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Bi-Hamiltonian structures of WDVV-type

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We study a class of nonlinear partial differential equations (PDEs) that admit the same bi-Hamiltonian structure as the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations: a Ferapontov-type first-order Hamiltonian operator and a homogeneous third-order Hamiltonian operator in a canonical Doyle–Potëmin form, which are compatible. Using various equivalence groups, we classify such equations in two-component and three-component cases. In a four-component case we add further evidence to the conjecture that there exists only one integrable system of the above type. Finally, we give an example of the six-component system with required bi-Hamiltonian structure. To streamline the symbolic computation, we develop an algorithm to find the aforementioned Hamiltonian operators, which includes putting forward a conjecture on the structure of the metric parameterizing the first-order Hamiltonian operator.

1. Introduction

(a) Integrable systems and the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations

In the area of infinite-dimensional integrable systems, bi-Hamiltonian systems of partial differential equations (PDEs) play a central role [1–6]. The presence of two independent compatible Hamiltonian structures ensures, under certain mild hypotheses [7], the existence of infinite sequences of commuting conserved quantities and commuting symmetries, thus mimicking the Liouville integrability of the finite-dimensional case.

It was proved by B. Dubrovin that the solutions of WDVV equations can be put in correspondence with a large class of bi-Hamiltonian structures (see Dubrovin [8]), thus providing a bridge between topological field theories and integrable systems. The consequences in mathematics have been far-reaching as solutions of the WDVV equations yield Dubrovin–Frobenius manifolds, which nowadays is an active research topic.

In dimension N, $F = F(t^1, \ldots, t^N)$, and the set of WDVV equations is a nonlinear overdetermined system of PDEs

$$\eta^{\lambda\mu} F_{\lambda\alpha\beta} F_{\mu\nu\gamma} = \eta^{\lambda\mu} F_{\lambda\alpha\nu} F_{\mu\beta\gamma}, \quad \text{where} \quad F_{\alpha\beta\gamma} := \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}.$$

The inverse $(\eta_{\alpha\beta})$ of the constant symmetric non-degenerate matrix $(\eta^{\alpha\beta})$ is by definition $\eta_{\alpha\beta} := F_{1\alpha\beta}$. The WDVV equations are equivalent to the requirement of associativity of a product operation with structure constants $c^\alpha_{\beta\gamma} = \eta^{\alpha\nu} F_{\nu\beta\gamma}$, and therefore they are sometimes also called associativity equations. Note that the requirements on F completely specify its dependence on t^1 up to second-degree polynomials

$$F = \frac{1}{6}\eta_{11}(t^1)^3 + \frac{1}{2}\sum_{k>1} \eta_{1k} t^k (t^1)^2 + \frac{1}{2}\sum_{k,s>1} \eta_{sk} t^s t^k t^1 + f(t^2, \ldots, t^N).$$

The fact that systems whose solutions yield integrable systems can themselves be formulated as integrable systems is a common phenomenon in the field of integrable systems. Thus, it was proved [9] that a system of WDVV equations, after being reformulated as a quasilinear first-order system of PDEs in conservative form

$$u^i_t = (V^i)_x, \tag{1.1}$$

admits a bi-Hamiltonian formulation

$$u^i_t = A^{ij}_k \frac{\delta H_k}{\delta u^j}, \qquad k = 1, 2 \tag{1.2}$$

(we use Einstein's summation convention throughout the paper). Here $u = (u^1, \ldots, u^n)$ is the vector of field variables, (t, x) are the independent variables and V^i are smooth functions of the dependent variables. In addition to its conservative form, the quasilinear system we consider has two further important properties: non-diagonalizability and linear degeneracy. Recall that a quasilinear system $u^i_t = V^i_j u^j_x$ is called diagonalizable if there exists a change of coordinates $r^i = r^i(u)$ such that the transformed system is diagonal, $r^i_t = \tilde{V}^i r^i_x$, and linearly degenerate if $L_p v = 0$ for every pair (v, p), where p is the right eigenvector of the matrix (V^i_j) corresponding to its eigenvalue v (L is the Lie derivative). The pair of compatible Hamiltonian operators A_1, A_2 (compatibility means $A_1 + \lambda A_2$ is a Hamiltonian operator for any $\lambda \in \mathbb{R}$ as well) is different from the pair of compatible Hamiltonian operators, which is determined by a solution of WDVV equations. Bi-Hamiltonianity was further confirmed for other simple cases of WDVV equations in Kalayci & Nutku [10,11]. After some years, advances both in the theory [12] and in symbolic computations [13–16] led to the discovery of new bi-Hamiltonian structures for more complicated WDVV equations [13,14,17,18].

(b) Bi-Hamiltonian structures for WDVV systems

A matrix differential operator $A^{ij} = a^{ij\sigma}(u_\tau)D_\sigma$, where $D_\sigma = D^\sigma_x$, u_τ stands for a finite collection of derivatives of the dependent variables and $\sigma, \tau \in \mathbb{N}_0$, is called Hamiltonian if the corresponding bracket

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$$\{F,G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta u^j}\,\mathrm{d}x$$

is Poisson. More precisely, skew symmetry of the bracket is equivalent to the skew-adjointness of A, and the Jacobi property is equivalent to differential conditions on the coefficients of A that have an intrinsic formulation, the vanishing of the Schouten bracket: $[A, A] = 0$. In turn, the compatibility of two Hamiltonian operators A_1, A_2 is equivalent to the vanishing of the Schouten bracket, $[A_1, A_2] = 0$.

Both Hamiltonian operators, A_1, A_2, admitted by a WDVV system are homogeneous, where homogeneity is defined with respect to the grading $\deg D_x = 1$. Such operators were introduced in Dubrovin & Novikov [19,20] as a family of operators that is form-invariant with respect of diffeomorphisms of the space of field variables.

In particular, the operator A_1 is a first-order homogeneous Hamiltonian operator $A_1 = P$ of Ferapontov type

$$P^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + c^{\alpha\beta} w^i_{\alpha h} u^h_x \mathrm{D}_x^{-1} w^j_{\beta k} u^k_x, \tag{1.3}$$

and the operator A_2 is the third-order Hamiltonian operator $A_2 = R$, where

$$R^{ij} = \mathrm{D}_x (f^{ij}\mathrm{D}_x + c^{ij}_k u^k_x)\mathrm{D}_x, \tag{1.4}$$

which is compatible with P. All coefficient functions g^{ij}, f^{ij}, Γ^{ij}_k, c^{ij}_k and $w^i_{\alpha h}$ mentioned above are functions of the field variables u only, and $c^{\alpha\beta}$ are constants.

For example, in the simplest case, $N = 3$ and

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

the WDVV equations reduce to the form

$$f_{ttt} = f^2_{txx} - f_{xxx} f_{ttx}. \tag{1.5}$$

Following Ferapontov & Mokhov [21], we introduce three new dependent variables $u^1 = f_{xxx}$, $u^2 = f_{txx}$, $u^3 = f_{ttx}$. Then, equation (1.5) can be rewritten in the form

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \quad u^3_t = ((u^2)^2 - u^1u^3)_x. \tag{1.6}$$

It was found in Ferapontov *et al*. [9] that the system (1.6) is bi-Hamiltonian with the local (i.e. $c^{\alpha\beta} = 0$) first-order homogeneous operator P and the third-order homogeneous operator R,

$$P = \begin{pmatrix} -\frac{3}{2}\mathrm{D}_x & \frac{1}{2}\mathrm{D}_x u^1 & \mathrm{D}_x u^2 \\ \frac{1}{2}u^1\mathrm{D}_x & \frac{1}{2}(\mathrm{D}_x u^2 + u^2\mathrm{D}_x) & \frac{3}{2}u^3\mathrm{D}_x + u^3_x \\ u^2\mathrm{D}_x & \frac{3}{2}\mathrm{D}_x u^3 - u^3_x & ((u^2)^2 - u^1u^3)\mathrm{D}_x + \mathrm{D}_x((u^2)^2 - u^1u^3) \end{pmatrix},$$

$$R = \mathrm{D}_x \begin{pmatrix} 0 & 0 & \mathrm{D}_x \\ 0 & \mathrm{D}_x & -\mathrm{D}_x u^1 \\ \mathrm{D}_x & -u^1\mathrm{D}_x & \mathrm{D}_x u^2 + u^2\mathrm{D}_x + u^1\mathrm{D}_x u^1 \end{pmatrix} \mathrm{D}_x.$$

The corresponding Hamiltonian densities are

$$h_P = u^3 \quad \text{and} \quad h_R = -\tfrac{1}{2}u^1\left(\mathrm{D}_x^{-1}u^2\right)^2 - (\mathrm{D}_x^{-1}u^2)(\mathrm{D}_x^{-1}u^3).$$

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Remark 1.1. We stress that the type of the bi-Hamiltonian pair for the WDVV equations is different from the bi-Hamiltonian pair determined by a solution of the WDVV equations. Indeed, the latter is constituted by two compatible local homogeneous Hamiltonian operators of order one [8,22].

