C}_8 = \mathcal{J}\big(-\tfrac{38944}{35}x-\tfrac{115424}{35}\tfrac{1}{x}+\dots\big). \tag{C.3}$$

Comparing the small x expansions of $\mathcal{C}_{2n}$ in (3.24), (C.2) in the folded membrane case with the large $\mathcal{J}$ expansions of the coefficients in the large k expansion of the 1-loop energy of the circular (S,J) membrane in (5.14) suggests a conjecture that for any n

$$\mathcal{C}^{\rm circular}_{2n} \overset{\mathcal{J}\gg 1}{=} \mathrm{c}_{2n}\,\frac{\mathcal{J}^{2n-1}}{[u(1+u)]^{n-1}} + \mathcal{O}(\mathcal{J}^{2n-3}), \qquad \mathcal{C}^{\rm folded}_{2n} \overset{x\ll 1}{=} \mathrm{c}_{2n}\,\frac{\mathcal{J}}{x^{2n-2}} + \mathcal{O}\Big(\frac{1}{x^{2n-4}}\Big), \tag{C.4}$$

where c_{2n} are the *same* numerical coefficients in both cases.

D Expansion of 1-loop energy of circular (S,J) string for small $\mathcal{J}$

In this appendix we study the small $\mathcal{J}$ expansion of the 1-loop string energy for the circular (S,J) solution discussed in section 4. The small $\mathcal{J}$ expansion of CP^3 fluctuation frequencies in (4.13) reads (here we set $p_1 = n$)[18]

$$CP^3: \qquad (p_0)_{1,2,3,4} = \tfrac{1}{2}\sqrt{4n^2-u^2}+\mathcal{O}(\mathcal{J}^2), \qquad (p_0)_5 = \sqrt{n^2-u^2}+\mathcal{O}(\mathcal{J}^2). \tag{D.1}$$

For $n=0$ the first four degenerate frequencies are imaginary, indicating an instability.[19] For the AdS_4 frequencies in (4.14) we get

$$AdS_4: \qquad (p_0)_1 = \sqrt{n^2+u^2}+\frac{u}{\sqrt{1+u^2}\sqrt{n^2+u^2}}\mathcal{J}+\mathcal{O}(\mathcal{J}^2), \tag{D.2}$$

$$(p_0)_{2,3} = \mp\frac{(n\pm1)^2+u^2+(-n\mp2)\sqrt{(1+u^2)\big[(n\pm1)^2+u^2\big]}}{(1+u^2)\big[(n\pm1)^2+u^2\big]}\mathcal{J}+\mathcal{O}(\mathcal{J}^2). \tag{D.3}$$

[18] For simplicity, we continue branches of solutions of algebraic equations from large n to small values of n. For a more careful discussion see [37].

[19] This is a reflection of the fact that the string wrapped on a circle in the internal space is unstable, unless the contraction due to its tension is balanced by its rotation. For a discussion of a similar instability in $AdS_5\times S^5$ context see [26].

The fermionic frequencies (4.15), (4.16) have the following expansions

$$(p_0)_1 = \begin{cases} \frac{1}{2}(1+2n-\sqrt{1+u^2}) + \big[\frac{u}{2\sqrt{1+u^2}} - \frac{u(-3-3u^2+2n\sqrt{1+u^2}}{2(1+u^2)^{3/2}(2n-\sqrt{1+u^2})}\big]\mathcal{J} + \mathcal{O}(\mathcal{J}^2), & n>0 \\ \frac{1}{2}(1-2n+\sqrt{1+u^2}) + \big[\frac{u}{2\sqrt{1+u^2}} - \frac{u(-3-3u^2+2n\sqrt{1+u^2}}{2(1+u^2)^{3/2}(-2n+\sqrt{1+u^2})}\big]\mathcal{J} + \mathcal{O}(\mathcal{J}^2), & n<0, \end{cases}$$

$$(p_0)_2 = \begin{cases} \frac{1}{2}(-1+2n+\sqrt{1+u^2}) + \big[-\frac{u}{2\sqrt{1+u^2}} + \frac{u(3+3u^2+2n\sqrt{1+u^2}}{2(1+u^2)^{3/2}(2n+\sqrt{1+u^2})}\big]\mathcal{J} + \mathcal{O}(\mathcal{J}^2), & n>0 \\ \frac{1}{2}(-1-2n-\sqrt{1+u^2}) + \big[-\frac{u}{2\sqrt{1+u^2}} - \frac{u(3+3u^2+2n\sqrt{1+u^2}}{2(1+u^2)^{3/2}(2n+\sqrt{1+u^2})}\big]\mathcal{J} + \mathcal{O}(\mathcal{J}^2), & n<0, \end{cases} \tag{D.4}$$

$$(p_0)_{3,4} = \tfrac{1}{2}(\sqrt{(2n\pm 1)^2+u^2} \mp \sqrt{1+u^2}) \mp \frac{u((2n\pm1)^2+u^2\mp 2(1\pm n)\sqrt{(1+u^2)[(2n\pm1)^2+u^2)]}}{2(1+u^2)[(2n\pm1)^2+u^2]}\mathcal{J} + \mathcal{O}(\mathcal{J}^2).$$

