



Further advances on isotropic correlation models with negative values

D. Posa¹

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Abstract

Most of the covariance models utilized in spatial and in spatio-temporal data analysis are non negative; at this purpose, covariance functions characterized by negative values need to be explored not only to analyze some particular case studies characterized by negative correlation, but also and most importantly to provide richer families of correlation models. In this paper new families of correlation functions will be proposed, among the others, a generalization of the Cauchy correlation family as well as a generalization of a subclass of Matérn family. Moreover, apart from the Bochner's spectral representation, in order to construct further families of correlation functions, a technique based on the differentiation of covariance models will be provided. Some of these new classes are characterized by a finite number of zeros as many as desired; at our knowledge, there seem not to exist correlation functions which present this relevant characteristic. Moreover, some of these new families can be built through differentiation by starting from covariance functions which are always positive in their domain. As it will be shown, the proposed classes are extremely flexible since they are able to model either correlation structures just characterized by positive values, either correlation structures which are negative in a subset of their domain: this is a relevant property of the proposed covariance models, essentially because most of the well known covariance functions, belonging to the same family, are not able to model both negative and positive correlation structures, according to their parameters values. A graphical representation and some general remarks on the proposed families of correlation models have been provided.

Keywords Spectral density function · Correlation functions · Negative correlation

1 Introduction

Most of the correlation functions provided in the literature are non negative: this aspect can also be clearly remarked in all the theoretical as well as in the practical situations found in several applications, such as in meteorological field (Bourotte et al. 2016; Cressie and Huang 1999; Gneiting 2002b), in Environmental Sciences (De Cesare et al. 2001; De Iaco et al. 2013), in research concerning machine learning (Garg et al. 2012), in forestry (Jost et al. 2005), in Atmospheric Sciences (Brown et al. 2001), in demographical case studies (De Iaco et al. 2015) and in financial applications (Porcu et al. 2012). This peculiarity distinguishes the Matérn class

(Matérn 1980), which is capable to model just non negative correlation structures. Although correlation functions characterized by negative values are not so common in the applications, however these models are required in various situations, such as many applications concerning turbulence, hydrological and biological analysis (Shkarofsky 1968; Xu et al. 2003a; Levinson et al. 1984; Yakhot et al. 1989; Xu et al. 2003b; Pomeroy et al. 2003; Xu et al. 2003b).

In spatial and spatio-temporal data analysis, correlation functions (c.f.s) characterized by negative values are usually modelled through the whole class of Bessel c.f.s, which include as special cases the cosine model, the Bessel J_0 c.f. and the cardinal sine model; in particular, most of these models are often utilized in spatial data analysis to model case studies characterized by hole effects (Ma and Jones 2001; Journel and Froidevaux 1982; Asghari 2015).

The cosine model is distinguished by some relevant drawbacks, because it is not admissible in $\mathbb{R}^m$ with $m > 1$ and the same model is not strictly positive definite (De Iaco and Posa 2018).

✉ D. Posa
donato.posa@unisalento.it

¹ Dipartimento di Scienze Economiche e Matematico-Statistiche, University of Salento, Complesso Ecotekne, Via per Monteroni, Lecce, Italy

The *cardinal sine* model, which belongs to the class of c.f.s characterized by an hole effect, is admissible in $\mathbb{R}^3$; however, these last correlation models present severe limitations because they are not able to model various behaviours modifying their parameters. For example, all the classical c.f.s belonging to the Bessel class exhibit a behaviour which is parabolic near the origin and they are characterized by a countable infinity of zeros.

In the literature, Ma (2005) and Gregori et al. (2008) have analyzed linear combinations, with negative weights, of real valued c.f.s; in particular, Gregori et al. (2008) remarked the importance of negative correlation for several applications concerning biological and medical problems. Gneiting (2002a) has proposed compactly supported c.f.s with ramifications to model negative correlations: indeed, compactly supported c.f.s can be useful for computationally efficient spatial prediction, fast and exact simulation. Apart from the above contributions, further relevant correlation models with negative holes (Hristopulos and Elogne 2007; Hristopulos 2015) deserve a peculiar attention. Recently Alegria and Emery (2024) have proposed new interesting parametric families of isotropic matrix-valued functions exhibiting non-monotonic behaviors, namely hole effects and cross-dimples, with an application on a bivariate data set consisting of concentrations of airborne particulate matter. Moreover, a comprehensive review concerning the role of negative values in c.f.s across multiple fields has been provided by Alegria and Emery (2026), which also analyzed the differences of covariance functions exhibiting geometric anisotropies. On the other hand, Laga and Kleiber (2017) examined spectral densities whose maximum is not located at zero frequency, which is a characteristic of processes with strong periodic behavior.

A wide class of new c.f.s, able to model negative correlation structures, has been recently built considering the difference of two c.f.s (Posa 2021, 2023). On the other hand, some relevant results which generalize these last models have been lately provided concerning the difference between two Matérn families (Posa 2025). All these last c.f.s present interesting features, because the same class can be always positive or negative in a subset of its field of definition, by suitably modifying the parameters on which the same class depends; moreover, these models present a complementary behaviour with respect to the Bessel correlation family, because they are characterized by just one zero and they can also provide c.f.s with a linear behaviour near the origin: this last feature cannot be described by the whole class of Bessel family.

In this paper further advances with respect to the just mentioned results have been provided. In particular, new families of correlation models, also characterized by negative values, valid in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, will be proposed: among

the others, a generalization of the Cauchy correlation family, as well as a generalization of a subclass of Matérn family. As it will be shown, the same families are able to model either correlation structures just characterized by positive values, either correlation structures which are negative in a subset of their domain: this is a relevant property of the proposed covariance models, essentially because most of the well known covariance functions, belonging to the same family, are not able to model both negative and positive correlation structures, according to their parameters values. Moreover, apart from the classical Bochner's representation which characterizes the whole class of continuous c.f.s in any dimensional space, new techniques to build negative correlation structures will be provided, starting from some classical results, in spite of this, scarcely used in the applications. It is suitable to underline that these new classes of c.f.s are characterized by some relevant properties which lack the classical hole effect models, neither the new models proposed recently by Posa (2023, 2025) and De Iaco and Posa (2025): in fact, some of these new classes are characterized by a finite number of zeros as many as desired; at our knowledge, there seem not to exist c.f.s which present this remarkable property. Moreover, it will be shown that, even starting from correlation models which are always positive in their domain, new valid families of c.f.s characterized by negative values can be obtained through differentiation.

These achievements are particularly interesting from a theoretical point of view since they provide a significant development and enlargement concerning the families of covariance models, on the other hand they are also helpful from a practical perspective since they furnish a guideline for practitioners on how to utilize c.f.s characterized by negative values, taking into account various aspects, such as the dimension of the Euclidean space, the behaviour in proximity of the origin and the number of zeros, among the others.

This paper is subdivided as follows: in Sect. 2, a brief summary concerning traditional classes of isotropic c.f.s, as well as the classical families of c.f.s belonging to the Bessel family, which present negative values and often utilized in the applications, has been provided; the most remarkable features and drawbacks of these families have been pointed out. In Sect. 3 new classes of c.f.s, such as a generalization of the Cauchy family and a generalization of a subclass of the Matérn family, as well as the classes obtained through the product of an exponential or a Gaussian model with a second degree polynomial, which are able to model both always positive correlation structures as well as correlation structures characterized by negative values, have been proposed; moreover, new families of c.f.s have been obtained through differentiation of positive covariance models, in order to enlarge the whole class of c.f.s with models which can be negative in a subset of their domain: the properties

of these families have been analyzed. In Sect. 4, a graphical representation and some general remarks on the proposed families of correlation models has been provided.

2 An overview on isotropic correlation functions

In the present Section a brief overview concerning the construction of isotropic c.f.s through their corresponding spectral density functions (if these last exist) has been provided. Moreover, it will be pointed out how to check, through the spectral density function, the admissibility of a c.f. in $\mathbb{R}^2$ and $\mathbb{R}^3$, starting from a c.f. which is valid in $\mathbb{R}$. It is well known, in the literature, that Bochner's theorem represents the cornerstone to build several correlation models.

In this paper isotropic correlation models will be considered, i.e., $\rho(\mathbf{h}) = \rho(h)$, with $h = \|\mathbf{h}\|$ and $\mathbf{h} \in \mathbb{R}^m$. If a continuous c.f. ρ is isotropic in $\mathbb{R}^m$ and if:

$$\int_0^\infty h^{m-1} |\rho(h)| dh < \infty, \quad m \in \mathbb{N}, m \geq 2,$$

then the existence of a spectral density function f is ensured, moreover ρ can be written as follows (Yaglom 1957):

$$\rho(h) = (2\pi)^{m/2} \int_0^\infty \frac{J_{(m/2-1)}(\omega h)}{h^{(m/2-1)}} \omega^{m/2} f(\omega) d\omega, \quad h > 0, \quad (1)$$

where J is a Bessel function of order $(m/2 - 1)$. In this case, the expression of the spectral density function is given hereafter:

$$f(\omega) = \frac{1}{(2\pi)^{m/2}} \int_0^\infty \frac{J_{(m/2-1)}(\omega h)}{\omega^{(m/2-1)}} h^{m/2} \rho(h) dh. \quad (2)$$

In the particular cases with $m = 2$ or $m = 3$, the relationships (1) and (2) can be better specified as follows:

- $m = 2$:

$$\begin{aligned} \rho(h) &= 2\pi \int_0^\infty J_0(\omega h) \omega f(\omega) d\omega \\ f(\omega) &= \frac{1}{2\pi} \int_0^\infty J_0(\omega h) h \rho(h) dh; \end{aligned} \quad (3)$$

- $m = 3$:

$$\begin{aligned} \rho(h) &= 4\pi \int_0^\infty \frac{\sin(\omega h)}{\omega h} \omega^2 f(\omega) d\omega, \\ f(\omega) &= \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(\omega h)}{\omega h} h^2 \rho(h) dh. \end{aligned} \quad (4)$$

The following relationships (Yaglom 1987) are displayed by recalling the link between the spectral density function f_1 (in $\mathbb{R}$) with the spectral density f (in $\mathbb{R}^2$ and in $\mathbb{R}^3$, respectively):

$$\begin{aligned} f(\omega) &= \frac{-1}{\pi} \int_\omega^\infty \frac{df_1(\omega_1)}{d\omega_1} \frac{1}{(\omega_1^2 - \omega^2)^{1/2}} d\omega_1, \\ f(\omega) &= \frac{-1}{2\pi\omega} \frac{df_1(\omega)}{d\omega}. \end{aligned} \quad (5)$$

In order to check if ρ is an admissible isotropic c.f. in $\mathbb{R}^m$, it is enough to utilize the relationship (2) to compute and to examine the non negativity of the spectral density f . Equivalently, the Fourier transform f_1 of the c.f. ρ in $\mathbb{R}$ can be found and subsequently to compute the spectral density f , through expression (5), in $\mathbb{R}^2$ or in $\mathbb{R}^3$ and verify the non negativity of f .

2.1 Classical correlation models with negative values

In the present Section a brief overview on the traditional classes of isotropic c.f.s which can present negative values will be provided, in particular the whole class of Bessel c.f.s (including the special cases, such as the cosine model, the Bessel J_0 c.f. and the cardinal sine model); moreover, some details concerning the behaviour of these last families and the specifics concerning the admissibility in various dimensional spaces will be given. It is suitable to underline that all the classical Bessel c.f.s exhibit a behaviour which is parabolic near the origin (i.e., in (7) $C'(0) = 0, C''(0^+) < 0$) and are characterized by a countable infinity of zeros.

2.2 Bessel correlation functions

The family of the Bessel covariance functions is given hereafter:

$$C(h; \alpha, \nu) = (\alpha h)^{-\nu} J_\nu(\alpha h), \quad \alpha > 0, \nu \geq -1/2, \quad (6)$$

where J_ν is the Bessel function of order ν and $\Gamma(\nu + 1/2)$ is the Γ -function.

Considering in (6) the special case $\nu = (m - 2)/2$, the following family of covariance functions is obtained:

$$C\left(h; \alpha, \frac{m-2}{2}\right) = (\alpha h)^{-(m-2)/2} J_{(m-2)/2}(\alpha h), \quad \alpha > 0, m \in \mathbb{N}. \quad (7)$$

Some special cases of (7) are provided below, corresponding to the three different values of the parameter m which refers to the dimension of the Euclidean space $\mathbb{R}^m$. As will

be shown, the well known cosine and the cardinal sine c.f.s are special cases of the Bessel family.

2.2.1 Cosine correlation function

By setting $m = 1$ in (7),

$$J_{(-1/2)}(\alpha h) = \left(\frac{2}{\pi \alpha h} \right)^{1/2} \cos(\alpha h),$$

then the classical cosine c.f. follows:

$$\rho(h; \alpha) = \cos(\alpha h), \quad \alpha > 0, \quad (8)$$

which takes any value in $[-1, 1]$.

