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REGULAR ARTICLE

The limiting distribution of a bivariate random vector under univariate truncation

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Abstract

The dependence structure in the tails of bivariate random vectors is studied by means of the copula representation. In particular, asymptotic results for the distribution of a random pair under univariate truncation is proved in the spirit of multivariate extensions of the Pickands-Balkema-de Haan Theorem.

Keywords Copula · Extreme-value distributions · Limit distribution

Mathematics Subject Classification 62H05 · 60G70

1 Introduction

Copula models are nowadays ubiquitous when the stochastic behaviour of a given phenomenon can be represented by multiple random variables. In fact, they have the great advantage of capturing both the marginal behaviour and the dependence structure with great flexibility. See, for instance, (Emura et al. 2019; Jaworski et al. 2013; Joe 2015; Mai and Scherer 2017) and references therein.

An understanding of the stochastic dependence structures has become very important in various applied fields, especially when quantitative methods for risk estimation are called for (see, for instance, Ibragimov and Prokhorov (2017)). In general, however, it may be difficult to fit an explicit model to real data, due to the presence of partial information about the joint distribution. Consider, for instance, the case when

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relevant data are not available or data are recorded at different time horizons. In such situations, asymptotic results can be applied in order to approximate the true distribution by its limiting behaviour, especially when the tail of the distribution is of interest and suitable quantiles need to be calculated.

In the univariate case, one of the milestone results in this setting is the celebrated Pickands-Balkema-De Haan Theorem (see Balkema and de Haan 1974; Pickands 1975). It states that the limit distribution of scaled excesses over high thresholds is represented by the generalized Pareto distribution (GPD) given by

$$V_{\xi,\sigma}(x) = \begin{cases} 1 - \left(1 - \dfrac{\xi x}{\sigma}\right)^{1/\xi}, & \text{if } \xi \neq 0, \\ 1 - e^{-x/\sigma}, & \text{if } \xi = 0, \end{cases} \tag{1.1}$$

where $\sigma > 0$, and $x \geq 0$ for $\xi \leq 0$, or $0 \leq x \leq \sigma/\xi$ for $\xi > 0$.

Two-dimensional extensions of this theorem have been provided in the last decades. For instance, (Juri and Wüthrich 2003,Theorem 4.1) and (Wüthrich 2004,Theorem 2.1) studied the limiting behaviour of bivariate excess distributions of (X, Y) given that both variables are below low (respectively, above high) thresholds; these results have been further extended in Di Bernardino et al. (2013). For some related results in the copula setting, see (Charpentier and Juri 2006,Theorem 3.1) (and also Hofert 2021; Charpentier and Segers 2009; Hua and Joe 2011; Ibragimov and Prokhorov 2017; Joe et al. 2010; Siburg et al. 2024). For a different approach, see also (Falk 2019).

Here, we aim at extending such studies by showing the limiting behaviour of a a random pair given that the first (respectively, the second) component is below a given low threshold. By adopting a copula approach, the limiting distribution will be obtained as a composition of: (a) a limiting univariate distribution in the spirit of the Pickands-Balkema-de Haan theorem (see, e.g., Theorems 3.2.3 and 3.4.13(b) in Embrechts et al. (1997)), (b) a copula that is invariant under truncation with respect to one variable (see, e.g., Durante and Jaworski 2012; Jaworski 2013).

This approximation result is of interest, for instance, in the study about conditional value-at-risk (i.e. CoVaR), defined as the change in the value at risk of a random variable conditional on another random variable being under distress; see, e.g., (Bernard and Czado 2015; Bernardi et al. 2017). The operation of truncation is important for (re)insurance as the copula allows one to study the dependence between the components of a truncated loss random vector (see, for instance, Hofert (2021)).

The paper is organized as follows. Sections 2, 3, 4 and 5 revisit and complement various results about the tail distribution of a copula under some mild assumptions that are satisfied by most of the families currently used. Specifically, Sect. 3 provides the applications to the risk measurement, to the extreme behaviour of the systemic risk measure called CoVaR. Section 4 deals with the concept of the tail function of degree d in the new framework and, finally, Sect. 5 is devoted to the study of the impact of some additional tame condition. In Sect. 6, instead, the main characterization theorem is proved in full detail.

2 Tail behaviour of a copula

We start by recalling some well known results, for which we refer the reader to Wüthrich (2004) and Durante and Jaworski (2012), and by introducing the notations that will be used throughout this paper.

Let (X, Y) be a continuous random vector with (unique) copula C. In the following, we shall deal with a key assumption, which can be considered a generalization of a similar property considered in Wüthrich (2004).

Condition 2.1 Let C be a copula such that $C(u, v) > 0$ whenever $\min(u, v) > 0$. Suppose that there exist two continuous functions $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for every $x \in \mathbb{R}_+$, the following limits exist with

$$\lim_{u \to 0^+} \frac{C(xu, u)}{C(u, u)} = g(x); \tag{2.1}$$

$$\lim_{u \to 0^+} \frac{C(u, xu)}{C(u, u)} = h(x). \tag{2.2}$$

Moreover, $g(x) > 0$ and $h(x) > 0$ whenever $x > 0$.

In the following, the function, defined on the unit interval, $\delta(u) = C(u, u)$ will be referred to as a diagonal section or diagonal for short.

Note that, if an exchangeable copula C satisfies Condition 2.1, then $g = h$ and Condition 2.1 is equivalent to Assumption 2.3 in Wüthrich (2004). Later on in this Section, we shall prove that the existence of g (respectively, h) implies the existence of h (respectively, g) so that, in Condition 2.1, it is enough to compute only one of the two limits (2.1) and (2.2), since the other one can be derived from the latter.

Example 2.1 Here we shall present some examples of copulas fulfilling Condition 2.1.

- *Independence copula.* Set $C := \Pi$. Then, for every $x \in \mathbb{R}_+$, eq. (2.1) becomes

  $$\lim_{u \to 0^+} \frac{\Pi(xu, u)}{\Pi(u, u)} = x,$$

  thus $g^{\Pi}(x) := x$ satisfies the above requirements, and of course $h^{\Pi} = g^{\Pi}$ since Π is an exchangeable copula.
- *Comonotonic copula.* Set $C := M$. For every $x \in \mathbb{R}_+$, eq. (2.1) becomes

  $$\lim_{u \to 0^+} \frac{M(xu, u)}{M(u, u)} = \min(x, 1),$$

  thus $g^M(x) := \min(x, 1)$ fulfills the above requirements, and $h^M = g^M$.

- *Mardia-Takahasi-Clayton copula with positive parameter* $\alpha > 0$. Let

$$C_\alpha^{\mathbf{MTC}}(u, v) := \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-1/\alpha} \tag{2.3}$$

be the Mardia-Takahasi-Clayton copula with positive parameter $\alpha > 0$. In this case, we reasonably expect to find a function $g_\alpha^{\mathbf{MTC}}$ that depends on the fixed parameter α. Indeed, for every $x \in \mathbb{R}_+$, eq. (2.1) becomes

$$\lim_{u\to 0^+} \frac{C_\alpha^{\mathbf{MTC}}(xu, u)}{C_\alpha^{\mathbf{MTC}}(u, u)} = \lim_{u\to 0^+} \left(\frac{(xu)^{-\alpha} + u^{-\alpha} - 1}{2u^{-\alpha} - 1}\right)^{-1/\alpha} = \left(\frac{x^{-\alpha} + 1}{2}\right)^{-1/\alpha},$$

hence $g_\alpha^{\mathbf{MTC}}(x) := \left(\frac{x^{-\alpha}+1}{2}\right)^{-1/\alpha}$ is the needed function, together with $h_\alpha^{\mathbf{MTC}} = g_\alpha^{\mathbf{MTC}}$.

- *Gumbel-Hougaard copula with parameter* $\alpha \geq 1$. Let

$$C_\alpha^{\mathbf{GH}}(u, v) := \exp\left(-\left((-\log(u))^\alpha + (-\log(v))^\alpha\right)^{1/\alpha}\right)$$

be the Gumbel-Hougaard copula with parameter $\alpha \geq 1$. Again, we expect to find a function $g_\alpha^{\mathbf{GH}}$ that depends on the fixed parameter α. Indeed, for every $x \in \mathbb{R}_+$, after setting $z := -\log(u) \to +\infty$ as $u \to 0^+$, eq. (2.1) becomes

$$\begin{aligned}\lim_{u\to 0^+} \frac{C_\alpha^{\mathbf{GH}}(xu, u)}{C_\alpha^{\mathbf{GH}}(u, u)} &= \lim_{u\to 0^+} \left(\frac{\exp\left(-\left((-\log(xu))^\alpha + (-\log(u))^\alpha\right)^{1/\alpha}\right)}{\exp\left(-\left(2(-\log(u))^\alpha\right)^{1/\alpha}\right)}\right)\\ &= \lim_{z\to+\infty} \exp\left(-\left((z - \log x)^\alpha + z^\alpha\right)^{1/\alpha} + 2^{1/\alpha} z\right)\\ &= \exp\left(2^{\frac{1}{\alpha}-1} \log(x)\right) = x^{2^{\frac{1}{\alpha}-1}},\end{aligned}$$

hence $g_\alpha^{\mathbf{GH}}(x) := x^{2^{\frac{1}{\alpha}-1}}$ is the required function, and again $h_\alpha^{\mathbf{GH}} = g_\alpha^{\mathbf{GH}}$, because of the exchangeability.