The 1-loop correction to the energy (2.11) can be written as

$$E_1 = \frac{1}{2\kappa}\sum_{n\in\mathbb{Z}}\hat{\omega}_n(\mathcal{J}) = \frac{1}{2\kappa}\hat{\omega}_0(\mathcal{J}) + \frac{1}{\kappa}\sum_{n=1}^{\infty}\bar{\omega}_n(\mathcal{J}), \qquad \bar{\omega}_n = \frac{1}{2}(\hat{\omega}_n + \hat{\omega}_{-n}), \tag{D.5}$$

$$\hat{\omega}_n(\mathcal{J}) \equiv \sum_{\{p_0\}}(-1)^F p_0 = \hat{\omega}_n^{(0)} + \hat{\omega}_n^{(1)}\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2), \tag{D.6}$$

$$\hat{\omega}_0(\mathcal{J}) = (1+3i)\,u + \frac{1}{\sqrt{1+u^2}}\mathcal{J} + \mathcal{O}(\mathcal{J}^2), \tag{D.7}$$

$$\begin{aligned} \bar{\omega}_n^{(0)} &= -4n + \sqrt{n^2-u^2} + 2\sqrt{4n^2-u^2} + \sqrt{n^2+u^2} + \sqrt{(n+1)^2+u^2} + \sqrt{(n-1)^2+u^2} \\ &\quad - \sqrt{(2n+1)^2+u^2} - \sqrt{(2n-1)^2+u^2}, \end{aligned} \tag{D.8}$$

$$\begin{aligned} \bar{\omega}_n^{(1)} &= \frac{8un}{\sqrt{1+u^2}(1-4n^2+u^2)} + \frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{n^2+u^2}} + \frac{u(2-n)}{\sqrt{1+u^2}\sqrt{(n-1)^2+u^2}} \\ &\quad + \frac{u(2+n)}{\sqrt{1+u^2}\sqrt{(n+1)^2+u^2}} + \frac{2u(n-1)}{\sqrt{1+u^2}\sqrt{(2n-1)^2+u^2}} - \frac{2u(n+1)}{\sqrt{1+u^2}\sqrt{(2n+1)^2+u^2}}. \end{aligned}$$

The sums over n are convergent since $\bar{\omega}_n^{(0)} = \frac{7u^2-4u^4}{8}\frac{1}{n^3} + \mathcal{O}(n^{-4})$, $\bar{\omega}_n^{(1)} = \frac{10u-u^3}{8\sqrt{1+u^2}}\frac{1}{n^3} + \mathcal{O}(n^{-4})$, and can be computed numerically at fixed u. For example, for $\mathcal{S} = \mathcal{J}$, i.e. $u=1$, we find

$$u = 1: \qquad \sum_{n=1}^{\infty}\bar{\omega}_n^{(0)} = -0.40652, \qquad \sum_{n=1}^{\infty}\bar{\omega}_n^{(1)} = -1.5001, \tag{D.9}$$

and as a result the real and imaginary parts of E_1 are given by

$$E_1 = 0.093 - 1.212\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2) + i\big[1.5 - 1.06\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2)\big]. \tag{D.10}$$

One may also consider the expansion of (D.5) in small u. Splitting the sum into $n=1$ and $n \geqslant 2$ parts we get

$$\bar{\omega}_1^{(0)} = -4 + u + \sqrt{1-u^2} + 2\sqrt{4-u^2} + \sqrt{4+u^2} - \sqrt{9+u^2} = u - \tfrac{11}{12}u^2 - \tfrac{289}{1728}u^4 + \ldots, \tag{D.11}$$

$$\begin{aligned} \sum_{n=2}^{\infty}\bar{\omega}_n^{(0)} &= \sum_{n=2}^{\infty}\Big[\frac{7n^2-1}{2n(1-5n^2+4n^4)}u^2 + \frac{-9+135n^2-807n^4+2005n^6-2028n^8+864n^{10}-1024n^{12}}{32(n-5n^3+4n^5)^3}u^4 + \ldots\Big] \\ &= \left(\frac{11}{12} - \log 2\right)u^2 + \left[\frac{289}{1728} - \frac{5}{16}\zeta(3)\right]u^4 + \ldots, \end{aligned} \tag{D.12}$$

$$\sum_{n=1}^{\infty}\bar{\omega}_n^{(0)} = u - \log 2\; u^2 - \tfrac{5}{16}\zeta(3)\,u^4 + \ldots. \tag{D.13}$$