However, one of the drawbacks of the c.f. (8) concerns the fact that it is not strictly positive definite (De Iaco et al. 2011), hence it is not adequate to be used for predictive purposes if it is not combined with strictly positive definite functions; moreover, model (8) is admissible only in $\mathbb{R}$.

Note that the cosine c.f. always presents a behaviour which is parabolic near the origin.

2.2.2 Bessel J_0 correlation function

By setting $m = 2$ in (7), then the classical Bessel J_0 c.f. is obtained:

$$\rho(h; \alpha) = J_0(\alpha h), \quad \alpha > 0, \quad (9)$$

where:

$$J_0(x) = \frac{1}{\pi} \int_{-1}^1 \frac{e^{itx}}{(1-t^2)^{1/2}} dt = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k! \Gamma(k+1)} \quad (10)$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

Note that the absolute minimum value of the Bessel c.f.:

$$\min_{h \in \mathbb{R}} J_0(\alpha h) \approx -0.403$$

does not depend on the parameter α ; moreover, model (9) is admissible in $\mathbb{R}^2$, hence also in $\mathbb{R}$, however it is not admissible in $\mathbb{R}^3$.

2.2.3 Cardinal sine correlation function

By setting $m = 3$ in (7), $J_{(1/2)} = \left(\frac{2}{\pi \alpha h} \right)^{1/2} \sin(\alpha h)$, then the cardinal sine c.f., extended for continuity, is obtained:

$$\rho(h; \alpha) = \begin{cases} \frac{\sin(\alpha h)}{\alpha h} & h \neq 0 \\ 1 & h = 0; \end{cases} \quad (11)$$

taking into account that the c.f. (11) is valid in $\mathbb{R}^3$, it is also valid in $\mathbb{R}$ and in $\mathbb{R}^2$.

The cardinal sine c.f. presents a behaviour which is parabolic near the origin; moreover, the absolute minimum αh_m of (11) can be computed by evaluating the extreme points of (11), which correspond to the solution of the equation $tg(\alpha h) = \alpha h$ and it is given hereafter:

$$\alpha h_m \approx 4.493, \quad \rho(h_m; \alpha) = \frac{-1}{\sqrt{1 + (\alpha h_m)^2}} \approx -0.217.$$

Note that the absolute minimum value of (11) does not depend on the parameter α and it assumes the same value in any dimensional space $\mathbb{R}^m$, $m = 1, 2, 3$; moreover, model (11) always presents a behaviour which is parabolic near the origin.

Remark Note that damped oscillations correlation models can be built by using the cosine function with the gaussian and with the rational model, respectively. In particular, the product between the cosine function and the gaussian correlation model has been described in Ma and Jones (2001): however, in this last paper some details, such as the validity and main features of the mentioned models, have not been provided.

3 New correlation models with negative values

In this Section new classes of c.f.s which present the relevant characteristic to be only positive in their domain, but also to assume negative values in a subset of their domain, according to the values of their parameters on which the same classes depend, will be presented; moreover, some special features concerning these families will be provided. As already underlined, most of the c.f.s with negative values, known in the literature, present a behaviour which is parabolic near the origin and are characterized by an infinite number of zeros, moreover the same classes are not able to model positive correlation structures; at this purpose, new families are required which exhibit different features, among the others, as many zeros as desired and a behaviour which could be linear near the origin. As detailed hereafter, these new classes of models can be built utilizing different techniques, starting from positive correlation models: in particular, not only spectral methods can be employed, but

also some known results, in spite of this, scarcely used in the applications.

3.1 Spectral methods

In the following Section, Bochner's spectral representation will be utilized to construct negative c.f.s: in particular, starting from well-known positive correlation models, families of c.f.s characterized by negative values and different behaviours at the origin, have been constructed utilizing the difference between the above mentioned positive correlation models. Apart from these last models, generalizations both of the Cauchy correlation family and of a subclass of Matérn family, as well as the product of a second degree polynomial with an exponential and a Gaussian model will be proposed, together with their possible extensions in a higher dimensional space ($\mathbb{R}^2$ and $\mathbb{R}^3$).

3.1.1 Difference of correlation models

Besides the classical Bessel c.f.s described in Sect. 2.1, new classes of correlation families characterized by negative values (Posa 2021) have been recently constructed through the Bochner's spectral representation, considering the difference of positive c.f.s, as outlined by the following result.

Theorem 1 (Posa 2021). *Let $C_k : \mathbb{R}^m \rightarrow \mathbb{C}, k = 1, 2$ be covariance functions and consider*

$$C(\mathbf{h}; \mathbf{\Lambda}) = AC_1(\mathbf{h}; \boldsymbol{\alpha}) - BC_2(\mathbf{h}; \boldsymbol{\beta}), \quad \mathbf{h} = (h_1, \dots, h_m) \in \mathbb{R}^m, \quad (12)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors of parameters, $A > 0, B > 0$ and $\mathbf{\Lambda} = (A, B, \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Suppose that the covariance functions $C_k, k = 1, 2$, can be expressed as follows:

$$C_1(\mathbf{h}; \boldsymbol{\alpha}) = \int_{\mathbb{R}^m} \exp(i\boldsymbol{\omega}^T \mathbf{h}) f_1(\boldsymbol{\omega}; \boldsymbol{\alpha}) d\boldsymbol{\omega},$$

$$C_2(\mathbf{h}; \boldsymbol{\beta}) = \int_{\mathbb{R}^m} \exp(i\boldsymbol{\omega}^T \mathbf{h}) f_2(\boldsymbol{\omega}; \boldsymbol{\beta}) d\boldsymbol{\omega},$$

with $f_i(\boldsymbol{\omega}; \cdot) \geq 0, \forall \boldsymbol{\omega} \in \mathbb{R}^m$ and $\int_{\mathbb{R}^m} f_i(\boldsymbol{\omega}; \cdot) d\boldsymbol{\omega} < \infty, i =$

1, 2; then

$$C(\mathbf{h}; \mathbf{\Lambda}) = \int_{\mathbb{R}^m} \exp(i\boldsymbol{\omega}^T \mathbf{h}) (Af_1(\boldsymbol{\omega}; \boldsymbol{\alpha}) - Bf_2(\boldsymbol{\omega}; \boldsymbol{\beta})) d\boldsymbol{\omega} \quad (13)$$

is a covariance function if and only if $\forall \boldsymbol{\omega} \in \mathbb{R}^m$,

$$Af_1(\boldsymbol{\omega}; \boldsymbol{\alpha}) - Bf_2(\boldsymbol{\omega}; \boldsymbol{\beta}) \geq 0. \quad (14)$$

Note that $(Af_1 - Bf_2)$ is integrable, since f_1 and f_2 are integrable.

In Posa (2023) some examples of Theorem 1 are given for the subset of the real covariances: in particular, it has been shown how the difference between two positive covariance functions, suitably chosen among the most utilized models, i.e. gaussian, exponential and rational model, can generate families of correlation structures which are extremely advantageous to model a wide range of case studies, since the same classes are able to model just positive as well as negative correlation structures, according to the values of the parameters on which the same models depend. In particular, the admissibility of the various families obtained through expression (13) has been analyzed in the Euclidean spaces $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^3$. A generalization of the previous results concerning the difference between two Matérn families has been recently provided (Posa 2025), as well as a detailed analysis on the proposed models has also been given. Moreover, as already underlined in the Introduction, an interesting analysis concerning the differences of covariance functions exhibiting geometric anisotropies has been provided by Alegria and Emery (2026).

3.1.2 A generalization of the Cauchy correlation family

The next result ensures to build a wide parametric class of c.f.s: according to the values of the parameter k in (15), the present class can assume always positive values or, alternatively, it can be negative in a subset of its domain; moreover, the same class can be valid in $\mathbb{R}^3$ if $n \geq 3$ in (15) and it is characterized by just one zero for $h > 0$, hence, it can be considered complementary to all the previous classical families, which always exhibit an infinite number of zeros. To some extent, the new class generalizes the classical Cauchy family (Chilès and Delfiner 1999) which can be obtained from (15) with $k = 0$: indeed, this last class is always positive in its domain. First of all, it is necessary to prove the following Theorem.

Theorem 2 *Given, in $\mathbb{R}$, the following class of parametric functions:*

$$\rho(h; k, n) = \frac{1 - kh^2}{(h^2 + 1)^n}, \quad h \in \mathbb{R}, \quad n \geq 2, \quad (15)$$

if $-1 \leq k \leq 2n - 3$, then (15) is a valid correlation family.

Proof (See the Appendix). It can be easily checked that the class (15) is always positive if $-1 \leq k \leq 0$, whereas it always assumes negative values if $0 < k \leq (2n - 3)$;

moreover, in this last case, the same class presents, for $h > 0$, just one minimum value:

$$h_m = \sqrt{\frac{n+k}{k(n-1)}}, \quad \rho(h_m) = -\left(\frac{k}{n}\right)^n \left(\frac{n-1}{k+1}\right)^{n-1}. \quad (16)$$

On the other hand, for a fixed value of n , the lowest minimum value h_m^* is obtained for $k = 2n - 3$; in this last case, expressions in (16) become:

$$h_m^* = \sqrt{\frac{3}{2n-3}}, \quad \rho(h_m^*) = -2\left(\frac{2n-3}{2n}\right)^n. \quad (17)$$

As a consequence:

$$\lim_{n \rightarrow \infty} \rho(h_m^*) = \lim_{n \rightarrow \infty} \left[-2\left(\frac{2n-3}{2n}\right)^n \right] = -\frac{2}{e^{1.5}} \approx -0.446, \quad (18)$$

which can be considered, in $\mathbb{R}$, the lowest minimum value for the class (15).

Some examples for the class (15) related to particular values of the parameter $n = 2, 3, 4$ are provided hereafter; moreover, in order to check if the parametric class (15) is a valid correlation model in a higher dimensional space ($\mathbb{R}^2$ or $\mathbb{R}^3$), relationships given in (5) can be utilized, as will be described in the following examples.

Example 1 Let $n = 2$ in (15), then

$$\rho(h; k) = \frac{1 - kh^2}{(h^2 + 1)^2}, \quad h \in \mathbb{R}, \quad (19)$$

is a valid correlation model in $\mathbb{R}$ if its Fourier transform is non negative and integrable; by setting $n = 2$ in (50):

$$\begin{aligned} f_1(\omega; k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(\omega t) \frac{1 - kt^2}{(t^2 + 1)^2} dt \\ &= \frac{e^{-|\omega|}}{4} [|\omega|(1 + k) + 1 - k], \end{aligned}$$

then f_1 is non negative and integrable if and only if $-1 \leq k \leq 1$.

Family (19) exhibits a behaviour which is always parabolic near the origin, moreover, if $-1 \leq k \leq 0$, it can accommodate just positive correlation structures; on the other hand, if $0 < k \leq 1$ correlation structures which exhibit negative values can be fitted.

In this last case, by using (16), $\rho(h_m) = \frac{-k^2}{4(k+1)}$ is the minimum value assumed by (19), where $h_m = \sqrt{1 + \frac{2}{k}}$.

Then, as k increases, $\rho(h_m)$ becomes more and more negative; in particular, according to the previous conditions on k , the minimum of ρ is obtained for $k = 1$, i.e., $\rho(h_m) = -0.125$.

In order to check if the correlation model (19) is valid in $\mathbb{R}^2$, the first relationship given in (5) can be utilized, i.e.,

$$f(\omega; k) = \int_{\omega}^{\infty} [(k+1)h - 2k] \frac{e^{-h}}{\sqrt{h^2 - \omega^2}} dh, \quad h \geq 0. \quad (20)$$

According to Gradshteyn and Ryzhik (2007), pag. 350, equ. 3.387.6 and pag. 351, equ. 3.389.4:

$$\int_{\omega}^{\infty} \frac{e^{-h}}{\sqrt{h^2 - \omega^2}} dh = K_0(\omega),$$

and

$$\int_{\omega}^{\infty} h \frac{e^{-h}}{\sqrt{h^2 - \omega^2}} dh = \omega K_1(\omega),$$

where K_0 and K_1 are the modified Bessel functions of order 0 and 1, respectively.

As a consequence, the function (20) becomes:

$$f(\omega; k) = [(k+1)\omega K_1(\omega) - 2kK_0(\omega)],$$

hence, in order to be a spectral density function, it must be non negative and integrable.

Note that, taking into account the properties of K_0 and K_1 , $f(\omega; k) \geq 0$ if and only if $-1 \leq k \leq 0$; moreover, according to Gradshteyn and Ryzhik (2007), pag. 660, equ. 6.511.12 and pag. 665, equ. 6.521.5:

$$\begin{aligned} &\int_0^{\infty} f(\omega; k) d\omega \\ &= (k+1) \int_0^{\infty} \omega K_1(\omega) d\omega - 2k \int_0^{\infty} K_0(\omega) d\omega = \frac{\pi}{2}(1 - k), \end{aligned}$$

hence f is a spectral density function if and only if: $-1 \leq k \leq 0$. As a consequence, in this last case the correlation model (19) is valid in $\mathbb{R}^2$ and it is always positive.

