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Schur line defect correlators and giant graviton expansion

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Abstract: We consider Schur line defect correlators in four dimensional $\mathcal{N} = 4$ U(N) SYM and their giant graviton expansion encoding finite N corrections to the large N limit. We compute in closed form the single giant graviton contribution to correlators with general insertions of $\frac{1}{2}$-BPS charged Wilson lines. For the 2-point function with fundamental and anti-fundamental Wilson lines, we match the result from fluctuations of two half-infinite strings ending on the giant graviton, recently proposed in arXiv:2403.11543. In particular, we prove exact factorization of the defect contribution with respect to wrapped D3 brane fluctuations representing the single giant graviton correction to the undecorated Schur index. This follows from a finite-difference representation of the Schur line defect index in terms of the index without defects, and similar factorization holds quite generally for more complicated defect configurations. In particular, the single giant graviton contribution to the 4-point function with two fundamental and two anti-fundamental lines is computed and discussed in this perspective.

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1 Introduction

The superconformal index introduced in [1–3] is the Witten index [4] in radial quantization and is a common device for the study of the BPS spectrum of superconformal theories. As a general fact, for $\mathrm{U}(N)$ gauge theories with a gravity dual, the superconformal index has a definite $N \to \infty$ limit matching the index of BPS supergravity states.[1] On the CFT side, corrections at finite N are due to gauge group trace relations taking into account the structure of multi-trace states. The dual gravity explanation of these corrections is in terms of giant graviton contributions [7] with charge $\sim N$. As suggested recently in [8, 9], finite N corrections may also be computed by analyzing the BPS geometries of supergravity bubbling solutions.

In the specific case of 4d $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM, dual to IIB superstring in $AdS_5 \times S^5$, the giant graviton expansion of the index involves the "brane index" of the theory living on the world-volume of multiply wrapped D3 brane configurations [10–12]. In this paper, we will consider the Schur specialization of the general index [13, 14]. It may be introduced in theories with at least $\mathcal{N} = 2$ supersymmetry where it reproduces the vacuum character of the chiral algebra characterizing a protected sector [15]. In $\mathcal{N} = 4$ $\mathrm{U}(N)$ SYM the definition of the Schur index is

$$\mathrm{I}^{\mathrm{U}(N)}(\eta;q) = \mathrm{Tr}_{\mathrm{BPS}}[(-1)^{\mathrm{F}}\, q^{H+J+\bar{J}}\eta^{R_1-R_2}]\,, \tag{1.1}$$

[1] This is a limit order by order in the charge of the contributing BPS states. It captures single trace states in the CFT, but should not be confused with the large N limit of the index, see in particular [5, 6].

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where H is the Hamiltonian, $J, \bar{J}$ are two spins, and the R_1, R_2 are two of the R-charge generators in the PSU(2, 2|4) superconformal group. The variable q is the universal fugacity, while η is usually referred to as a flavor fugacity. The Schur index admits an explicit holonomy integral representation that reads (PE stands for plethystic exponentiation), see for instance [16],

$$\mathrm{I}^{\mathrm{U}(N)}(\eta;q) = \oint_{|\boldsymbol{z}|=1} D^N\boldsymbol{z}\ \mathrm{PE}[f(\eta;q)\chi_\square(\boldsymbol{z})\chi_\square(\boldsymbol{z}^{-1})]\,, \tag{1.2}$$

where the measure $D^N\boldsymbol{z}$ and the character $\chi_\square(\boldsymbol{z})$ of the U(N) fundamental representation are

$$D^N\boldsymbol{z} = \frac{1}{N!}\prod_{n=1}^{N}\frac{dz_n}{2\pi i\, z_n}\prod_{n\neq m}\left(1-\frac{z_n}{z_m}\right), \qquad \chi_\square(\boldsymbol{z}) = \sum_{n=1}^{N} z_n\,. \tag{1.3}$$

The function $f(\eta;q)$ in (1.2) is the single particle Schur index and is given by the simple function

$$f(\eta;q) = \frac{(\eta+\eta^{-1})\,q-2q^2}{1-q^2}\,. \tag{1.4}$$

Exact results for the Schur index at specific values of N have been obtained in [17–20] in the case of U(N) gauge group, and generalized to B_n,C_n, D_n, G_2 groups in [21].

As discussed in [22], the giant graviton expansion of the Schur index is simpler than that of the general index and can be written in terms of the $\mathcal{N}=4$ SYM index itself. One has indeed the remarkable relation

$$\mathrm{I}^{\mathrm{U}(N)}(\eta;q) = \mathrm{I}^{\mathrm{KK}}(\eta;q)\sum_{n=0}^{\infty}\sum_{p=0}^{n}(\eta q)^{(n-p)N}\mathrm{I}^{\mathrm{D3}}_{n-p}(\eta;q)q^{2(n-p)p}(\eta^{-1}q)^{pN}\mathrm{I}^{\mathrm{D3}}_{p}(\eta^{-1};q)\,, \tag{1.5}$$

where $\mathrm{I}^{\mathrm{KK}}(\eta;q)$ is the large N Kaluza-Klein supergravity contribution and the brane indices $\mathrm{I}^{\mathrm{D3}}_n(\eta;q)$ are obtained by analytic continuation of the $\mathcal{N}=4$ U(n) SYM index [23]

$$\mathrm{I}^{\mathrm{D3}}_n(\eta;q) = \mathrm{I}^{\mathrm{U}(n)}(\eta^{-1/2}q^{-3/2};\eta^{-1/2}\,q^{1/2})\,. \tag{1.6}$$

The terms in the sum (1.5) are organized in contributions with weight $\sim q^{nN}$ with the index n being the wrapping number of D3 branes with topology $S^1\times S^3$, where $S^1\subset AdS_5$ and $S^3\subset S^5$. The approach based on the analytic continuation (1.6) was successfully applied in many other instances [24–30] and was confirmed by complete fluctuation analysis in [31–34].[2] Recently, the brane indices $\mathrm{I}^{\mathrm{D3}}_n(\eta;q)$ were computed in closed form in [37].

A natural extension of the Schur index consists in decorating it by inserting defect lines [38, 39] and exact results have been obtained for the associated Schur correlators involving an arbitrary number of defect operator ('t Hooft or Wilson lines) insertions [40–46]. When the index is regarded as a supersymmetric partition function on $S^1\times S^3$, the defect lines are wrapping S^1 and are placed on a great circle of S^3 to preserve supersymmetry [41]. Schur line defect correlators are topological and do not depend on the distance between the inserted Wilson lines. Here, we consider the insertion of $\frac{1}{2}$-BPS Wilson lines with generic charges.

[2] An alternative approach based on localization of the theory on the brane world-volume has been recently discussed in [35] and later developed in [36] for the giant graviton expansion of the $\frac{1}{2}$-BPS index in $\mathcal{N}=4$ U(N) SYM.

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Insertion of Wilson line defects in representations $R_1, R_2, \ldots$ is computed by the following modification of the holonomy integral in (1.2)

$$\mathrm{I}^{\mathrm{U}(N)}_{R_1,R_2,\ldots}(\eta;q) = \oint_{|\boldsymbol{z}|=1} D^N\boldsymbol{z} \prod_{n\geq 1} \chi_{R_n}(\boldsymbol{z})\ \mathrm{PE}[f(\eta;q)\chi_{\Box}(z)\chi_{\Box}(z^{-1})]\,. \tag{1.7}$$

Due to its basic role in the following discussion, we introduce a special notation for the Schur line defect 2-point function with a fundamental and an anti-fundamental

$$\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}}(\eta;q) \equiv \mathrm{I}^{\mathrm{U}(N)}_{\Box,\overline{\Box}}(\eta;q) = \mathrm{I}^{\mathrm{U}(N)}_{\text{adjoint}}(\eta;q)\,. \tag{1.8}$$

When the Wilson lines in fundamental representation have charges Q_n, the corresponding expression involves multiple power symmetric characters $\prod_n \chi_{\Box}(\boldsymbol{z}^{Q_n})$.

In this paper, we consider the giant graviton expansion of the above Schur line defect correlators, working at single giant graviton level. In the case of the 2-point function $\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}}(\eta;q)$, the large N limit is known to take the factorized form [39]

$$\mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{F}}(\eta;q) = \mathrm{I}_{\mathrm{F}1}(\eta;q)\times \mathrm{I}^{\mathrm{U}(\infty)}(\eta;q), \qquad \mathrm{I}_{\mathrm{F}1}(\eta;q) = \frac{1}{1-f(\eta;q)}\,. \tag{1.9}$$

From the AdS/CFT perspective, the factor $\mathrm{I}_{\mathrm{F}1}(\eta;q)$ corresponds to fluctuations of a fundamental string along $AdS_2 \subset AdS_5$ [47, 48] meeting the boundary of AdS_2 at the two poles of S^3 in $\partial AdS_5 = \mathbb{R}\times S^3$, where the line operators are placed. The detailed analysis of fluctuations was performed in [39][3] confirming that the expression $f_{\mathrm{F}1}(\eta;q) = -q^2 + (\eta+\eta^{-1})\,q$ in the formula

$$\mathrm{I}_{\mathrm{F}1}(\eta;q) = \mathrm{PE}[f_{\mathrm{F}1}(\eta;q)]\,, \tag{1.10}$$

matches the single particle index of fluctuations of the fundamental string.