(c) Problem and results

Definition 1.1. We define a *bi-Hamiltonian structure of WDVV-type* to be a pair of compatible Hamiltonian operators P, R as in relations (1.3) and (1.4), respectively.

We say a quasilinear first-order system of PDEs in conservative form (1.1) is a *bi-Hamiltonian system of WDVV-type* if it is endowed with a bi-Hamiltonian structure of WDVV-type.

The problem: In this paper, we aim at classifying (when possible) bi-Hamiltonian equations of WDVV-type and at the same time, introducing new bi-Hamiltonian equations of WDVV-type, which are not necessarily related to the WDVV equations.

As this task presents several theoretical and computational challenges, our paper contains both the theoretical advances and the computational algorithms that made this research possible.

The main results of the paper are listed below, following the number of unknown functions u^i. Note that we discard linear bi-Hamiltonian systems of WDVV-type as we regard them as trivial.

- $n=2$: We have an affine classification of bi-Hamiltonian structures of WDVV-type, according to which there exist two classes of nonlinear bi-Hamiltonian systems of WDVV-type. Both cases are linearizable if we enlarge the group action to the group of projective reciprocal transformations that preserve t.
- $n=3$: In this case, under the action of the group of projective reciprocal transformations that preserve t, we have five non-trivial cases. Three of them are particular types of WDVV equations known in the literature, while two of them are new. To compute the leading coefficient of the operator P, we put forward conjecture 2.1 in §2*c* that proved to be true in all known cases. Then, under the action of the full group of projective reciprocal transformations, it can be shown that all non-trivial cases reduce to one, the simplest WDVV equation. Unfortunately, the corresponding transformation is practically impossible to be found with current computational tools. This implies that our methods, which allow us to find the bi-Hamiltonian structure, are still highly valuable.
- $n=4$: In this case, systems with third-order Hamiltonian structures have been classified; however, only one of them is known to be integrable. The system is obtained within the geometric theory of linearly degenerate systems in the Temple class [23] (no relationship with WDVV). We found a new bi-Hamiltonian structure of WDVV-type for this system. In some other interesting cases, we proved that such structures do not exist, although we cannot guarantee this in general.
- $n=6$: We proved that two commuting systems of the type (6.1) are bi-Hamiltonian systems of WDVV-type. These are two integrable systems that arise from integrability conditions of a certain class of Lagrangians [24]. The systems are not related to the WDVV equations; this, together with the previous item, proves that the class of bi-Hamiltonian systems of WDVV-type does not reduce to WDVV equations only.

It is worth comparing bi-Hamiltonian structures in this paper with another family of bi-Hamiltonian structures that have been introduced in Lorenzoni *et al.* [25]. Such structures are of the form $A_1=P$, $A_2=Q+\epsilon^2 R$ and are said to be bi-Hamiltonian structures of KdV-type. The operators (P,Q,R) are a compatible triple of homogeneous Hamiltonian operators, with P, Q being of the form (1.3) and R being a higher order homogeneous Hamiltonian operator. Note that ϵ is introduced as a perturbative parameter. For example, the triple

$$P = D_x, \quad Q = \frac{2}{3}u\mathrm{D}_x + \frac{1}{3}u_x, \quad R = \mathrm{D}_x^3$$

provides a bi-Hamiltonian structure for the KdV equation. Many integrable systems have this form (see Lorenzoni *et al*. [25] for a detailed list of examples and references).

Remark 1.2. When comparing bi-Hamiltonian structures of KdV- and WDVV-type, it is evident that *we can treat the bi-Hamiltonian structures of WDVV-type as a singular case of bi-Hamiltonian structures of KdV-type*. The 'singularity' of the bi-Hamiltonian pair goes in the opposite direction with respect to the usual dispersionless limit $\epsilon \to 0$.

The results obtained in this article show that the class of bi-Hamiltonian structures of WDVV-type is rich and interesting, and deserves further investigation.

(d) Computational problems and their solutions

Generally speaking, finding Hamiltonian operators for systems of PDEs of the type 1.1 in high ($n \geq 3$) dimension, and proving their compatibility is not an easy task. The computational problems to be solved are summarized below:

- Finding a Hamiltonian formulation (1.2) with a first-order operator P as in relation (1.3) amounts to solving a complicated system of nonlinear PDEs. Such a system is not easily solvable even when $n = 3$.
- On the other hand, third-order operators are classified [26,27] in low dimensions, as well as the systems of the form 1.1 that admit a corresponding Hamiltonian formulation [12]. Hence, we can start from a third-order operator R from the classification and the corresponding systems and find first-order Hamiltonian formulations (1.2).
- First-order operators P for the aforementioned systems of PDEs are found by solving the system mentioned above, with the additional requirement of compatibility: $[P, R] = 0$.

Compatibility is the major computational problem here. First, until recently, there did not exist a way to bring the Schouten bracket between non-local operators in a canonical form and require its vanishing. In Casati *et al*. [28] an algorithm was presented. However, finding a *minimal* set of conditions that are equivalent to $[P, R] = 0$ is the problem that is still unresolved.

In Casati *et al*. [29] the aforementioned algorithm was implemented in the computer algebra systems Maple, Mathematica and Reduce. Thus, in principle, we can compute $[P, R]$ and require its vanishing if one of the two operators is unknown. However, the computational complexity is sometimes overwhelming, even for dedicated compute servers, and the main problem is to keep the complexity within the limit of what our computers can do.

The requirement $[P, R] = 0$ for unknown P leads to a complicated overdetermined nonlinear system of PDEs that we are able to solve only in dimension $n = 2$ and in the simplest case in dimension 3. Working in the Maple computer algebra system, we used `rifsimp` to reduce the equations to an involutive system with a known dimension of finite-dimensional solution space and then `pdsolve` to solve it.

For more general calculations, we used conjecture 2.1; it yields a condition that we always verified in concrete compatible operators P and R. Such a condition is very likely to be an important part of the compatibility conditions and brings again the problem of finding P to a system of algebraic equations, which becomes manageable also in high dimensions.

However, that is not all: we still need to check that $[P, R] = 0$, although at this point that is only a straightforward computation as P and R are completely specified. Not all such calculations can be performed on a modern laptop: primarily, huge amounts of RAM are needed as the formula of the Schouten bracket (see §2*c*), and the algorithm for bringing the expressions to a canonical form [28] implies the calculation of iterated derivatives of large rational expressions, leading to expression swell that only at the very end simplifies to zero (if that is the case).

In particular, the largest successful direct calculations have been performed for the calculations of $[P, Q]$ in the case $R^{(2)}$ $(n = 3)$ (15 GB of RAM and 90 h of computing time) and in the case $n = 4$. In the case $R^{(1)}$ $(n = 3)$, the calculation of $[P, Q]$ failed for lack of RAM. This led to a re-analysis of the algorithm in Casati *et al.* [28] and a work-around has been found. Since in the computation all denominators are of the form σ^n, where σ is the singular variety of the corresponding Monge metric, while n varies, an expression σ^{-1} was given a new notation, effectively making all rational expressions polynomials. This modification led to a much-reduced RAM consumption in case $R^{(2)}$ (only 2 GB) and manyfold reduction of computing time and allowed us to prove the compatibility of the Hamiltonian operators $R^{(1)}$ and $P^{(1)}$ using 45 GB of RAM and 54 h of computing time.

Surprisingly, the case $n = 6$ was manageable on a laptop with 16 GB of RAM, mostly owing to its locality.

All large calculations have been performed on a compute server of the Istituto Nazionale di Fisica Nucleare (INFN), Section of Lecce. The server has a processor AMD EPYC 7282 and 256 GB of RAM. The most important computations programmed by us are available at Opanasenko & Vitolo [30], as well as in the form of Electronic Supplementary Material that accompanies this paper; we will be happy to help other researchers who are interested in our software or similar computational tasks.