Example 2 Let $n = 3$ in (15), then

$$\rho(h; k) = \frac{1 - kh^2}{(h^2 + 1)^3}, \quad h \in \mathbb{R}, \quad (21)$$

is a valid correlation model if its Fourier transform is non negative and integrable; by setting $n = 3$ in (50):

$$\begin{aligned} f_1(\omega; k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(\omega t) \frac{1 - kt^2}{(t^2 + 1)^3} dt \\ &= \frac{e^{-|\omega|}}{16} [\omega^2(1 + k) + |\omega|(3 - k) + 3 - k]; \end{aligned}$$

if $-1 \leq k \leq 3$, then f_1 is non negative and integrable.

Family (21) exhibits a behaviour which is parabolic near the origin; moreover, if $-1 \leq k \leq 0$, it can accommodate just positive correlation structures, on the other hand, if $0 < k \leq 3$, correlation structures which exhibit negative values can be fitted.

In this last case, by using (16), $\rho(h_m) = \frac{-4k^3}{27(k+1)^2}$ is the minimum value assumed by (21), where $h_m = \sqrt{\frac{1}{2} + \frac{3}{2k}}$. Then, as k increases, $\rho(h_m)$ becomes more and more negative; in particular, according to the previous conditions on k , the minimum of ρ is obtained for $k = 3$, i.e., $\rho(h_m) = -0.25$.

In order to check if the correlation model (21) is valid in $\mathbb{R}^3$, the second relationship given in (5) can be utilized, i.e.,

$$f(\omega; k) = \frac{1}{32\pi} e^{-|\omega|} [|\omega|(1 + k) + 1 - 3k]. \quad (22)$$

Hence, the function (22) is a spectral density and, as a consequence, (21) is a valid correlation model in $\mathbb{R}^3$ if and only if: $-1 \leq k \leq \frac{1}{3}$.

In particular, the class (21) is always positive if $-1 \leq k \leq 0$ and it always assumes negative values if:

$$0 \leq k \leq \frac{1}{3}.$$

In this last case, the minimum value assumed by (21) is obtained for $k = \frac{1}{3}$, then:

$$\rho(h_m) = -1/324.$$

Example 3 Let $n = 4$ in (15), then

$$\rho(h; k) = \frac{1 - kh^2}{(h^2 + 1)^4}, \quad h \in \mathbb{R}, \quad (23)$$

is a valid correlation model if its Fourier transform is non negative and integrable; by setting $n = 4$ in (50):

$$\begin{aligned} f_1(\omega; k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(\omega t) \frac{1 - kt^2}{(t^2 + 1)^4} dt \\ &= \frac{e^{-|\omega|}}{48} [|\omega|^3(k + 1) + 6\omega^2 + |\omega|(15 - 3k) + 15 - 3k]; \end{aligned}$$

if $-1 \leq k \leq 5$, then f_1 is non negative and integrable.

Family (23) exhibits a behaviour which is parabolic near the origin; moreover, if $-1 \leq k \leq 0$, it can accommodate just positive correlation, on the other hand, if $0 < k \leq 5$, correlation structures which exhibit negative values can be fitted.

In this last case, by using (16), $\rho(h_m) = \frac{-27k^4}{256(k+1)^3}$ is the minimum value assumed by (23), where $h_m = \sqrt{\frac{1}{3} + \frac{4}{3k}}$. As a consequence, as k increases, $\rho(h_m)$ becomes more and more negative; in particular, according to the previous conditions on k , the minimum of ρ is obtained for $k = 5$, i.e.,

$$\rho(h_m) = -\frac{625}{2048} \approx -0.30.$$

In order to check if the correlation model (23) is valid in $\mathbb{R}^3$, the second relationship given in (5) can be utilized, i.e.,

$$f(\omega; k) = \frac{1}{48\pi} e^{-|\omega|} [\omega^2(k + 1) + 3|\omega|(1 - k) + 3(1 - k)]. \quad (24)$$

Hence, if $-1 \leq k \leq 1$, the function (24) is a spectral density and, as a consequence, (23) is a valid correlation model in $\mathbb{R}^3$.

In particular, the class (23) is always positive if: $-1 \leq k \leq 0$ and it always assumes negative values if: $0 < k \leq 1$. In this last case, the minimum value assumed by (21) is obtained for $k = 1$, then

$$\rho(h_m) = -27/2048 \approx -0.01.$$

3.1.3 A generalization of a subclass of Matérn family

It is well known that the following parametric family:

$$\rho_M(h; a) = (1 + ah) \exp(-ah), \quad h \geq 0, \quad a > 0, \quad (25)$$

is a particular member of the subset of the Matérn family; some details on this special class are well described in Hristopulos (2020). Indeed, as already underlined, the whole Matérn family is characterized just by positive correlation models. In particular, the correlation family (25) is a valid family of c.f.s and it is always positive, moreover it can be

easily checked that it always presents a parabolic behaviour at the origin since $\rho'_M(0) = 0$.

The next result shows that it is possible to generalize the previous class of c.f.s.

Theorem 3 *Let*

$$\rho(h; A, a) = (1 + Ah) \exp(-ah), \quad h \geq 0, \quad a > 0; \quad (26)$$

then ρ is a c.f. in $\mathbb{R}$ if and only if $-a \leq A \leq a$.

Proof (See the Appendix). Note that the subclass of the Matérn family (25) is a special case of the class (26) for $A = a > 0$; in particular, the class (26) is always positive if $0 \leq A \leq a$ and it always assumes negative values if $-a < A < 0$. With respect to the class (25), the family (26) always presents a linear behaviour at the origin if: $A \neq a$ since $\rho'(0) = A - a$.

Moreover, if the class (26) presents negative values, i.e. $-a \leq A < 0$, then there exists just one zero, for $h > 0$ and $h_m = \frac{1}{a} - \frac{1}{A}$ is the absolute minimum of (26), with

$$C(h_m; A, a) = \frac{A}{a} \exp \left[- \left(1 - \frac{a}{A} \right) \right];$$

in particular, the function $G(z) = -z \exp \left[- \left(1 + \frac{1}{z} \right) \right]$, with $z = -\frac{A}{a} > 0$ presents, taking into account the constraint, an absolute minimum value $z_m = -\frac{A}{a} = 1$, hence:

$$G(1) = -\frac{1}{e^2} \approx -0.135.$$

The following result provides the conditions which must be satisfied for the correlation model (26) to be valid in $\mathbb{R}^2$.

Theorem 4 *Let*

$$\rho(h; A, a) = (1 + Ah) \exp(-ah), \quad a > 0, \quad h = \|\mathbf{h}\|, \quad \mathbf{h} \in \mathbb{R}^2; \quad (27)$$

then ρ is a c.f. in $\mathbb{R}^2$ if and only if: $-\frac{a}{2} \leq A \leq a$.

Proof (See the Appendix). In particular, the class (27) is always positive if $0 \leq A \leq a$ and it always assumes negative values if: $-\frac{a}{2} \leq A < 0$.

The following result provides the conditions which must be satisfied for the correlation model (26) to be valid in $\mathbb{R}^3$.

Theorem 5 *Let*

$$\rho(h; A, a) = (1 + Ah) \exp(-ah), \quad a > 0, \quad h = \|\mathbf{h}\|, \quad \mathbf{h} \in \mathbb{R}^3; \quad (28)$$

then ρ is a c.f. in $\mathbb{R}^3$ if and only if: $-\frac{a}{3} \leq A \leq a$.

Proof (See the Appendix). In particular, the class (28) is always positive if $0 \leq A \leq a$ and it always assumes negative values if: $-\frac{a}{3} \leq A < 0$.

3.1.4 Product of an exponential with a second degree polynomial

A further class of models can be obtained through the product of an exponential with a second degree polynomial, as specified in the following results.

Theorem 6 *Let*

$$\rho(h; A, a) = (1 + Ah^2) \exp(-ah), \quad a > 0, \quad h \in \mathbb{R}; \quad (29)$$

then ρ is a c.f. in $\mathbb{R}$ if and only if $-\frac{a^2}{2} \leq A \leq \frac{a^2}{3}$.

Proof (See the Appendix). In particular, the class (29) is always positive if: $0 \leq A \leq \frac{a^2}{3}$ and it always assumes negative values if: $-\frac{a^2}{2} \leq A < 0$, moreover the same class always presents a linear behaviour at the origin since $\rho'(0) = -a$.

The following result provides the conditions which must be satisfied for the correlation model (29) to be valid in $\mathbb{R}^2$.

Theorem 7 *Let*

$$\rho(h; A, a) = (1 + Ah^2) \exp(-ah), \quad a > 0, \quad h = \|\mathbf{h}\|, \quad (30)$$

with $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$; then ρ is a c.f. in $\mathbb{R}^2$ if and only if: $-\frac{a^2}{6} \leq A \leq \frac{2a^2}{9}$.

Proof (See the Appendix). In particular, the class (30) is always positive if: $0 \leq A \leq \frac{2a^2}{9}$ and it always assumes negative values if: $-\frac{a^2}{6} \leq A < 0$.

The following result provides the conditions which must be satisfied for the correlation model (29) to be valid in $\mathbb{R}^3$.

Theorem 8 *Let*

$$\rho(h; A, a) = (1 + Ah^2) \exp(-a|h|), \quad a > 0, \quad h = \|\mathbf{h}\|, \quad (31)$$

with $\mathbf{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$; then ρ is a valid family of isotropic c.f.s in $\mathbb{R}^3$ if and only if:

$$-\frac{a^2}{12} \leq A \leq \frac{a^2}{6}.$$

Proof (See the Appendix). In particular, the class (31) is always positive if: $0 \leq A \leq \frac{a^2}{6}$ and it always assumes negative values if: $-\frac{a^2}{12} \leq A < 0$.

3.1.5 Product of a Gaussian model with a second degree polynomial

In the present Section the product of a Gaussian model with a second degree polynomial will be examined in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem 9 *Let:*

$$\rho(h; A, a) = (1 - Ah^2) \exp(-ah^2), \quad a > 0, \quad (32)$$

then ρ is a valid c.f., in $\mathbb{R}$, if and only if: $0 \leq A \leq 2a$.

Proof (See the Appendix). According to (56), the class (32) cannot be a correlation family if $A < 0$, as a consequence the same class is always negative in a subset of its domain and it always presents just one zero for $h > 0$.

Note that $h_m = \sqrt{\frac{1}{a} + \frac{1}{A}}$ is the absolute minimum of (32) for $h > 0$, with

$$C(h_m) = -\frac{A}{a} \exp \left[-\left(1 + \frac{a}{A}\right) \right];$$

in particular, the function $G(z) = -z \exp \left[-\left(1 + \frac{1}{z}\right) \right]$,

with $z = \frac{A}{a} > 0$ presents, taking into account the constraint

$A \leq 2a$, an absolute minimum value $z_m = 2$, i.e. $\frac{A}{a} = 2$,

hence: $G(2) = -\frac{2}{e^{1.5}} \approx -0.446$.

The following result provides the conditions which must be satisfied for the correlation model (32) to be valid in $\mathbb{R}^2$.

Theorem 10 *The following family*

$$\rho(h; A, a) = (1 - Ah^2) \exp(-ah^2), \quad a > 0, \quad h = \|\mathbf{h}\|, \quad (33)$$

with $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$ is a valid family of isotropic c.f. s in $\mathbb{R}^2$ if and only if: $0 \leq A \leq a$.

Proof (See the Appendix). The following result provides the conditions which must be satisfied for the correlation model (32) to be valid in $\mathbb{R}^3$.

Theorem 11 *The following family*

$$\rho(h; A, a) = (1 - Ah^2) \exp(-ah^2), \quad a > 0, \quad h = \|\mathbf{h}\|, \quad (34)$$

with $\mathbf{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$ is a valid family of isotropic c.f.s in $\mathbb{R}^3$ if and only if: $0 \leq A \leq \frac{2a}{3}$.

Proof (See the Appendix). All the results of the Sects. 3.1.3, 3.1.4 and 3.1.5, concerning the admissibility of the proposed models, their behaviour near the origin and the sign of the same models, have been properly summarized in Tables 1, 2 and 3 for the Euclidean spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

3.2 Negative correlation models through differentiable correlation functions

Utilizing the result given by Ushakov (1999), some interesting models of c.f.s, characterized by negative values, can be built by computing the derivatives of even order of a differentiable c.f.. In particular it will be exhibited that, even starting from a c.f. which is always positive in its domain, correlation families characterized by negative values can be obtained through differentiation. It will be shown that these new classes of c.f.s are characterized by a finite number of zeros as many as desired; at our knowledge, there seem not to exist c.f.s which present this relevant property.