Quite naturally, one expects that for the 2-point function $\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}}(\eta;q)$ one should also have a leading giant graviton contribution due to the contribution from two semi-infinite strings attached to the Wilson lines and ending on the giant graviton. This proposal and its quantitative verification appeared very recently in [50] (with extension to multi-graviton contributions). In particular, it was found that[4]

$$\frac{\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}} - I_{\mathrm{F}1}\,\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}} = 1 + \left(\mathcal{G}^+_{\mathrm{F}}(\eta;q)\,\eta^N + \mathcal{G}^-_{\mathrm{F}}(\eta;q)\,\eta^{-N}\right) q^N + \mathcal{O}(q^{2N})\,, \tag{1.11}$$

$$\mathcal{G}^+_{\mathrm{F}}(\eta;q) = G^+_{\mathrm{D3}}(\eta;q)\times\frac{1}{\eta q}\,\mathrm{PE}[f_{\mathrm{F}}(\eta;q)]\,, \qquad f_{\mathrm{F}}(\eta;q) = 2\eta^{-1}q - 2q^2\,, \tag{1.12}$$

where $G^+_{\mathrm{D3}}(\eta;q)$ is the single giant graviton contribution to the undecorated Schur index coming from D3 brane fluctuations [22, 37] and $f_{\mathrm{F}}(\eta;q)$ in (1.12) agrees with the single particle index from fluctuations of the two semi-infinite strings ending on the giant graviton. The origin of the prefactor $1/(\eta q)$ is at the moment unclear. It also affects higher order giant graviton

[3]Fluctuations are in multiplets of $\mathrm{SO}(2,1)\times\mathrm{SO}(3)\times\mathrm{SO}(5)$. The last two factors correspond to rotations of the remaining AdS_5 coordinates giving SO(3) symmetry, and rotations of the fixed point in S^5 giving SO(5) [49].

[4]In the ratio (1.11) we subtract the contribution $\mathrm{I}_{\mathrm{F}1}\mathrm{I}^{(N)}$ corresponding to the case when the two half-infinite strings do not end on the giant graviton, and the difference is divided by the supergravity contribution.

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contributions and was suggested to be a back-reaction effect in [50]. Relations (1.11), (1.12) were confirmed by comparing with the first terms in the small q expansion of the 2-point function at large N.

In this paper, we derive the exact form of the single giant graviton expansion of various Schur line defect correlators. In particular, for $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta; q)$ we prove the exact result

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta; q)}{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(\eta; q)} = 1 + \left(G_{\mathrm{F}}^{+}(\eta; q)\, \eta^{N} + G_{\mathrm{F}}^{-}(\eta; q)\, \eta^{-N}\right) q^{N} + \mathcal{O}(q^{2N})\,, \tag{1.13}$$

with (see appendix A for notation)

$$G_{\mathrm{F}}^{+}(\eta; q) = -\eta^{2}\, q \left(1 + \frac{1-q^{2}}{\eta\, q}\frac{1-\eta\, q}{1-\eta^{-1}\, q}\right) \frac{\left(\frac{q}{\eta}\right)^{3}_{\infty}}{\vartheta\left(\eta^{2}, \frac{q}{\eta}\right)}, \qquad G_{\mathrm{F}}^{-}(\eta; q) = G_{\mathrm{F}}^{+}(\eta^{-1}; q). \tag{1.14}$$

Expression (1.14) is equivalent to (1.11), (1.12) and has a remarkable factorized form. Indeed, the effect of the two Wilson lines insertion is fully captured by the second term in round bracket. In other words, one has the exact simple relation

$$G_{\mathrm{F}}^{+}(\eta; q) = \left(1 + \frac{1}{\eta\, q}\frac{(1-q^{2})(1-\eta\, q)}{1-\eta^{-1}\, q}\right) G_{\mathrm{D3}}^{+}(\eta; q)\,. \tag{1.15}$$

The correction factor in brackets has the factor $1/(\eta q)$, as in (1.12), while the rest coincides with the two half strings fluctuations. Our derivation builds on the results of [51] for general multi-coupling unitary matrix models. We will illustrate how to reduce the calculation of the Schur line defect correlators to finite differences of the undecorated Schur index with respect to the gauge group rank. The factorization property in (1.15) will then follow as a simple consequence.

By our approach, it will be possible to generalize results like (1.15) to a large extent. To give an example, for the 4-point function with two Wilson lines in the fundamental representation and two in the anti-fundamental, we obtain for the ratio similar to (1.13) the exact result

$$G_{\square,\square,\overline{\square},\overline{\square}}^{+}(\eta; q) = \left[1 + \frac{1-5q^{2}+3\eta q+\eta q^{3}}{2\eta^{2}\, q^{2}}\frac{(1-q^{2})(1-\eta\, q)}{(1-\eta^{-1}\, q)^{2}}\right] G_{\mathrm{D3}}^{+}(\eta; q)\,. \tag{1.16}$$

For the subtracted ratio similar to (1.11), this gives

$$\mathcal{G}_{\square,\square,\overline{\square},\overline{\square}}^{+}(\eta; q) = G_{\mathrm{D3}}^{+}(\eta; q) \times \frac{1-5q^{2}+3\eta q+\eta q^{3}}{\eta^{2} q^{2}}\, \mathrm{PE}[-3q^{2}+4\eta^{-1}q+\eta q]\,. \tag{1.17}$$

Comparing with (1.12), we see that it takes a factorized form with a more complicated prefactor which is however still a sum of monomials, times plethystic of a three terms combination. This should be the single particle index for fluctuations of the worldsheet attached to the four Wilson lines and ending on the giant graviton.

Our methods may provide exact predictions for many Schur line defect correlators to be hopefully compared with the analysis of explicit string fluctuations.

Plan of the paper. The plan of the paper is the following. In section 2, we discuss the large N limit of Schur line defect correlators. In section 3, we present our main results. In particular, we derive the exact result (1.14) for the Schur defect 2-point function with two Wilson lines in the fundamental and anti-fundamental. The derivation includes the case of a pair of oppositely charged Wilson lines. In section 4 we obtain the single giant graviton correction to the 4-point function with two fundamental and two anti-fundamental lines. Section 5 discusses the relation between our closed formulas and their interpretation in terms of string fluctuations. The case of general charge assignments with a vanishing large N limit, but admitting a non-trivial single giant graviton correction, is presented in appendix B.

2 Schur line defect correlators and the large N limit

In this section, we begin by discussing the $N \to \infty$ limit of Schur line defect correlators. Let us start from the multi-coupling unitary matrix integral [51] with $\boldsymbol{g} = (g_1, g_2, \dots)$

$$Z_N(\boldsymbol{g}) = \int_{\mathrm{U}(N)} dU \, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} g_n \operatorname{Tr} U^n \operatorname{Tr} U^{-n}\right). \tag{2.1}$$

The Schur index is obtained by specialization

$$\mathrm{I}^{\mathrm{U}(N)}(\eta; q) = Z_N(\boldsymbol{g}), \qquad g_n = f(\eta^n; q^n)\,. \tag{2.2}$$

An important result of [51] is the following large N limit

$$Z_\infty(\boldsymbol{g}) = \prod_{n=1}^{\infty} \frac{1}{1-g_n}\,. \tag{2.3}$$

From this result, we can obtain the large N limit of correlators with any number of pairs of oppositely charged Wilson lines. For instance, we have

$$Z_N^{\mathrm{F}}(\boldsymbol{g}) = \frac{\partial}{\partial g_1} Z_N(\boldsymbol{g})\,, \qquad \to \qquad Z_\infty^{\mathrm{F}}(\boldsymbol{g}) = \frac{1}{1-g_1} \prod_{n=1}^{\infty} \frac{1}{1-g_n}\,, \tag{2.4}$$

which agrees with (1.9). Its string derivation from fluctuations of a fundamental string along $AdS_2 \subset AdS_5$ was given in [39].

Similar expressions can be obtained in more general cases by repeated differentiation. The fact that other charge assignments have vanishing large N limit can be proved by group representation theory. An alternative direct derivation is possible by the methods of [51]. To this aim, let us consider insertions of multiple Wilson lines with arbitrary charges and the Schur line defect correlator

$$\mathrm{I}_{\boldsymbol{Q}}^{\mathrm{U}(N)}(\eta; q) = \oint_{|\boldsymbol{z}|=1} D^N \boldsymbol{z} \, \prod_{n=1}^{\infty} \chi_\square(\boldsymbol{z}^{q_n}) \;\; \mathrm{PE}[f(\eta; q) \chi_\square(z) \chi_\square(z^{-1})]\,, \tag{2.5}$$

$$\boldsymbol{Q} = (q_1, q_2, \dots), \qquad \sum_n q_n = 0\,. \tag{2.6}$$

Following [51], we introduce the generating functional

$$\widetilde{Z}_N(\boldsymbol{t}^+, \boldsymbol{t}^-) = \int_{\mathrm{U}(N)} dU \, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (t_n^+ \operatorname{Tr} U^n + t_n^- \operatorname{Tr} U^{-n})\right). \tag{2.7}$$

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For a function $f(\boldsymbol{t}^+,\boldsymbol{t}^-)$, we define

$$\langle f\rangle_{\boldsymbol{g}} = \prod_{n=1}^{\infty}\int \frac{dt_n^+dt_n^-}{2\pi\, n g_n} e^{-\frac{1}{n g_n}t_n^+t_n^-}\, f(\boldsymbol{t}^+,\boldsymbol{t}^-)\,, \qquad \int \frac{dt^+dt^-}{2\pi g} e^{-\frac{1}{g}t^+t^-}(t^+)^{n_+}(t^-)^{n_-} = n!\, g^n\, \delta_{n^+,n^-}\,. \tag{2.8}$$

The relation between $Z_N(\boldsymbol{g})$ and $\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-)$ is

$$Z_N(\boldsymbol{g}) = \langle \widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-)\rangle_{\boldsymbol{g}}\,. \tag{2.9}$$

The $N\to\infty$ limit of the generating functional $\widetilde{Z}_N$ is

$$\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}t_n^+t_n^-\right). \tag{2.10}$$