Similarly, we find that

$$\sum_{n=1}^{\infty} \bar{\omega}_n^{(1)} = 1 + 3(1 - 2\log 2)u - \tfrac{1}{2}u^2 + \big[-\tfrac{5}{2} + 3\log 2 - \tfrac{5}{8}\zeta(3)\big]u^3 + \tfrac{3}{8}u^4 + \dots . \tag{D.14}$$

Using that (cf. (4.6)) $\frac{1}{\kappa} = \frac{1}{u} - \frac{1}{u^2\sqrt{1+u^2}}\mathcal{J} + \mathcal{O}(\mathcal{J}^2)$, we obtain the following expression for the real part of E_1 (the imaginary part comes from $3i\,u$ term in (D.7))[20]

$$\begin{aligned}\operatorname{Re} E_1 = {} & \big[\tfrac{3}{2} + (3 - 5\log 2)\mathcal{J} + \mathcal{O}(\mathcal{J}^2)\big] - \big[\log 2 + \mathcal{O}(\mathcal{J}^2)\big]\,u \\ & + \big[(-\tfrac{5}{2} + \tfrac{5}{2}\log 2 - \tfrac{5}{16}\zeta(3))\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2)\big]\,u^2 - \big[\tfrac{5}{16}\zeta(3) + \mathcal{O}(\mathcal{J}^2)\big]\,u^3 + \mathcal{O}(u^4). \end{aligned} \tag{D.15}$$

Comparison with the $AdS_5 \times S^5$ case. It is of interest to compare the above expressions with E_1 for a similar circular (S, J) solution in $AdS_5 \times S^5$ (see [25]). Using the same notation here one has 4 + 2 + 2 bosonic fluctuation frequencies (cf. (4.14)) and 4 + 4 fermionic frequencies given by

$$\text{B}: \qquad (p_0)_{1,2,3,4} = \sqrt{n^2+\mathcal{J}^2-u^2}, \qquad (p_0)_{5,6} = \sqrt{n^2+\kappa^2}, \tag{D.16}$$

$$(p_0^2-n^2)^2+4r_1^2\kappa^2p_0^2-4(1+r_1^2)(\sqrt{1+\kappa^2}p_0-p_1)^2 = 0, \qquad p_0 = (p_0)_{7,8}\,, \tag{D.17}$$

$$\text{F}: \qquad (p_0)_{1,2} = \sqrt{(n\pm c)^2+a^2}, \tag{D.18}$$

$$a^2 = \tfrac{1}{2}(\kappa^2+\mathcal{J}^2-u^2), \qquad c = \tfrac{1}{2}\kappa\big[1+\tfrac{2(1+r_1^2)}{\kappa^2-\mathcal{J}^2+u^2}\big]\sqrt{\tfrac{\kappa^2-\mathcal{J}^2+u^2-2r_1^2}{2(\kappa^2+1)}}. \tag{D.19}$$

Their small $\mathcal{J}$ expansion reads

$$\begin{aligned} \text{B}: \quad & (p_0)_{1,2,3,4} = \sqrt{n^2-u^2}+\mathcal{O}(\mathcal{J}^2), \qquad (p_0)_{5,6} = \sqrt{n^2+u^2}+\tfrac{u}{\sqrt{1+u^2}\sqrt{n^2+u^2}}\mathcal{J}+\mathcal{O}(\mathcal{J}^2), \\ & (p_0)_{7,8} = \sqrt{(n\pm 1)^2+u^2}\mp\sqrt{1+u^2}\mp\tfrac{(n\pm 1)^2+u^2+(-n\mp 2)\sqrt{(1+u^2)((n\pm 1)^2+u^2}}{(1+u^2)((n\pm 1)^2+u^2)}\mathcal{J}+\mathcal{O}(\mathcal{J}^2), \\ \text{F}: \quad & (p_0)_{1,2} = \tfrac{1}{2}\sqrt{1+u^2+4n(n\pm\sqrt{1+u^2})}+\tfrac{u(4n^2\sqrt{1+u^2}\pm 8n(1+u^2)+3(1+u^2)^{3/2})}{2(1+u^2)^{3/2}(\pm 2n+\sqrt{1+u^2})\sqrt{1+u^2+4n(n\pm\sqrt{1+u^2})}}\mathcal{J}+\mathcal{O}(\mathcal{J}^2). \end{aligned} \tag{D.20}$$