Theorem 12 (Ushakov 1999). *If a covariance function C is $2n$ times differentiable in $\mathbb{R}$ then the function*

$$\rho(h) = \frac{C^{(2n)}(h)}{C^{(2n)}(0)}$$

is a c.f..

The following result establishes that the second derivative of a positive and twice differentiable covariance function vanishes at least in one point, of the positive semiaxis, if the same covariance function goes to zero at infinity and presents a parabolic behaviour at the origin.

Theorem 13 *Let C be, in $\mathbb{R}$, a twice differentiable covariance function, such that:*

Table 1 Summary of the results of the Sects. 3.1.3, 3.1.4 and 3.1.5 where the Euclidean space is $\mathbb{R}$. L.B. and P.B. stand for Linear Behaviour and Parabolic Behaviour, respectively

$\rho(h; A, a) =$	$(1 + A h) \exp(-a h)$	$(1 + Ah^2) \exp(-a h)$	$(1 - Ah^2) \exp(-ah^2)$
ρ is a c.f	$-a \leq A \leq a$	$\frac{-a^2}{2} \leq A \leq \frac{a^2}{3}$	$0 \leq A \leq 2a$
L.B	$-a \leq A < a$	$\frac{-a^2}{2} \leq A \leq \frac{a^2}{3}$	
P. B	$a = A$		$0 \leq A \leq 2a$
$\rho(h) > 0, \forall h \in \mathbb{R}$	$0 \leq A \leq a$	$0 \leq A \leq \frac{a^2}{3}$	$A = 0$
$\rho(h) \leq 0$ in a	$-a \leq A < 0$	$\frac{-a^2}{2} \leq A < 0$	$0 < A \leq 2a$
subset of $\mathbb{R}$			

Table 2 Summary of the results of the Sects. 3.1.3, 3.1.4 and 3.1.5 where the Euclidean space is $\mathbb{R}^2$. L.B. and P.B. stand for Linear Behaviour and Parabolic Behaviour, respectively

$\rho(h; A, a) =$	$(1 + A h) \exp(-a h)$	$(1 + Ah^2) \exp(-a h)$	$(1 - Ah^2) \exp(-ah^2)$
ρ is a c.f	$\frac{-a}{2} \leq A \leq a$	$\frac{-a^2}{6} \leq A \leq \frac{2a^2}{9}$	$0 \leq A \leq a$
L.B	$\frac{-a}{2} \leq A < a$	$\frac{-a^2}{6} \leq A \leq \frac{2a^2}{9}$	
P. B	$a = A$		$0 \leq A \leq a$
$\rho(h) > 0, \forall h \in \mathbb{R}$	$0 \leq A \leq a$	$0 \leq A \leq \frac{2a^2}{9}$	$A = 0$
$\rho(h) \leq 0$ in a	$\frac{-a}{2} \leq A < 0$	$\frac{-a^2}{6} \leq A < 0$	$0 < A \leq a$
subset of $\mathbb{R}$			

Table 3 Summary of the results of the Sects. 3.1.3, 3.1.4 and 3.1.5 where the Euclidean space is $\mathbb{R}^3$. L.B. and P.B. stand for Linear Behaviour and Parabolic Behaviour, respectively

$\rho(h; A, a) =$	$(1 + A h) \exp(-a h)$	$(1 + Ah^2) \exp(-a h)$	$(1 - Ah^2) \exp(-ah^2)$
ρ is a c.f	$\frac{-a}{3} \leq A \leq a$	$\frac{-a^2}{12} \leq A \leq \frac{a^2}{6}$	$0 \leq A \leq \frac{2a}{3}$
L.B	$\frac{-a}{3} \leq A < a$	$\frac{-a^2}{12} \leq A \leq \frac{a^2}{6}$	
P. B	$a = A$		$0 \leq A \leq \frac{2a}{3}$
$\rho(h) > 0, \forall h \in \mathbb{R}$	$0 \leq A \leq a$	$0 \leq A \leq \frac{a^2}{6}$	$A = 0$
$\rho(h) \leq 0$ in a	$\frac{-a}{3} \leq A < 0$	$\frac{-a^2}{12} \leq A < 0$	$0 < A \leq \frac{2a}{3}$
subset of $\mathbb{R}$			

1) $C(h) > 0, \forall h \in \mathbb{R}$, 2) $\lim_{h \rightarrow +\infty} C(h) = 0$, 3) $C''(0) < 0$,

then there exists $h_0 > 0$ such that $C''(h_0) = 0$.

Proof (See the Appendix).

Corollary 1 If the hypotheses of Theorem (13) are satisfied and there exists just one inflection point $h_0 > 0$ for the covariance function C , then

$$\rho(h) = \frac{C''(h)}{C''(0)}, \quad h \in \mathbb{R},$$

is a valid c.f. with just one zero, i.e., $\rho(h_0) = 0$, for $h_0 > 0$.

Proof The result easily follows from Theorem 12 and Theorem 13. $\square$

The next result points out that the second derivative of a twice differentiable c.f. with one zero for $h > 0$, a parabolic behaviour at the origin and vanishing at infinity, presents at least two zeros for $h > 0$.

Theorem 14 Let ρ be a twice differentiable c.f. with just one zero, i.e., $\rho(h_0) = 0, h_0 > 0$ and such that:

1) $\lim_{h \rightarrow +\infty} \rho(h) = 0$, 2) $\rho'(0) = 0$, 3) $\rho''(0) < 0$,

then ρ'' presents 2 zeros for $h > 0$.

Proof (See the Appendix).

Corollary 2 If the hypotheses of Theorem (14) are satisfied for the c.f. ρ , then

$$\rho_1(h) = \frac{\rho''(h)}{\rho''(0)}, \quad h \in \mathbb{R},$$

is a valid c.f. with 2 zeros for $h > 0$.

Proof It easily follows from Theorem 12 and Theorem 14. $\square$

As will be shown hereafter, it is possible to construct correlation models characterized by negative values, starting from positive correlation families. Indeed, taking into account all the previous results, the following general Theorem can be introduced.

Theorem 15 Let C be a $2n$ times differentiable covariance function, in $\mathbb{R}$, such that:

1) $C(h) > 0, \forall h \in \mathbb{R}$; 2) $\lim_{h \rightarrow +\infty} C^{(2n)}(h) = 0$,
 $\forall n \in \mathbb{N}$; 3) $C'(0) = 0$; 4) $C''(0) < 0$,

then the function

$$\rho(h) = \frac{C^{(2n)}(h)}{C^{(2n)}(0)},$$

is a c.f., in $\mathbb{R}$, with n zeros for $h > 0$.

Proof The result follows, in an iterative way, taking into account Theorems 12, 13, and 14 as well as Corollaries 1 and 2. $\square$

3.2.1 A special example

Taking into account the results of the previous Section, a special family of c.f.s is introduced hereafter as a significant example.

Theorem 16 Let:

$$C_0(h; A, a) = (1 - Ah^2) \exp(-ah^2), \quad h \in \mathbb{R}, \quad A > 0, a > 0, \quad (35)$$

and, more generally,

$$\rho_n(h; A, a) = \frac{C_0^{(2n)}(h; A, a)}{C_0^{(2n)}(0; A, a)} \quad n = 1, 2, \dots; \quad (36)$$

then $\rho_n, \forall n \in \mathbb{N}$ are valid c.f.s, in $\mathbb{R}$, if $A \leq 2a$. Moreover, the parametric family (36), for a fixed value of $n \in \mathbb{N}$, presents $(n + 1)$ zeros for $h > 0$.

Proof First of all, note that (35) is infinitely differentiable and:

$$\lim_{h \rightarrow \infty} \rho_n(h; A, a) = 0, \quad n = 1, 2, \dots$$

It has been already proved in Theorem 9 that (35) is a c.f. if $A \leq 2a$. Hence, since C_0 presents just one zero for $h > 0$ and:

$$\rho'_n(0; A, a) = 0, \quad \rho''_n(0; A, a) < 0, \quad \forall n \in \mathbb{N},$$

because of the previous results, it comes out that $\rho_n, \forall n \in \mathbb{N}$, are valid c.f.s; moreover, for a fixed value of $n \in \mathbb{N}$, ρ_n presents $(n + 1)$ zeros for $h > 0$. $\square$

Remark The Gaussian c.f.: $\rho(h; a) = \exp(-ah^2)$, $h \in \mathbb{R}$, $a > 0$, satisfies the hypotheses of Theorem 15, hence $\rho^*(h; a) = \frac{\rho''(h; a)}{\rho''(0; a)}$ is a c.f.; in particular:

$$\rho^*(h; a) = (1 - 2ah^2) \exp(-ah^2), \quad (37)$$

is a subclass of the family given in (32) or equivalently of the family (35), with $A = 2a$.

3.2.2 Models derived from the difference of Gaussian covariances

It has been shown (Posa 2023) that, in $\mathbb{R}$, the following class:

$$\rho(h; \Theta) = A \exp(-ah^2) - B \exp(-bh^2), \quad (38)$$

$$A > 0, B > 0, a > 0, b > 0,$$

where $\Theta = (A, B, a, b)$, is a valid family of c.f.s if $1 < \frac{a}{b} < \left(\frac{A}{B}\right)^2$, $A - B = 1$, with just one zero for $h > 0$. Hence, according to Theorem 14, the following c.f.:

$$\rho_D(h; \Theta) = \frac{Aa(1 - 2ah^2) \exp(-ah^2) - Bb(1 - 2bh^2) \exp(-bh^2)}{Aa - Bb}, \quad (39)$$

presents 2 zeros for $h > 0$, whereas the second derivative of (39) presents 3 zeros for $h > 0$; more generally, the derivative of order $2n$ of (38) presents $(n + 1)$ zeros for $h > 0$.

3.2.3 Models derived from the difference of exponential models

It has been shown (Posa 2023) that, in $\mathbb{R}$, the following class:

$$\rho(h; \Theta) = A \exp(-\alpha|h|) - B \exp(-\beta|h|), \quad A > 0, B > 0, \alpha > 0, \beta > 0, \quad (40)$$

where $\Theta = (A, B, \alpha, \beta)$, is always a positive family of c.f.s if $1 < \frac{\beta}{\alpha} = \frac{A}{B}$, with $A - B = 1$.

Since the class (40) is differentiable, then the function:

$$\frac{\rho''(h; \Theta)}{\rho''(0; \Theta)} = \frac{B\beta^2 \exp(-\beta|h|) - A\alpha^2 \exp(-\alpha|h|)}{B\beta^2 - A\alpha^2}, \quad (41)$$

is a valid correlation family. Taking into account the constraint $\frac{\beta}{\alpha} = \frac{A}{B}$, expression (41) can be easily simplified as follows:

$$\rho(h; A, B, \alpha, \beta) = \frac{\beta \exp(-\beta|h|) - \alpha \exp(-\alpha|h|)}{(\beta - \alpha)}. \quad (42)$$

The previous correlation family presents just one zero for $h > 0$:

$$h_0 = \frac{1}{\beta - \alpha} \ln\left(\frac{\beta}{\alpha}\right),$$

and just one minimum value for $h > 0$:

$$\rho(h_m) = \frac{1}{\beta - \alpha} \left[\beta \left(\frac{\alpha}{\beta}\right)^{\frac{2\beta}{\beta - \alpha}} - \alpha \left(\frac{\alpha}{\beta}\right)^{\frac{2\alpha}{\beta - \alpha}} \right],$$

with:

$$h_m = \frac{2}{\beta - \alpha} \ln\left(\frac{\beta}{\alpha}\right);$$

it can be easily checked that ρ presents an absolute minimum value:

$$\rho(h_m^*) \approx -0.135.$$

3.2.4 Models derived from the logarithmic model

It can be easily proved that the following family:

$$\rho(h; a, b) = \frac{0.5}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{h^2 + a^2}{h^2 + b^2}\right), \quad h \in \mathbb{R}, \quad 0 < b < a, \quad (43)$$

is a valid class of correlation models which are always positive in $\mathbb{R}$.

In fact the Fourier transform of (43):

$$f(\omega; a, b) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho(h; a, b) e^{-i\omega h} dh = \frac{1}{\ln\left(\frac{a}{b}\right)} \frac{e^{-b|\omega|} - e^{-a|\omega|}}{|\omega|}, \quad \omega \in \mathbb{R},$$

is a valid spectral density function since it is non negative and integrable in $\mathbb{R}$, if $0 < b < a$. Moreover, since (43) is differentiable in $\mathbb{R}$, then

$$\rho_1(h; a, b) = \frac{\rho''(h; a, b)}{\rho''(0; a, b)} = \frac{a^2 b^2 (a^2 b^2 - h^2 (a^2 + b^2) - 3h^4)}{(h^2 + a^2)^2 (h^2 + b^2)^2}, \quad (44)$$

is a valid correlation family in $\mathbb{R}$ and, according to Theorem 15, the same class always presents just one zero for $h > 0$.