2. A pair of Hamiltonian structures

Our goal is to classify quasilinear first-order systems of evolutionary PDEs of the type 1.1 that are endowed with a pair of Hamiltonian operators (1.3) and (1.4).

(a) First-order Hamiltonian operator of Ferapontov type

A Hamiltonian operator P,

$$P^{ij} \;=\; g^{ij}\mathrm{D}_x + \Gamma^{ij}_s u^s_x + c^{\alpha\beta} w^i_{\alpha s} u^s_x \mathrm{D}_x^{-1} w^j_{\beta t} u^t_x,$$

is a non-local generalization of a classical Dubrovin–Novikov Hamiltonian operator ($c^{\alpha\beta} = 0$), which was introduced by Ferapontov in Ferapontov [31]. The operator P is Hamiltonian if and only if the following properties are fulfilled (we assume the non-degeneracy condition $\det(g^{ij}) \neq 0$):

$$g^{ij} = g^{ji}, \quad g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k, \quad g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s, \tag{2.1a}$$

$$g^{is} w^j_{\alpha s} = g^{js} w^i_{\alpha s}, \tag{2.1b}$$

$$\nabla_k w^i_{\alpha j} = \nabla_j w^i_{\alpha k}, \tag{2.1c}$$

$$[w_\alpha, w_\beta] = 0, \tag{2.1d}$$

$$c^{\alpha\beta} = c^{\beta\alpha}, \tag{2.1e}$$

$$R^{ij}_{kl} = c^{\alpha\beta}\left(w^i_{\alpha l} w^j_{\beta k} - w^i_{\alpha k} w^j_{\beta l}\right). \tag{2.1f}$$

Here and below, $g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial u^k}$ and $w_\alpha = w^i_{\alpha j} u^j_x \frac{\partial}{\partial u^i}$. We observe that the conditions (2.1e) and (2.1f) differ from those found in the literature, since the operator P is a slightly generalized form of an operator originally introduced in Ferapontov [31], where the matrix $(c^{\alpha\beta})$ is diagonal. Nonetheless, they can easily be deduced from the calculations in Casati *et al.* [28] after replacing the ansatz for the first-order operator with the ansatz (1.3).

The number m of the commuting vector fields (and hence the dimension m of the symmetric matrix $(c^{\alpha\beta})$) is not known *a priori*. However, an important computational feature[1] of linearly degenerate non-diagonalizable systems is that in low dimension ($n \leq 5$), they only admit two commuting flows of first order, namely the x- and t-translations, represented by

$$w_1 = V^i_j u^j_x \frac{\partial}{\partial u^i} \quad \text{and} \quad w_2 = u^i_x \frac{\partial}{\partial u^i}.$$

All but one of our examples fall within this category of equations, and the one that does not is local, see §6, and therefore we shall restrict ourselves to this case hereafter.

The aforementioned operator P is clearly invariant with respect to local diffeomorphisms of the dependent variables: $\tilde{u} = \tilde{U}(u)$. Under such transformations, g^{ij} transform as a contravariant 2-tensor and Γ^{ij}_k transform as the contravariant Christoffel symbols of a linear connection. Thus, geometrically the conditions (2.1) mean that if $(g_{ij}) = (g^{ij})^{-1}$, then $\Gamma^i_{jk} = -g_{jh}\Gamma^{hi}_k$ are the Christoffel symbols of the Levi–Civita connection of the metric g_{ij}. Its curvature, expressed via the Kobayashi–Nomizu convention

$$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^i_{kr}\Gamma^r_{lj} - \Gamma^i_{lr}\Gamma^r_{kj},$$

admits the expansion (2.1f), where

$$R^{ijk}_l = g^{is}g^{jt}R^k_{tsl} = g^{is}(\partial_l\Gamma^{jk}_s - \partial_s\Gamma^{jk}_l) + \Gamma^{ij}_s\Gamma^{sk}_l - \Gamma^{sj}_l\Gamma^{ik}_s$$

and $R^{jk}_{hl} = g_{hi}R^{ijk}_l$. The equations (2.1b)–(2.1f) are nothing else but the Gauss–Peterson–Codazzi equations for submanifolds M^n with a flat normal connection in a (pseudo-) Euclidean space of dimension $n + N$, where the metric g plays the role of the first quadratic form of M^n, and w_α are the Weingarten operators corresponding to the field of pairwise orthogonal unit normals.

It was also proved [32] that the non-locality type of P is preserved under linear transformations of the independent variables (t, x).

Finally, see Ferapontov [31] and Maltsev & Novikov [33] for a discussion on how to define a Hamiltonian for the operator P in the non-local case.

(b) Third-order Hamiltonian operator in the Doyle–Potëmin canonical form

The most general form of a third-order homogeneous Hamiltonian operator, in accordance with the definition given in Dubrovin & Novikov [19], is

$$R^{ij} = g^{ij}_3 D^3_x + b^{ij}_{3\,s} u^s_x D^2_x + [c^{ij}_{3\,s} u^s_{xx} + c^{ij}_{3\,st} u^s_x u^t_x] D_x + d^{ij}_{3\,s} u^s_{xxx} + d^{ij}_{3\,st} u^s_x u^t_{xx} + d^{ij}_{3\,srt} u^s_x u^r_x u^t_x . \tag{2.2}$$

Nevertheless, in view of the computational difficulties, a minimal set of conditions on the coefficient of the above-mentioned operator R that is equivalent to its skew-adjointness and $[R, R] = 0$ is not known.

It was independently proved by Doyle [34] and Potëmin [35,36] that there always exists a change of dependent variables that brings a Hamiltonian operator of the form (2.2) into the *Doyle–Potëmin canonical form* (1.4). This form has a drastically reduced set of coefficients, and the Hamiltonian property of such an operator is equivalent to

$$c_{ijk} = \frac{1}{3}(f_{ik,j} - f_{ij,k}), \tag{2.3a}$$

$$f_{ij,k} + f_{jk,i} + f_{ki,j} = 0, \tag{2.3b}$$

[1]It is not a theorem, but an experimental observation communicated to us by E.V. Ferapontov.

$$c_{ijk,\,l} = -f^{pq} c_{pil} c_{qjk}. \tag{2.3c}$$

Here, $c_{ijk} = f_{iq} f_{jp} c^{pq}_k$ and, of course, $(f_{ij}) = (f^{ij})^{-1}$.

A covariant 2-tensor f satisfying equation (2.3b) must be a Monge metric of a quadratic line complex (see Ferapontov *et al*. [26] and reference therein). This is an algebraic variety in the Plucker embedding of the projective space for which the affine chart is (u^i). In particular, being Monge, f_{ij} is a quadratic polynomial in the field variables u.

It was further proved in refs. [34–37] that the tensor f_{ij} can be factorized as

$$f_{ij} = \phi_{\alpha\beta} \psi^{\alpha}_i \psi^{\beta}_j, \quad \left(\text{or, in a matrix form,} \quad f = \Psi\Phi\Psi^{\top}\right), \tag{2.4}$$

where ϕ is a constant non-degenerate symmetric matrix of dimension n, and

$$\psi^{\gamma}_k = \psi^{\gamma}_{ks} u^s + \omega^{\gamma}_k$$

with the constants ψ^{γ}_{ij} and ω^{γ}_k satisfying the relations

$$\begin{aligned}
\psi^{\gamma}_{ij} &= -\psi^{\gamma}_{ji},\\
\phi_{\beta\gamma}(\psi^{\beta}_{il}\psi^{\gamma}_{jk} + \psi^{\beta}_{jl}\psi^{\gamma}_{ki} + \psi^{\beta}_{kl}\psi^{\gamma}_{ij}) &= 0,\\
\phi_{\beta\gamma}(\omega^{\beta}_{i}\psi^{\gamma}_{jk} + \omega^{\beta}_{j}\psi^{\gamma}_{ki} + \omega^{\beta}_{k}\psi^{\gamma}_{ij}) &= 0.
\end{aligned}$$

The interested reader can find the general expression of the Hamiltonian for the operator R (provided it exists for a given system) in Ferapontov *et al*. [12].

The system that admits the operator R in a canonical form is evidently conservative. On the other hand, if a system admits such an operator, then different conservative forms thereof may not necessarily admit third-order Hamiltonian operator in a canonical form, see discussion in Ferapontov *et al*. [9, p. 664].