The corresponding 1-loop energy is given by (D.6) and (D.5) where now

$$\begin{aligned} \hat{\omega}_0(\mathcal{J}) &= 2(1+2i)u-2\sqrt{1+u^2}+\frac{2(1+u^2-4u\sqrt{1+u^2})}{(1+u^2)^{3/2}}\mathcal{J}+\mathcal{O}(\mathcal{J}^2), \\ \bar{\omega}_n^{(0)} &= 4\sqrt{n^2-u^2}+2\sqrt{n^2+u^2}+\sqrt{(n-1)^2+u^2}+\sqrt{(n+1)^2+u^2} \\ & \quad -2\sqrt{4n^2-4n\sqrt{1+u^2}+1+u^2}-2\sqrt{4n^2+4n\sqrt{1+u^2}+1+u^2}, \\ \bar{\omega}_n^{(1)} &= \tfrac{2u}{\sqrt{1+u^2}\sqrt{n^2+u^2}}+\tfrac{u(2-n)}{\sqrt{1+u^2}\sqrt{(n-1)^2+u^2}}+\tfrac{u(2+n)}{\sqrt{1+u^2}\sqrt{(n+1)^2+u^2}} \\ & \quad +2u\big(\tfrac{2n}{\sqrt{1+u^2}}-3\big)\tfrac{1}{\sqrt{1+u^2}\sqrt{4n^2-4n\sqrt{1+u^2}+1+u^2}}-2u\big(\tfrac{2n}{\sqrt{1+u^2}}+3\big)\tfrac{1}{\sqrt{1+u^2}\sqrt{4n^2+4n\sqrt{1+u^2}+1+u^2}}. \end{aligned} \tag{D.21}$$

The sums are again convergent and can be computed numerically at fixed u. At the special point $\mathcal{S} = \mathcal{J}$, i.e. $u = 1$, we find $\sum_{n=1}^{\infty}\bar{\omega}_n^{(0)} = -1.9621$, $\sum_{n=1}^{\infty}\bar{\omega}_n^{(1)} = -3.1034$, and thus (cf. (D.10))

$$E_1 = -2.376 - 2.716\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2) + i\big[2 - 1.414\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2)\big]. \tag{D.22}$$

[20] Note that here the term $-u\log 2$ can be again be interpreted as coming from the redefinition $\sqrt{\bar{\lambda}} \to 2\bar{h}(\bar{\lambda})$ in the classical part of the energy (cf. (1.19)). Other $\log 2$ terms do not have this origin.

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Expanding in small u we get (cf. (D.15))

$$\begin{aligned}\operatorname{Re} E_1 = &-\big[u^{-1} + \mathcal{O}(\mathcal{J}^2)\big] + \big[2 + (\tfrac{1}{2} - 8\log 2)\mathcal{J} + \mathcal{O}(\mathcal{J}^2)\big] - \big[1 + \mathcal{O}(\mathcal{J}^2)\big]\, u \\ &+ \big[(\tfrac{5}{8} + 4\log 2 - \zeta(3))\,\mathcal{J} + \mathcal{O}(\mathcal{J}^2)\big]\, u^2 + \big[\tfrac{1}{4} - \zeta(3) + \mathcal{O}(\mathcal{J}^2)\big]\, u^3 + \mathcal{O}(u^4). \qquad \text{(D.23)}\end{aligned}$$

The u^{-1} term (that was absent in (D.7)) comes from the mode $n = 0$ that has a $\mathcal{O}(\mathcal{J}^0)$ part which is not vanishing for $u = 0$.

To make the comparison between the $\mathrm{AdS}_4 \times \mathrm{CP}^3$ and $\mathrm{AdS}_5 \times S^5$ cases more transparent, we may remove the contribution of $\hat{\omega}_0(\mathcal{J})$ in both cases to get

$$\begin{aligned}E_1^{\mathrm{AdS}_4\times \mathrm{CP}^3}\big|_{n\neq 0} = &\big[1+(3-5\log 2)\mathcal{J}+\mathcal{O}(\mathcal{J}^2)\big]-\big[\log 2+\mathcal{O}(\mathcal{J}^2)\big]\, u \\ &+\big[(-\tfrac{5}{2}+\tfrac{5}{2}\log 2-\tfrac{5}{16}\zeta(3))\,\mathcal{J}+\mathcal{O}(\mathcal{J}^2)\big]\, u^2+\big[-\tfrac{5}{16}\zeta(3)+\mathcal{O}(\mathcal{J}^2)\big]\, u^3+\mathcal{O}(u^4), \\ E_1^{\mathrm{AdS}_5\times S^5}\big|_{n\neq 0} = &\big[1+(\tfrac{9}{2}-8\log 2)\mathcal{J}+\mathcal{O}(\mathcal{J}^2)\big]-\big[\tfrac{1}{2}+\mathcal{O}(\mathcal{J}^2)\big]\, u \\ &+\big[(-\tfrac{27}{8}+4\log 2-\zeta(3))\,\mathcal{J}+\mathcal{O}(\mathcal{J}^2)\big]\, u^2+\big[\tfrac{1}{8}-\zeta(3)+\mathcal{O}(\mathcal{J}^2)\big]\, u^3+\mathcal{O}(u^4). \qquad \text{(D.24)}\end{aligned}$$