- *Marshall–Olkin copula.* Let

$$C_{\alpha,\beta}^{\mathbf{MO}}(u, v) := \min\left(u^{1-\alpha} v, u v^{1-\beta}\right)$$

be the Marshall–Olkin copula with parameters $\alpha, \beta \in [0, 1]$. Note that this copula is not exchangeable provided that $\alpha \neq \beta$, but

$$C_{\alpha,\beta}^{\mathbf{MO}}(u, v) = C_{\beta,\alpha}^{\mathbf{MO}}(v, u).$$

Thus

$$h_{\beta,\alpha}^{\mathbf{MO}}(x) = g_{\alpha,\beta}^{\mathbf{MO}}(x).$$

For every $x \in \mathbb{R}_+$, we have, for $\alpha < \beta$,

$$\begin{aligned}\lim_{u \to 0^+} \frac{C^{\mathbf{MO}}_{\alpha,\beta}(xu, u)}{C^{\mathbf{MO}}_{\alpha,\beta}(u, u)} &= \lim_{u \to 0^+} \frac{\min\left(x^{1-\alpha}u^{2-\alpha}, xu^{2-\beta}\right)}{u^{2-\alpha}} \\ &= \lim_{u \to 0^+} \min\left(x^{1-\alpha}, xu^{\alpha-\beta}\right) = x^{1-\alpha},\end{aligned}$$

hence $g^{\mathbf{MO}}_{\alpha,\beta}(x) := x^{1-\alpha}$ is the first function (note that it only depends on the parameter α and not on β). In a similar way we show that, for $\alpha = \beta$, $g^{\mathbf{MO}}_{\alpha,\beta}(x) := \min(x, x^{1-\alpha})$ and for $\alpha > \beta$, $g^{\mathbf{MO}}_{\alpha,\beta}(x) := x$. As far as the second function, it holds, for $\alpha < \beta$,

$$\begin{aligned}\lim_{u \to 0^+} \frac{C^{\mathbf{MO}}_{\alpha,\beta}(u, xu)}{C^{\mathbf{MO}}_{\alpha,\beta}(u, u)} &= \lim_{u \to 0^+} \frac{\min\left(xu^{2-\alpha}, x^{1-\beta}u^{2-\beta}\right)}{u^{2-\alpha}} \\ &= \lim_{u \to 0^+} \min\left(x, x^{1-\beta}u^{\alpha-\beta}\right) = x,\end{aligned}$$

hence $h^{\mathbf{MO}}_{\alpha,\beta}(x) := x$ is the second function (which does not even depend on α). Similarly, for $\alpha = \beta$, $h^{\mathbf{MO}}_{\alpha,\beta}(x) := \min(x, x^{1-\alpha})$ and, for $\alpha > \beta$, $h^{\mathbf{MO}}_{\alpha,\beta}(x) := x^{1-\beta}$. In such case, the two functions are different for any $\alpha \neq 0$, but Condition 2.1 is again satisfied.

We shall reconsider the above examples after recalling the main properties of the functions g and h, which we now state. The following proposition collects together several results that can be found in (Wüthrich 2004, Sect. 4) or are slightly modified version thereof; for completeness, we also include the (slightly modified) proofs.

Proposition 2.1 *Assume that a copula C satisfies Condition 2.1. Then it holds:*

(i) *g and h are increasing functions with $g(0) = h(0) = 0$ and $g(1) = h(1) = 1$. Furthermore, the convergence in Condition 2.1 is uniform on the interval $[0, x]$, for every $x > 0$.*

(ii) *There exists $\theta \geq 1$ such that:*

  (a) *the diagonal section of C, i.e. $\delta(u) := C(u, u)$ for every $u \in [0, 1]$, is regularly varying at the point 0 with index θ, that is*

$$\lim_{\alpha \to 0} \frac{\delta(\alpha y)}{\delta(\alpha)} = y^{\theta}, \quad \textit{for } y > 0. \tag{2.4}$$

  Moreover, the convergence is uniform on the interval $[0, x]$, for every $x > 0$.

  (b) *For every $y > 0$,*

$$g(y) = y^{\theta} h\left(\frac{1}{y}\right). \tag{2.5}$$

(iii) There exists a function $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that, for every* $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\lim_{u \to 0^+} \frac{C(xu, yu)}{C(u, u)} = G(x, y) \tag{2.6}$$

where, for every $x \geq 0$, $g(x) = G(x, 1)$ *and* $h(x) = G(1, x)$. *Moreover, the convergence is uniform on every bounded subset of* $\mathbb{R}_+ \times \mathbb{R}_+$.

(iv) The function G *can also be expressed, for every* $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, *in terms of* g *and* h *by the following formulas*

$$G(x, y) = \begin{cases} y^\theta g\left(\dfrac{x}{y}\right), & \text{if } x > 0 \text{ and } y > 0; \\ 0, & \text{if } x = 0 \text{ or } y = 0; \end{cases} \tag{2.7}$$

or

$$G(x, y) = \begin{cases} x^\theta h\left(\dfrac{y}{x}\right), & \text{if } x > 0 \text{ and } y > 0; \\ 0, & \text{if } x = 0 \text{ or } y = 0. \end{cases} \tag{2.8}$$

(v) G *is strictly positive, when both arguments are strictly positive, continuous, 2–increasing and a* θ*-homogeneous function, i.e., for every* $x, y, t \geq 0$, *it holds*

$$G(tx, ty) = t^\theta G(x, y).$$

(vi) If C *is exchangeable, then* G *is symmetric in its arguments.*

Proof Point (i). The increasingness of g and h comes from the increasingness of copulas (in both arguments); also, (2.1) and (2.2) evaluated in $x = 0$ and $x = 1$ easily yield $g(0) = h(0) = 0$ and $g(1) = h(1) = 1$. The increasingness of copulas implies also that the convergence in (2.1) and (2.2) is uniform on closed intervals (see (Łojasiewicz 1988,Theorem 1.2.2) or (Resnick 2007,Proposition 2.1)).

Point (ii). Next, let

$$U(t) = \delta(\min(1, 1/t)), \quad t > 0.$$

Then

$$\begin{aligned} \lim_{t \to +\infty} \frac{U(tx)}{U(t))} &= \lim_{\alpha \to 0} \frac{\delta(\alpha x^{-1})}{\delta(\alpha)} \\ &= \lim_{\alpha \to 0} \frac{C(\alpha x^{-1}, \alpha x^{-1})}{C(\alpha, \alpha)} \\ &= \lim_{\alpha \to 0} \frac{C(\alpha x^{-1}, \alpha x^{-1})}{C(\alpha x^{-1}, x\alpha x^{-1})} \cdot \frac{C(\alpha x^{-1}, \alpha)}{C(\alpha, \alpha)} \\ &= \frac{g(1/x)}{h(x)}. \end{aligned}$$

Hence, due to Proposition 2.3(i) in Resnick (2007), it follows that $U(t)$ is regularly varying at ∞ with index ρ and

$$\frac{g(1/x)}{h(x)} = x^\rho \quad \text{and} \quad \lim_{\alpha \to 0} \frac{\delta(\alpha x^{-1})}{\delta(\alpha)} = x^\rho .$$

To conclude we put $\theta = -\rho$ and $y = x^{-1}$. Note that, since the diagonal $\delta(t)$ is bounded by t, θ must be greater or equal to 1. The increasingness of δ implies that the above convergence is uniform on closed intervals.