(c) Compatibility and a conjecture

The only missing components of the picture are the conditions under which the operators P and R are compatible, i.e. their Schouten bracket vanishes: $[P, R] = 0$. We use the definition of the Schouten bracket in Casati *et al*. [28]

$$[A_1, A_2](\psi^1, \psi^2, \psi^3) = \left[(\ell_{A_1,\psi^1}(A_2(\psi^2)))\psi^3 + (\ell_{A_2,\psi^1}(A_1(\psi^2)))\psi^3\right] + \text{cyclic}(\psi^1, \psi^2, \psi^3),$$

where $\psi^a = (\psi^a_i(x, u_\sigma))$, $a = 1,2,3$, are covectors and the square brackets on the right-hand side mean that the result is to be considered up to the image of D_x (i.e. 'total divergencies'). In the aforementioned formula, we use the linearization of an operator A [1]: if, in local coordinates, one has $A(\psi)^i = a^{ij\sigma} D_\sigma \psi_j$, then

$$\ell_{A,\psi}(\phi)^i = \frac{\partial a^{it\sigma}}{\partial u^s_\tau} D_\sigma \psi_t D_\tau \phi^s, \quad \phi = (\phi^s(x, u_\sigma))\,.$$

Casati *et al*. [28] presented an algorithm to compute a divergence-free form of the bracket $[A_1, A_2]$ for a wide class of non-local operators (weakly non-local operators), to which Ferapontov-type operators belong. Based on this algorithm, in Casati *et al*. [29], software packages were developed for Mathematica, Maple and Reduce.

A minimal set of conditions that are equivalent to $[P, R] = 0$ would be very useful, but so far a solution to this problem has been out of reach. An alternative is to determine a suitable ansatz for Hamiltonian operators. Known examples of bi-Hamiltonian systems of WDVV-type [14] allow us to formulate the following conjecture.

Conjecture 2.1. Let a Ferapontov-type operator P parameterized by a metric g_{ij} and a Doyle–Potëmin operator R parameterized by a Monge metric f_{ij} with the Monge decomposition $f_{ij} = \psi_i^\alpha \phi_{\alpha\beta} \psi_j^\beta$ be compatible. Then the symmetric matrix

$$Q^{\alpha\beta} = \psi_i^\alpha g^{ij} \psi_j^\beta$$

has entries that are second-degree polynomials in the field variables.

In turn, it means that a suitable ansatz for g is

$$g^{ij} = \psi_\alpha^i Q^{\alpha\beta} \psi_\beta^j \quad \text{or, in a matrix form,} \quad g = \Psi^{-1} Q (\Psi^{-1})^\top.$$

Therefore, whenever finding a Ferapontov-type operator with a generic ansatz is not feasible, we resort to the aforementioned ansatz. Luckily, when we are able to make a general computation without resorting to an ansatz, the result is always within the ansatz, thus indicating that the ansatz might be a feature of all first-order operators P that are compatible with a third-order operator R as mentioned above.

Finally, note that although $(Q^{\alpha\beta})$ is quadratic in its entries, it is not a Monge metric in general.

(d) Classification of bi-Hamiltonian equations of WDVV-type

We classify quasilinear first-order systems of PDEs in a conservative form (1.1),

$$u_t = (V^i)_x = V_s^i u_x^s,$$

that are bi-Hamiltonian with respect to a pair of operators P, R. Here, P is a Ferapontov operator of the form

$$P^{ij} = g^{ij} \mathrm{D}_x + \Gamma_s^{ij} u_x^s + c^{11} V_s^i u_x^s \mathrm{D}_x^{-1} V_r^j u_x^r + c^{12} \left(V_s^i u_x^s \mathrm{D}_x^{-1} u_x^j + u_x^i \mathrm{D}_x^{-1} V_s^j u_x^s \right) + c^{22} u_x^i \mathrm{D}_x^{-1} u_x^j;$$

Note that we used only two commuting vector fields in P, in view of the discussion in §2*a*. The operator R is a third-order homogeneous local Hamiltonian operator, R, and can always be brought to the Doyle–Potëmin canonical form by a transformation of the dependent variables. Such a transformation neither changes the locality (resp., non-locality) of the Ferapontov operator, P, nor the shape of the system (1.1).

In turn, Doyle–Potëmin operators are classified for $n = 1, \dots, 4$ with respect to several group actions [26,27]:

- the maximal group of transformations of the dependent variables that preserves the Doyle–Potëmin canonical form of R—the group of affine transformations $\tilde{u}^i = A_j^i u^j + A_0^i$ with constant A_j^i and A_0^i;
- the group of projective reciprocal transformations that fix t (as it was found in Ferapontov *et al.* [26])

$$\tilde{u}^i = \frac{A_j^i u^j + A_0^i}{\Delta}, \qquad \Delta = A_j^0 u^j + A_0^0,$$
$$\mathrm{d}\tilde{x} = \Delta\, \mathrm{d}x + (A_i^0 V^i + A_0^0) \mathrm{d}t, \qquad \mathrm{d}\tilde{t} = \mathrm{d}t; \tag{2.5}$$

- the group of general projective reciprocal transformations [27], which is generated by a t-fixing projective reciprocal transformation as mentioned above and an $x \leftrightarrow t$ inversion.

It is clear that each of the above-mentioned groups is a subgroup of the next one on the list. We will use the groups to classify bi-Hamiltonian systems of WDVV-type in low dimensions.

It was proved by Ferapontov *et al*. [12] that a first-order system of conservation laws, system 1.1, admits a third-order homogeneous Hamiltonian operator (in a canonical form (1.4)) if and only if

$$f_{is}V^s_j = f_{js}V^s_i, \tag{2.6a}$$

$$c_{skj}V^s_i + c_{sik}V^s_j + c_{sji}V^s_k = 0, \tag{2.6b}$$

$$f_{ks}V^s_{ij} = c_{ksj}V^s_i + c_{ksi}V^s_j. \tag{2.6c}$$

It is important to recall that systems of the type 1.1 that are Hamiltonian with respect to third-order operators are linearly degenerate (or weakly nonlinear, in another terminology) and non-diagonalizable [12].

The system (2.6) can be used in two ways. Given a conservative system defined by (V^i), one can determine the matrix f parameterizing a third-order operator R. Since the coefficients of f are second-degree polynomials, the problem becomes algebraic in nature. On the other hand, given the matrix f one can determine all the systems that are Hamiltonian with respect to the corresponding operator R. This problem is completely solved in Ferapontov *et al*. [12]: we have $V^i = \psi^i_\gamma W^\gamma$, where $W^\gamma = \eta^\gamma_s u^s + \xi^\gamma$, and the constants η^γ_m and ξ^γ fulfil a certain linear algebraic system.

Therefore, given a Doyle–Potëmin operator R, it is an algebraic problem given by the system (2.6) to find all quasilinear systems in a conservative form that are Hamiltonian with respect to the operator R. We simplify the obtained family of systems by equivalence transformations to get a subfamily $\mathcal{S}$ of nonlinear systems, $u^i_t = V^i_j u^j_x$. In particular, every third-order homogeneous Hamiltonian operator in dimension one can be transformed to D^3_x using transformations of the dependent variables only. But the quasilinear conservative systems that are Hamiltonian with respect to D^3_x are linear owing to system (2.6). Therefore, we omit altogether the dimension one in the classification below.

Note also that systems as mentioned above do not change their form when subject to a general projective reciprocal transformation (see Ferapontov *et al*. [12]).

Finally, we check if the family $\mathcal{S}$ admits also a Ferapontov operator P. This is done by direct computation, assisted by computer algebra.

We use the packages from the paper Casati *et al*. [29] to compute the Schouten bracket $[P, R]$, with given R and unknown P (when possible). That is, we collect all coefficients of the Schouten bracket as a three-vector and require them to vanish; add to this system the system of conditions under which P is Hamiltonian. If we can solve this system, we have an answer.

Otherwise, we assume conjecture 2.1 to hold. Finding a Ferapontov operator thus reduces to the three following steps. Firstly, we solve the commutativity condition $g^{is}V^j_s = g^{js}V^i_s$. Next, we solve the commutativity condition $\Gamma^{sij}V^k_s = \Gamma^{skj}V^i_s$ coming from system (2.1b) and the fact that the system $\mathcal{S}$ is in a conservative form, where $\Gamma^{lij} = \frac{1}{2}(g^{is}g^{jl}_{,s} + g^{ls}g^{ji}_{,s} - g^{js}g^{il}_{,s})$. At this point, the matrix Q and therefore g is completely determined and to find the matrix $(c^{\alpha\beta})$ we solve the system (2.1f).