For a set of charges $\boldsymbol{Q} = (q_1,q_2,\dots;-q_1',-q_2',\dots)$, $q_i,q_j'>0$, we have

$$\widetilde{Z}_\infty^{\boldsymbol{Q}}(\boldsymbol{t}^+,\boldsymbol{t}^-) = D_{\boldsymbol{Q}}\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}t_n^+t_n^-\right), \qquad D_{\boldsymbol{Q}} = \prod_{i,j} q_i q_j'\partial_{t_{q_i}^+}\partial_{t_{q_j'}^-}\,. \tag{2.11}$$

Thus,

$$Z_\infty^{\boldsymbol{Q}}(\boldsymbol{g}) = \langle D_{\boldsymbol{Q}}\widetilde{Z}_\infty^{\boldsymbol{Q}}(\boldsymbol{t}^+,\boldsymbol{t}^-)\rangle_g = \int\prod_{k=1}\frac{dt_k^+dt_k^-}{2\pi k g_k}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k g_k}t_k^+t_k^-\right) D_{\boldsymbol{Q}}\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}t_k^+t_k^-\right). \tag{2.12}$$

If $\boldsymbol{Q}$ is not made of pairs of opposite charges, i.e. is not symmetric under a change of sign of all charges, we cannot end up with contributions with the same number of t_k^+ and t_-^k and we get zero due to (2.8). This, together with the previous discussion of the opposite charge cases, proves the conjectures in section 5.1.1 of [43].

We remark that for general charges $\boldsymbol{Q}$ not of the form $(q_1,q_2,\dots;-q_1,-q_2,\dots)$, the fact that $Z_\infty^{\boldsymbol{Q}}(\boldsymbol{g})=0$ means that the expression of $Z_N^{\boldsymbol{Q}}(\boldsymbol{g})$ starts with a term which is q^N times a non-trivial function of η and q that may be computed as discussed in appendix B.

3 Leading giant graviton correction to the $(\square,\overline{\square})$ 2-point function

Let us now move to the finite N corrections to the 2-point function in the fundamental, i.e. $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)$. We begin by recalling what is known in the case of the undecorated Schur index. Its leading giant graviton expansion was derived in closed form in [37]. It reads (see appendix A for our conventions)[5]

$$\frac{\mathrm{I}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}^{\mathrm{U}(\infty)}(\eta;q)} = 1-\left[\eta^{N+2}\frac{\left(\frac{q}{\eta}\right)_\infty^3}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)}+\eta^{-N-2}\frac{(\eta\, q)_\infty^3}{\vartheta(\eta^{-2},\eta\, q)}\right]q^{N+1}+\mathcal{O}(q^{2N})\,. \tag{3.1}$$

The first terms of its expansion in small q are

$$\frac{\mathrm{I}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}^{\mathrm{U}(\infty)}(\eta;q)} = 1+\left[\frac{\eta}{1-\eta^2}(\eta^{N+2}-\eta^{-N-2})+\frac{1}{\eta}(1-\eta^2)(\eta^{N+1}-\eta^{-N-1})\,q+\cdots\right]q^{N+1}+\mathcal{O}(q^{2N})\,. \tag{3.2}$$

[5] Here and in the following, we will denote by $\mathcal{O}(q^{2N})$ the double giant graviton contribution. Strictly speaking, its q expansion starts at $q^{2N+\delta}$ for some integer δ. We will not discuss it and just split out all contributions with an explicit q^{2N} factor.

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3.1 Finite N analysis of the holonomy matrix integrals

Let us examine the structure of the N dependence of $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)$ by computing explicitly the associated matrix integrals at finite N. We computed explicit series expansions at order q^{N+1} for various N and results are collected in appendix C. From the pattern guessed in (C.12), we have

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(\eta;q)} = 1 + \left[\frac{\eta}{1-\eta^2}(\eta^{N+1}-\eta^{-N-1}) + \frac{1-\eta^2+\eta^4}{\eta(1-\eta^2)}(\eta^N-\eta^{-N})\,q+\cdots\right] q^N + \mathcal{O}(q^{2N})\,. \tag{3.3}$$

This can be written

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(\eta;q)} = 1 + \Big(G_{\mathrm{F}}^{+}(\eta;q)\,\eta^N + G_{\mathrm{F}}^{-}(\eta;q)\,\eta^{-N}\Big)\,q^N + \mathcal{O}(q^{2N})\,, \tag{3.4}$$

with

$$\begin{aligned} G_{\mathrm{F}}^{+}(\eta;q) &= \frac{\eta^2}{1-\eta^2} + \frac{1-\eta^2+\eta^4}{\eta(1-\eta^2)}\,q + \frac{1-\eta^4-2\eta^6+\eta^8}{\eta^4(1-\eta^2)}\,q^2 + \mathcal{O}(q^3), \\ G_{\mathrm{F}}^{-}(\eta;q) &= G_{\mathrm{F}}^{+}(\eta^{-1};q)\,. \end{aligned} \tag{3.5}$$

Comparing (3.3) with (3.2), we notice that the insertion of the adjoint character in the Schur index has the effect of giving a first giant graviton correction that starts at order $\mathcal{O}(q^N)$ instead of $\mathcal{O}(q^{N+1})$.

In the next section, we will compute the first term in (3.5) which turns out to a simple calculation. This will confirm the pattern conjectured in (C.12) and leading to (3.4), (3.5). A full calculation of the function $G_{\mathrm{F}}^{+}(\eta;q)$ will be presented later.

As a remark, in the unflavored limit $\eta\to 1$ limit, we get from (3.2)

$$\frac{\mathrm{I}^{\mathrm{U}(N)}(1;q)}{\mathrm{I}^{\mathrm{U}(\infty)}(1;q)} = 1-(N+2)\,q^{N+1} + \mathcal{O}(q^{2N})\,, \tag{3.6}$$

in agreement with the exact results in [18]. For the Schur line defect, the unflavored limit can also be read from the above expressions and takes the form

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(1;q)}{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(1;q)} = 1 - \Big[N+1+N\,q-(N+3)\,q^2+\cdots\Big]\,q^N + \mathcal{O}(q^{2N})\,. \tag{3.7}$$

We will see that this result is actually exact, i.e. there are no higher order corrections in the square bracket beyond the shown three terms. The explicit factors of N in (3.7) are somehow expected as a general feature of unrefined indices with algebraic constraints on fugacities. This was discussed as a wall-crossing effect in [11, 12, 52]. On gravity side, these factors come from zero modes of wrapped branes fluctuations [32, 33, 53].

3.2 Determination of the q^N correction from Young tableaux expansion

The result

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(\eta;q)} = 1 + \left[\frac{\eta}{1-\eta^2}(\eta^{N+1}-\eta^{-N-1}) + \mathcal{O}(q)\right] q^N + \mathcal{O}(q^{2N})\,, \tag{3.8}$$

may be obtained in a straightforward way by Young tableaux expansion methods [54]. We illustrate this approach because of its simplicity and general applicability. It may compute systematically the corrections in (3.8), but we will not delve into this extension, since we will later resum the full set of contributions by a different method.

Introducing holonomies $\boldsymbol{z} = (z_1, \dots z_n)$, we have

$$Z_N(\boldsymbol{g}) = \oint_{|\boldsymbol{z}|=1} D^N\boldsymbol{z}\ \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}g_n\chi_{\square}(\boldsymbol{z}^n)\chi_{\square}(\boldsymbol{z}^{-n})\right). \tag{3.9}$$

For two symmetric functions $a(\boldsymbol{z}), b(\boldsymbol{z})$, following [54], we define the product

$$\langle a, b\rangle_N = \oint_{|\boldsymbol{z}|=1} D^N\boldsymbol{z}\ a(\boldsymbol{z})\, b(\boldsymbol{z}^{-1}). \tag{3.10}$$

A partition λ can be represented as $(\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots$ or in frequency representation $1^{r_1}2^{r_2}\cdots$. The number of parts of λ is $\ell(\lambda) = \sum_n r_n$ (the number of non-zero λ_i). It is the number of rows in the associated Young tableau. The weight of the partition λ is $|\lambda| = \sum_n \lambda_n = \sum_n n r_n$, the number of blocks in the Young tableau. The known relation between plethystic and Young tableaux gives the expansion

$$Z_N(\boldsymbol{g}) = \sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{\boldsymbol{g}^{\lambda}}{\rho_\lambda}\,\langle\chi_{\square}^{\lambda}(\boldsymbol{z}), \chi_{\square}^{\lambda}(\boldsymbol{z})\rangle_N\,, \tag{3.11}$$

where

$$\boldsymbol{g}^{\lambda} = \prod_{n=1}^{\infty} g_{\lambda_n} = \prod_{n=1}^{\infty} g_n^{r_n}, \qquad \rho_\lambda = \prod_{n=1}^{\infty} r_n!\, n^{r_n}, \qquad \phi^{\lambda}(\boldsymbol{z}) = \prod_{n=1}^{\infty}\phi(\boldsymbol{z}^{\lambda_n}). \tag{3.12}$$