E Large $\mathcal{J}$ expansion of circular M2 bosonic fluctuation frequencies

The frequencies corresponding to zeros of P_4 polynomials (5.6) given in (5.7) have the following large $\mathcal{J}$ expansion

$$\begin{aligned}\mathrm{B}: \qquad (p_0)_{1,2} &= \mathcal{J} + \big[p_1^2 + u p_1 + \tfrac{1}{4}u(1+u)kp_2(\pm 2 + kp_2)\big]\,\frac{1}{\mathcal{J}} + \dots, \\ (p_0)_{3,4} &= -\mathcal{J} - \big[p_1^2 - u p_1 + \tfrac{1}{4}u(1+u)kp_2(\pm 2 + kp_2)\big]\,\frac{1}{\mathcal{J}} + \dots, \\ (p_0)_{5,6} &= \big[p_1^2 - u p_1 + \tfrac{1}{4}u(1+u)kp_2(\pm 2 + kp_2)\big]\,\frac{1}{\mathcal{J}} + \dots, \\ (p_0)_{7,8} &= -\big[p_1^2 + u p_1 + \tfrac{1}{4}u(1+u)kp_2(\pm 2 + kp_2)\big]\,\frac{1}{\mathcal{J}} + \dots. \qquad \text{(E.1)}\end{aligned}$$

For the frequencies corresponding to the zeros of the polynomial P_8 in (5.6) one finds

$$\begin{aligned}(p_0)_9 &= 2\mathcal{J}+\big[p_1^2-2(1+u)p_1+2(1+3u+u^2)+\tfrac{1}{2}u(1+u)k^2p_2^2\big]\frac{1}{2\mathcal{J}}+\dots, \\ (p_0)_{10} &= -2\mathcal{J}-\big[p_1^2+2(1+u)p_1+2(1+3u+u^2)+\tfrac{1}{2}u(1+u)k^2p_2^2\big]\frac{1}{2\mathcal{J}}+\dots, \\ (p_0)_{11} &= \mathcal{J}+\big[p_1^2-u^2+\tfrac{1}{2}u(1+u)k^2p_2^2\big]\frac{1}{2\mathcal{J}}+\dots, \\ (p_0)_{12} &= -\mathcal{J}-\big[p_1^2-u^2+\tfrac{1}{2}u(1+u)k^2p_2^2\big]\frac{1}{2\mathcal{J}}+\dots, \\ (p_0)_{13} &= \mathcal{J}+\big[p_1^2+u(2+u)-\tfrac{1}{2}u(1+u)k^2p_2^2\big]\frac{1}{2\mathcal{J}}+\dots, \qquad \text{(E.2)} \\ (p_0)_{14} &= -\mathcal{J}-\big[p_1^2+u(2+u)-\tfrac{1}{2}u(1+u)k^2p_2^2\big]\frac{1}{2\mathcal{J}}+\dots, \\ (p_0)_{15,16} &= \Big[2(1+u)p_1\pm\sqrt{p_1^4+4u(1+u)p_1^2(1-\tfrac{1}{4}k^2p_2^2)-u^2(1+u^2)k^2p_2^2(1-\tfrac{1}{4}k^2p_2^2)}\;\Big]\frac{1}{2\mathcal{J}}+\dots.\end{aligned}$$

F 1-loop energy of circular M2 brane in small $\mathcal{J}$ limit

One can obtain the leading $\sim u\,\mathcal{J}^0$ terms in the coefficients (5.15) of the $1/k^2$ corrections from the small $\mathcal{J}$ expansion of the corresponding membrane fluctuation frequencies

$$
\begin{aligned}
\mathrm{B}:\quad (p_0)_{1,2,3,4} &= \tfrac{1}{2}\sqrt{4n^2+4s_1nu+ku^2\ell(2s_2+k\ell)}+\mathcal{O}(\mathcal{J}), \qquad s_1,s_2=\pm 1\\
(p_0)_{5,6} &= \tfrac{1}{2}\sqrt{4n^2+u^2(\pm 4+k^2\ell^2)}+\mathcal{O}(\mathcal{J}),\\
(p_0)_{7,8} &= \pm\sqrt{1+u^2}+\tfrac{1}{2}\sqrt{4(n\mp 1)^2+u^2(4+k^2\ell^2)}+\mathcal{O}(\mathcal{J}),\\
\mathrm{F}:\quad (p_0)_{1,2,3,4} &= \tfrac{1}{2}\big(s_1\sqrt{1+u^2}+\sqrt{(s_2-2s_1s_2n+u)^2+k^2u^2\ell^2}\big)+\mathcal{O}(\mathcal{J}), \qquad s_1,s_2=\pm 1,\\
(p_0)_{5,6,7,8} &= \tfrac{1}{2}\big(s_1\sqrt{1+u^2}+\sqrt{1+4(-s_1+n)n+u^2(s_2+k\ell)^2}\big)+\mathcal{O}(\mathcal{J})\,,
\end{aligned}
\tag{F.1}
$$

where we renamed the integer mode numbers as $p_1\equiv n,\ p_2\equiv \ell$. Replacing the sum over n with an integral we get

$$
E_1^{\rm M2}=\frac{1}{2\kappa}\sum_{\ell\in\mathbb{Z}-\{0\}}\sum_{n\in\mathbb{Z}}\sum(-1)^F p_0\to -6u\frac{\zeta(2)}{k^2}-\frac{3}{2}u\frac{\zeta(4)}{k^4}+\cdots+\mathcal{O}(\mathcal{J})\,, \tag{F.2}
$$

reproducing the leading terms in (5.15). One can similarly find the order $\mathcal{J}^2$ terms in (5.15).