Points (iii) and (iv). Now, fix $(x, y) \in (0, +\infty)^2$, for every $u \in (0, 1]$ we have

$$\frac{C(xu, yu)}{C(u, u)} = \frac{C\left(xu, \frac{y}{x} xu\right)}{C(xu, xu)} \cdot \frac{C(xu, xu)}{C(u, u)}$$

and

$$\frac{C(xu, yu)}{C(u, u)} = \frac{C\left(\frac{x}{y} yu, yu\right)}{C(yu, yu)} \cdot \frac{C(yu, yu)}{C(u, u)}.$$

Taking the limit for $u \to 0^+$ in both of the above equalities, we find out that the limit in (2.6) exists and the function G can be expressed in terms of g and h in two ways, as

$$G(x, y) = x^\theta h\left(\frac{y}{x}\right)$$

and as

$$G(x, y) = y^\theta g\left(\frac{x}{y}\right).$$

Moreover, since the limits (2.1), (2.2) and (2.4) are uniform on compacts, so is the convergence in (2.6). Specifically, we fix $q \geq 1$ and for every $\varepsilon > 0$ select τ such that for $0 < u < \tau$ and $\xi \in [0, q]$

$$\left| \frac{\delta(\xi u)}{\delta(u)} - \xi^\theta \right| \leq \frac{1}{3} \min(\varepsilon, 1), \tag{2.9}$$

$$\left| \frac{C(\xi u, u)}{\delta(u)} - g(\xi) \right| \leq \frac{1}{3q^\theta} \min(\varepsilon, 1), \tag{2.10}$$

$$\left| \frac{C(u, \xi u)}{\delta(u)} - h(\xi) \right| \leq \frac{1}{3q^\theta} \min(\varepsilon, 1). \tag{2.11}$$

Then for $0 \leq y \leq x \leq q$, $(x, y) \neq (0, 0)$ and $0 < u < \tau/q$ we get

$$\left| \frac{C(xu, yu)}{\delta(u)} - G(x, y) \right| = \left| \frac{C(xu, yu)}{\delta(xu)} \frac{\delta(xu)}{\delta(u)} - x^\theta h\left(\frac{y}{x}\right) \right| \tag{2.12}$$

$$\le \left|\frac{C(xu,yu)}{\delta(xu)} - h\left(\frac{y}{x}\right)\right| \cdot \left|\frac{\delta(xu)}{\delta(u)} - x^\theta\right| \tag{2.13}$$

$$+ \left|\frac{C(xu,yu)}{\delta(xu)} - h\left(\frac{y}{x}\right)\right| \cdot x^\theta + \left|\frac{\delta(xu)}{\delta(u)} - x^\theta\right| \cdot h\left(\frac{y}{x}\right)$$

$$\le \frac{\varepsilon}{9q^\theta} + \frac{\varepsilon}{3q^\theta}q^\theta + \frac{\varepsilon}{3} \le \varepsilon. \tag{2.14}$$

In a similar way, we show the same bound for $0 \le x \le y \le q$ and $(x, y) \ne (0, 0)$. Since for $(x, y) = (0, 0)$ the bound is 0, we conclude the proof of the uniform convergence on the square $[0, q]^2$.

Point (v). As far as the properties of the function G, its positivity and continuity come from the same properties held by g (or h), whereas the 2-increasing property is inherited from the 2-increasingness of the copula C. Moreover, for every $x, y, t > 0$, by using (2.7) it holds

$$G(tx, ty) = (ty)^\theta g\left(\frac{tx}{ty}\right) = t^\theta y^\theta g\left(\frac{x}{y}\right) = t^\theta G(x, y),$$

which shows that $G(\cdot, \cdot)$ is a θ-homogeneous function.

Point (vi). Finally, if the copula C is exchangeable, we have $g = h$ and then, using again (2.8) and (2.7), it holds

$$G(y, x) = y^\theta h\left(\frac{x}{y}\right) = y^\theta g\left(\frac{x}{y}\right) = G(x, y),$$

namely G is symmetric if the copula C is. This concludes the proof. □

Remark 2.1 The regular variation of δ implies that the function G introduced in Proposition 2.1, eq. (2.6), equals to the lower tail order function b studied in Hua and Joe (2011). Such a b is an homogeneous function of order k defined by

$$C(uw_1, uw_2) \sim u^\kappa b(w_1, w_2)\ell(u)$$

for $u \to 0^+$, $w_1 > 0$, $w_2 > 0$, $\kappa \ge 1$ and a slowly varying function ℓ. However our Condition 2.1 is stronger than the assumptions formulated in (Hua and Joe 2011, Definition 3), i.e. the assumption that the diagonal section δ is regularly varying with index κ and the assumption that the following limit exists with

$$\lim_{u\to 0^+} \frac{C(uw_1, uw_2)}{\delta(u)} = b(w_1, w_2; C, \kappa).$$

Indeed. Point (iii) of Proposition 2.1 and (2.4) imply the regular variation of δ and the existence of b. But Condition 2.1 implies also that G, hence b as well, is continuous and the convergence is almost uniform, which is neither required in (Hua and Joe 2011, Definition 3) nor implied by it. The counterexample may be constructed with the help of the so-called diagonal copulas - compare Proposition 8.3.1 and Corollary 8.3.1 in Jaworski (2010).

3 Applications to risk management

The previously stated convergence results can be very useful in many applications, for instance in determining the limiting behaviour of the modified Conditional Value at Risk (mCoVaR). We recall that for two random variables X, Y modelling financial positions (aka profit/loss distributions), we define

$$\mathrm{mCoVaR}_{\alpha,\beta}(Y|X) = \mathrm{VaR}_{\beta}(Y|X + \mathrm{VaR}_{\alpha}(X) \leq 0), \quad \alpha, \beta \in (0, 1). \tag{3.1}$$

We recall that VaR_α denotes the Value at Risk at the significance level α, $\alpha \in (0, 1)$. It is the smallest amount of money need to keep the position save and can be expressed in terms of quantiles

$$\mathrm{VaR}_\alpha(X) = \inf\{v \in \mathbb{R} : \mathbb{P}(X + v < 0) \leq \alpha\} = -Q_\alpha^+(X).$$

As was proved in Bernardi et al. (2017), if C is the copula of random variables X and Y which have continuous distribution functions, then

$$\mathrm{mCoVaR}_{\alpha,\beta}(Y|X) = \mathrm{VaR}_{w_*}(Y),$$

where the implied significance level w_* is the solution of the equation

$$C(\alpha, w_*) = \alpha\beta. \tag{3.2}$$

Now, let us consider the limiting case, α is tending to 0 and β depends on α, $\beta = \beta(\alpha)$. Then the implied significance level w_* is a function of α as well.

Proposition 3.1 *Assume that the copula C satisfies Condition 2.1 and that the following limit exists*

$$\lim_{\alpha\to 0} \frac{\alpha\beta(\alpha)}{\delta(\alpha)} = \rho_0. \tag{3.3}$$

If the preimage of ρ_0, $h^{-1}(\{\rho_0\})$ is not empty and consists exactly of one point, then the implied significance level $w_*(\alpha)$ tends to 0 *and*

$$\lim_{\alpha\to 0} \frac{w_*(\alpha)}{\alpha} = h^{-1}(\rho_0). \tag{3.4}$$

Proof Let w' be any cluster point of $\frac{w_*(\alpha)}{\alpha}$. Then there exists a sequence α_n such that

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{w_*(\alpha_n)}{\alpha_n} = w'.$$

Thanks to the uniform convergence to h on $[0, 2w']$, we obtain (compare Łojasiewicz (1988) Theorem 3.1.9)

$$h(w') = \lim_{n\to\infty} \frac{C(\alpha_n, \frac{w_*(\alpha_n)}{\alpha_n}\alpha_n}{\delta(\alpha_n)} = \lim_{n\to\infty} \frac{\alpha_n \beta(\alpha_n)}{\delta(\alpha_n)} = \rho_0. \tag{3.5}$$

Hence every cluster point equals $h^{-1}(\rho_0)$. This implies that the limit of the quotient $\frac{w_*(\alpha)}{\alpha}$ exists and

$$\lim_{\alpha\to 0} \frac{w_*(\alpha)}{\alpha} = h^{-1}(\rho_0).$$

Since $\alpha \to 0$, the same is valid for $w_*(\alpha)$. □

4 The lower tail expansion of a copula

The functions g and h from Condition 2.1 are also closely related with the notion of lower tail expansion. We say that a copula has a lower tail expansion if, near the origin, it can be approximated by a homogeneous function (compare Jaworski 2010, 2006, 2004; Iwaniec 2009). In particular, we shall prove that the presence of the lower tail expansion with a continuous leading term, implies the existence of the continuous functions g and h. First, we recall the following definition.

Definition 4.1 A copula $C : [0, 1]^2 \longrightarrow [0, 1]$ has a *lower tail expansion* of *degree* d if there exist a homogeneous function $L : [0, +\infty)^2 \longrightarrow \mathbb{R}$ of degree d, i.e., for every $t \geq 0$ and for every $(x, y) \in [0, +\infty)^2$

$$L(tx, ty) = t^d L(x, y),$$

and a function $R : [0, 1]^2 \longrightarrow \mathbb{R}$ with

$$\lim_{t\to 0} R(tu, tv) = 0, \qquad \text{for every } (u, v) \in [0, 1]^2,$$

such that, for every $(u, v) \in [0, 1]^2$

$$C(u, v) = L(u, v) + R(u, v)(u + v)^d. \tag{4.1}$$

Furthermore, we say that the expansion is *uniform* when R is bounded and

$$\lim_{(u,v)\to(0,0)} R(u, v) = 0.$$

The function L will be called *the leading part* of the expansion. When $L \equiv 0$ we shall say that the expansion is *trivial*.