Irreducible representations of the symmetric group S_N are also labeled by a Young tableau. If $\sigma \in S_N$ and $X^{\lambda}(\sigma)$ is the associated matrix, we define $\chi^{\lambda}(\sigma) = \operatorname{Tr} X^{\lambda}(\sigma)$. Conjugacy classes in S_N are also labeled by a Young tableau, and corresponds to the cycle structure of a class representative. Let they be K_μ. We define $\widehat{\chi}^{\lambda}_{\mu} = \chi^{\lambda}(\sigma)$ with $\sigma \in K_\mu$. These object can be computed by the Murnaghan-Nakayama rule, see for instance [55]. One has the two completeness relations

$$\begin{gathered}\frac{1}{N!}\sum_{\sigma\in S_N}\chi^{\lambda}(\sigma)\chi^{\mu}(\sigma) = \sum_{|\nu|=N}\frac{1}{\rho_\nu}\widehat{\chi}^{\lambda}_{\nu}\widehat{\chi}^{\mu}_{\nu} = \delta_{\lambda\mu},\\ \sum_{|\nu|=N}\widehat{\chi}^{\nu}_{\lambda}\widehat{\chi}^{\nu}_{\mu} = \rho_\lambda\,\delta_{\lambda\mu}\,,\end{gathered} \tag{3.13}$$

and the important formula [55]

$$\langle\chi^{\lambda}_{\square}, \chi^{\mu}_{\square}\rangle_N = \delta_{|\lambda|,|\mu|}\sum_{\substack{|\nu|=|\lambda|\\ \ell(\nu)\leq N}}\widehat{\chi}^{\nu}_{\lambda}\widehat{\chi}^{\nu}_{\mu}\,. \tag{3.14}$$

Hence, (3.11) can be written as

$$Z_N(\boldsymbol{g}) = \sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{\boldsymbol{g}^{\lambda}}{\rho_\lambda}\sum_{\substack{|\nu|=d\\ \ell(\nu)\leq N}}(\widehat{\chi}^{\nu}_{\lambda})^2\,. \tag{3.15}$$

The Schur line defect 2-point function in the fundamental $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)$ is obtained from an object similar to (3.11), i.e.

$$Z_N^{\mathrm{F}}(\boldsymbol{g})=\sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{\boldsymbol{g}^{\lambda}}{\rho_\lambda}\,\langle\chi_\square(\boldsymbol{z})\chi_\square^{\lambda}(\boldsymbol{z}),\chi_\square(\boldsymbol{z}^{-1})\chi_\square^{\lambda}(\boldsymbol{z})\rangle_N\,. \tag{3.16}$$

Let us denote by λ' the Young Tableaux λ with the addition of one block at the bottom, so with one more row of length 1. Obviously

$$\chi_\square(\boldsymbol{z})\chi_\square^{\lambda}(\boldsymbol{z})=\chi_\square^{\lambda'}(\boldsymbol{z}),\qquad \rho_{\lambda'}=(r_1+1)\,\rho_\lambda\,, \tag{3.17}$$

and thus

$$Z_N^{\mathrm{F}}(\boldsymbol{g})=\sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{r_1+1}{\rho'_\lambda}\boldsymbol{g}^{\lambda}\sum_{\substack{|\nu|=d+1\\ \ell(\nu)\le N}}(\widehat{\chi}_{\lambda'}^{\nu})^2\,. \tag{3.18}$$

The large N limits discussed previously are clearly reproduced by these expressions. For $Z_N(\boldsymbol{g})$ one has indeed [51]

$$Z_\infty(\boldsymbol{g})=\sum_\lambda\boldsymbol{g}^\lambda=\prod_{n=1}^{\infty}\sum_{r_n=0}^{\infty}g_n^{r_n}=\prod_{n=1}^{\infty}\frac{1}{1-g_n}\,, \tag{3.19}$$

and this is easily generalized to the 2-point function of the fundamental representation (or other cases)

$$Z_\infty^{\mathrm{F}}(\boldsymbol{g})=\sum_\lambda(r_1+1)\boldsymbol{g}^\lambda=\sum_{r_1=0}^{\infty}(r_1+1)g_1^{r_1}\times\prod_{n=2}^{\infty}\sum_{r_n=0}^{\infty}g_n^{r_n}=\frac{1}{1-g_1}\prod_{n=1}^{\infty}\frac{1}{1-g_n}\,. \tag{3.20}$$

To go beyond the large N limit, we start from the representation (3.18) for the index, cf. (2.2),

$$\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}(\eta;q)=\sum_{d=0}^{\infty}\sum_{|\lambda|=d}\frac{r_1+1}{\rho'_\lambda}f^{\lambda}(\eta;q)\sum_{\substack{|\nu|=d+1\\ \ell(\nu)\le N}}(\widehat{\chi}_{\lambda'}^{\nu})^2, \tag{3.21}$$

with, cf. (3.12),

$$f^{\lambda}(\eta;q)=\prod_{n=1}^{\infty}f(\eta^{\lambda_n};q^{\lambda_n})\,. \tag{3.22}$$

The first correction with respect to the $N\to\infty$ limit (3.20) is due to Young tableaux with $|\lambda|=N$. Recall also that $f^{\lambda}(\eta;q)=\mathcal{O}(q^{|\lambda|})$. We have thus

$$\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}-\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}=\sum_{d=N}^{\infty}\sum_{|\lambda|=d}\frac{r_1+1}{\rho_{\lambda'}}f_\lambda(\eta;q)\Bigg[\sum_{\substack{|\nu|=d+1\\ \ell(\nu)\le N}}(\widehat{\chi}_{\lambda'}^{\nu})^2-\sum_{|\nu|=d+1}(\widehat{\chi}_{\lambda'}^{\nu})^2\Bigg]=\sum_{d=N}^{\infty}J_d, \tag{3.23}$$

with

$$J_d=-\sum_{|\lambda|=d}\frac{r_1+1}{\rho_{\lambda'}}f_\lambda(\eta;q)\sum_{\substack{|\nu|=d+1\\ \ell(\nu)\ge N+1}}(\widehat{\chi}_{\lambda'}^{\nu})^2. \tag{3.24}$$

The leading term has $d = N$ and therefore corresponds to the unique partition $\nu = (1, \dots, 1) = (1^{N+1})$ for which $(\widehat{\chi}^{\nu}_{\lambda'})^2 = 1$ [54]. Thus

$$J_N = -\sum_{|\lambda|=N} \frac{1}{\rho_\lambda} f_\lambda(\eta; q) = -\,\mathrm{PE}[\varepsilon f(\eta; q)]\big|_{\varepsilon^N}, \qquad \mathrm{PE}[\varepsilon f(\eta; q)] = \exp\left[\sum_{n=1}^{\infty} \frac{\varepsilon^n}{n} f(\eta^n; q^n)\right]. \tag{3.25}$$

In our specific case, we have

$$f(\eta; q) = (\eta + \eta^{-1})\, q + \mathcal{O}(q^2)\,, \tag{3.26}$$

and therefore

$$\frac{\mathrm{I}^{\mathrm{U}(N)}(\eta; q)}{\mathrm{I}^{\mathrm{U}(\infty)}(\eta; q)} = 1 - \mathrm{PE}[\varepsilon(\eta + \eta^{-1})]\big|_{\varepsilon^N}\, q^N + \mathcal{O}(q^{N+1}). \tag{3.27}$$

The coefficient ε^N in the plethystic is easily computed from

$$-\,\mathrm{PE}[\varepsilon(\eta + \eta^{-1})]\big|_{\varepsilon^N} = -\oint_{|z|=r} \frac{dz}{2\pi i z^{N+1}} \frac{1}{1 - \varepsilon\eta} \frac{1}{1 - \varepsilon\eta^{-1}} = \frac{\eta}{1 - \eta^2}(\eta^{N+1} - \eta^{-N-1}), \tag{3.28}$$

where the circle radius r is taken small enough to exclude the poles $\varepsilon = \eta^{\pm 1}$ and the integral is then done by picking up residues. The above is in agreement with the finite N analysis, cf. (3.8).

3.3 Exact single giant graviton correction $G^{+}_{\mathrm{F}}(\eta; q)$

Let us now compute the exact sum of all missing contributions in (3.8), i.e. the exact function $G^{+}_{\mathrm{F}}(\eta; q)$ in (3.4) whose first terms in the small q expansion were given in (3.5). We consider again the multi-coupling matrix integral $Z_N(\boldsymbol{g})$ introduced in (2.1). The leading single giant graviton correction has been worked out in general in [51] and reads

$$\frac{Z_N(\boldsymbol{g})}{Z_\infty(\boldsymbol{g})} = 1 + G_N(\boldsymbol{g}) + \cdots\,, \tag{3.29}$$

with

$$G_N(\boldsymbol{g}) = -\frac{\zeta}{(1-\zeta)^2} \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \frac{g_n}{1 - g_n}(1 - \zeta^n)(1 - \zeta^{-n})\right]\Bigg|_{\zeta^{-N}}. \tag{3.30}$$

This formula is the first contribution from a determinantal expansion and is expected to be valid up to terms where two gravitons contributions are important, i.e. generically $\sim q^{2N}$, see footnote 5. The full higher order giant graviton expansion was discussed in [51] and interpreted as an instanton expansion in [56]. The two giant graviton contribution was worked out in [57]. These works examined the unitary matrix integral representation of the index. A discussion of the comparison with wrapped brane expansion beyond the single giant graviton contribution was presented in [58].