Strictly speaking, we do not have a proof of the fact that the operator P preserves its shape under a general projective reciprocal transformation, thus remaining of Ferapontov type. However, the transformed operator will be again homogeneous of degree 1, and computational experiments show that it will be again of the same type. We conjecture that this is a general property.

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3. Two-component systems

(a) Affine classification

It was shown by Ferapontov *et al*. [26] that up to affine transformations $\tilde{u}^i = A^i_j u^j + A^i_0$, $i = 1, \ldots, n$, there are three distinct third-order homogeneous Hamiltonian operators

$$R^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_x^3, \quad R^{(2)} = D_x \begin{pmatrix} 0 & D_x \dfrac{1}{u^1} \\ \dfrac{1}{u^1} D_x & \dfrac{u^2}{(u^1)^2} D_x + D_x \dfrac{u^2}{(u^1)^2} \end{pmatrix} D_x,$$

$$R^{(3)} = D_x \begin{pmatrix} D_x & D_x \dfrac{u^2}{u^1} \\ \dfrac{u^2}{u^1} D_x & \dfrac{(u^2)^2+1}{2(u^1)^2} D_x + D_x \dfrac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} D_x.$$

The operator $R^{(1)}$ is admitted by linear systems of PDEs, and therefore we pay no attention to it here.

The operator $R^{(2)}$ is admitted by a family of quasilinear systems

$$u^1_t = (\alpha u^1 + \beta u^2)_x, \quad u^2_t = \left(\alpha u^2 + \frac{\beta (u^2)^2 + \gamma}{u^1}\right)_x.$$

There are two inequivalent cases: $(\beta, \alpha) = (1,0)$ and $(\beta, \alpha) = (0,1)$. The second case is degenerate in the sense that the corresponding system is partially coupled and one equation can be solved explicitly, effectively making the second one linear. So, we will focus on the first case only.

Theorem 3.1. *The system*

$$u^1_t = u^2_x, \quad u^2_t = \left(\tfrac{(u^2)^2+\gamma}{u^1}\right)_x$$

is Hamiltonian with respect to three first-order local homogeneous Hamiltonian operators $P^{(2,i)}$, $i = 1, 2, 3$, *parameterized by the metrics*

$$g^{(2,1)} = \begin{pmatrix} -u^1 & 0 \\ 0 & \dfrac{(u^2)^2+\gamma}{u^1} \end{pmatrix}, \quad g^{(2,2)} = \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix}, \quad g^{(2,3)} = \begin{pmatrix} 2u^2 & \dfrac{(u^2)^2+\gamma}{u^1} \\ \dfrac{(u^2)^2+\gamma}{u^1} & 0 \end{pmatrix}.$$

The first-order operators mentioned above are mutually compatible as well as compatible with the operator $R^{(2)}$.

The operators $P^{(2,i)}$ belong to the list [25] of all first-order local homogeneous Hamiltonian operators that are compatible with $R^{(2)}$.

The operator $R^{(3)}$ is admitted by a family of quasilinear systems

$$u^1_t = (\alpha u^1 + \beta u^2)_x, \quad u^2_t = \left(\alpha u^2 + \frac{\beta (u^2)^2 + \gamma u^2 - \beta}{u^1}\right)_x.$$

Again, there are two inequivalent cases: $(\beta, \alpha) = (1, 0)$ and $(\beta, \alpha) = (0, 1)$, and we will only deal with the first one.

Theorem 3.2. *The system*

$$u^1_t = u^2_x, \quad u^2_t = \left(\tfrac{(u^2)^2+\gamma u^2-1}{u^1}\right)_x$$

is Hamiltonian with respect to three first-order local homogeneous Hamiltonian operators $P^{(3,i)}$, $i = 1,2,3$, parameterized by the metrics

$$g^{(2,1)} = \begin{pmatrix} -u^1 & 0 \\ 0 & \dfrac{(u^2)^2+\gamma}{u^1} \end{pmatrix}, \quad g^{(2,2)} = \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix}, \quad g^{(2,3)} = \begin{pmatrix} 2u^2 & \dfrac{(u^2)^2+\gamma}{u^1} \\ \dfrac{(u^2)^2+\gamma}{u^1} & 0 \end{pmatrix}.$$

The first-order operators mentioned above are mutually compatible as well as compatible with the operator $R^{(3)}$.

Again, the operators $P^{(3,i)}$ belong to the list [25] of all first-order local homogeneous Hamiltonian operators that are compatible with $R^{(3)}$. To the best of our knowledge, the multi-Hamiltonian systems in theorems 3.1 and 3.2 are not known in the literature.

Proposition 3.1. *Each of the metrics $(g^{ij})_{i,j=1}^{2}$ of first-order local homogeneous Hamiltonian operators P that are compatible with a third-order local homogeneous Hamiltonian operator R (with Monge metric $f_{ij} = \phi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$), listed in Lorenzoni et al. [25], can be factorized as in conjecture 2.1:*

$$g^{ij} = \psi^i_\alpha Q^{\alpha\beta}\psi^j_\beta,$$

where $Q = (Q^{\alpha\beta})$ is a symmetric matrix whose entries are quadratic polynomials of field variables.

As an example, the Monge metric $f^{(3)}$ of the operator $R^{(3)}$ admits the decomposition (2.4) with

$$h_3 = \begin{pmatrix} (u^2)^2+1 & -u^1u^2 \\ -u^1u^2 & (u^1)^2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} -u^2 & 1 \\ u^1 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

while the Hamiltonian operator of Dubrovin–Novikov type parameterized by the metric $g := c_1g^{(3,1)} + c_2g^{(3,2)} + c_3g^{(3,3)}$ admits the decomposition (2.1) with

$$Q = \begin{pmatrix} \gamma(c_2-c_3)(u^2)^2 + (c_1\gamma u^2 - c_1)u^1 + 2c_3u^2 & c_2(u^1)^2 + c_1u^1u^2 - c_3(u^2)^2 - c_3 \\ c_2(u^1)^2 + c_1u^1u^2 - c_3(u^2)^2 - c_3 & -c_1u^1 + 2c_3u^2 + c_3\gamma \end{pmatrix}$$

(b) Projective classification

It follows from the works of Agafonov & Ferapontov [38–40] that the quasilinear first-order systems that are Hamiltonian with respect to a third-order homogeneous Hamiltonian operator [12] correspond to linear line congruences. These are algebraic varieties in the Plucker embedding of the Grassmannian of lines in $\mathbb{P}^{n+1}$.

The classical works of Castelnuovo imply that all such congruences can be transformed to a single one (see Ferapontov *et al*. [12]) when $n = 2$, which implies that both operators $R^{(2)}$ and $R^{(3)}$ can be transformed to $R^{(1)}$ by a reciprocal projective transformation. Thus, there is no interesting case in dimension 2 under the action of the above-mentioned group.

More precisely, we have the following result.

Proposition 3.2. *Any two-component quasilinear system in a conservative form is linearizable by a reciprocal projective transformation.*

It can be very difficult to find the above-mentioned linearizing reciprocal projective transformation explicitly, even in this low-dimensional case. Alternatively, as a canonical case of the classification one can take a physically relevant system. Thus, the Chaplygin gas system

$$u_t + uu_x + \frac{v_x}{v^3} = 0, \quad v_t + (uv)_x = 0,$$

is known [41] to admit three first-order Dubrovin–Novikov Hamiltonian operators. Moreover, as a subcase of a polytropic gas dynamics system

$$u_t + uu_x + v^\gamma v_x = 0, \quad v_t + (uv)_x = 0,$$

it admits [42] a non-homogeneous third-order Hamiltonian operator. But it also admits a homogeneous one. Indeed, the diagonalized form of the Chaplygin gas system [41] is

$$U_t = VU_x, \quad V_t = UV_x, \quad \left(\text{with} \quad U = u - \frac{1}{v}, \quad V = u + \frac{1}{v}\right)$$

to which the system

$$u^1_t = u^2_x, \quad u^2_t = \left(\frac{(u^2)^2 - a^2}{u^1}\right)_x$$

is reduced with the help of the point transformation of the dependent variables

$$u^1 = \frac{2a}{U - V}, \quad u^2 = \frac{a(U+V)}{U - V},$$

and the system

$$u^1_t = u^2_x, \quad u^2_t = \left(\frac{(u^2)^2 + \gamma u^2 - 1}{u^1}\right)_x$$

is reduced with the help of the point transformation of the dependent variables

$$u^1 = -\frac{\sqrt{\gamma^2 + 4}}{U - V}, \quad u^2 = -\frac{\sqrt{\gamma^2+4}}{2}\frac{U+V}{U-V} - \frac{\gamma}{2}.$$

Since both pairs of transformations preserve the homogeneity of Hamiltonian operators, the Chaplygin gas system possesses a homogeneous third-order Hamiltonian operator (which is not in Doyle–Potëmin canonical form).