By fixing $b^2 = 1$, the absolute minimum value of (44) can be easily obtained:

$$\rho_{1,m}(1, 1) = -0.25.$$

The above procedure can be iterated; indeed, since ρ_1 is differentiable in $\mathbb{R}$, then:

$$\rho_2(h; a, b) = \frac{\rho_1''(h; a, b)}{\rho_1''(0; a, b)},$$

is a valid correlation family in $\mathbb{R}$, that is:

$$\rho_2(h; a, b) = \frac{a^4 b^4 [10h^{10} + 5\alpha h^8 + 4\beta h^6 + \gamma h^4 - 2\delta a^2 b^2 h^2 + \alpha a^4 b^4]}{\alpha(h^2 + a^2)^4(h^2 + b^2)^4} \quad (45)$$

where:

$$\alpha = (a^2 + b^2), \quad \beta = (a^4 - 9a^2 b^2 + b^4), \\ \gamma = (a^6 - 23a^4 b^2 - 23a^2 b^4 + b^6), \quad \delta = (3a^4 + a^2 b^2 + 3b^4);$$

according to Theorem 15, this last class presents two zeros for $h > 0$.

Some remarks 1. It has been proved (Gneiting et al. 2001) that if γ is twice differentiable in $\mathbb{R}$, with $\gamma(0) = 0$, then γ is a variogram if and only if γ'' is a c.f.. According to this last result, it can be easily verified that, starting from the variogram function corresponding to some positive correlation models, negative correlation families can be obtained as special cases of some classes previously discussed, as illustrated in the following examples.

- Consider, in $\mathbb{R}$, the following Gaussian variogram model:

$$\gamma(h; k) = 1 - e^{-kh^2}, \quad h \in \mathbb{R}, \quad k > 0, \quad (46)$$

then $C = \gamma''$ is a covariance function in $\mathbb{R}$, whose corresponding c.f. is the following:

$$\rho(h; k) = e^{-kh^2}(1 - 2kh^2), \quad k > 0, \quad (47)$$

which is a special case of the family (35).

- Consider the variogram function corresponding to the Cauchy correlation family:

$$\gamma(h; n) = 1 - \frac{1}{(h^2 + 1)^n}, \quad h \in \mathbb{R}, \quad n \geq 2;$$

the correlation class ρ^* related to the covariance function $C = \gamma''$ is given hereafter:

$$\rho^*(h; n) = \frac{1 - h^2(2n - 3)}{(h^2 + 1)^n},$$

which is a special case of the family (15).

- It is suitable to underline that the classes of correlation models introduced in this paper highlight the capability of the same models for developing hypothesis testing based on the parameters involved, such as the parameter k (e.g., Equation (15)) and the parameter A (e.g., Equation (26) and Equation (29)). Indeed, the threshold values, $k = 0$ and $A = 0$ can be utilized to separate the null hypothesis of positive correlation, from the alternative hypothesis involving negative correlation.

4 Representation of the results

The results of the previous Section, concerning the behaviour of the proposed correlation models, have been properly represented according to the parameters values on which the same models depend.

The graphical representation and discussion of the results follows hereafter taking into account the same sequence detailed in the previous Section.

- As already underlined, the generalized Cauchy correlation family in (15) is able to describe both positive and negative correlation structures, according to the values of the parameter k . It has been shown that the generalized Cauchy correlation family is a valid class, in $\mathbb{R}$, if $-1 \leq k \leq (2n - 3)$ and, for a fixed value of n , the absolute minimum value is obtained for $k = 2n - 3$. Figure 1 shows the behaviour of the family (15) for different values of the parameter n , with $k = 2n - 3$. Note that as n increases the sequence of the absolute minimum values approaches the lowest minimum value -0.446 of the class (15), as described in expression (18). It has been shown that the c.f.s in (15) for $n = 2$ are also valid in $\mathbb{R}^2$ if $-1 \leq k \leq 0$, although in this last case the correlation family can only assume positive values. On the other hand, it has been shown that the same family is admissible in $\mathbb{R}^3$ (as a consequence, it is also valid in $\mathbb{R}^2$) for $n = 3$ and $n = 4$, for suitable values of the parameter k : in these last cases the correlation family in (15) can assume just positive values, as well as negative values in a subset of its domain.

Fig. 1 Behaviour of the family (15) for different values of the parameter n , with $k = 2n - 3$. Note that as n increases the sequence of the absolute minimum values approaches the lowest minimum value -0.446 of the class (15), as described in expression (18)

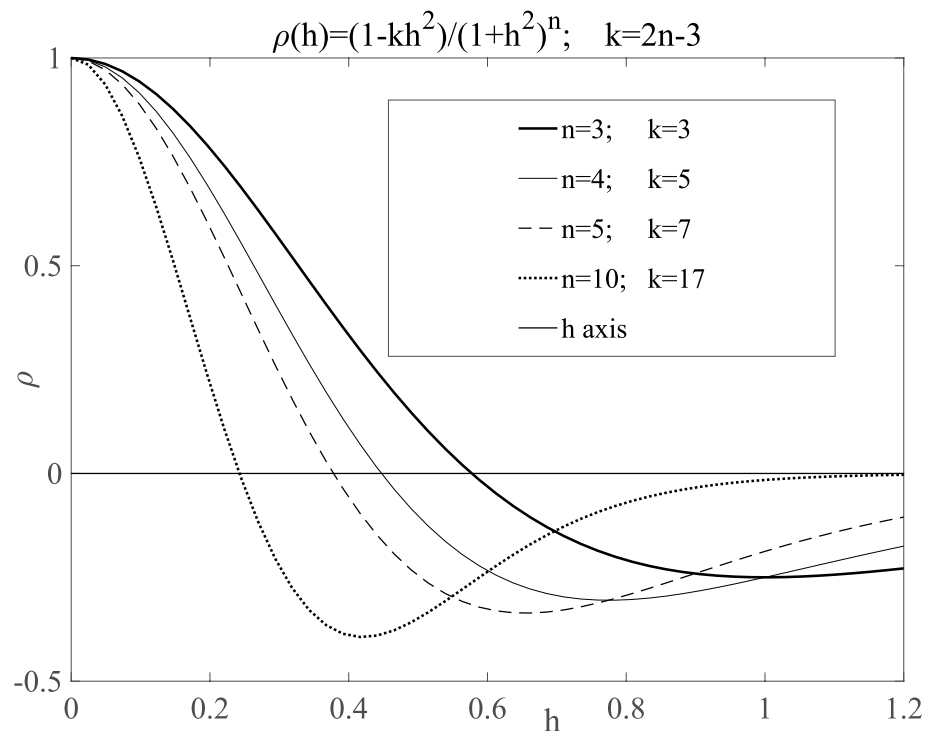
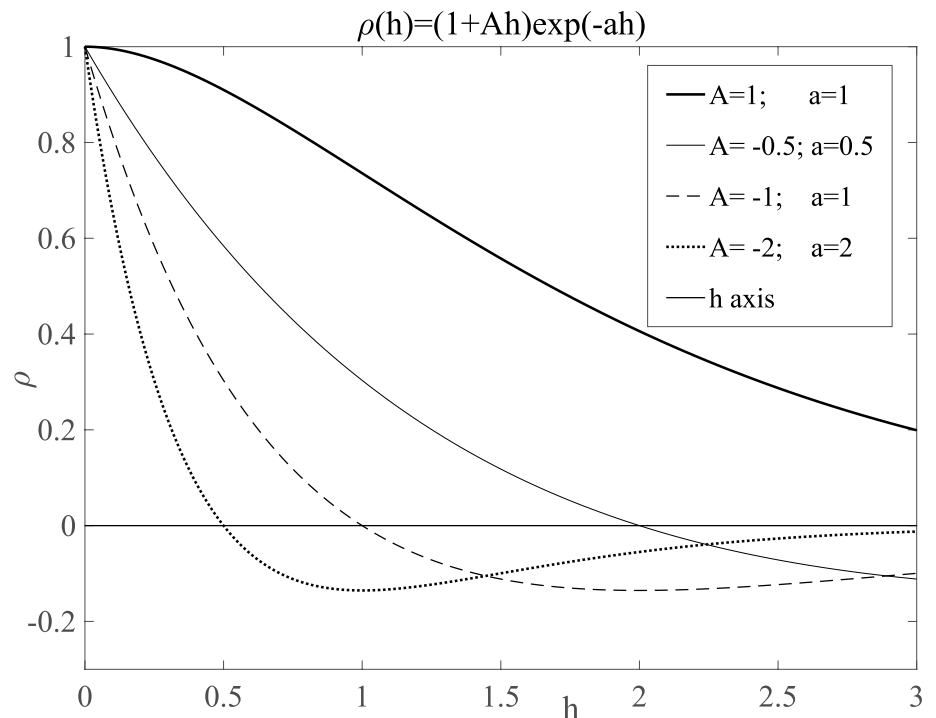


Fig. 2 Behaviour of the class (26) for different values of its parameters, in particular for $A = a$ and for some values of A and a such that $A = -a$; indeed, in this last case, the absolute minimum value assumed by the ρ does not change



- The class (26) provides a generalization of a subclass of Matérn family: indeed, the class (26) is able to model both positive and negative correlation structures, with respect to the traditional subclass of Matérn family which is able to model just positive correlation structures. In $\mathbb{R}$, it is a valid class of c.f.s if $-a \leq A \leq a$. In particular, if $A = a$ the class presents a parabolic behaviour

near the origin, apart from the fact that the correlation family is always positive if $0 \leq A \leq a$; on the other hand, it presents just one zero, it is always negative in a subset of its domain and it presents a linear behaviour near the origin if $-a \leq A < 0$, moreover, the lowest minimum value is reached for $A = -a$. Figure 2 shows the behaviour of the class (26) for different values of its

parameters, in particular for $A = a$ and for some values of A and a such that $A = -a$; indeed, in this last case, the absolute minimum value does not change, as it has been detailed in Sect. 3.1.3. It has been shown that the c.f.s in (26) is also valid in $\mathbb{R}^2$ and in $\mathbb{R}^3$ and the same family can assume just positive values, as well as negative values in a subset of its domain for suitable values of its parameters.

3. The class (29) is a valid correlation family, in $\mathbb{R}$, if $-\frac{a^2}{2} \leq A \leq \frac{a^2}{3}$ and it is able to model both positive and negative correlation structures in any dimensional space ($\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$). If $-\frac{a^2}{2} \leq A < 0$ it presents just one zero, it is always negative in a subset of its domain and it always presents a linear behaviour near the origin. Figure 3 shows the behaviour of the class (29) in $\mathbb{R}$ for the value $a = 1$ and different values of A : note that the absolute minimum value of (29) is obtained for $A = -\frac{a^2}{2}$, moreover, if $-\frac{a^2}{2} \leq A < 0$ and A increases, the minimum value becomes less negative, whereas the point of intersection with the h axis increases. Of course, if $A \geq 0$ the family (29) is always positive. It has been shown that the c.f.s in (29) is also valid in $\mathbb{R}^2$ and in $\mathbb{R}^3$ and the same family can assume just positive values, as well as negative values in a subset of its domain for suitable values of its parameters.
4. The class (32) is a valid c.f., in $\mathbb{R}$, if and only if: $0 \leq A \leq 2a$. According to (56), the class (32) cannot

be a correlation family if $A < 0$, as a consequence the same class is always negative in a subset of its domain, it always presents just one zero for $h > 0$ and it always presents a parabolic behaviour near the origin. As already underlined, $C(h_m) \approx -0.446$ is the absolute minimum value of (32) for $h > 0$. Figure 4 shows the behaviour of the class (32) by fixing the parameter $a = 1$: the minimum value becomes more and more negative as A gets closer to $2a = 2$, whereas the absolute minimum value is obtained for $A = 2$. It has been shown that the c.f.s in (32) is also valid in $\mathbb{R}^2$ and in $\mathbb{R}^3$ and the same family always assumes negative values in a subset of its domain for suitable values of its parameters. As already underlined, all the results of the last three items, which refer to the Sections 3.1.3, 3.1.4 and 3.1.5, concerning the admssibility of the proposed models, their behaviour near the origin and the sign of the same models, have been properly summarized in Tables 1, 2 and 3 for the Euclidean spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

5. Starting from the model (32), it has been shown through Theorem 16 that a countable family of c.f.s can be obtained and the same family presents as many zeros as desired. For example, the class (32) presents just one zero for $h > 0$, whereas its second derivative presents two zeros, for $h > 0$, moreover, its fourth derivative presents 3 zeros, for $h > 0$. Hence, starting from the model (32) and utilizing expression (36) for $n = 1$ and $n = 2$, respectively, the following models have been obtained:

Fig. 3 Behaviour of the class (29) in $\mathbb{R}$ for the value $a = 1$ and different values of A : note that the absolute minimum value of (29) is obtained for $A = -\frac{a^2}{2}$, moreover, if $-\frac{a^2}{2} \leq A < 0$ and A increases, the minimum value becomes less negative, whereas the point of intersection with the h axis increases. If $A \geq 0$ the family (29) is always positive