For the Schur line defect 2-point function in fundamental representation, we need

$$Z^{\mathrm{F}}_N(\boldsymbol{g}) = \int_{\mathrm{U}(N)} dU\ \mathrm{Tr}\, U\, \mathrm{Tr}\, U^{-1} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} g_n\, \mathrm{Tr}\, U^n\, \mathrm{Tr}\, U^{-n}\right), \tag{3.31}$$

and we recall relations (2.4). The giant graviton expansion of $Z_N^{\rm F}(\boldsymbol{g})$ is then given by

$$\frac{Z_N^{\rm F}(\boldsymbol{g})}{Z_\infty^{\rm F}(\boldsymbol{g})} = \frac{\partial_{g_1} Z_N(\boldsymbol{g})}{\frac{1}{1-g_1} Z_\infty(\boldsymbol{g})} = (1-g_1)\frac{1}{Z_\infty(\boldsymbol{g})}\partial_{g_1}[Z_\infty(\boldsymbol{g})(1+G_N(\boldsymbol{g})+\cdots)]$$
$$= (1-g_1)\partial_{g_1}\log Z_\infty(\boldsymbol{g})\,(1+G_N(\boldsymbol{g})+\cdots)+(1-g_1)\partial_{g_1}G_N(\boldsymbol{g})+\cdots$$
$$= 1+G_N(\boldsymbol{g})+(1-g_1)\partial_{g_1}G_N(\boldsymbol{g})+\cdots\,. \tag{3.32}$$

Thus, we may write

$$\frac{Z_N^{\rm F}(\boldsymbol{g})}{Z_\infty^{\rm F}(\boldsymbol{g})} = 1+G_N^{\rm F}(\boldsymbol{g})+\cdots\,, \qquad G_N^{\rm F}(\boldsymbol{g}) = G_N(\boldsymbol{g}) - \frac{1}{1-g_1}\,E_N(\boldsymbol{g})\,, \tag{3.33}$$

where, using (3.30), we find

$$E_N(\boldsymbol{g}) = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\frac{g_n}{1-g_n}(1-\zeta^n)(1-\zeta^{-n})\right]\bigg|_{\zeta^{-N}}. \tag{3.34}$$

This is the same plethystic as in (3.30) up to the missing prefactor $-\zeta/(1-\zeta)^2$ and thus can be evaluated as a finite difference of $G_N(\boldsymbol{g})$ functions. To streamline notation, let us denote

$$H(\boldsymbol{g},\zeta) = \frac{\boldsymbol{g}}{1-\boldsymbol{g}}(1-\zeta)(1-\zeta^{-1}), \qquad G(\boldsymbol{g},\zeta) = {\rm PE}[-H(\boldsymbol{g},\zeta)], \qquad G_N(\boldsymbol{g}) = G(\boldsymbol{g},\zeta)|_{\zeta^{-N}}\,. \tag{3.35}$$

We have simply

$$E_N(\boldsymbol{g}) = \oint \frac{d\zeta}{\zeta^{N+1}}\,{\rm PE}[-H(\boldsymbol{g},\zeta)]$$
$$= -\oint \frac{d\zeta}{\zeta^{N+1}}\frac{(1-\zeta)^2}{\zeta}\,G(\boldsymbol{g},\zeta)$$
$$= -G_{N+1}(\boldsymbol{g})+2G_N(\boldsymbol{g})-G_{N-1}(\boldsymbol{g})\,, \tag{3.36}$$

which is valid for $N\geq 2$. This means that we have obtained the single giant graviton correction to ${\rm I}_{\rm F}^{{\rm U}(N)}(\eta;q)$ in the form

$$\frac{{\rm I}_{\rm F}^{{\rm U}(N)}(\eta;q)}{{\rm I}_{\rm F}^{{\rm U}(\infty)}(\eta;q)} = 1+G_N^{\rm F}(\eta;q)+\cdots\,, \tag{3.37}$$

with the following finite-difference expression

$$G_N^{\rm F}(\eta;q) = G_N(\eta;q)+\frac{1}{1-f(\eta;q)}\,[G_{N+1}(\eta;q)-2G_N(\eta;q)+G_{N-1}(\eta;q)]\,. \tag{3.38}$$

From (3.1), the exact expression of $G_N(\eta;q)$ is

$$G_N(\eta;q) = -q^{N+1}\left[\eta^{N+2}\frac{\left(\frac{q}{\eta}\right)_\infty^3}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)}+\eta^{-N-2}\frac{(\eta\,q)_\infty^3}{\vartheta(\eta^{-2},\eta\,q)}\right], \tag{3.39}$$

and this gives

$$G^{\mathrm{F}}_N(\eta;q) = \\ -q^{N+1}\left[\eta^{N+2}\left(1+\frac{1-q^2}{\eta\,q}\frac{1-\eta\,q}{1-\eta^{-1}q}\right)\frac{\left(\frac{q}{\eta}\right)^3_\infty}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)}+\eta^{-N-2}\left(1+\frac{1-q^2}{\eta^{-1}q}\frac{1-\eta^{-1}q}{1-\eta\,q}\right)\frac{(\eta\,q)^3_\infty}{\vartheta(\eta^{-2},\eta\,q)}\right]. \tag{3.40}$$

Comparing with (3.4), the exact expression of the function G^+_{F} is thus

$$G^+_{\mathrm{F}}(\eta;q) = -\eta^2\,q\left(1+\frac{1-q^2}{\eta\,q}\,\frac{1-\eta\,q}{1-\eta^{-1}\,q}\right)\frac{\left(\frac{q}{\eta}\right)^3_\infty}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)}. \tag{3.41}$$

As a check, one can expand it in small q and reproduce (3.5). In the unflavored limit we have, cf. (3.6),

$$G_N(1;q) = -(N+2)\,q^{N+1}. \tag{3.42}$$

Using (3.38) gives then

$$G^{\mathrm{F}}_N(1;q) = -\Big[N+1+N\,q-(N+3)\,q^2\Big]\,q^N\,, \tag{3.43}$$

which is exact and shows that (3.7) has actually three terms without further corrections. As a final comment, it is clear that the factorization in (3.41) is a simple consequence of the finite-difference structure in (3.38).

3.4 Schur line defect correlators of charged Wilson lines

Generalization to the case of insertion of a pair of Wilson lines with opposite charges $\pm Q$ is straightforward by the same method. Now, we differentiate with respect to g_Q and start from

$$Z^Q_N(\boldsymbol{g}) = Q\frac{\partial}{\partial g_Q}Z_N(\boldsymbol{g})\,. \tag{3.44}$$

For $N=\infty$, we get[6]

$$Z^Q_\infty(\boldsymbol{g}) = Q\frac{\partial}{\partial g_Q}\prod_{n=1}^\infty\frac{1}{1-g_n} = \frac{Q}{1-g_Q}\prod_{n=1}^\infty\frac{1}{1-g_n}\,. \tag{3.45}$$

The single giant graviton correction is given by

$$\begin{aligned}\frac{Z^Q_N(\boldsymbol{g})}{Z^Q_\infty(\boldsymbol{g})} &= \frac{Q\partial_{g_Q}Z_N(\boldsymbol{g})}{\frac{Q}{1-g_Q}Z_\infty(\boldsymbol{g})} = (1-g_Q)\frac{1}{Z_\infty(\boldsymbol{g})}\partial_{g_Q}[Z_\infty(\boldsymbol{g})(1+G_N(\boldsymbol{g})+\cdots)] \\ &= 1+G_N(\boldsymbol{g})+(1-g_Q)\partial_{g_Q}G_N(\boldsymbol{g})+\cdots\,.\end{aligned} \tag{3.46}$$

[6]Our formalism provides immediately the large N limit of Schur defect line correlators with general insertions proving easily the conjectures in section 5.1.1 of [43].

We now observe that

$$\begin{aligned}(1-g_Q)\partial_{g_Q}G_N(\boldsymbol{g}) &= -\frac{1}{Q}\frac{1}{1-g_Q}\frac{\zeta^{1-Q}(1-\zeta^Q)^2}{(1-\zeta)^2}\,\mathrm{PE}[-H(\boldsymbol{g},\zeta)] \\ &= \frac{1}{Q}\frac{1}{1-g_Q}\zeta^{-Q}(1-\zeta^Q)^2\left[-\frac{\zeta}{(1-\zeta)^2}\,\mathrm{PE}[-H(\boldsymbol{g},\zeta)]\right].\end{aligned} \tag{3.47}$$

By the same manipulations as in the $Q=1$ case, it follows

$$(1-g_Q)\partial_{g_Q}G_N(\boldsymbol{g}) = \frac{1}{Q}\frac{1}{1-g_Q}[G_{N+Q}(\boldsymbol{g})-2G_N(\boldsymbol{g})+G_{N-Q}(\boldsymbol{g})]\,, \tag{3.48}$$

and thus

$$\frac{\mathrm{I}_Q^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}_Q^{\mathrm{U}(\infty)}(\eta;q)} = 1+G_N^Q(\eta;q)+\cdots\,, \tag{3.49}$$

with

$$G_N^Q(\eta;q) = G_N(\eta;q)+\frac{1}{Q}\frac{1}{1-f(\eta^Q;q^Q)}\,[G_{N+Q}(\eta;q)-2G_N(\eta;q)+G_{N-Q}(\eta;q)]\,. \tag{3.50}$$

Using again (3.39), we obtain

$$\begin{aligned}G_N^Q(\eta;q) = -q^{N+1}\Bigg[&\eta^{N+2}\left(1+\frac{1}{Q}\frac{1-q^{2Q}}{\eta^Q\,q^Q}\frac{1-\eta^Q\,q^Q}{1-\eta^{-Q}\,q^Q}\right)\frac{\left(\frac{q}{\eta}\right)^3_\infty}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)} \\ &+\eta^{-N-2}\left(1+\frac{1}{Q}\frac{1-q^{2Q}}{\eta^{-Q}\,q^Q}\frac{1-\eta^{-Q}\,q^Q}{1-\eta^Q\,q^Q}\right)\frac{(\eta\,q)^3_\infty}{\vartheta(\eta^{-2},\eta\,q)}\Bigg].\end{aligned} \tag{3.51}$$

The first correction appears now at order q^{N+1-Q}. In the unflavored limit $\eta\to1$ it is exactly

$$G_N^Q(1;q) = -q^{N+1-Q}\left[\frac{N+2}{Q}-1+N\,q^Q-\left(\frac{N+2}{Q}+1\right)q^{2Q}\right]. \tag{3.52}$$