4. Three-component systems

An affine classification is no longer feasible when $n > 2$, owing to the large amount of cases and subcases that would result. We resort to two distinct projective classifications.

(a) Partial projective classification

Under the action of projective reciprocal transformations that fix t, there are six classes of third-order homogeneous Hamiltonian operators $R^{(i)}$ (see Ferapontov *et al.* [26]). They are defined by the following Monge metrics:

$$f^{(1)} = \begin{pmatrix} (u^2)^2 + \mu & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + \mu(u^3)^2 & -\mu u^2u^3 - u^1 \\ 2u^2 & -\mu u^2 u^3 - u^1 & \mu(u^2)^2 + 1 \end{pmatrix}, \quad f^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix},$$
$$f^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding quasilinear first-order systems of evolutionary PDEs have been found and described in Ferapontov *et al.* [12,27].

Case $R^{(6)}$. The system which is Hamiltonian with respect to $R^{(6)}$ is linear, hence it is out of consideration here.

Case $R^{(5)}$. The conservative quasilinear system of PDEs that is determined by the operator $R^{(5)}$ is system (1.6), which is equivalent to a WDVV equation. The bi-Hamiltonian pair for such a system was found in Ferapontov *et al*. [9] (see §1).

Case $R^{(4)}$. The system of PDEs that is determined by $R^{(4)}$ is

$$u^1_t = u^2_x, \quad u^2_t = \left(\frac{(u^2)^2 + u^3}{u^1}\right)_x, \quad u^3_t = u^1_x,$$

Setting $u^1 = f_{xxt}$, $u^2 = f_{xtt}$, $u^3 = f_{xxx}$, we obtain $f_{xxx} = f_{ttt}f_{xxt} - f^2_{xtt}$, which is equivalent to WDVV equation (1.5) under the interchange of x and t. Its bi-Hamiltonian representation by means of a compatible pair P and $R^{(4)}$ as in this paper was constructed in Kalayci & Nutku [10,11].

Case $R^{(3)}$. The integrability of the system of PDEs determined by $R^{(3)}$,

$$u^1_t = (u^2 + u^3)_x, \quad u^2_t = \left(\frac{u^2(u^2 + u^3) - 1}{u^1}\right)_x, \quad u^3_t = u^1_x,$$

was first determined in Agafonov [43]. Its third-order Hamiltonian structure was found in Ferapontov *et al*. [12]. In Vašíček & Vitolo [14], a criterion from Bogoyavlenskij [44] was used to find the metric g of a first-order operator. That is, for non-diagonalizable quasilinear systems of PDEs, the metric g is proportional to a contraction of the square of the Haantjes tensor of the velocity matrix (V^i_j) of the system

$$g_{ij} = f(u)H^{\alpha}_{i\beta}H^{\beta}_{j\alpha}$$

(see Bogoyavlenskij [44] for the definition of the Haantjies tensor).

In this case the first-order homogeneous Hamiltonian operator P was non-local, $c^{11} = c^{22} = -1$, $c^{12} = c^{21} = 0$. The metric is written down in Casati *et al*. [29], and here we present it in accordance with conjecture 2.1. The Monge metric $f^{(3)}$ admits a Monge decomposition $(f^{(3)}_{ij}) = \Psi\Phi\Psi^\top$, where the metric $g = \Psi^{-1}Q(\Psi^{-1})^\top$, where

$$Q^{11} = 4(u^1)^2 + (u^2)^2 + 1, \quad Q^{12} = -3u^1, \quad Q^{13} = -2u^2 - u^3, \quad Q^{22} = (u^1)^2 + (u^3)^3 + 4,$$

$$Q^{23} = u^1(u^2 + 2u^3), \quad Q^{33} = (u^1)^2 + (u^2 + 2u^3)^2 + 1,$$

$$\Psi = (\psi^\alpha_i) = \begin{pmatrix} -u^2 & 0 & 1 \\ u^1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Phi = (\phi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The system of PDEs that is determined by $R^{(3)}$ is equivalent to a particular WDVV equation [8] obtained when $\eta = \mathrm{Id}$, the identity matrix. Setting $u^1 = f_{xxt}$, $u^2 = f_{xtt} - f_{xxx}$, $u^3 = f_{xxx}$, we have

$$f^2_{xxt} - f_{xxx}f_{xtt} + f^2_{xtt} - f_{xxt}f_{ttt} - 1 = 0.$$

At this point, it is natural to ask the question, in case $n = 3$, if all the items in the projective classification of operators R determine the WDVV systems of the PDEs; the answer to which is no. Indeed, the WDVV equations in dimension three were classified in Mokhov & Pavlenko [45], and in Vašíček & Vitolo [14] the bi-Hamiltonian pairs for each member of the classification were found to be the pairs considered in this paper. It transpired that the projective classes of the third-order homogeneous Hamiltonian operators R were that of $R^{(3)}$, $R^{(4)}$ and $R^{(5)}$. This means that the systems of PDEs that are determined by $R^{(1)}$ and $R^{(2)}$ are not WDVV systems.

This is one of the main motivations for this paper: there are more WDVV-type equations than WDVV systems.

Case $R^{(2)}$. A Monge decomposition $(f^{(2)}_{ij}) = \Psi\Phi\Psi^{\top}$ of the Monge metric $f^{(2)}$ is given by

$$\Psi = (\psi^{\alpha}_{i}) = \begin{pmatrix} u^2 & 0 & 1 \\ -u^1 & -u^3 & 0 \\ 1 & u^2 & 0 \end{pmatrix}, \quad \Phi = (\phi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The system $u^i_t = (V^i)_x$ admitted by $R^{(2)}$ is (up to a transformation in the stabilizer of $R^{(2)}$ that reduces the number of constants)

$$\begin{aligned} u^1_t &= (\alpha u^2 + \beta u^3)_x, \\ u^2_t &= \left(\frac{((u^2)^2 - 1)(\alpha u^2 + \beta u^3) - (\gamma + \delta u^1)}{S}\right)_x, \\ u^3_t &= \left(\frac{(u^2u^3 - u^1)(\alpha u^2 + \beta u^3) - u^1(\gamma + \delta u^1)}{S}\right)_x, \end{aligned} \tag{4.1}$$

where $S = u^1u^2 - u^3$ is proportional to $\sqrt{\det(f^{(2)})}$ and $\alpha, \beta, \gamma, \delta$ are arbitrary constants. The system admits a Hamiltonian formulation through $R^{(2)}$ and the non-local Hamiltonian

$$H = \int \left(\frac{\alpha}{2}u^3(\mathrm{D}_x^{-1}u^2)^2 + \beta u^3(\mathrm{D}_x^{-1}u^2)(\mathrm{D}_x^{-1}u^3) - \frac{\gamma}{2}x^2u^1 - \delta x u^1(\mathrm{D}_x^{-1}u^1)\right)\mathrm{d}x\,.$$

Again, this system is linearly degenerate and non-diagonalizable for generic values of parameters; it is diagonalizable if and only if $\alpha\delta - \beta\gamma = 0$.