One can obtain the generating function for all $1/k^{2r}$ coefficients at leading order in small $\mathcal{J}$ as[21]

$$
\sum_{n=-\infty}^{\infty}\sum(-1)^F p_0\overset{k\ell\gg 1}{=}u^2F(k\ell)+\mathcal{O}(\mathcal{J}), \tag{F.3}
$$

$$
\begin{aligned}
F(y) &= \tfrac{1}{2}(y-1)^2\log(y-1)+\tfrac{1}{2}(y+1)^2\log(y+1)+y^2\log y+\tfrac{1}{8}(4-y^2)\log(y^2-4)\\
&\quad -\tfrac{3}{8}(4+y^2)\log(y^2+4)+\tfrac{1}{4}(1+2y-y^2)\log(-1-2y+y^2)\\
&\quad +\tfrac{1}{4}(1-2y-y^2)\log(-1+2y+y^2),\\
&\overset{y\gg 1}{=} -\tfrac{6}{y^2}-\tfrac{3}{2}\tfrac{1}{y^4}-\tfrac{70}{3}\tfrac{1}{y^6}-\tfrac{391}{12}\tfrac{1}{y^8}+\dots .
\end{aligned}
\tag{F.4}
$$

The first two terms in (F.4) are seen to be in agreement with the corresponding coefficients in (5.15) after using that $1/\kappa = 1/u+\mathcal{O}(\mathcal{J})$.

G Details of 1-loop M2 brane computations in flat space

Circular $J_1 = J_2$ brane. In the case of the solution in (6.3) we may choose the frame in the normal bundle so that $\mathrm{n}_r = \partial_{X^r}$ for $r=5,\dots,9$ and ($\xi^i=(\tau,\sigma,\sigma')$)

$$
\mathrm{n}_1=-\partial_{X^0}+\sin(\tau+\sigma)\partial_{X^1}-\cos(\tau+\sigma)\partial_{X^2}+\sin(\tau-\sigma)\partial_{X^3}-\cos(\tau-\sigma)\partial_{X^4}, \tag{G.1}
$$

$$
\mathrm{n}_2=\cos(\tau+\sigma)\partial_{X^1}+\sin(\tau+\sigma)\partial_{X^2},\qquad \mathrm{n}_3=\cos(\tau-\sigma)\partial_{X^3}+\sin(\tau-\sigma)\partial_{X^4}\,. \tag{G.2}
$$

Then the fluctuation operator in $r=5,\dots,9$ directions is given by the massless Laplacian for the metric $\bar g$ in (6.4). The "mass matrix" in the fluctuation operator (6.1) in $\mathrm{n}_{1,2,3}$ directions is

$$
\mathcal{M}_{pq}=-(\mathcal{K}_{ij})_p(\mathcal{K}^{ij})_q=\begin{pmatrix}0&0&0\\0&0&-\frac{4}{\kappa^2}\\0&-\frac{4}{\kappa^2}&0\end{pmatrix}_{pq}, \qquad p,q=1,2,3. \tag{G.3}
$$

[21] It is interesting to note a similarity between (F.4) and eq. (2.16) appearing in the expression for the cusp anomalous dimension in [3].

The covariant derivative part of the operator in (6.1) $\Delta^\perp = \frac{1}{\sqrt{-\bar g}}(\nabla_i^\perp)^\dagger(\sqrt{-\bar g}\bar g^{ij}\nabla_j^\perp)$ can be found using that the connection on the normal bundle is given by

$$\Omega_{pq} = \langle n_p, \partial n_q\rangle = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} d\tau + \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} d\sigma\,. \tag{G.4}$$

As a result,[22]

$$\Delta^\perp + \mathcal{M} = \begin{pmatrix} \partial_0^2 - \partial_1^2 - \Lambda^2\partial_2^2 & -2(\partial_0 - \partial_1) & -2(\partial_0 + \partial_1) \\ 2(\partial_0 - \partial_1) & \partial_0^2 - \partial_1^2 - \Lambda^2\partial_2^2 & -4 \\ 2(\partial_0 + \partial_1) & -4 & \partial_0^2 - \partial_1^2 - \Lambda^2\partial_2^2 \end{pmatrix}, \qquad \Lambda^2 \equiv \frac{\kappa^2}{2R_{11}^2}\,. \tag{G.5}$$

Expanding in Fourier modes $(p_0,\ p_1,\ p_2)$ (cf. (3.7)), the characteristic frequencies for $q = 1,2,3$ directions can be determined from the polynomial equation[23] $\det(\Delta^\perp + \mathcal{M}) = 0$. In addition, there are 5 "massless" modes $p_0^2 = p_1^2 + \Lambda^2 p_2^2$. In total, we get the agreement with the expression for $\mathcal{D}_\mathrm{B}$ in (6.6).