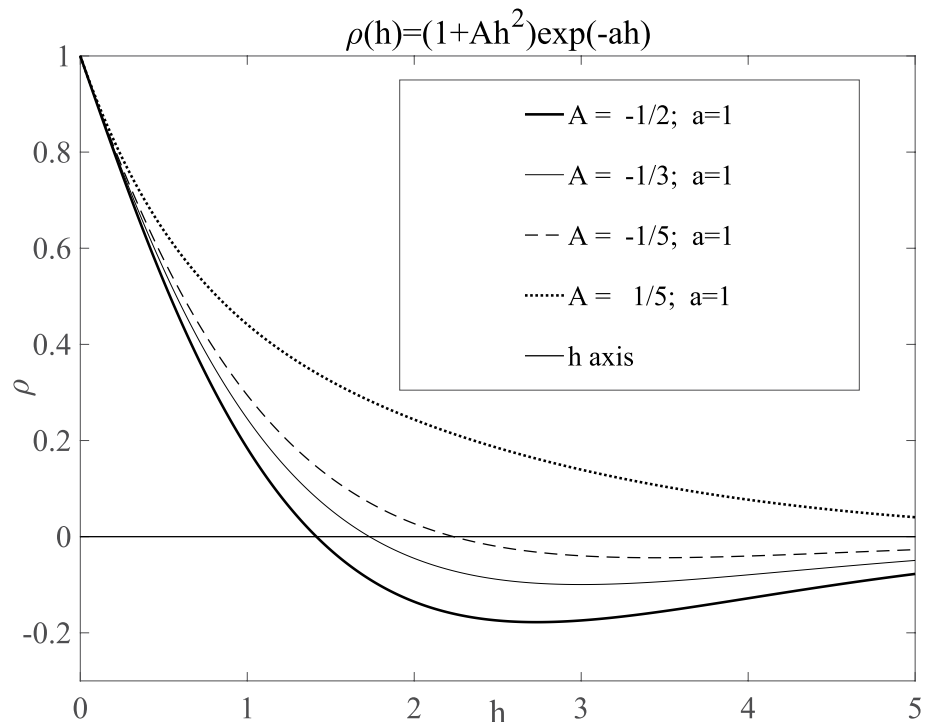
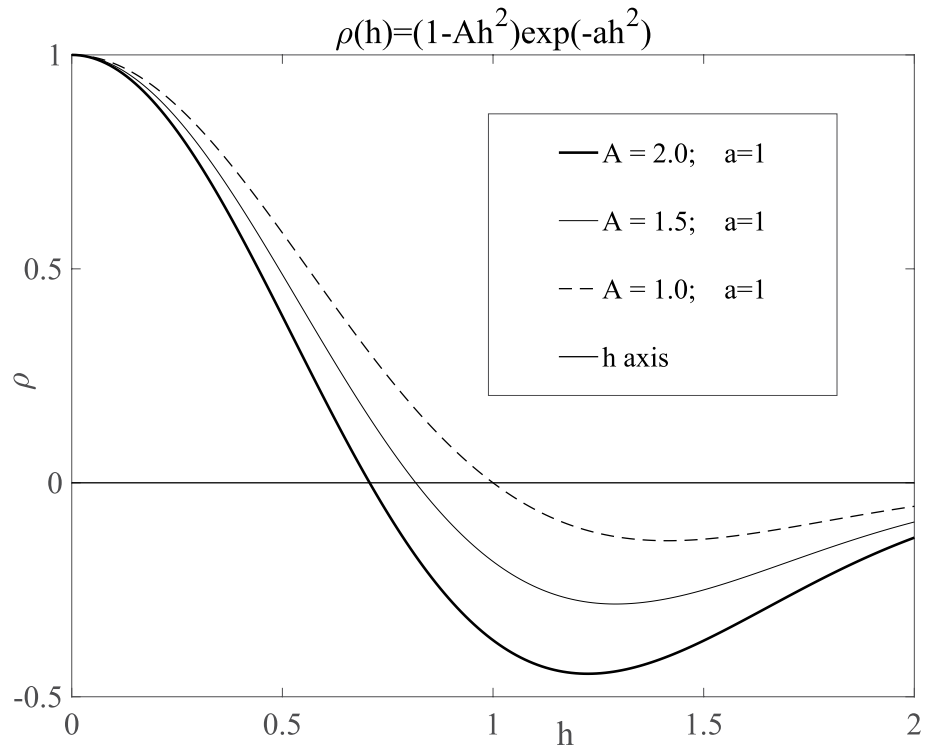


Fig. 4 Behaviour of the class (32) by fixing the parameter $a = 1$: the minimum value becomes more and more negative as A gets closer to $2a = 2$, whereas the absolute minimum value is obtained for $A = 2$



$$\rho_1(h; A, a) = \frac{(2a^2Ah^4 - (2a^2 + 5Aa)h^2 + a + A)e^{-ah^2}}{a + A}, \quad (48)$$

$$\rho_2(h; A, a) = \frac{[(3a + 6A) - (12a^2 + 39aA)h^2 + (4a^3 + 28Aa^2)h^4 - 4Aa^3h^6]e^{-ah^2}}{3a + 6A}, \quad (49)$$

these two last classes of models are valid in $\mathbb{R}$ if $0 \leq A \leq 2a$. Figure 5a) shows the behaviour of model (48) for $a = 1$ and different values of A : note that as A gets closer to the upper bound ($A = 2a$), the minimum values become more and more negative. On the other hand, Fig. 5b) shows the behaviour of model (48) for a fixed value of the ratio $\frac{A}{a} = 2$ corresponding to the absolute minimum value reached by the same class. Indeed, it can be checked that the absolute minimum value of ρ_1 does not change and it just depends on the above ratio. The family (48) always presents two zeros for $h > 0$ and a parabolic behaviour near the origin. Figure 6 shows the behaviour of model (49) for different values of the parameters a and A such that $A = 2a$. Indeed, even in this case, it can be checked that the absolute minimum value of ρ_2 does not change and it just depends on the ratio $\frac{A}{a}$: in particular, the absolute minimum value corresponds to $A = 2a$. The family (49) always presents three zeros for $h > 0$ and a parabolic behaviour near the origin.

6. As already underlined, the class (38), in $\mathbb{R}$, is a valid family of c.f.s if

$$1 < \frac{a}{b} < \left(\frac{A}{B}\right)^2, \quad A - B = 1,$$

with just one zero for $h > 0$ (Posa 2023). Hence, according to Theorem 14, the class (39):

$$\rho_D(h; \Theta) = \frac{Aa(1 - 2ah^2)\exp(-ah^2) - Bb(1 - 2bh^2)\exp(-bh^2)}{Aa - Bb},$$

presents 2 zeros for $h > 0$, whereas the second derivative of (39) presents 3 zeros for $h > 0$; more generally, the derivative of order $2n$ of (38) presents $(n + 1)$ zeros for $h > 0$. Fig. 7 represents the behaviour of the class (39) by fixing $b = B = 1$. Note that, as the difference between $\left(\frac{A}{B}\right)^2$ and $\frac{a}{b}$ increases the minimum value becomes less negative; on the other hand, as the ratio $\frac{a}{b}$ becomes closer to the ratio $\left(\frac{A}{B}\right)^2$ and both ratios are very close to 1, the minimum value becomes more negative.

7. It has been detailed that the model derived from the difference of exponential c.f.s, as described in (40), is always a positive and differentiable family of c.f.s if $1 < \frac{\beta}{\alpha} < \frac{A}{B}$, with $A - B = 1$. Then, taking into account the constraint $\frac{\beta}{\alpha} < \frac{A}{B}$, the function:

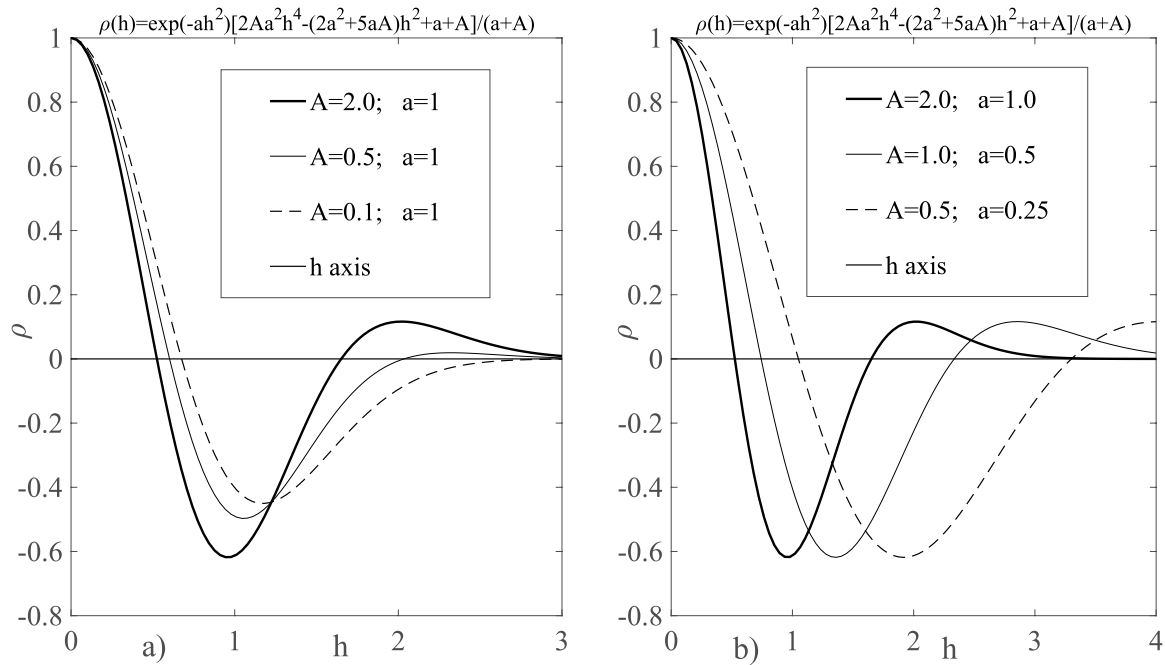


Fig. 5 **a** Behaviour of model (48) for $a = 1$ and different values of A : note that as A gets closer to the upper bound ($A = 2a$), the minimum values become more and more negative. **b** Behaviour of model (48) for

a fixed value of the ratio $\frac{A}{a} = 2$ corresponding to the absolute minimum value reached by the same class. Note that the absolute minimum value does not change and it just depends on the above ratio. The family (48) always presents two zeros for $h > 0$ and a parabolic behaviour near the origin

Fig. 6 Behaviour of model (49) for different values of the parameters a and A such that $A = 2a$. Indeed, even in this case, it can be checked that the absolute minimum value of ρ_2 does not change and it just depends on the ratio $\frac{A}{a}$: in

particular, the absolute minimum value corresponds to $A = 2a$. The family (49) always presents three zeros for $h > 0$ and a parabolic behaviour near the origin

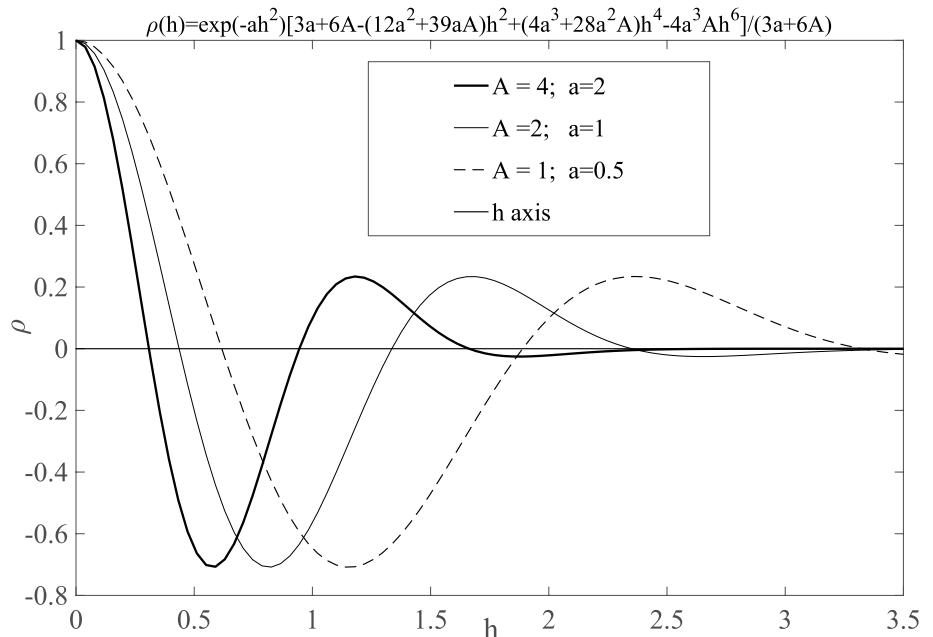


Fig. 7 Behaviour of the class (39) by fixing $b = B = 1$. Note that, as the difference between $\left(\frac{A}{B}\right)^2$ and $\frac{a}{b}$ increases the minimum value becomes less negative; on the other hand, as the ratio $\frac{a}{b}$ becomes closer to the ratio $\left(\frac{A}{B}\right)^2$ and both ratios are very close to 1, the minimum value becomes more negative