Theorem 4.1. *The system (4.1) has a unique Ferapontov-type Hamiltonian operator P, which is compatible with* $R^{(2)}$*, fulfils conjecture 2.1 with the metric* $(g^{ij}) = \Psi^{-1}Q(\Psi^{-1})^{\top}$, $c^{11} = 3$, $c^{12} = c^{21} = 0$, $c^{22} = -\beta^2$, *where*

$$\begin{gathered} Q^{11} = 2(A^2 + B^2 + 4BC + 2AC), \quad Q^{12} = 2(3AD - BC), \quad Q^{13} = 2B(2A + 3C), \\ Q^{22} = -2(2A + C)(2A + 3C), \quad Q^{23} = 8A^2 + 10AC + 2BD \quad Q^{33} = -6A^2 + 2B^2, \\ A = \alpha u^2 + \beta u^3, \quad B = \beta u^1 + \alpha, \quad C = \delta u^1 + \gamma, \quad D = \delta u^3 + \gamma u^2. \end{gathered}$$

Case $R^{(1)}$. In this subsection, $\mu^2 \neq 1$. The Monge metric $f^{(1)}$ admits a Monge decomposition $(f^{(1)}_{ij}) = \Psi\Phi\Psi^{\top}$, where

$$(\psi^{\alpha}_{i}) = \begin{pmatrix} u^2 & 0 & 1 \\ -u^1 & -u^3 & 0 \\ 1 & u^2 & 0 \end{pmatrix}, \quad (\phi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 1 & \mu \end{pmatrix}.$$

The system $u^i_t = (V^i)_x$ admitted by $R^{(1)}$ is (up to a transformation in the stabilizer of $R^{(2)}$ that reduces the number of constants) is

$$\begin{aligned} u^1_t &= (\alpha u^2 + \beta u^3)_x, \\ u^2_t &= \left(\frac{\big((u^2)^2 - \mu\big)(\alpha u^2 + \beta u^3) + \gamma(1 - \mu(u^2)^2) + \delta(u^1 - \mu u^2u^3)}{S}\right)_x, \\ u^3_t &= \left(\frac{\alpha u^3\big((u^2)^2 - \mu\big) + \beta u^3(u^2u^3 - \mu u^1)\gamma(u^1 - \mu u^2u^3) + \delta\big((u^1)^2 - \mu(u^3)^2\big)}{S}\right)_x, \end{aligned} \tag{4.2}$$

where $S = u^1u^2 - u^3$ is proportional to $\sqrt{\det(f^{(1)})}$, and $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Its non-local Hamiltonian is

$$H = \int \Big(\frac{\alpha}{2}(2\mu x u^1(\mathrm{D}_x^{-1}u^2) + u^3(\mathrm{D}_x^{-1}u^2)^2 + \mu x^2 u^3) + \beta u^3(1-\mu^2)(\mathrm{D}_x^{-1}u^2)(\mathrm{D}_x^{-1}u^3) + \frac{\gamma}{2}(\mu u^1(\mathrm{D}_x^{-1}u^2)^2 + x^2u^1 + 2\mu x u^3(\mathrm{D}_x^{-1}u^2))$$
$$+ \delta(x u^1(\mathrm{D}_x^{-1}u^1) + \mu u^3(\mathrm{D}_x^{-1}u^1)(\mathrm{D}_x^{-1}u^2) + \mu u^1(\mathrm{D}_x^{-1}u^2)(\mathrm{D}_x^{-1}u^3) + \mu x u^3(\mathrm{D}_x^{-1}u^3))\Big)\, dx$$

The system (4.2) is linearly degenerate and non-diagonalizable for generic values of parameters ($\alpha\delta - \beta\gamma \neq 0$).

Theorem 4.2. *The system* (4.2) *has a unique first-order non-local Hamiltonian operator P that fulfils conjecture 2.1 with the metric* $(g^{ij}) = \Psi^{-1}Q(\Psi^{-1})^{\top}$ *and is compatible with* $R^{(1)}$*,*

$$c^{11} = \mu^2 + 3, \quad c^{12} = c^{21} = -4\mu\delta, \quad c^{22} = \mu^3\beta^2 + 4\mu^2\delta^2 - \mu\beta^2,$$
$$Q^{11} = -(\mu^2-1)\big(\mu^2(A+C)^2 + \mu(B^2+D^2) - 2BD - 4EF\big), \quad Q^{12} = -(\mu^2-1)(\mu ED - FB),$$
$$Q^{13} = -(\mu^2-1)(\mu B(2E+F) - 3DE), \quad Q^{22} = -F^2\mu^3 - \mu^2(4A^2+D^2) + \mu(8BD+F^2) - 3D^2,$$
$$Q^{23} = -\mu^2(2BD + (u^1u^2-u^3)(\alpha\delta - \beta\gamma)) + 4\mu(B^2+D^2) - 5BD - EF$$
$$Q^{33} = -\mu^3E^2 - \mu^2(B^2+4D^2) + \mu(E^2+8BD) - 3B^2,$$
$$A = \beta u^1 + \delta u^3, \quad B = \alpha u^2 + \beta u^3, \quad C = \gamma u^2 + \alpha, \quad D = \delta u^1 + \gamma, \quad E = \beta u^1 + \alpha, \quad F = \delta u^3 + \gamma u^2.$$

(b) General projective classification

In Ferapontov *et al.* [12], it was proved that third-order homogeneous Hamiltonian operators in the Doyle–Potëmin canonical form are invariant also with respect to transformations that exchange t and x. This, together with projective reciprocal transformations that fix t, generate a larger group of reciprocal transformations of the following types

$$\mathrm{d}\tilde{x} = (A_iu^i + A_0)\mathrm{d}x + (A_iV^i + C_0)\mathrm{d}t,$$
$$\mathrm{d}\tilde{t} = (B_iu^i + B_0)\mathrm{d}x + (B_iV^i + D_0)\mathrm{d}t,$$

coupled with affine transformations of the dependent variables [38,40]. Such transformations act as $\mathrm{SL}(n+2)$ transformations on the linear line congruence that corresponds to the quasilinear first-order systems of PDEs determined by third-order homogeneous Hamiltonian operators.

According to the classical results by Castelnuovo, there are four distinct classes of linear line congruences in $\mathbb{P}^4$ under the action of SL(5): only two of them are endowed with third-order homogeneous Hamiltonian operator, they correspond to $R^{(5)}$ and $R^{(6)}$ (see Ferapontov *et al.* [12]).

The six classes discussed in the previous section can be transformed to two classes, one of which contains a linear system, and the other contains the simplest WDVV equation (1.6). This provides a proof that all systems that we discussed in the previous section are bi-Hamiltonian and integrable.

However, we stress that finding equivalence transformations between a given system or a Hamiltonian operator and a representative of a corresponding equivalence class can be extremely challenging, even with the most advanced computer algebra systems. So, having a direct proof and methods to efficiently compute systems and Hamiltonian operators proves to be an invaluable set of tools.

5. Four-component systems

There exists [27] a projective classification of third-order homogeneous Hamiltonian operators in $n = 4$. The group acting on the operators is that of t-fixing projective reciprocal transformations.

Unfortunately, unlike cases with $n \leq 3$, not all classes of third-order homogeneous Hamiltonian operators and associated systems admit a compatible first-order local or non-local Hamiltonian operator that fulfil the criteria of conjecture 2.1. As an important example, let us consider systems of the type

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \quad u^3_t = u^4_x, \quad u^4_t = (f(u))_x. \tag{5.1}$$

Linearly degenerate systems of the above type have been studied in Agafonov [43]. In Ferapontov *et al*. [27], it is proved that the above-mentioned system is Hamiltonian with respect to a third-order homogeneous Hamiltonian operator R only for two values of f:

$$f_1(u) = (u^2)^2 - u^1u^3, \qquad f_2(u) = (u^3)^2 - u^2u^4 + u^1. \tag{5.2}$$

Proposition 5.1. *There does not exist a matrix* (g^{ij}) *fulfilling the conjecture 2.1 for the systems* (5.1) and (5.2).

The aforementioned proposition does not exclude the possibility that an operator P that does not fulfil the conjecture, and is still compatible with the operator R, exists for the above-mentioned system; however, given the fact that the conjecture has been verified in a substantial number of cases (and so far not disproven), we think that such a possibility has little chance.