As a matter of fact, the lower tail expansion is related to the *lower tail dependence coefficient* (LTDC) of a copula (see, e.g., Durante and Sempi 2016) defined as

$$\lambda_L(C) := \lim_{t\to 0^+} \frac{C(t,t)}{t}, \tag{4.2}$$

provided that the above limit exists. In fact, when the copula C admits the lower tail expansion L_C of degree 1, then $\lambda_L(C) = L_C(1,1)$. If the degree d of the lower tail expansion is, instead, strictly greater than 1, then one easily proves that $\lambda_L(C) = 0$.

Remark 4.1 If a copula C has a lower tail expansion of degree d, i.e., (4.1) is satisfied for suitable functions L and R as in Definition 4.1, then, for every $(x, y) \in [0, +\infty)^2$, the ray-like limit

$$\lim_{t\to 0^+} \frac{C(tx,ty)}{t^d}$$

exists and it is equal to $L(x, y)$. Indeed,

$$\begin{aligned}\lim_{t\to 0^+} \frac{C(tx,ty)}{t^d} &= \lim_{t\to 0^+} \frac{t^d L(x.y) + t^d R(tx,ty)(x+y)^d}{t^d}\\ &= L(x,y) + (x+y)^d \lim_{t\to 0^+} R(tx,ty) = L(x,y).\end{aligned}$$

Conversely, let us assume that, for every $(x, y) \in [0, +\infty)^2$, the limit

$$L(x,y) := \lim_{t\to 0^+} \frac{C(tx,ty)}{t^d} \tag{4.3}$$

exists for a fixed d. Then, it is straightforward to verify that the function L is d-homogeneous. Moreover, one can set, for every $(u, v) \in [0, 1]^2$:

$$R(u,v) := \begin{cases} 0, & \text{if } (u,v) = (0,0);\\ \dfrac{C(u,v) - L(u,v)}{(u+v)^d}, & \text{if } (u,v) \neq (0,0).\end{cases}$$

Clearly, with these definitions, (4.1) is satisfied. Moreover, for every $(u, v) \in (0, 1)^2$:

$$\begin{aligned}\lim_{t\to 0^+} R(tu,tv) &= \lim_{t\to 0^+} \frac{C(tu,tv) - t^d L(u,v)}{t^d (u+v)^d}\\ &= \frac{1}{(u+v)^d}\left(\lim_{t\to 0^+} \frac{C(tu,tv)}{t^d} - L(u,v)\right) = 0,\end{aligned}$$

hence C admits a lower tail expansion of degree d. We shall make use of this equivalence in the proofs of our further results. Moreover, in the case $d = 1$, it is known that the expansion is *uniform* and the leading part L is concave and Lipschitz continuous:

we refer the reader to (Jaworski 2006,Theorem 1) for a detailed proof of this stronger result.

In particular, we can derive the copulas satisfying Condition 2.1 from the analysis of the lower tail expansion of a copula. On the other hand, Condition 2.1 implies that: if a copula admits lower tail expansion, then this expansion is uniform.

Proposition 4.1 *Let C be a copula that admits a non-trivial lower tail expansion of degree d with continuous leading part L (as in (4.1)). Then $L(1,1) > 0$ and the diagonal section $\delta(t) = C(t,t)$ is equivalent to t^d at the origin*

$$\lim_{t\to 0} \frac{\delta(t)}{t^d} = L(1,1). \tag{4.4}$$

Furthermore, the following statements holds:

(i) The functions g and h of Condition 2.1 exist and are given, for every $x \geq 0$, by

$$g(x) = \frac{L(x,1)}{L(1,1)} \quad \text{and} \quad h(x) = \frac{L(1,x)}{L(1,1)}, \tag{4.5}$$

respectively. Moreover, if L is symmetric, then $g = h$.

(ii) The function G introduced in Proposition 2.1 equals

$$G(x,y) = \frac{L(x,y)}{L(1,1)} \quad \text{and} \quad \theta = d. \tag{4.6}$$

(iii) The lower tail expansion of C is uniform.

Proof First of all, it must be shown that $L(1,1) \neq 0$. Let us assume $L(1,1) = 0$. We recall that, by Remark 4.1, the leading part of the expansion is given by (4.3), hence the function L is 2-increasing since the copula C is. Therefore, for any fixed $(u,v) \in]0,+\infty[^2$, we would have

$$L(u,v) \leq L(u \vee v, u \vee v) = (u \vee v)^d L(1,1) = 0,$$

which would be a contradiction because L is supposed to be non-trivial by hypothesis. Then $L(1,1) \neq 0$. Moreover, the decomposition (4.1) implies

$$\lim_{t\to 0^+} \frac{\delta(t)}{t^d} = \lim_{t\to 0^+} \frac{L(t,t) + R(t,t)(2t)^d}{t^d} = \lim_{t\to 0^+} \Big(L(1,1) + 2^d R(t,t)\Big) = L(1,1).$$

Analogously, for fixed $x, y \geq 0$, we have

$$
\begin{aligned}
g(x) &= \lim_{t \to 0^+} \frac{C(xt,t)}{C(t,t)} = \lim_{t \to 0^+} \frac{C(xt,t)}{t^d} \frac{t^d}{\delta(t)} = \frac{L(x,1)}{L(1,1)},\\
h(y) &= \lim_{t \to 0^+} \frac{C(t,yt)}{C(t,t)} = \lim_{t \to 0^+} \frac{C(t,yt)}{t^d} \frac{t^d}{\delta(t)} = \frac{L(1,y)}{L(1,1)},\\
G(x,y) &= \lim_{t \to 0^+} \frac{C(xt,yt)}{C(t,t)} = \lim_{t \to 0^+} \frac{C(xt,yt)}{t^d} \frac{t^d}{\delta(t)} = \frac{L(x,y)}{L(1,1)},
\end{aligned}
$$

From (4.5), the continuity of g and h follows since L is continuous, as well as the fact that $g = h$ whenever L is symmetric.
Moreover, the decomposition (4.1) implies

$$
\begin{aligned}
R(u,v) &= \frac{C(u,v) - L(u,v)}{(u+v)^d} = \frac{C(u,v) - G(u,v)L(1,1)}{(u+v)^d}\\
&= \frac{C\left(\frac{u}{u+v}(u+v), \frac{v}{u+v}(u+v)\right)}{\delta(u+v)} \frac{\delta(u+v)}{(u+v)^d} - G\left(\frac{u}{u+v}, \frac{v}{u+v}\right) L(1,1).
\end{aligned}
$$

We apply the limit $u + v \to 0$. Since the convergence to G is uniform on the unit square, we get

$$
\lim_{(u,v)\to(0,0)} R(u,v) = 0
$$

and C admits a uniform lower tail expansion of degree d. □

Proposition 4.1 allows us to extend the examples 2.1 to other (and more general) classes of copulas.

Example 4.1 (Absolutely continuous copulas) Let C be an absolutely continuous copula with density c. The tail behaviour of C depends heavily on the tail behaviour of c. If c is continuous at the origin $(0,0)$ and $c(0,0) = c_0 > 0$ then, in view of (Jaworski 2010,Proposition 8.3.2) for $l(u) := 1$ and $d = n := 2$, C admits a non-trivial lower tail expansion of degree 2, given by

$$
L(u,v) := c_0 uv.
$$

Note that such a L is symmetric. Hence, by Proposition 4.1, C satisfies Condition 2.1 with the functions

$$
g(x) = h(x) := \frac{L(x,1)}{L(1,1)} = x.
$$

Moreover, note that, because of (4.3) for $(u,v) := (1,1)$ and $d := 2$, it holds

$$
L(1,1) = c_0 = \lim_{t \to 0^+} \frac{C(t,t)}{t^2},
$$

hence

$$\lim_{t\to 0^+}\frac{C(t,t)}{t}=\lim_{t\to 0^+}\frac{C(t,t)}{t^2}t=c_0\cdot 0=0,$$

i.e., C admits a lower tail dependence coefficient $\lambda_L=0$, in accordance with points *1.* and *2.* of Proposition 4.2, since the index of g and h is $\theta=2>1$ and since $g(+\infty)=h(+\infty)=+\infty$.

Example 4.2 (Archimedean copulas) We recall that a copula $C:[0,1]^2\longrightarrow[0,1]$ is called *Archimedean* if it can be expressed in the form

$$C_\varphi(u,v)=\psi\left(\varphi(u)+\varphi(v)\right),$$

where $\psi:[0,+\infty]\longrightarrow[0,1]$ and $\varphi:[0,1]\longrightarrow[0,+\infty]$ are continuous convex nonincreasing functions such that

$$\psi\circ\varphi=id\quad\text{and}\quad\varphi(1)=0.$$

Depending on the source, either ψ (see, e.g., Durante and Sempi 2016) or φ (see, e.g., Nelsen 2007) is called *the generator*. Furthermore, it is known that if ψ and φ are as above then C_φ is a copula.