4 The $(\square,\square,\overline{\square},\overline{\square})$ 4-point function

The previous methods can be clearly applied to more complicated cases. In particular we can consider the 4-point function with insertion of two lines in the fundamental and two in the anti-fundamental. In this case, we need

$$Z_N^{\mathrm{FF}}(\boldsymbol{g}) = \int_{\mathrm{U}(N)} dU\,(\mathrm{Tr}\,U\,\mathrm{Tr}\,U^{-1})^2\exp\left(\sum_{n=1}^\infty\frac{1}{n}g_n\,\mathrm{Tr}\,U^n\,\mathrm{Tr}\,U^{-n}\right) = \frac{\partial^2}{\partial g_1^2}Z_N(\boldsymbol{g})\,. \tag{4.1}$$

The large N limit is

$$Z_\infty^{\mathrm{FF}}(\boldsymbol{g}) = \frac{2}{(1-g_1)^2}\prod_{n=1}^\infty\frac{1}{1-g_n}\,. \tag{4.2}$$

The single giant graviton correction is computed from

$$\begin{aligned}\frac{Z_N^{\rm FF}(\boldsymbol{g})}{Z_\infty^{\rm FF}(\boldsymbol{g})} &= \frac{\partial_{g_1}^2 Z_N(\boldsymbol{g})}{\frac{2}{(1-g_1)^2}Z_\infty(\boldsymbol{g})} = \frac{1}{2}(1-g_1)^2\frac{1}{Z_\infty(\boldsymbol{g})}\partial_{g_1}^2[Z_\infty(\boldsymbol{g})(1+G_N(\boldsymbol{g})+\cdots)]\\ &= 1+G_N(\boldsymbol{g})+(1-g_1)\partial_{g_1}G_N(\boldsymbol{g})+\frac{1}{2}(1-g_1)^2\partial_{g_1}^2 G_N(\boldsymbol{g})+\cdots\,. \end{aligned} \tag{4.3}$$

This gives

$$\frac{Z_N^{\rm FF}(\boldsymbol{g})}{Z_\infty^{\rm FF}(\boldsymbol{g})} = 1+G_N^{\rm FF}(\boldsymbol{g})+\cdots\,, \tag{4.4}$$

with

$$\begin{aligned} G_N^{\rm FF}(\boldsymbol{g}) &= G_N(\boldsymbol{g}) - \left[\frac{1-2g_1}{(1-g_1)^2}+\frac{1}{(1-g_1)^2}\frac{1+\zeta^2}{2\zeta}\right]{\rm PE}[-H(\boldsymbol{g},\zeta)]\\ &= G_N(\boldsymbol{g}) + \left[\frac{(1-\zeta)^2}{\zeta}\frac{1-2g_1}{(1-g_1)^2}+\frac{1}{(1-g_1)^2}\frac{(1+\zeta^2)(1-\zeta)^2}{2\zeta^2}\right]\frac{-\zeta}{(1-\zeta)^2}{\rm PE}[-H(\boldsymbol{g},\zeta)]\,. \end{aligned} \tag{4.5}$$

For the index, this implies

$$\frac{{\rm I}_{\rm FF}^{{\rm U}(N)}(\eta;q)}{{\rm I}_{\rm FF}^{{\rm U}(\infty)}(\eta;q)} = 1+G_N^{\rm FF}(\eta;q)+\cdots = 1+[\eta^N G_{\rm FF}^{+}(\eta;q)+\eta^{-N}G_{\rm FF}^{-}(\eta;q)]q^N+\cdots \tag{4.6}$$

with the following exact finite-difference expression ($f\equiv f(\eta,q), G_N\equiv G_N(\eta;q)$)

$$G_N^{\rm FF}(\eta;q) = \frac{f\,(2+f)}{(1-f)^2}G_N+\frac{1}{2(1-f)^2}(G_{N+2}+G_{N-2})-\frac{2f}{(1-f)^2}(G_{N+1}+G_{N-1})\,. \tag{4.7}$$

Using (3.39), we get

$$G_{\rm FF}^{+}(\eta;q) = -\eta^2\,q\left[1+\frac{1-q^2}{2\eta^2\,q^2}\frac{1-\eta\,q}{(1-\eta^{-1}\,q)^2}(1-5q^2+3\eta q+\eta q^3)\right]\frac{\left(\frac{q}{\eta}\right)_\infty^3}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)}\,, \tag{4.8}$$

with explicit expansion

$$G_{\rm FF}^{+}(\eta;q) = \frac{\eta}{2\,(1-\eta^2)}\frac{1}{q}+\frac{1+3\eta^4}{2\eta^2(1-\eta^2)}+\left(\frac{1}{2\eta^5}+\frac{1}{\eta^3}+\frac{3}{2\eta}-\eta\right)q+\cdots\,. \tag{4.9}$$

This expression reproduces the matrix integral series up to the order q^{2N} where double giant graviton contributions appear. In the unflavored limit $\eta\to 1$, we get the polynomial correction

$$G_N^{\rm FF}(1;q) = -[N+4N\,q-8q^2-4\,(N+2)\,q^3+(N+4)\,q^4]\,q^{N-1}\,. \tag{4.10}$$

Again, the factorized structure of (4.8) is a direct consequence of the finite-difference representation (4.7).

5 Single giant graviton correction from string fluctuations

For the Schur index without insertions, the giant graviton expansion (3.1) may be written in the form (3.4) as

$$\frac{\mathrm{I}^{\mathrm{U}(N)}(\eta;q)}{\mathrm{I}^{\mathrm{U}(\infty)}(\eta;q)} = 1 + \left[\eta^N G^+_{\mathrm{D3}}(\eta;q) + \eta^{-N} G^-_{\mathrm{D3}}(\eta;q)\right] q^N + \mathcal{O}(q^{2N})\,, \tag{5.1}$$

where

$$G^+_{\mathrm{D3}}(\eta;q) = -\eta^2 q\,\frac{\left(\frac{q}{\eta}\right)^3_\infty}{\vartheta\left(\eta^2,\frac{q}{\eta}\right)} = \mathrm{PE}\left[\frac{\frac{1}{\eta q}-\frac{2}{\eta}q+q^2}{1-\frac{q}{\eta}}\right], \tag{5.2}$$

is the single giant graviton contribution from wrapped D3 brane [22, 37].

The 2-point function with representations $(\square,\overline{\square})$. As we mentioned in the introduction, the large N limit of $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(\eta;q)$ may be written in the factorized form

$$\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}(\eta;q) = \mathrm{I}_{\mathrm{F1}}(\eta;q)\;\mathrm{I}^{\mathrm{U}(\infty)}(\eta;q), \tag{5.3}$$

where

$$I_{\mathrm{F1}}(\eta;q) = \frac{1}{1-f(\eta;q)} = \frac{1-q^2}{(1-\eta q)(1-\eta^{-1}q)} = \mathrm{PE}[-q^2+(\eta+\eta^{-1})\,q]\,, \tag{5.4}$$

is the index of fluctuations of a fundamental string along $AdS_2 \subset AdS_5$. Indeed the simple three term argument of the plethystic is the associated single particle index as shown in [39].

As suggested in [50], in the analysis of finite N corrections to $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}$, one should consider two possibilities for the coexistence of the string worldsheet and a single giant graviton. The first case is when it does not end on the giant world-volume of the giant. The second is when the string world-sheet is separated by the giant and the two semi-infinite strings end on it. Subtraction of the contribution from the first case isolates the latter possibility. This leads to consider the ratio

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)} - I_{\mathrm{F1}}\,\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}} = I_{\mathrm{F1}}\left(\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)}}{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}} - \frac{\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}}\right)\,, \tag{5.5}$$

where the difference has been divided by the supergravity contribution in absence of the defect, i.e. we do not divide by $\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)}$ as we did so far. This gives

$$\frac{\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)} - I_{\mathrm{F1}}\,\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}} = 1 + \left(\mathcal{G}^+_{\mathrm{F}}(\eta;q)\,\eta^N + \mathcal{G}^-_{\mathrm{F}}(\eta;q)\,\eta^{-N}\right) q^N + \mathcal{O}(q^{2N})\,, \tag{5.6}$$

with

$$\begin{aligned}\mathcal{G}^+_{\mathrm{F}} = I_{\mathrm{F1}}(G^+_{\mathrm{F}} - G^+_{\mathrm{D3}}) &= \frac{1}{\eta q}\frac{(1-q^2)^2}{(1-\eta^{-1}q)^2}\,\mathrm{PE}\left[\frac{\frac{1}{\eta q}-\frac{2}{\eta}q+q^2}{1-\frac{q}{\eta}}\right]\\ &= \frac{1}{\eta q}\,\mathrm{PE}\left[\frac{\frac{1}{\eta q}-\frac{2}{\eta}q+q^2}{1-\frac{q}{\eta}} + 2\eta^{-1}q - 2q^2\right]\,.\end{aligned} \tag{5.7}$$

In the conventions of [50] we have $q = \sqrt{xy}$ and $\eta = \sqrt{x/y}$. The extra single particle index in (5.7) is thus

$$2\eta^{-1}q - 2q^2 = 2\,(1-x)\,y\,, \tag{5.8}$$

in agreement with the analysis in [50] of the fluctuation modes on a world-sheet along $AdS_2 \subset AdS_5$ studied in [49, 59]. In more details, without defect lines, there are five bosonic scalar fluctuations in the fundamental of SO(5) corresponding to S^5 coordinates. Together with the three scalar fluctuations in the remaining coordinates of AdS_5 plus fermionic states, these are part of a supermultiplet with $8_\mathrm{B} + 8_\mathrm{F}$ states with $\mathrm{SO}(1,2)\times\mathrm{SO}(3)\times\mathrm{SO}(5)$ quantum numbers

$$\mathrm{B}:\ (1,0,\mathbf{5})\oplus(2,1,\mathbf{1}), \qquad \mathrm{F}:\ (\tfrac{3}{2},\tfrac{1}{2},\mathbf{4})\,. \tag{5.9}$$

The defect lines break it to $\mathrm{SO}(1,2)\times\mathrm{SO}(2)\times\mathrm{SO}(3)$ and the five scalars split into ϕ_3 in the triplet of SO(3) plus two scalars φ_i, $i=1,2$ in the singlet. BPS states contributing the index are three, i.e. one component of ϕ_3, one of the two scalars φ_i and one fermionic state. Dirichlet boundary conditions on the giant graviton remove one of the scalars and leave two BPS states corresponding to the two contributions in (5.8).