On the other hand, it is believed that there is a unique integrable case within the class of systems of conservation laws that admits a Hamiltonian formulation through a third-order homogeneous Hamiltonian operator (see the discussion at the end of Ferapontov *et al*. [12]). This is represented by the system

$$\begin{aligned}
u^1_t &= u^3_x,\\
u^2_t &= u^4_x,\\
u^3_t &= \left(\frac{u^1u^2u^4 + u^3((u^3)^2 + (u^4)^2 - (u^2)^2 - 1)}{u^1u^3 + u^2u^4}\right)_x,\\
u^4_t &= \left(\frac{u^1u^2u^3 + u^4((u^3)^2 + (u^4)^2 - (u^1)^2 - 1)}{u^1u^3 + u^2u^4}\right)_x,
\end{aligned} \tag{5.3}$$

which is known to possess a Lax pair and a Doyle–Potëmin Hamiltonian operator R parameterized by a Monge metric $f = (f_{ij})$ [12],

$$(f_{ij}) = \begin{pmatrix} (u^2)^2 + (u^3)^2 + 1 & -u^1u^2 + u^3u^4 & -u^1u^3 + u^2u^4 & -2u^2u^3 \\ -u^1u^2 + u^3u^4 & (u^1)^2 + (u^4)^2 + 1 & -2u^1u^4 & u^1u^3 - u^2u^4 \\ -u^1u^3 + u^2u^4 & -2u^1u^4 & (u^1)^2 + (u^4)^2 & u^1u^2 - u^3u^4 \\ -2u^2u^3 & u^1u^3 - u^2u^4 & u^1u^2 - u^3u^4 & (u^2)^2 + (u^3)^2 \end{pmatrix}.$$

The above-mentioned Monge metric can be factorized as $f = \Psi\Phi\Psi^\top = \psi^\alpha_i \phi_{\alpha\beta} \psi^\beta_j$,

$$\Psi = \begin{pmatrix} -u^2 & -u^3 & 1 & 0 \\ u^1 & -u^4 & 0 & 1 \\ -u^4 & u^1 & 0 & 0 \\ u^3 & u^2 & 0 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and is an original result of the fact that the above-mentioned system is bi-Hamiltonian of the type under consideration.

Theorem 5.1. *The system* (5.3) *is Hamiltonian with respect to a first-order non-local Hamiltonian operator P that is compatible with R and is defined by the metric* $g = (g^{ij})$ *splitting as in conjecture 2.1,* $c^{11} = c^{22} = 1$, $c^{12} = c^{21} = 0$. *Here,*

$$Q^{11}=(u^1)^2+(u^2)^2+(u^3)^2+(u^4)^2,\quad Q^{12}=-2u^1u^4+2u^2u^3,\quad Q^{13}=-u^2,$$
$$Q^{14}=u^1,\quad Q^{22}=(u^1)^2+(u^2)^2+(u^3)^2+(u^4)^2+1,\quad Q^{23}=-2u^3,$$
$$Q^{24}=-2u^4,\quad Q^{33}=(u^1)^2+(u^3)^2+1,\quad Q^{34}=u^1u^2+u^3u^4,\quad Q^{44}=(u^2)^2+(u^4)^2+1.$$

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6. And beyond

To the best of our knowledge, there is only one known example of a system with a WDVV-type bi-Hamiltonian structure for $n>4$. It comes from the WDVV equations in $N=4$ dimensions [17] (here, $n=6$). A new example not related to the WDVV equations is given below.

In [24, Eq. (11)] the problem of finding integrable Lagrangians within a certain class is reformulated as the problem of finding solutions to two *commuting* quasilinear systems of first-order PDEs of the type $u^i_t=V^i_j(u^k)u^j_x$.

Here, we rewrite the above-mentioned two systems in conservative form; one of them is

$$\begin{aligned}
u^1_t&=\left(\frac{4(u^1)^3u^3-4(u^1)^2u^6+2u^1u^2u^4-(u^4)^2}{2(2u^1u^2u^3-u^1u^5-u^3u^4)}\right)_x,\\
u^2_t&=u^1_x,\\
u^3_t&=\left(\frac{4(u^1)^2(u^3)^2-4u^1u^3u^6+u^4u^5}{2(2u^1u^2u^3-u^1u^5-u^3u^4)}\right)_x,\\
u^4_t&=\left(\frac{2(u^1)^3u^5-4(u^1)^2u^2u^6+2(u^1)^2u^3u^4+2u^1(u^2)^2u^4-u^2(u^4)^2}{2(2u^1u^2u^3-u^1u^5-u^3u^4)}\right)_x,\\
u^5_t&=\left(\frac{2(u^1)^2u^3u^5+2u^1(u^3)^2u^4-2u^1u^5u^6+u^2u^4u^5-2u^3u^4u^6}{2(2u^1u^2u^3-u^1u^5-u^3u^4)}\right)_x,\\
u^6_t&=\left(\frac{1}{2}u^4\right)_x.
\end{aligned}\tag{6.1}$$

Proposition 6.1. *System (6.1) possesses a third-order Hamiltonian operator R parameterized by a Monge metric $f=(f_{ij})$, where*

$$(f_{ij})=\begin{pmatrix}
(u^3)^2 & -\frac{u^5+u^2u^3}{2} & -u^1u^3+\frac{(u^2)^2}{2} & 0 & -\frac{u^2}{2} & 0\\
-\frac{u^5+u^2u^3}{2} & u^1u^3+u^6 & -\frac{u^4+u^1u^2}{2} & u^3 & u^1 & -\frac{u^2}{2}\\
-u^1u^3+\frac{(u^2)^2}{2} & -\frac{u^4+u^1u^2}{2} & (u^1)^2 & -\frac{u^2}{2} & 0 & 0\\
0 & u^3 & -\frac{u^2}{2} & 0 & \frac{1}{2} & 0\\
-\frac{u^2}{2} & u^1 & 0 & \frac{1}{2} & 0 & 0\\
0 & -\frac{u^2}{2} & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Proof. We make use of equation (2.5) for the above-mentioned system, with an unknown Monge metric. We find the above-mentioned Monge metric as the unique solution (up to constant multiples). ∎

The above-mentioned Monge metric can be factorized as $f=\Psi\Phi\Psi^\top=\psi^\alpha_i\phi_{\alpha\beta}\psi^\beta_j$, where

$$\Psi = \begin{pmatrix} -u^3 & -u^2 & u^5 & 0 & 0 & 0 \\ 0 & u^1 & -u^6 & u^3 & 1 & 0 \\ u^1 & 0 & u^4 & -u^2 & 0 & 0 \\ 0 & 1 & -u^3 & 0 & 0 & 0 \\ 0 & 0 & -u^1 & 1 & 0 & 0 \\ 0 & 0 & u^2 & 0 & 0 & 1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 6.1. *The system* (6.1) *is Hamiltonian with respect to a first-order local Hamiltonian operator P that is defined by the metric* $g = (g^{ij})$ *splitting as in conjecture 2.1 and compatible with R. Here,*

$$Q^{11} = 2u^1u^3, \quad Q^{12} = u^1u^2 - u^4, \quad Q^{13} = -2u^1u^5 + 2u^3u^4, \quad Q^{14} - u^2u^3 + u^5, \quad Q^{15} = 0, \quad Q^{16} = 0,$$
$$Q^{22} = 2(u^1)^2, \quad Q^{23} = -4u^1u^6 + 2u^2u^4, \quad Q^{24} = 2u^1u^3 - (u^2)^2 + 4u^6, \quad Q^{25} = 4u^1, \quad Q^{26} = 2u^4,$$
$$Q^{33} = -2u^4u^5 + 2(u^6)^2, \quad Q^{34} = 2u^2u^5 - 4u^3u^6, \quad Q^{35} = -4u^1u^3 + (u^2)^2 - 2u^6,$$
$$Q^{36} = u^2u^6 - u^1u^5 - u^3u^4, \quad Q^{44} = 2(u^3)^2, \quad Q^{45} = 4u^3, \quad Q^{46} = 2u^5, \quad Q^{55} = 2, \quad Q^{56} = u^2, \quad Q^{66} = 2u^6.$$

Remark 6.1. It is interesting to observe that there is another quasilinear first-order system in a conservative form that can be deduced from the two quasilinear systems in Ferapontov *et al.* [24, Eq. (11)]. The above-mentioned bi-Hamiltonian structure of WDVV-type holds also for this system, a phenomenon that has already been observed for the four-component WDVV systems in Ferapontov & Mokhov [46].

Remark 6.2. A further example of bi-Hamiltonian structure of WDVV-type is incomplete but interesting. Indeed, the simplest case of oriented associativity equation has a first-order local operator *P* and a third-order *non-local* operator *R*, which suggests the possibility that the class of bi-Hamiltonian structures of WDVV-type can be further enlarged.

It must be stressed that in the above-mentioned case, compatibility of the two operators *P*, *R* has never been proved, largely in view of the computational complexity of the problem.

Data accessibility. The program files used in the paper are available as part of the supplementary material [47].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. S.O.: investigation, methodology, software, supervision, validation, writing—original draft, writing—review and editing; R.V.: investigation, methodology, project administration, software, supervision, validation, writing—original draft, writing—review and editing.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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