The lower tail expansion of an Archimedean copula depends on the value of its generator φ at 0 and on the *limit elasticity* of φ at 0:

$$\varphi_0:=\varphi(0)=\lim_{x\to 0^+}\varphi(x),\quad \mathcal{E}_x(0):=\lim_{x\to 0^+}\frac{x\varphi'(x)}{\varphi(x)},$$

and also on the index κ of regular variation of the function $-\frac{\psi(t)}{\psi'(t)}$ at infinity (assuming that it exists). Note that ψ and φ, being convex, are differentiable except at most in a countable set of points.

If φ_0 is finite, then C_φ vanishes in some neighbourhood of the origin, so $L\equiv 0$ (for any degree d). Indeed, by continuity, there exists some $x_0>0$ such that

$$\forall x\in[0,x_0]\qquad \varphi(x)\geq\frac{\varphi_0}{2}.$$

Then we have

$$\forall(u,v)\in[0,x_0]^2\qquad \varphi(u)+\varphi(v)\geq\varphi_0,$$

which implies, since ψ is decreasing,

$$\forall(u,v)\in[0,x_0]^2\qquad C_\varphi(u,v)=\psi\left(\varphi(u)+\varphi(v)\right)\leq\psi(\varphi_0)=0,$$

so that $C_\varphi(u,v)=0$ for every $(u,v)\in[0,x_0]^2$. This easily yields $L=0$, by (4.3).

If $\varphi_0 = +\infty$ and φ is regularly varying at 0 with index α then $\mathcal{E}_x(0) = \alpha$ and vice versa (see (Resnick 2007,Proposition 2.5) and (Charpentier and Segers 2009,Lemma A.1)).

If $\varphi_0 = +\infty$ and the limit $\mathcal{E}_x(0)$ exists and and is negative, i.e.

$$\mathcal{E}_x(0) = -\alpha, \qquad 0 < \alpha < +\infty,$$

then, as a consequence of (Jaworski 2010,Proposition 8.3.5), the Archimedean copula C_φ admits a uniform lower tail expansion of degree 1, with leading part

$$L(x, y) := \left(x^{-\alpha} + y^{-\alpha}\right)^{-1/\alpha},$$

for $(x, y) \in \,]0, +\infty[^2$. Clearly, the leading part L is symmetric since every Archimedean copula is exchangeable and since (4.3) holds. Hence, the functions g and h associated to C_φ can be written from (4.5) as:

$$g(x) = h(x) := \left(\frac{x^{-\alpha} + 1}{2}\right)^{-1/\alpha}. \tag{4.7}$$

Moreover, if $\mathcal{E}_x(0) = -\infty$, then $L(x, y) = \min(x, y)$, and if $\mathcal{E}_x(0) = 0$ then $L(x, y) = 0$, i.e., the expansion is trivial. In this last case, assuming again $\varphi_0 = +\infty$, we cannot use (4.5) to compute the function g (or h), but, as a direct corollary from (Jaworski 2010,Proposition 8.3.6) or (Charpentier and Segers 2009,Theorem 3.3), we obtain that the function g exists, provided that the function $-\frac{\psi(t)}{\psi'(t)}$ is regularly varying at infinity of finite index κ. If this is the case, then it can be shown that $\kappa \leq 1$ and

$$g(x) = h(x) = x^{2^{-\kappa}}. \tag{4.8}$$

Example 4.3 (Survival BEV copulas) In the bivariate case, Pickands (1981) showed that every max-stable (or Bivariate Extreme Value, BEV for short) copula can be written in terms of a convex function $A : [0, 1] \to [1/2, 1]$ such that, for all $t \in [0, 1]$, $A(t) \geq \max(t, 1 - t)$. Specifically, one has, for all $u, v \in (0, 1)$,

$$C_A(u, v) = \exp\left(\ln(uv) A\left(\ln(v)/\ln(uv)\right)\right) = (uv)^{A(\ln(v)/\ln(uv))}. \tag{4.9}$$

The mapping A is called a Pickands dependence function and plays an important role in modeling the dependence between two extreme risks. See for example (Gudendorf and Segers 2009; Genest and Nešlehová 2012; Joe et al. 2010; Jaworski 2019; Kamnitui et al. 2019) for more details. The survival BEV copula is given by

$$\hat{C}_A(u, v) = u + v - 1 + \exp\left(\ln((1 - u)(1 - v)) A\left(\ln(1 - v)/\ln((1 - u)(1 - v))\right)\right). \tag{4.10}$$

Note that for all BEV copulas the leading term of the survival copula $\hat{C}_A$ (ie. the upper tail leading term of C_A) is well defined (Jaworski 2004 Ex.5.4) and equals

$$L_{\hat{C}_A}(x,y) = (x+y)\left(1 - A\left(\frac{y}{x+y}\right)\right).$$

Subsequently,

$$\lambda_L(\hat{C}_A) = L_{\hat{C}_A}(1,1) = 2(1 - A(1/2)). \tag{4.11}$$

$$g(x) = \frac{(1+x)(1 - A(1/(1+x))}{2(1 - A(1/2))}, \tag{4.12}$$

$$h(x) = \frac{(1+x)(1 - A(x/(1+x))}{2(1 - A(1/2))}, \tag{4.13}$$

$$G(x,y) = \frac{(x+y)(1 - A(y/(x+y))}{2(1 - A(1/2))}. \tag{4.14}$$

Below, we collect some facts about the lower tail dependence coefficient $\lambda_L(C)$ and some of the concepts presented here.

Proposition 4.2 *Assume that a copula C satisfies Condition 2.1 with associated functions g and h and with θ from (2.5). Then the following results hold:*

(i) If $\theta > 1$, then $\lambda_L(C)$ exists and equals 0.
(ii) If either $g(+\infty) = +\infty$ or $h(+\infty) = +\infty$, then $\lambda_L(C)$ exists and equals 0.
(iii) If $\lambda_L(C)$ exists and $\lambda_L(C) \neq 0$, then

$$g(+\infty) \leq \frac{1}{\lambda_L(C)} \quad \textit{and} \quad h(+\infty) \leq \frac{1}{\lambda_L(C)}.$$

(iv) If $\lambda_L(C)$ exists, then C admits the lower tail uniform expansion of degree 1 with the leading term

$$L_C(u,v) = \lambda_L(C) v g\left(\frac{u}{v}\right), \ \textit{for } v > 0, \tag{4.15}$$

which can also be written as either

$$L_C(u,v) = \lambda_L(C) u h\left(\frac{v}{u}\right), \ \textit{for } u > 0, \tag{4.16}$$

or

$$L_C(u,v) = \lambda_L(C) G(u,v), \tag{4.17}$$

where G is the function defined as in (2.6). Furthermore, if $\lambda_L(C) \neq 0$, then

$$g(+\infty) = \frac{1}{\lambda_L(C)} \lim_{x\to\infty} L_C(x,1), \quad h(+\infty) = \frac{1}{\lambda_L(C)} \lim_{x\to\infty} L_C(1,x) \tag{4.18}$$

and both functions g and h are concave.

Proof Part (i) follows from (2.4). Since the function $f(u) := \delta(u)/u$, $u \in (0,1]$, is regularly varying at 0 with index $\theta - 1$, which is positive, the limit of f at 0 must be 0 (compare (Resnick 2007,Proposition 2.6), for $U(x) := f(1/x)$ and $\rho := 1-\theta$). Since this limit coincides with $\lambda_L(C)$, the claim follows.

Parts (ii) and (iii) follow from the estimates $C(xu,u) \le u$ and $C(u,xu) \le u$ for $x > 1$. Indeed, if, for instance, $g(+\infty) = +\infty$ (the same argument can be applied when $h(+\infty) = +\infty$ by simply replacing $C(xu,u)$ by $C(u,xu)$ in the following inequality):

$$\limsup_{u\to 0^+} \frac{C(u,u)}{u} = \limsup_{u\to 0^+} \frac{C(u,u)}{C(xu,u)} \frac{C(xu,u)}{u} \le \frac{1}{g(x)}.$$

Since $(g(x))^{-1} \searrow 0$ as $x \to \infty$, one has $\lambda_L(C) = 0$.