Our derivation reproduces the peculiar prefactor $1/(\eta q)$ in (5.7), whose origin is at the moment unclear from the point of view of string fluctuations and has been suggested to be related to a back-reaction of the fundamental strings in [50].

When the pair of Wilson lines have charges $Q, -Q$, the expression (5.7) has to be simply changed to

$$\mathcal{G}^+_Q = G^+_{\mathrm{D3}}(\eta;q) \times \frac{1}{(\eta q)^Q}\,\mathrm{PE}\left[2\eta^{-Q}q^Q - 2q^{2Q}\right], \tag{5.10}$$

as follows from (3.51).

The 4-point function with representations $(\square,\square,\overline{\square},\overline{\square})$. It is interesting to examine what we get in the case of the 4-point function with two fundamental lines and two anti-fundamental lines. The large N limit of the index is $2\mathrm{I}^2_{\mathrm{F1}}$, cf. (4.2). So, the natural generalization of (5.5) reads

$$\frac{\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{FF}} - 2I^2_{\mathrm{F1}}\,\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}} = 2I^2_{\mathrm{F1}}\left(\frac{\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{FF}}}{\mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{FF}}} - \frac{\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}}\right), \tag{5.11}$$

and its expansion takes the form

$$\frac{\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{FF}} - 2I^2_{\mathrm{F1}}\,\mathrm{I}^{\mathrm{U}(N)}}{\mathrm{I}^{\mathrm{U}(\infty)}} = 1 + \left(\mathcal{G}^+_{\mathrm{FF}}(\eta;q)\,\eta^N + \mathcal{G}^-_{\mathrm{FF}}(\eta;q)\,\eta^{-N}\right)q^N + \mathcal{O}(q^{2N})\,, \tag{5.12}$$

with

$$\begin{aligned}\mathcal{G}^+_{\mathrm{FF}} = 2I^2_{\mathrm{F1}}(G^+_{\mathrm{FF}} - G^+_{\mathrm{D3}}) &= \frac{1-5q^2+3\eta q+\eta q^3}{\eta^2 q^2}\,\frac{(1-q^2)^3}{(1-\eta^{-1}q)^4(1-\eta q)}\,\mathrm{PE}\left[\frac{\frac{1}{\eta q} - \frac{2}{\eta}q + q^2}{1-\frac{q}{\eta}}\right]\\ &= \frac{1+3\eta q-5q^2+\eta q^3}{\eta^2 q^2}\,\mathrm{PE}\left[\frac{\frac{1}{\eta q} - \frac{2}{\eta}q + q^2}{1-\frac{q}{\eta}} - 3q^2 + 4\eta^{-1}q + \eta q\right].\end{aligned} \tag{5.13}$$

Comparing with (5.7), we have now a more complicated prefactor which however is still a sum of monomials. The extra contribution in the plethystic exponential should come from fluctuations of a world-sheet attached to the four lines and ending on the giant graviton. Here, the geometry is more complicated and additional subtractions could be needed in (5.11) to simplify the result. Still, it seems clear that a better deeper understanding of the prefactor origin is definitely worth.

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A Special functions

We collect in this appendix the definition of special functions appearing in the text.

q-Pochhammer symbol

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-a\,q^k)\,, \qquad (a^{\pm};q)_\infty = (a;q)_\infty(a^{-1};q)_\infty\,, \tag{A.1}$$

$$(q)_\infty \equiv (q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k)\,. \tag{A.2}$$

q-theta function. The q-theta function is defined as

$$\vartheta(x,q) = -x^{-\frac{1}{2}}(q)_\infty(x;q)_\infty(qx^{-1};q)_\infty\,, \tag{A.3}$$

with

$$\vartheta(x;q) = -\vartheta(x^{-1};q), \qquad \vartheta(q^m x;q) = (-1)^m q^{-\frac{m^2}{2}}x^{-m}\vartheta(x;q)\,. \tag{A.4}$$

B Correlators with general insertions and their single giant graviton correction

As we have seen, for general charge assignment of the Wilson lines, the large N limit of the Schur defect correlator has a vanishing large N limit. This happens when the charge set $\boldsymbol{Q}$ is not symmetric under a global change of sign of charges. In this case, $\mathrm{I}^{\mathrm{U}(N)}_{\boldsymbol{Q}}(\eta;q)$ has a small q expansion that starts at order q^N, up to an N independent non-trivial function of the fugacities. The aim of this appendix is to derive this function.

As discussed in [51], using free fermion methods, one computes the following first determinantal correction to $\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-)$ in (2.7)

$$\frac{\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-)}{\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)} = 1 - K_N(\boldsymbol{t}^+,\boldsymbol{t}^-)+\cdots\,, \qquad K_N(\boldsymbol{t}^+,\boldsymbol{t}^-) = \sum_{\substack{N<r\\ r\in\mathbb{Z}+\frac{1}{2}}}\widetilde{K}(r,r;\boldsymbol{t}^+,\boldsymbol{t}^-)\,, \tag{B.1}$$

with

$$\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \widetilde{K}(r,s;\boldsymbol{t}^+,\boldsymbol{t}^-)\, z^r w^{-s} = \frac{J(z;\boldsymbol{t}^+,\boldsymbol{t}^-)}{J(w;\boldsymbol{t}^+,\boldsymbol{t}^-)}\, \frac{\sqrt{zw}}{z-w}, \qquad |w|<|z|,$$

$$J(z;\boldsymbol{t}^+,\boldsymbol{t}^-) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}(t_n^+ z^n - t_n^- z^{-n})\right). \tag{B.2}$$

Including $\boldsymbol{Q}$ charged insertions, we get

$$\begin{aligned} D_{\boldsymbol{Q}}\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-) &= D_{\boldsymbol{Q}}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-) - \widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)K_N(\boldsymbol{t}^+,\boldsymbol{t}^-) + \cdots] \\ &= \widetilde{Z}_\infty^{\boldsymbol{Q}}(\boldsymbol{t}^+,\boldsymbol{t}^-) - D_{\boldsymbol{Q}}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)K_N(\boldsymbol{t}^+,\boldsymbol{t}^-)] + \cdots, \end{aligned} \tag{B.3}$$

where $D_{\boldsymbol{Q}}$ was defined in (2.11). Then, assuming $Z_\infty^{\boldsymbol{Q}}(\boldsymbol{g})=0$, the first term drops and we have

$$Z_N^{\boldsymbol{Q}}(\boldsymbol{g}) = \langle D_{\boldsymbol{Q}}\widetilde{Z}_N(\boldsymbol{t}^+,\boldsymbol{t}^-)\rangle_{\boldsymbol{g}} = -\langle D_{\boldsymbol{Q}}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)\,K_N(\boldsymbol{t}^+,\boldsymbol{t}^-)]\rangle_{\boldsymbol{g}} + \cdots. \tag{B.4}$$

The right hand side can be evaluated by computing the r.h.s. of

$$\begin{aligned} &\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} z^r w^{-s}\langle D_{\boldsymbol{Q}}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)\,\widetilde{K}(r,s;\boldsymbol{t}^+,\boldsymbol{t}^-)]\rangle_{\boldsymbol{g}} \\ &\quad= \frac{\sqrt{zw}}{z-w}\int\prod_{k=1}\left[\frac{dt_k^+dt_k^-}{2\pi k g_k}\exp\left(-\frac{1}{kg_k}t_k^+t_k^-\right)\right] \\ &\qquad\times D_{\boldsymbol{Q}}\prod_{k=1}^{\infty}\exp\frac{1}{k}\Big(t_k^+t_k^- + t_k^+(z^k-w^k) - t_k^-(z^{-k}-w^{-k})\Big), \end{aligned} \tag{B.5}$$

which is a straightforward calculation given the precise charges $\boldsymbol{Q}$. For the purpose of illustration, let us examine as a specific example the charge configuration $\boldsymbol{Q}=(1,1;-2)$ or

$$D_{\boldsymbol{Q}} = 2\partial_{t_1^+}^2\partial_{t_2^-}. \tag{B.6}$$

Integrating by parts gives

$$-2\partial_{t_1^+}^2\partial_{t_2^-}\exp\left(-\sum_k\frac{1}{kg_k}t_k^+t_k^-\right) = \frac{t_2^+(t_1^-)^2}{g_1^2 g_2}\exp\left(-\sum_k\frac{1}{kg_k}t_k^+t_k^-\right). \tag{B.7}$$

Thus,

$$\begin{aligned} &\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} z^r w^{-s}\langle D_{(1,1;-2)}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)\,\widetilde{K}(r,s;\boldsymbol{t}^+,\boldsymbol{t}^-)]\rangle_{\boldsymbol{g}} \\ &= \frac{1}{g_1^2 g_2}\frac{\sqrt{zw}}{z-w}\int\prod_{k=1}\frac{dt_k^+dt_k^-}{2\pi k g_k}\,t_2^+(t_1^-)^2\exp\frac{1}{k}\Big(-\frac{1-g_k}{g_k}t_k^+t_k^- + t_k^+(z^k-w^k) - t_k^-(z^{-k}-w^{-k})\Big). \end{aligned} \tag{B.8}$$