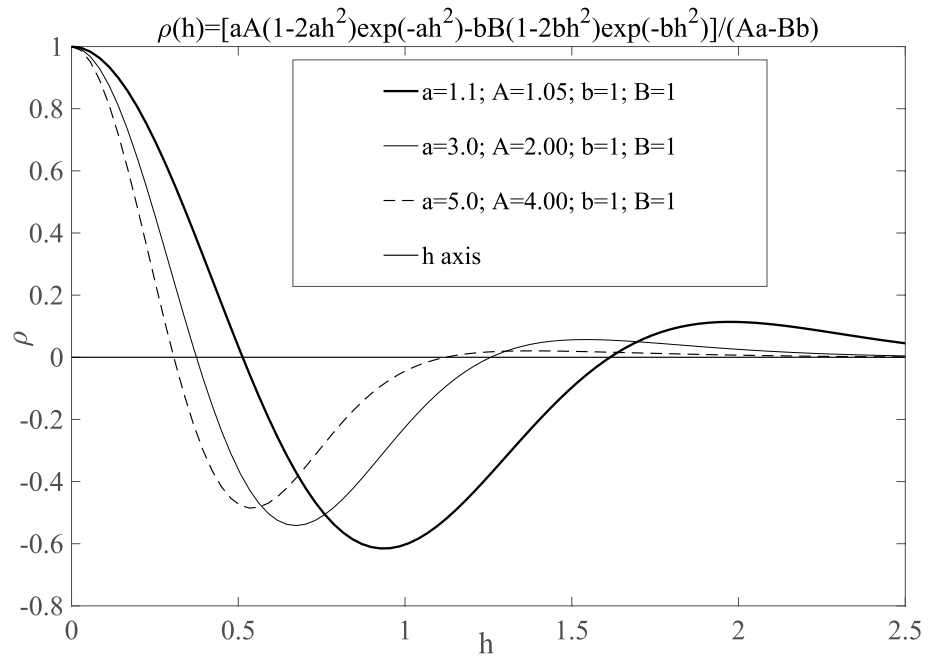
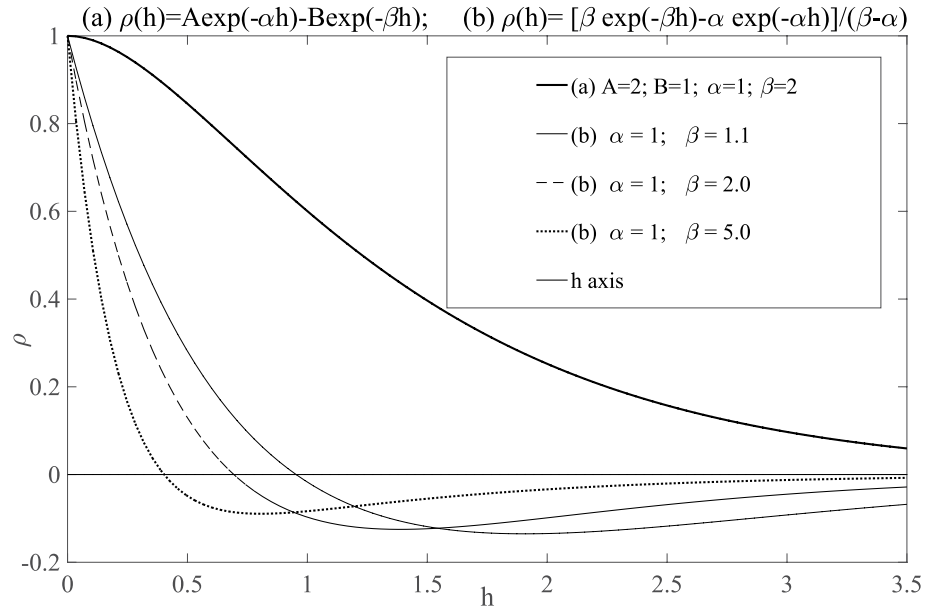


Fig. 8 Behaviour of model (40), for $A = 2, B = 1, \alpha = 1, \beta = 2$, together with model (41) by fixing $\alpha = 1$ and different values of β . The minimum value becomes ever more negative as $\beta \rightarrow 1^+$, up to reach the absolute minimum value

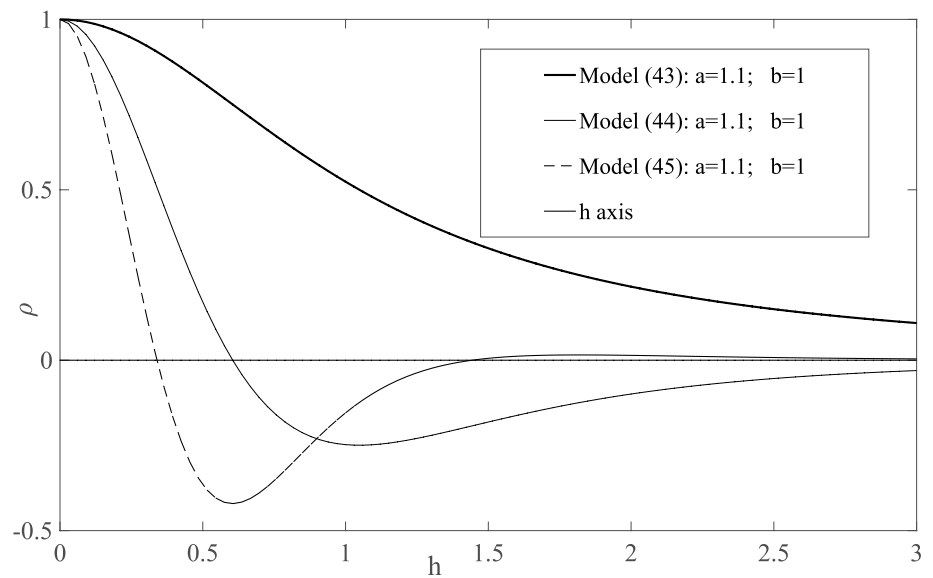


$$\frac{\rho''(h; \Theta)}{\rho''(0; \Theta)} = \frac{\beta \exp(-\beta|h|) - \alpha \exp(-\alpha|h|)}{(\beta - \alpha)},$$

is a valid correlation family. As already underlined, this last correlation family presents just one zero for $h > 0$ and just one minimum value for $h > 0$. Moreover, as $\frac{\beta}{\alpha} \rightarrow 1^+$, the same correlation family reaches the absolute minimum value $\rho(h_m^*) \approx -0.135$. Fig. 8 shows the behaviour of model (40), for $A = 2, B = 1, \alpha = 1, \beta = 2$, together with model (41) by fixing $\alpha = 1$ and different values of β . The minimum value becomes ever more negative as $\beta \rightarrow 1^+$, up to reach the absolute minimum value.

8. It has been proved that the family (43), in $\mathbb{R}$ if $0 < b < a$, is a valid class of correlation models and it is always positive and differentiable; then, the correlation family (44) is a valid class of c.f.s and, according to Theorem 15, the same class always presents just one zero for $h > 0$. Moreover, since (44) is differentiable in $\mathbb{R}$, then (45) is also a valid correlation family in $\mathbb{R}$ and, according to Theorem 15, this last class presents two zeros for $h > 0$. Fig. 9 shows the behaviour of the class (43), (44) and (45), respectively, for the values $a = 1.1$ and $b = 1$: note that the first model is always positive, the second presents just one zero, whereas the third

Fig. 9 Behaviour of the class (43), (44) and (45), respectively, for the values $a = 1.1$ and $b = 1$: note that the first model is always positive, the second presents just one zero, whereas the third presents 2 zeros, as already proved through Theorem 15



presents 2 zeros, as already proved through Theorem 15.

5 Conclusions

In this paper further developments and a comprehensive analysis concerning the behaviour and the admissibility of c.f.s characterized by negative values has been provided; in particular, new families of models have been proposed, in order to get wider classes of correlation models and to overcome some of the shortcomings which characterize some of the well-known correlation models with negative values encountered in the literature. It has been underlined that all the classical models are not able to describe some behaviors of correlation structures, hence the demand to get new flexible models which can be adequate to most of the case studies is advisable. In particular, apart from the classical Bochner's representation which characterizes the whole class of continuous c.f.s in any dimensional space, new techniques to build negative correlation structures will be provided, starting from some classical results, in spite of this, scarcely used in the applications. It is suitable to underline that these new classes of c.f.s are characterized by some relevant properties which lack the classical hole effect models, neither the new models recently proposed by (Posa 2023, 2025): in fact, some of these new classes are characterized by a finite number of zeros as many as desired; at our knowledge, there seem not to exist c.f.s which present this remarkable property. This kind of survey is particularly interesting from a theoretical point of view, but it is also helpful from a practical perspective since it provides a guideline for practitioners on how to utilize c.f.s characterized by negative values, taking into account various aspects,

such as the number of zeros which characterize the correlation models, the dimension of the Euclidean space, as well as the behaviour in proximity of the origin, among the others.

Appendix

Proof of Theorem 2 The Fourier transform of (15) is given hereafter:

$$\begin{aligned} f_1(\omega; k, n) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(\omega h) \frac{1 - kh^2}{(h^2 + 1)^n} dh \\ &= (1 + k) \int_{-\infty}^{+\infty} \frac{\cos(\omega h)}{(h^2 + 1)^n} dh - k \int_{-\infty}^{+\infty} \frac{\cos(\omega h)}{(h^2 + 1)^{n-1}} dh; \end{aligned}$$

hence, according to Gradshteyn and Ryzhik (2007), pag. 430, equ. 3.737.1:

$$\begin{aligned} f_1(\omega; k, n) &= \frac{(1 + k)e^{-|\omega|}}{(n - 1)!2^{2n-1}} \sum_{j=0}^{n-1} \frac{(2n - j - 2)!(2|\omega|)^j}{j!(n - j - 1)!} \\ &\quad - \frac{ke^{-|\omega|}}{(n - 2)!2^{2n-3}} \sum_{j=0}^{n-2} \frac{(2n - j - 4)!(2|\omega|)^j}{j!(n - j - 2)!}. \end{aligned} \quad (50)$$

As a consequence, the class (15) is a valid correlation family, in $\mathbb{R}$, if (50) is a spectral density function, i.e., it is non negative and integrable. At this purpose, note that (50) can be expressed through the product of an exponential function with a polynomial of degree $(n - 1)$, as specified hereafter:

$$f_1(\omega; k, n) = e^{-|\omega|} \sum_{j=0}^{n-1} \beta_j |\omega|^j; \quad (51)$$

hence (50) is integrable and it is non negative if $\beta_j \geq 0, j = 0, \dots, n-1$.

Note that for $j = 0$:

$$\beta_0 = \frac{(2n-4)!(2n-3-k)}{2^{2n-2}[(n-2)!]^2(n-1)},$$

which is non negative if: $k \leq (2n-3)$, for $n \geq 2$. For $j = 1$:

$$\beta_1 = \frac{(2n-5)!(2n-3-k)}{2^{2n-3}(n-2)!(n-3)!(n-1)},$$

which is non negative if: $k \leq (2n-3)$, for $n > 2$. For $j = n-1$:

$$\beta_{n-1} = \frac{(1+k)}{(n-1)!2^n},$$

which is non negative if: $k \geq -1$, for $n \geq 2$.

For an arbitrary value of j , such that $2 \leq j < (n-2)$:

$$\beta_j = \frac{(2n-j-4)!}{2^{2n-j-3}j!(n-2)!(n-j-2)!} \cdot \left[\frac{(1+k)(2n-j-3)(2n-j-2)}{4(n-1)(n-j-1)} - k \right];$$

if:

$$\delta(n, k, j) = 4n^2 - 2n(2j+5) + j^2 + 5j + 6 + k(j^2 + j + 2 - 2n) \geq 0, \quad (52)$$

then β_j is non negative. Note that:

$$4n^2 - 2n(2j+5) + j^2 + 5j + 6 = (2n-j-2)(2n-j-3) > 0,$$

hence two cases, according to the sign of $(j^2 + j + 2 - 2n)$, will be discussed in (52):

- if $(j^2 + j + 2 - 2n) > 0$, then it must be checked that:

$$-\frac{4n^2 - 2n(2j+5) + j^2 + 5j + 6}{(j^2 + j + 2 - 2n)} < -1;$$

this last inequality is always satisfied since $j < (n-1)$.