Point (iv) is a consequence of the fact that, if $v > 0$, then

$$\frac{C(tu,tv)}{t} = \frac{C(tu,tv)}{C(tv,tv)} \cdot \frac{C(tv,tv)}{tv} \cdot v \overset{t\to 0^+}{\longrightarrow} g\left(\frac{u}{v}\right) \lambda_L(C) v;$$

analogously, if $u > 0$, then

$$\frac{C(tu,tv)}{t} = \frac{C(tu,tv)}{C(tu,tu)} \cdot \frac{C(tu,tu)}{tu} \cdot u \overset{t\to 0^+}{\longrightarrow} h\left(\frac{v}{u}\right) \lambda_L(C) u,$$

where we also used (4.3) for $d = 1$. This proves formulas (4.15) and (4.16). As far as (4.17), note that, if $\lambda_L(C) = 0$, then all of the right hand sides of the three equations vanish. If, instead, $\lambda_L(C) \neq 0$, part (i) yields $\theta = 1$ and then (4.17) follows from (4.15) and (2.7) or from (4.16) and (2.8). Moreover, if $\lambda_L(C) \neq 0$, then (4.17) yields

$$g(+\infty) = \lim_{x\to+\infty} G(x,1) = \frac{1}{\lambda_L(C)} \lim_{x\to+\infty} L_C(x,1),$$

and using a similar argument we obtain an analogous formula for $h(+\infty)$. The concavity of g and h follows from the fact that L_C is concave, see (Jaworski 2010,Theorem 8.2.1) or (Jaworski 2006). This concludes the proof. □

Remark 4.2 Note that point (iii) from Proposition 4.2 cannot be improved, since there are copulas with

$$g(+\infty) < \frac{1}{\lambda_L(C)} \qquad \text{and/or} \qquad h(+\infty) < \frac{1}{\lambda_L(C)}.$$

Consider, for instance, the convex sum $(M+W)/2$ of the comonotonicity and countermonotonicity copulas, for which we have $g(+\infty) = h(+\infty) = 1$ and $\lambda_L(C) = 1/2$. The other examples come from the family of survival BEV copulas. We observe that

$$\lambda_L(\hat{C}_A) g(+\infty) = \lim_{x\to\infty} \frac{1 - A(1/(1+x))}{1/(1+x)} = -A'(0^+).$$

Similarly,

$$\lambda_L(\hat{C}_A)h(+\infty) = A'(1^-).$$

Since A might be any convex function such that $1 \geq A(u) \geq \min(u, 1-u)$, we may choose A with a right-sided derivative at 0 equal to $-1/2$ or left-sided derivative at 1 equal to $1/2$.

Now, let C be any copula from Example 2.1. It can be checked that C admits a lower tail dependence coefficient. However, we shall point out that the existence of the functions g and h from Condition 2.1 does not guarantee the existence of a lower tail dependence coefficient $\lambda_L(C)$, as the following example shows.

Example 4.4 Consider the function $f : (0, 1] \to \mathbb{R}$ defined as

$$f(x) := x \exp\left[\sin\left(\sqrt{-\ln(x)}\right)\right],$$

A long and tedious calculation based on the trigonometric identity

$$\sin(A) - \sin(B) = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

and the well-known limit

$$\sqrt{-\ln(tx)} - \sqrt{-\ln(t)} \xrightarrow{t \to 0^+} 0,$$

shows that f is regularly varying at 0 with index 1. Moreover, it is easy to see that the function $f(x)/x$ does not have a limit as $x \to 0^+$. Next, we define a function $\delta : [0, 1] \to [0, 1]$ as follows:

$$\delta(x) := \begin{cases} 0, & \text{if } x = 0, \\ \max\left(\frac{f(x)}{5}, 2x - 1\right), & \text{if } x \in (0, 1]. \end{cases} \tag{4.19}$$

After some calculations it can be showed that δ is a diagonal section of a copula, i.e. (see, e.g., Durante and Sempi 2016) it satisfies the following conditions:

(a) $\delta(0) = 0$ and $\delta(1) = 1$;
(b) $\delta(x) \leq x$ for every $x \in [0, 1]$;
(c) δ is increasing;
(d) δ is 2−Lipschitz, i.e., for every $x, y \in [0, 1]$, it holds $|\delta(x) - \delta(y)| \leq 2|x - y|$.

Now, let $C := C_\delta^{Ber}$ be the *Bertino copula* (see Fredricks and Nelsen 2003) with diagonal section equal to δ, where we recall that the Bertino copula associated with

δ is the smallest copula, in the concordance (or point wise) ordering, having δ as its diagonal section. Such a copula is given, for every $(u, v) \in [0, 1]^2$, by

$$C(u, v) := \min(u, v) - \min\{x - \delta(x) : x \in [u \wedge v, u \vee v]\}.$$

Note that the Bertino copula is exchangeable. We now claim that C satisfies Condition 2.1 with the functions

$$g(x) = h(x) := \min(x, 1), \qquad x \geq 0.$$

First, note that, if $x, u \in (0, 1)$ and u is sufficiently small, then $xu < u$ and we have

$$C(xu, u) = xu - \min_{t \in [xu, u]} (t - \delta(t)) = xu - (xu - \delta(xu)) = \frac{f(xu)}{5}.$$

If, instead, $x > 1$ and $u \in (0, 1)$ is sufficiently small, then $xu > u$ and we have

$$C(xu, u) = u - \min_{t \in [u, xu]} (t - \delta(t)) = u - (u - \delta(u)) = \frac{f(u)}{5}.$$

Moreover, note that

$$C(u, u) = \delta(u) = \frac{f(u)}{5}$$

for small values of u. Putting all together, we can now compute the function g, which is given, for $x \in (0, 1)$, by

$$g(x) = \lim_{u \to 0^+} \frac{C(xu, u)}{C(u, u)} = \lim_{u \to 0^+} \frac{f(xu)}{f(u)} = x = \min(x, 1),$$

since f is regularly varying at 0 with index 1. And for $x > 1$ we have

$$g(x) = \lim_{u \to 0^+} \frac{C(xu, u)}{C(u, u)} = 1 = \min(x, 1),$$

hence $g(x) = h(x) = \min(x, 1)$ for every $x \geq 0$.
In order to prove that C does not admit an LTDC, note that

$$\frac{C(t, t)}{t} = \frac{f(t)}{5t}$$

for sufficiently small values of $t > 0$, and we have already pointed out that the function $f(t)/t$ does not have a limit as $t \to 0^+$, hence $\lambda_L(C)$ does not exist for the copula C.

Remark 4.3 After proving that the existence, for a certain copula C, of the functions g and h as in Condition 2.1 does not imply the existence of a lower tail dependence coefficient $\lambda_L(C)$, we now focus on a different problem, that is still an *open problem* for us.

Specifically, since the function h is increasing, the limit of $(h(x))^{-1}$ as $x \to +\infty$ exists, and in all of the cases presented in Example 2.1 it holds

$$\lim_{x\to+\infty} \frac{1}{h(x)} = \lambda_L(C). \tag{4.20}$$

However, to the best of our knowledge, even assuming that Condition 2.1 is satisfied and that the LTDC $\lambda_L(C)$ exists, we do not know yet whether (4.20) holds or does not hold. Notice that, since

$$\lim_{x\to+\infty} \frac{1}{h(x)} = \lim_{x\to+\infty} \lim_{u\to0^+} \frac{C(u,u)}{C(u,xu)},$$

the validity of (4.20) is related to the fact that, in general, we could not exchange the order of the iterated limits, i.e

$$\lim_{x\to+\infty} \frac{1}{h(x)} \overset{???}{=} \lim_{u\to0^+} \lim_{x\to+\infty} \frac{C(u,u)}{C(u,xu)} = \lim_{u\to0^+} \frac{C(u,u)}{C(u,1)} = \lim_{u\to0^+} \frac{C(u,u)}{u} = \lambda_L(C).$$

5 The tame condition

Taking into account Remark 4.3, in this section we will also assume the following condition.

Condition 5.1 Under Condition 2.1 we assume that C admits a lower tail dependence coefficient $\lambda_L(C)$ and it holds

$$\frac{1}{h(x)} \searrow \lambda_L(C), \qquad \text{as } x \to +\infty. \tag{5.1}$$

Moreover, h is strictly increasing on $\left[0, h^{(-1)}\left(1/\lambda_L(C)\right)\right]$, where $h^{(-1)}$ is the quasi-inverse of h.

Note that, under the same notations of Condition 5.1, if $\lambda_L(C) > 0$, then $\theta = 1$ from Proposition 4.2, where θ denotes the index of g and h, so that, for every $x > 0$

$$h(x) = xg\left(\frac{1}{x}\right).$$

Then the assumption

$$\frac{1}{h(x)} \searrow \lambda_L(C), \qquad \text{as } x \to +\infty$$

is easily seen to be equivalent to

$$xg\left(\frac{1}{x}\right) \nearrow \frac{1}{\lambda_L(C)}, \qquad x \to +\infty.$$

Moreover, notice that, under the Condition 5.1, if $\lambda_L(C) > 0$, then the "tail" probability mass of the copula C is concentrated close to the origin (0,0): see (Jaworski 2013,Theorem 7.4) for more details. Generally speaking, Condition 5.1 is a very restrictive one. For example, most of the survival BEV copulas is not fulfilling it (compare Remark 4.2). Only the ones generated by Pickands functions A, which graphs are tangent to the diagonal $v = u$, may fulfill it. On the other hand, Condition 5.1 simplifies the study of the limiting properties of the mCoVaR.