Since the connection along the membrane surface is flat, the square of the Dirac operator may be written as (cf. (B.1))

$$\not\nabla^2 = -\partial_0^2 + \partial_1^2 + \Lambda^2\partial_2^2 + \gamma_1\gamma_2(\partial_0 - \partial_1) + \gamma_1\gamma_3(\partial_0 + \partial_1) + \rho^0\rho^1\gamma_2\gamma_3\,. \tag{G.6}$$

One can choose a κ-symmetry gauge and the γ-matrices basis such that 11d Majorana fermion can be split as $\theta = \chi \otimes \eta,\ \rho^0\rho^1\eta = \eta\,,$ where χ is a 2d Majorana fermion so that

$$\not\nabla^2\chi = \begin{pmatrix} -\partial_0^2 + \partial_1^2 + \Lambda^2\partial_2^2 + i & (\partial_0 - \partial_1) - i(\partial_0 + \partial_1) \\ -(\partial_0 - \partial_1) - i(\partial_0 + \partial_1) & -\partial_0^2 + \partial_1^2 + \Lambda^2\partial_2^2 - i \end{pmatrix}\chi\,. \tag{G.7}$$

As a result, the 4+4 fermionic frequencies are found to be the same as the zeroes of $\mathcal{D}_\mathrm{F}$ in (6.6)

$$p_0^2(p_1,p_2) = p_1^2 + \Lambda^2 p_2^2 + 1 \pm \sqrt{4p_1^2 + \Lambda^2 p_2^2}\,. \tag{G.8}$$

Exact formula for the coefficients c_n in (6.11). One may represent c_n with $n > 1$ as $(c_1 = -4)$

$$c_n = \tfrac{1}{2}(\alpha_n + \beta_n)\,, \qquad \alpha_n \equiv (-1)^n \frac{4^n\Gamma(n-\frac{1}{2})\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\, n\, \Gamma(2n+\frac{1}{2})}\,, \tag{G.9}$$

where β_n are given by the solution of the second order difference equation

$$\begin{aligned} &64n^2(1+n)(-1+2n)(1+2n)(4+3n)(31+42n+14n^2)\beta_n \\ &-8(1+n)(918+9705n+41860n^2+94960n^3+122292n^4+89510n^5+34580n^6+5460n^7)\beta_{n+1} \\ &+(2+n)(3+2n)^2(1+3n)(5+4n)(7+4n)(3+14n+14n^2)\beta_{n+2} = 0, \end{aligned} \tag{G.10}$$

with the initial values $\beta_2 = -\frac{2176}{105},\ \ \beta_3 = -\frac{493568}{3465}$. Explicitly, we get

$$\begin{aligned} &\alpha_n = \tfrac{16}{35}, -\tfrac{128}{693}, \tfrac{128}{1287}, -\tfrac{14336}{230945}, \ldots, \qquad \beta_n = -\tfrac{2176}{105}, -\tfrac{493568}{3465}, -\tfrac{56639488}{45045}, -\tfrac{185098305536}{14549535}, \ldots \\ &c_n = \tfrac{1}{2}(\alpha_n + \beta_n) = -\tfrac{152}{15},\ -\tfrac{2496}{35},\ -\tfrac{9439168}{15015}, \ldots, \qquad n = 2, 3, \ldots\,. \end{aligned} \tag{G.11}$$

[22]We ignore the overall constant factor $2\kappa^{-2}$ that can be absorbed into the fluctuation fields.
[23]In the string case $p_2 = 0$ one gets 3+3 modes with $p_0^2(p_1, 0) = \{(p_1+2)^2,\ p_1^2,\ (p_1-2)^2\}$.

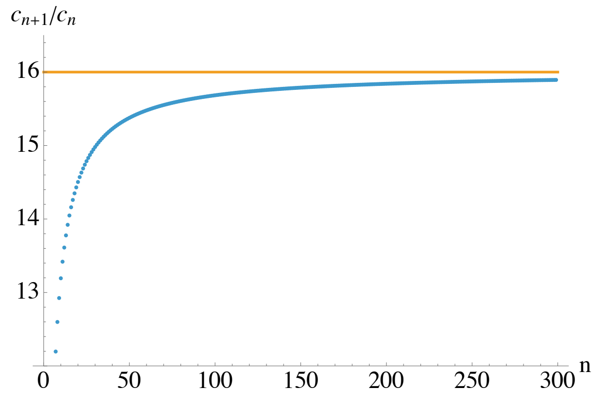

Figure 3. Ratio test for the coefficients of the function in (6.11).