- if $(j^2 + j + 2 - 2n) < 0$, then it must be checked that:

$$\frac{4n^2 - 2n(2j+5) + j^2 + 5j + 6}{-(j^2 + j + 2 - 2n)} > 2n-3;$$

this last inequality is always satisfied since $(j-1)(n-1) > 0$. Hence, if $-1 \leq k \leq (2n-3)$ then (50) is a spectral density function. $\square$

Proof of Theorem 3 According to Bochner's Theorem, a real function C is a covariance function, in $\mathbb{R}$, if:

$$f_1(\omega) = \int_{-\infty}^{+\infty} C(h) \cos(\omega h) dh$$

is a spectral density function, i.e., it will be non negative and integrable. As a consequence, the function ((Gradshteyn and Ryzhik 2007), pag. 499, equ. 3.944.12):

$$f_1(\omega) = \int_{-\infty}^{+\infty} (1 + A|h|) \exp(-a|h|) \cos(\omega h) dh \\ = k \frac{[(a-A)\omega^2 + a^2(a+A)]}{(\omega^2 + a^2)^2}, \quad k > 0,$$

is integrable, hence it is a spectral density function if and only if $-a \leq A \leq a$. In particular, (26) is a c.f. since $\rho(0) = 1$. $\square$

Proof of Theorem 4 In order to check if the correlation model (27) is valid in $\mathbb{R}^2$, the second relationship of (3) can be utilized and prove that f is a spectral density function. Taking into account the following ((Gradshteyn and Ryzhik 2007), pag. 700, equ. 6.621.4):

$$\int_0^{+\infty} h^{n+1} \exp(-\alpha h) J_0(\omega h) dh = (-1)^{n+1} \frac{d^{(n+1)}}{d\alpha^{(n+1)}} (\omega^2 + \alpha^2)^{-1/2},$$

according to the second relationship given in (3), the function:

$$f(\omega) = \int_0^{+\infty} h(1 + Ah) \exp(-ah) J_0(\omega h) dh \\ = \frac{\omega^2(a-A) + a^2(a+2A)}{(\omega^2 + a^2)^{\frac{5}{2}}}, \quad (53)$$

is a spectral density if and only if $-\frac{a}{2} \leq A \leq a$. $\square$

Proof of Theorem 5 In order to check if the correlation model (28) is valid in $\mathbb{R}^3$, the second relationship of (5) can be utilized and prove that f is a spectral density function. Indeed:

$$\begin{aligned} & \frac{-1}{2\pi\omega} \frac{d}{d\omega} \left[\frac{(a-A)\omega^2 + a^2(a+A)}{(\omega^2 + a^2)^2} \right] \\ &= k \frac{\omega^2(a-A) + a^2(a+3A)}{(\omega^2 + a^2)^3}, \end{aligned}$$

is a spectral density function if and only if: $-\frac{a}{3} \leq A \leq a$. $\square$

Proof of Theorem 6 According to Bochner's Theorem, a real function ρ is a c.f., in $\mathbb{R}$, if:

$$f(\omega) = \int_{-\infty}^{+\infty} \rho(h) \cos(\omega h) dh$$

is a spectral density function, i.e., it will be non negative and integrable. At this purpose, the function ((Gradshteyn and Ryzhik 2007), pag. 499, equ. 3.944.12):

$$\begin{aligned} f_1(\omega) &= \int_{-\infty}^{+\infty} (1 + Ah^2) \exp(-a|h|) \cos(\omega h) dh \\ &= 2a \frac{[\omega^4 + (2a^2 - 6A)\omega^2 + a^4 + 2Aa^2]}{(\omega^2 + a^2)^3} \end{aligned} \quad (54)$$

is integrable, hence it is a spectral density function if and only if $-\frac{a^2}{2} \leq A \leq \frac{a^2}{3}$. $\square$

Proof of Theorem 7 It is well known that, in $\mathbb{R}^2$, ρ in (30) is a c.f. if the following function

$$f(\omega) = \frac{1}{2\pi} \int_0^\infty J_0(\omega h) h \rho(h) dh \quad (55)$$

is a spectral density. Note that f is integrable, hence it must be proved that it is non negative, i.e.,

$$\int_0^\infty J_0(\omega h) h (1 + Ah^2) \exp(-a|h|) dh \geq 0, \quad \forall \omega.$$

According to Gradshteyn and Ryzhik (2007), pag. 700, equ. 6.621.4:

$$\begin{aligned} & \int_0^\infty J_0(\omega h) h (1 + Ah^2) \exp(-a|h|) dh \\ &= \frac{a}{(\omega^2 + a^2)^{3/2}} + \frac{3Aa(2a^2 - 3\omega^2)}{(\omega^2 + a^2)^{7/2}} \\ &= \frac{a}{(\omega^2 + a^2)^{3/2}} \left[1 + \frac{3Aa(2a^2 - 3\omega^2)}{(\omega^2 + a^2)^2} \right] \\ &= \frac{\omega^4 + \omega^2(2a^2 - 9A) + a^2(a^2 + 6A)}{(\omega^2 + a^2)^{7/2}}, \end{aligned}$$

which is always non negative if and only if: $-\frac{a^2}{6} \leq A \leq \frac{2a^2}{9}$. $\square$

Proof of Theorem 8 It is well known that, in $\mathbb{R}^3$, ρ is a c.f. if the following function

$$f(\omega) = \frac{-1}{2\pi\omega} \frac{df_1(\omega)}{d\omega}$$

is a spectral density function, where f_1 is the corresponding spectral density function in $\mathbb{R}$, already defined in (54).

At this purpose,

$$\begin{aligned} f(\omega) &= \frac{-1}{2\pi\omega} \frac{d}{d\omega} \left[k \frac{[\omega^4 + (2a^2 - 6A)\omega^2 + a^4 + 2Aa^2]}{(\omega^2 + a^2)^3} \right] \\ &= \frac{\omega^4 + \omega^2(2a^2 - 12A) + a^4 + 12a^2A}{(\omega^2 + a^2)^4}, \end{aligned}$$

is integrable and it is non negative if and only if: $-\frac{a^2}{12} \leq A \leq \frac{a^2}{6}$. $\square$

Proof of Theorem 9 According to Bochner's Theorem, a continuous, integrable and real function C is a c.f., in $\mathbb{R}$, if:

$$f_1(\omega) = \int_{-\infty}^{+\infty} C(h) \cos(\omega h) dh$$

is a spectral density function, i.e., it will be non negative and integrable. The function (32) is continuous and integrable, as a consequence, the function:

$$\begin{aligned} f_1(\omega) &= \int_{-\infty}^{+\infty} (1 - Ah^2) \exp(-ah^2) \cos(\omega h) dh \\ &= k \exp\left(\frac{-\omega^2}{4a}\right) [A\omega^2 + (4a^2 - 2Aa)], \quad k > 0, \end{aligned} \quad (56)$$

is integrable, hence it is a spectral density function if and only if: $0 \leq A \leq 2a$. $\square$

Proof of Theorem 10 It is well known that, in $\mathbb{R}^2$, ρ is a c.f. if the following function

$$f(\omega) = \frac{1}{2\pi} \int_0^\infty J_0(\omega h) h \rho(h) dh \quad (57)$$

is a spectral density. Note that f is integrable, hence it must be proved that it is non negative, i.e.,

$$\int_0^\infty J_0(\omega h) h (1 - Ah^2) \exp(-ah^2) dh \geq 0, \quad \forall \omega.$$

Taking into account that $J_0'(x) = -J_1(x)$, the expression of the lower incomplete gamma function:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt,$$

and some relationships of the Bessel functions ((Gradshteyn and Ryzhik 2007), pag. 706, equ. 6.631.4 and equ. 6.631.5,) after integration by parts the previous function:

$$\begin{aligned} & \int_0^\infty J_0(\omega h) h (1 - Ah^2) \exp(-ah^2) dh \\ &= \lim_{t \rightarrow +\infty} \frac{(Aat^2 + A - a) \exp(-at^2)}{2a^2} J_0(\omega t) + \frac{(a - A)}{2a^2} \\ &+ \frac{\omega}{2a^2} \left[\int_0^\infty J_1(\omega h) Aah^2 \exp(-ah^2) dh \right. \\ &\left. + (A - a) \int_0^\infty J_1(\omega h) \exp(-ah^2) dh \right] \\ &= \frac{(a - A)}{2a^2} + \frac{\omega}{2a^2} \left[\frac{Aa\omega}{4a^2} \exp(-\omega^2/4a) \right. \\ &\left. + \frac{(A - a)}{\omega} (1 - \gamma(1, \omega^2/4a)) \right] \\ &= \frac{(a - A)}{2a^2} + \frac{A\omega^2}{8a^3} \exp(-\omega^2/4a) + \frac{(A - a)}{2a^2} \exp(-\omega^2/4a) \\ &= \frac{(a - A)}{2a^2} [1 - \exp(-\omega^2/4a)] + \frac{A\omega^2}{8a^3} \exp(-\omega^2/4a), \end{aligned}$$

is always non negative if and only if: $0 \leq A \leq a$. $\square$

Proof of Theorem 11 It is well known that, in $\mathbb{R}^3$, ρ is a c.f. if the following function

$$f(\omega) = \frac{-1}{2\pi\omega} \frac{df_1(\omega)}{d\omega}$$

is a spectral density function, where f_1 is the corresponding spectral density function in $\mathbb{R}$, already defined in (56).

At this purpose,

$$\begin{aligned} f(\omega) &= \frac{-1}{2\pi\omega} \frac{d}{d\omega} \left[\exp\left(\frac{-\omega^2}{4a}\right) (A\omega^2 + (4a^2 - 2Aa)) \right] \\ &= \frac{(A\omega^2 + 4a^2 - 6Aa)}{2a} \exp\left(\frac{-\omega^2}{4a}\right), \end{aligned}$$

is integrable and it is non negative if and only if:
 $0 \leq A \leq \frac{2a}{3}$. $\square$

Proof of Theorem 13 By Schoenberg's Theorem, if $C'''(0)$ exists then it must be negative. By contradiction, if such a point h_0 does not exist, C must be a concave function; as a consequence, for any $a, b > 0$:

$$C\left(\frac{a+b}{2}\right) \geq \frac{C(a) + C(b)}{2}. \quad (58)$$

Hence, for a fixed $a \in I_{0+}$:

$$\lim_{b \rightarrow +\infty} C\left(\frac{a+b}{2}\right) \geq \lim_{b \rightarrow +\infty} \frac{C(a) + C(b)}{2} \iff 0 \geq C(a),$$

which is an absurd because of the continuity of C since $C(0) > 0$. $\square$

Proof of Theorem 14 Taking into account the hypotheses, it results that:

$$\rho(h) > 0, \quad \forall h \in]0, h_0[, \quad \text{whereas} \quad \rho(h) < 0, \quad \forall h \in]h_0, +\infty[.$$

Since $\rho''(0) < 0$, ρ'' is continuous in $h = 0$ then $\exists a > 0 : \rho''(h) < 0 \quad \forall h \in [0, a]$. Moreover, since $\rho(h_0) = 0$, $\lim_{h \rightarrow +\infty} \rho(h) = 0$, $\rho(h) < 0 \quad \forall h \in]h_0, +\infty[$ and ρ is continuous, because of Weierstrass Theorem $\exists h_m > h_0 : \rho(h_m) \leq \rho(h) \quad \forall h > h_0$. Then it results: $\rho'(h_m) = 0$ and $\rho''(h_m) \geq 0$.

Let's apply Rolle's Theorem to the function ρ' , which is differentiable, in the interval $[0, h_m]$: hence

$$\exists h_1 \in]0, h_m[: \rho''(h_1) = 0.$$

By contradiction, suppose that $\mu = +\infty$, then $\rho''(h) > 0 \quad \forall h \in [h_m, +\infty[$, that is ρ is characterized by upwards concavity in the set $[h_m, +\infty[$, as a consequence

$$\forall h \geq h_m, \quad \forall u \geq h_m : \rho(h) \geq \rho(u) + \rho'(u)(h - u).$$

Since h_m is a strict relative minimum point ($\rho''(h_m) > 0$)

$$\exists \bar{u} > h_m : \rho(\bar{u}) > \rho(h_m),$$

then because of Lagrange Theorem

$$\exists u' \in]h_m, \bar{u}[: \rho(\bar{u}) - \rho(h_m) = \rho'(u')(\bar{u} - h_m),$$

that is $\rho'(u') > 0$. Let's consider the inequality

$$\rho(h) \geq \rho(u') + \rho'(u')(h - u') :$$

it results $\lim_{h \rightarrow +\infty} \rho(h) = +\infty$, however $\lim_{h \rightarrow +\infty} \rho(h) = 0$.

Then $\mu \in \mathbb{R}$ and because of the continuity of ρ'' , $\rho''(\mu) = 0$, with $h_2 = \mu$. $\square$

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Declarations

Conflict of interest The authors declare no conflict of interest.

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