Proposition 5.1 *Suppose that C satisfies Conditions 2.1 and 5.1 with related functions g and h. If $\lambda_L(C) > 0$, then for any constant $\beta \in (0, 1)$ the implied significance level w_* fulfills*

$$\lim_{\alpha\to 0} w_*(\alpha) = 0 \quad and \quad \lim_{\alpha\to 0} \frac{w_*(\alpha)}{\alpha} = h^{-1}\left(\frac{\beta}{\lambda_L(C)}\right). \tag{5.2}$$

Proof The existence of the nonzero λ_L implies that

$$\rho_0 = \lim_{\alpha\to 0} \frac{\alpha\beta}{\delta(\alpha)} = \frac{\beta}{\lambda_L(C)}.$$

Furthermore Condition 2.1 implies that $\theta = 1$ and h is concave. By Condition 5.1 we get that

$$\rho_0 < \frac{1}{\lambda_L(C)} = \sup(h(x) : x \in [0, \infty)).$$

Thus Proposition 3.1 implies the thesis of the above Proposition. □

Now, we recall the definition of the truncated (conditional) copula (Durante and Jaworski 2012; Jaworski 2013). Let (U, V) be a pair of [0, 1]-uniform random variables distributed according the copula C. For $\alpha \in (0, 1]$ we denote by $C_{[\alpha]}$ the copula of the conditional distribution of (U, V) with respect to the condition $U \le \alpha$ (see, e.g., Durante and Jaworski 2012). Under the previous assumptions, the limiting behavior of $C_{[\alpha]}$ can be described via the following result.

Proposition 5.2 *Suppose that C satisfies Conditions 2.1 and 5.1 with related functions g and h. If $\lambda_L(C) > 0$, then, as $\alpha \to 0^+$, the limit of $C_{[\alpha]}$ exists and the limiting copula $C_{[0]}$ is given, for every $(u, v) \in (0, 1)^2$, by*

$$C_{[0]}(u, v) = uh\left(\frac{1}{u}h^{(-1)}\left(\frac{v}{\lambda_L(C)}\right)\right)\lambda_L(C). \tag{5.3}$$

Proof Proposition 4.2 implies that C admits a non-trivial lower tail expansion of degree 1 with leading part L. Condition 5.1 together with Proposition 4.1 implies that the limit $L(1, +\infty) = 1$. Indeed

$$h(x) = \frac{L(1, x)}{\lambda_L} \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{h(x)} = \lambda_L,$$

thus

$$\lim_{x \to +\infty} L(1, x) = \lim_{x \to +\infty} h(x)\lambda_L = 1.$$

Next, due to Theorem 7.4 Jaworski (2013) applied for $v = (0, 0)$ the limiting copula $C_{[0]}$ exists and is given by

$$C_{[0]}(u, L(1, y)) = L(u, y).$$

Substituting $L(1, y) = \lambda_L h(y)$ and $L(u, y) = \lambda_L u h(y/u)$ we get

$$C_{[0]}(u, \lambda_L h(y)) = \lambda_L u h(y/u).$$

Since by Condition 5.1 h is strictly increasing, it is invertible and putting $v = \lambda_L h(y)$ we get formula (5.3). □

6 Limiting theorems

From now on, let (X, Y) be a continuous random pair with copula C and univariate marginals that are identically distributed as F. We denote by x_F the left-endpoint of F, i.e., $x_F := \inf\{x \in \mathbb{R} : F(x) > 0\}$.

First, we recall that a function h (not otherwise specified) is said to be *regularly varying at* $-\infty$ *with index* α if, for every $t > 0$:

$$\lim_{x \to -\infty} \frac{h(xt)}{h(x)} = t^{\alpha}.$$

If this is the case, following the same notations as in Wüthrich (2004), we shall write $h \in \mathcal{R}_\alpha$. Depending on its rate of decay at $-\infty$, we assume that F is of one of the following types:

- *Frèchet class*: F belongs to the Fréchet class if $x_F = -\infty$ and there exists $\beta > 0$ such that $F \in \mathcal{R}_{-\beta}$, that is,

$$\forall t > 0 \qquad \lim_{x \to -\infty} \frac{F(xt)}{F(x)} = t^{-\beta}; \tag{6.1}$$

- *Weibull class*: F belongs to the Weibull class if $x_F > -\infty$ and there exists $\beta > 0$ such that $F(x_F - 1/\cdot) \in \mathcal{R}_{-\beta}$, that is,

$$\forall t > 0 \qquad \lim_{x \to -\infty} \frac{F\left(x_F - \frac{1}{xt}\right)}{F\left(x_F - \frac{1}{x}\right)} = t^{-\beta}; \tag{6.2}$$

- *Gumbel class*: F belongs to the Gumbel class if $x_F \geq -\infty$ and there exists a positive function $a(\cdot)$ such that

$$\forall t \in \mathbb{R} \qquad \lim_{u \to x_F^+} \frac{F\left(u + t a(u)\right)}{F\left(u\right)} = e^t. \tag{6.3}$$

Below we provide a weak convergence result for the distribution of (X, Y) given that X is below low threshold. Obviously, similar results can be found when Y is below a threshold. We remark that from a technical point of view it is easier to state results for low thresholds than results for high thresholds, although these latter can be found analogously.

Theorem 6.1 *Let* (X, Y) *be a continuous random pair distributed according to* $C(F, F)$*. Suppose that* C *satisfies Conditions 2.1 and 5.1 with related functions g and h, and let the function* $G(\cdot, \cdot)$ *be defined as in (2.6). The following results hold:*

Fréchet case *If* $F \in \mathcal{R}_{-\beta}$ *for some* $\beta > 0$*, then, for every* $x_1 > 0$ *and for every* $(x_2, y_1) \in (0, x_1] \times (0, +\infty)$*,*

$$\lim_{u \to -\infty} \mathbb{P}\left[X \leq \frac{u}{x_1}, Y \leq \frac{u}{y_1} \,\middle|\, X \leq \frac{u}{x_2}\right] = G\left(x_1^{\beta}, y_1^{\beta}\right) \frac{\lambda_L(C)}{x_2^{\beta}} = \left(\frac{x_1}{x_2}\right)^{\beta} h\left(\left(\frac{y_1}{x_1}\right)^{\beta}\right) \lambda_L(C). \tag{6.4}$$

Weibull case *If* $x_F > -\infty$ *and* $F(x_F - 1/\cdot) \in \mathbb{R}_{-\beta}$ *for some* $\beta > 0$*, then, for every* $x_1 > 0$ *and for every* $(x_2, y_1) \in (0, x_1] \times (0, +\infty)$*,*

$$\lim_{u \to -\infty} \mathbb{P}\left[X \leq x_F - \frac{x_1}{u}, Y \leq x_F - \frac{y_1}{u} \,\middle|\, X \leq x_F - \frac{x_2}{u}\right] = G\left(x_1^{\beta}, y_1^{\beta}\right) \frac{\lambda_L(C)}{x_2^{\beta}} = \left(\frac{x_1}{x_2}\right)^{\beta} h\left(\left(\frac{y_1}{x_1}\right)^{\beta}\right) \lambda_L(C). \tag{6.5}$$

Gumbel case *If* $x_F \geq -\infty$ *and there exists a positive function* $a(\cdot)$ *such that (6.3) is satisfied, then, for every* $x_2 \in \mathbb{R}$ *and for every* $(x_1, y_1) \in (-\infty, x_2] \times \mathbb{R}$*,*

$$\lim_{u \to -\infty} \mathbb{P}\left[X \leq u + x_1 a(u), Y \leq u + y_1 a(u) \,\middle|\, X \leq u + x_2 a(u)\right] = G\left(e^{x_1}, e^{y_1}\right) \frac{\lambda_L(C)}{e^{x_2}} = e^{x_1 - x_2} h\left(e^{y_1 - x_1}\right) \lambda_L(C). \tag{6.6}$$

Proof First, note that the conditional probabilities in the statement of the Theorem are well-defined, because we are assuming the existence of the functions g and h fulfilling (2.1) and (2.2), hence we are implicitly assuming $C(u, v) > 0$ for every $u, v > 0$.
Fréchet case: Since F satisfies (6.1), for every $x > 0$ and every $\epsilon \in (0, x)$ there exists $u_0 < 0$ such that

$$\forall u < u_0 : \qquad (x-\epsilon)^\beta F(u) \le F\left(\frac{u}{x}\right) \le (x+\epsilon)^\beta F(u).$$