The recursion relation for β_n allows one to compute the coefficients c_n to very high order. In particular, one can check that the series (6.11) for $F(x)$ has the radius of convergence $\frac{1}{x^4} < \frac{1}{16}$, i.e. $x > 2$, as presented in figure 3. This is also the radius of convergence of $\bar{\mathrm{E}}_1(\Lambda)$ in (6.9) since $\zeta(n) \to 1$ for large n, consistently with the data presented in figure 2.

Folded spinning brane. In the case of the solution in (6.12), we get 7 modes with the "massless" Laplace operator Δ in (6.15) and one mode with the operator $\Delta + R^{(2)}$ in (6.16). To get the latter one may use the Gauss equation to express the second term in the operator in (6.1) in terms of the scalar curvature of the metric in (6.13) as $-\mathcal{K}^{ij}\,\mathcal{K}_{ij} = R^{(3)} = R^{(2)}$.

The fermionic operator in (6.2) simplifies to $\nabla\!\!\!/ = \rho^i\big(\partial_{\mathrm{e}_i} + \frac{1}{4}\Omega_i^{jk}\rho_{jk}\big)$, $\Omega_{jk;i} = \langle \mathrm{e}_j, \nabla_{\mathrm{e}_i}\mathrm{e}_k\rangle$. A convenient choice of the gamma matrices basis is such that $\Gamma = \rho_0\rho_1\rho_2 = \sigma^0 \otimes \gamma_9$. Imposing the κ-symmetry gauge $(1+\Gamma)\theta = 0$, the 11d Majorana fermion can be written in terms of the 3d and 8d Majorana fermions $\theta = \psi \otimes \eta$, $\;(1+\gamma_9)\eta = 0$. Representing the metric in (6.13) as $ds^2 = \kappa^2\, e^{2\lambda(\sigma)}(-d\tau^2 + d\sigma^2) + R_{11}^2 d\sigma'^2$ where $e^\lambda = \cos\sigma$ we get for the Dirac operator (after expanding in modes in $\xi^2 = \sigma'$)

$$\nabla\!\!\!/ = -e^{-\lambda}\rho_0\partial_\tau + e^{-\lambda}\rho_1\partial_\sigma + ip_2\Lambda\rho_2 + \tfrac{1}{2}e^{-\lambda}\big(-\partial_\tau\lambda\rho_0 + \partial_\sigma\lambda\rho_1\big). \tag{G.12}$$

After the conformal transformation of the fermions $\theta \to e^{-\lambda/2}\theta$ it may be written as $\nabla\!\!\!/(e^{-\lambda/2}\theta) = e^{-3\lambda/2}\big(\rho^\alpha\partial_\alpha + ip_2\Lambda\, e^\lambda\rho_2\big)\theta$, with the non-trivial part of the determinant equivalent to the one of the operator in (6.17).

In the string theory limit, i.e. $p_2 = 0$, the rescaled fermionic operator becomes the flat space one and thus has the same spectrum as the massless bosonic operator $\partial_\tau^2 - \partial_\sigma^2$ in (6.15). The "massive" operator in (6.16) may be written as $\Delta + R^{(2)} = \partial_\tau^2 - \partial_\sigma^2 - 2\partial_\sigma^2(\log\cos\sigma)$. A peculiar property of $e^\lambda = \cos\sigma$ is that it is the same as the ground state wave function of the operator $-\partial_\sigma^2$ in the case of the Dirichlet boundary conditions at $\sigma = \pm\frac{1}{2}\pi$. This implies that the massless and "massive" operators are related by the Darboux transformation and have the same spectrum except for the ground state (see, e.g., [39], section 7.1). Explicitly, their eigenfrequencies are, respectively, $\omega_n = n+1$, $\;n = 1, 2, \ldots$ and $\omega_n = n+1$, $\;n = 0, 1, 2, \ldots$. This implies the expected triviality of the 1-loop correction to the energy coming from the string theory modes only.

In the case when $p_2 \neq 0$, we note that both operators Δ in (6.15) and $\Delta + R^{(2)}$ in (6.16) acting on $\mathrm{X}(\tau,\sigma)$ can be written in the form that appears in the angular oblate spheroidal equation[24]

$$\frac{d}{dz}\Big[(1-z^2)\frac{d}{dz}f(z)\Big] + \Big[\omega^2 - \tfrac{1}{4} - \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\Big]f(z) = 0\,, \qquad \text{(G.13)}$$
$$\mathrm{X} = e^{i\omega\tau}(1-z^2)^{1/4}f(z)\,, \qquad z = \sin\sigma,$$

where $\gamma^2 = \Lambda^2 p_2^2$ and $\mu^2 = \frac{1}{4}$ in the case of Δ while $\mu^2 = 2 + \frac{1}{4}$ in the case of $\Delta + R^{(2)}$. For a recent discussion on the eigenvalues of the spheroidal equation, see [40] and references therein. The Fredholm determinant of a more general Heun differential operator was also recently studied in [41].

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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