Thus, we can write, if $u < u_0$ and ϵ is sufficiently small:

$$\begin{aligned}\mathbb{P}\left[X \le \frac{u}{x_1}, Y \le \frac{u}{y_1} \,\middle|\, X \le \frac{u}{x_2}\right] &= \frac{\mathbb{P}\left[X \le \frac{u}{x_1}, Y \le \frac{u}{y_1}\right]}{F\left(\frac{u}{x_2}\right)} = \frac{C\left(F\left(\frac{u}{x_1}\right), F\left(\frac{u}{y_1}\right)\right)}{F\left(\frac{u}{x_2}\right)} \\ &\le \frac{C\left((x_1+\epsilon)^\beta F(u), (y_1+\epsilon)^\beta F(u)\right)}{(x_2-\epsilon)^\beta F(u)}.\end{aligned}$$

Now, having set $v := F(u) \to 0^+$ as $u \to -\infty$, we have

$$\limsup_{u\to-\infty} \mathbb{P}\left[X \le \frac{u}{x_1}, Y \le \frac{u}{y_1} \,\middle|\, X \le \frac{u}{x_2}\right] \le \limsup_{v\to 0^+} \frac{C\left((x_1+\epsilon)^\beta v, (y_1+\epsilon)^\beta v\right)}{(x_2-\epsilon)^\beta v}.$$

Recall that, by Condition 5.1, C admits a lower tail dependence coefficient $\lambda_L(C)$. Then we fall into the hypotheses of Proposition 4.2, part (iv), which states that the copula C admits the lower tail uniform expansion of degree 1 with the leading term L_C given by any of the three formulas (4.15), (4.16) or (4.17). This allows us to rewrite the above inequality as

$$\begin{aligned}\limsup_{u\to-\infty} \mathbb{P}\left[X \le \frac{u}{x_1}, Y \le \frac{u}{y_1} \,\middle|\, X \le \frac{u}{x_2}\right] &\le \lim_{v\to 0^+} \frac{C\left((x_1+\epsilon)^\beta v, (y_1+\epsilon)^\beta v\right)}{(x_2-\epsilon)^\beta v} \\ &= \frac{L_C\left((x_1+\epsilon)^\beta, (y_1+\epsilon)^\beta\right)}{(x_2-\epsilon)^\beta} = \lambda_L(C)\frac{G\left((x_1+\epsilon)^\beta, (y_1+\epsilon)^\beta\right)}{(x_2-\epsilon)^\beta}.\end{aligned}$$

On the other hand, we can analogously write, for $u < u_0$ and ϵ sufficiently small:

$$\mathbb{P}\left[X \le \frac{u}{x_1}, Y \le \frac{u}{y_1} \,\middle|\, X \le \frac{u}{x_2}\right] \ge \frac{C\left((x_1-\epsilon)^\beta F(u), (y_1-\epsilon)^\beta F(u)\right)}{(x_2+\epsilon)^\beta F(u)},$$

and then we can find a similar lower bound for the lim inf. Indeed, we obtain:

$$\liminf_{u\to-\infty} \mathbb{P}\left[X \le \frac{u}{x_1}, Y \le \frac{u}{y_1} \,\middle|\, X \le \frac{u}{x_2}\right] \ge \lambda_L(C)\frac{G\left((x_1-\epsilon)^\beta, (y_1-\epsilon)^\beta\right)}{(x_2+\epsilon)^\beta}.$$

Considering that ϵ can be chosen arbitrarily small and that G is continuous, it turns out that the lim inf and the lim sup coincide and (6.4) holds, hence the assertion follows for the Fréchet case.

Weibull case: Since F satisfies (6.2), for every $x > 0$ and every $\epsilon \in (0, x)$ there exists $u_0 < 0$ such that

$$\forall u < u_0 \qquad (x-\epsilon)^\beta F\left(x_F - \frac{1}{u}\right) \le F\left(x_F - \frac{x}{u}\right) \le (x+\epsilon)^\beta F\left(x_F - \frac{1}{u}\right).$$

But then the assertion follows as in the Fréchet case, having set $v := F\left(x_F - \frac{1}{u}\right) \to 0^+$ as $u \to -\infty$. We obtain (6.5) in the Weibull case.

Gumbel case: Since F satisfies (6.3), for every $x \le x_2$ and every $\epsilon > 0$ there exists $u_0 > x_F$ such that

$$\forall u \in (x_F, u_0) \qquad e^{x-\epsilon} F(u) \le F(u + xa(u)) \le e^{x+\epsilon} F(u).$$

But then the assertion follows as in the Fréchet case, having set $v := F(u) \to 0^+$ as $u \to x_F^+$, and having replaced x^β by e^x. Hence we obtain (6.6) in the Gumbel case, and this concludes the proof of the whole Theorem. □

When $\lambda_L(C) = 0$, all the three limiting distributions in Theorem 6.1 are vanishing. Otherwise, $\lambda_L(C) > 0$ and, by using Proposition 5.2, we can explicitly compute the limiting dependence structure.

Corollary 6.1 *Let (X, Y) be a continuous random pair distributed according to $C(F, F)$. Suppose that C satisfies Conditions 2.1 and 5.1 with related functions g and h, and let the function $G(\cdot, \cdot)$ be defined as in (2.6). Furthermore, assume that $\lambda_L(C) > 0$. Then, the limits in (6.4), (6.5) and (6.6) define proper bivariate distribution functions. Moreover, since the distribution functions are also continuous, their copula C^G is unique and, in all three cases, is given, for every $(u, v) \in (0, 1)^2$, by*

$$C^G(u, v) = uh\left(\frac{1}{u} h^{(-1)}\left(\frac{v}{\lambda_L(C)}\right)\right)\lambda_L(C). \qquad (6.7)$$

Note that C_G being a limiting copula is invariant with respect to conditioning: see (Jaworski 2013,Proposition 7.9) and Jágr et al. (2010).

In view of Corollary 6.1, we shall reconsider the copulas presented in Examples 2.1 for which the lower tail dependence coefficient is not zero.

Example 6.1

- *Comonotonic copula:* If $C = M$, then $\lambda_L(C) = 1$ and $h(x) = g(x) = \min(x, 1)$. Note that $h(x) = 1 = \lambda_L(C)^{-1}$ for every $x \ge 1$ and that $h(x)$ is strictly increasing from $[0, 1]$ to $\left[0, \lambda_L(C)^{-1}\right] = [0, 1]$, hence both Conditions 2.1 and 5.1 are satisfied. Moreover, for every $t \in [0,1]$, one has $h^{(-1)}(t) = h^{-1}(t) = t$. So, let us calculate the copula C^G associated to this copula, by using (6.7). We have, for every $(u, v) \in (0,1)^2$,

$$C^G(u,v) = uh\left(\frac{v}{u}\right) = u \min\left(\frac{v}{u}, 1\right) = \min(u,v) = M(u,v) = C(u,v),$$

hence it turns out that the copula of the distributions of the excesses under low thresholds actually coincides with the copula of (X, Y), when X and Y are comonotonic, i.e., their copula is M. This is not really surprising, because it is well known that the copula M is itself left truncation invariant, as shown in, e.g., Durante and Jaworski (2012).

- *Mardia-Takahasi-Clayton copula:* Let C be the copula of type (2.3) with positive parameter $\alpha > 0$. We know that $\lambda_L(C) = 2^{-1/\alpha}$ and that

$$h(x) = g(x) = \left(\frac{x^{-\alpha}+1}{2}\right)^{-1/\alpha} \nearrow 2^{1/\alpha} = \lambda_L(C)^{-1}, \quad x \to +\infty.$$

Moreover, $h(x)$ is strictly increasing from $[0, +\infty)$ to $[0, 2^{1/\alpha})$, hence both Conditions 2.1 and 5.1 are fulfilled. As far as the inverse of h, it holds

$$h^{(-1)}(t) = h^{-1}(t) = \left(2t^{-\alpha} - 1\right)^{-1/\alpha}, \quad t \in [0, 2^{1/\alpha}),$$

and $h^{(-1)}(2^{1/\alpha}) = +\infty$. So, we are ready to compute the copula C^G associated to the Mardia-Takahasi-Clayton copula C, by using (6.7). Fix $(u, v) \in (0,1)^2$. We have

$$C^G(u,v) = 2^{-1/\alpha} uh\left(\frac{h^{-1}\left(2^{1/\alpha}v\right)}{u}\right) = 2^{-1/\alpha} u \left(\frac{v^{-\alpha}-1}{2u^{-\alpha}} + \frac{1}{2}\right)^{-1/\alpha} =$$

$$= 2^{-1/\alpha} u \left(\frac{u^{-\alpha}+v^{-\alpha}-1}{2u^{-\alpha}}\right)^{-1/\alpha} = \left(u^{-\alpha}+v^{-\alpha}-1\right)^{-1/\alpha} = C(u,v),$$

hence we again obtain that the copula of the distributions of the excesses under low thresholds coincides with the copula of (X, Y). As for the copula M, the Clayton copula is also known to be left truncation invariant (see, again, Durante and Jaworski 2012), thus the result is what we actually expected.

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