We have

$$\begin{aligned} &\int\prod_{k=1}\frac{dt_k^+dt_k^-}{2\pi k g_k}\;\exp\frac{1}{k}\Big(-\frac{1-g_k}{g_k}t_k^+t_k^- + a_k t_k^+(z^k-w^k) - b_k t_k^-(z^{-k}-w^{-k})\Big) \\ &= \prod_{n=1}^{\infty}\frac{1}{1-g_n}\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}\frac{g_k}{1-g_k}a_k b_k(z^k-w^k)(z^{-k}-w^{-k})\right) \\ &= \prod_{n=1}^{\infty}\frac{1}{1-g_n}\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\frac{g_k}{1-g_k}a_k b_k(-2+(w/z)^{-k}+(w/z)^k)\right), \end{aligned} \tag{B.9}$$

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and we can bring down $t_2^+(t_1^-)^2$ by applying to the above expression the operator

$$\frac{2\cdot 1^2}{(z^2-w^2)(z^{-1}-w^{-1})^2}\partial_{a_2}\partial_{b_1}^2\,, \tag{B.10}$$

and setting $\boldsymbol{a},\boldsymbol{b}\to\mathbf{1}$. We thus obtain (introducing $\gamma_k=g_k/(1-g_k)$)

$$\begin{aligned}&\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} z^r w^{-s}\langle D_{(1,1;-2)}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)\,\widetilde{K}(r,s;\boldsymbol{t}^+,\boldsymbol{t}^-)]\rangle_{\boldsymbol{g}}\\ &=-\frac{1}{g_1^2g_2}\gamma_1^2\gamma_2\prod_{n=1}^\infty\frac{1}{1-g_n}\frac{\left(1-\frac{w}{z}\right)^3\left(1+\frac{w}{z}\right)}{\left(\frac{w}{z}\right)^2}\frac{\sqrt{\frac{w}{z}}}{1-\frac{w}{z}}\exp\left(\sum_{k=1}^\infty\frac{\gamma_k}{k}(-2+(w/z)^{-k}+(w/z)^k)\right).\end{aligned} \tag{B.11}$$

The r.h.s. is a function of w/z and thus the l.h.s. is diagonal in r,s and can be written

$$\begin{aligned}&\sum_{s\in\mathbb{Z}+\frac{1}{2}}\zeta^{-s}\langle D_{(1,1;-2)}[\widetilde{Z}_\infty(\boldsymbol{t}^+,\boldsymbol{t}^-)\,\widetilde{K}(s,s;\boldsymbol{t}^+,\boldsymbol{t}^-)]\rangle_{\boldsymbol{g}}\\ &=-\frac{1}{(1-g_1)^2}\frac{1}{1-g_2}\prod_{n=1}^\infty\frac{1}{1-g_n}\frac{(1-\zeta)^3(1+\zeta)}{\zeta^2}\frac{\sqrt{\zeta}}{1-\zeta}\exp\left(\sum_{k=1}^\infty\frac{\gamma_k}{k}(-2+\zeta^k+\zeta^{-k})\right).\end{aligned} \tag{B.12}$$

Now, if we have

$$\sum_{s\in\mathbb{Z}+\frac{1}{2}}\zeta^{-s}H_s=f(\zeta) \tag{B.13}$$

we obtain

$$\begin{aligned}\sum_{\substack{N<s\\ s\in\mathbb{Z}+\frac{1}{2}}}H_s&=\sum_{\substack{N<s\\ s\in\mathbb{Z}+\frac{1}{2}}}\oint d\zeta\,\zeta^{s-1}f(\zeta)=\oint d\zeta f(\zeta)\sum_{n=0}^\infty\zeta^{N+\frac{1}{2}+n-1}=\int d\zeta\zeta^{N-1}\frac{\sqrt{\zeta}}{1-\zeta}f(\zeta)\\ &=\frac{\sqrt{\zeta}}{1-\zeta}f(\zeta)\Big|_{\zeta^{-N}}.\end{aligned} \tag{B.14}$$

This gives the final expression

$$Z_N^{(1,1;-2)}(\boldsymbol{g})=\left[\frac{1}{(1-g_1)^2}\frac{1}{1-g_2}\prod_{n=1}^\infty\frac{1}{1-g_n}\right]\frac{1-\zeta^2}{\zeta}\exp\left(\sum_{k=1}^\infty\frac{\gamma_k}{k}(-2+\zeta^k+\zeta^{-k})\right)\Big|_{\zeta^{-N}}. \tag{B.15}$$

It is straightforward to write this in exact form as a finite sum over shifted $G_N(\boldsymbol{g})$. To test (B.15), we computed the exact matrix integral expansion of $\mathrm{I}^{\mathrm{U}(N)}_{(1,1;-2)}(\eta;q)$ at finite $N=2,3,4,\dots$. With $s_p=\eta^p+\eta^{-p}$, the first three cases are

$$\begin{aligned}I^{\mathrm{U}(2)}_{(1,1;-2)}(\eta;q)&=s_1\,q+2(s_2+1)\,q^2+(3s_3+s_1)\,q^3+(4s_4+2)\,q^4+\cdots,\\ I^{\mathrm{U}(3)}_{(1,1;-2)}(\eta;q)&=(s_2+1)\,q^2+(2s_3+3s_1)\,q^3+(4s_4+3s_2+4)\,q^4+(6s_5+4s_3+4s_1)\,q^5\\ &\quad+(9s_6+3s_4+5s_2+4)\,q^6+\cdots,\\ I^{\mathrm{U}(4)}_{(1,1;-2)}(\eta;q)&=(s_3+s_1)\,q^3+(2s_4+3s_2+4)\,q^4+(4s_5+4s_3+6s_1)\,q^5+(7s_6+6s_4+9s_2+6)\,q^6\\ &\quad+(11s_7+8s_5+10s_3+9s_1)\,q^7+(16s_8+9s_6+14s_4+7s_2+14)\,q^8+\cdots,\end{aligned} \tag{B.16}$$

and are fully reproduced by the perturbative series of (B.15). In particular, the leading term is $\mathcal{O}(q^{N-1})$ and its coefficient is a function of η that follows from

$$\mathrm{I}^{\mathrm{U}(N)}_{(1,1;-2)}(\eta;q) = \mathrm{PE}[\varepsilon\gamma]|_{\varepsilon^{N-1}} = \mathrm{PE}[\varepsilon(\eta+\eta^{-1})]|_{\varepsilon^{N-1}}\, q^{N-1} + \cdots . \tag{B.17}$$

Using (3.28), we obtain

$$\mathrm{I}^{\mathrm{U}(N)}_{(1,1;-2)}(\eta;q) = -\frac{\eta}{1-\eta^2}(\eta^N - \eta^{-N})\, q^{N-1} + \cdots = \sum_{\substack{p=-(N-1)\\ \Delta p=2}}^{N-1} \eta^p\, q^{N-1} + \cdots , \tag{B.18}$$

in agreement with the corresponding terms in (B.16). The procedure we have illustrated in this example can be easily applied to any other charge assignment.

C Explicit series expansion for the index $\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}}(\eta;q)$

We introduce the notation $s_p = \eta^p + \eta^{-p}$. The explicit series expansion of the Schur line defect 2-point function $\mathrm{I}^{\mathrm{U}(N)}_{\mathrm{F}}$ at finite N are

$$\mathrm{I}^{\mathrm{U}(2)}_{\mathrm{F}}(\eta;q) = 1 + 2\,s_1\,q + (3\,s_2+1)\,q^2 + 4\,s_3\,q^3 + \cdots \tag{C.1}$$

$$\mathrm{I}^{\mathrm{U}(3)}_{\mathrm{F}}(\eta;q) = 1 + 2\,s_1\,q + (4\,s_2+2)\,q^2 + 3\,(2\,s_3+s_1)\,q^3 + (9\,s_4+2\,s_2+3)\,q^4 + \cdots \tag{C.2}$$

$$\begin{aligned}\mathrm{I}^{\mathrm{U}(4)}_{\mathrm{F}}(\eta;q) = {}& 1 + 2\,s_1\,q + (4\,s_2+2)\,q^2 + 3\,(7\,s_3+4\,s_1)\,q^3 + (11\,s_4+5\,s_2+7)\,q^4 \\ &+ (16\,s_5+6\,s_3+7\,s_1)\,q^5 + \cdots \end{aligned} \tag{C.3}$$

$$\begin{aligned}\mathrm{I}^{\mathrm{U}(5)}_{\mathrm{F}}(\eta;q) = {}& 1 + 2\,s_1\,q + (4\,s_2+2)\,q^2 + 3\,(7\,s_3+4\,s_1)\,q^3 + (12\,s_4+6\,s_2+8)\,q^4 \\ &+ (18\,s_5+9\,s_3+11\,s_1)\,q^5 + (27\,s_6+10\,s_4+15\,s_2+13)\,q^6 + \cdots \end{aligned} \tag{C.4}$$

$$\begin{aligned}\mathrm{I}^{\mathrm{U}(6)}_{\mathrm{F}}(\eta;q) = {}& 1 + 2\,s_1\,q + (4\,s_2+2)\,q^2 + 3\,(7\,s_3+4\,s_1)\,q^3 + (12\,s_4+6\,s_2+8)\,q^4 \\ &+ (19\,s_5+10\,s_3+12\,s_1)\,q^5 + (29\,s_6+13\,s_4+19\,s_2+17)\,q^6 \\ &+ (42\,s_7+18\,s_5+23\,s_2+25\,s_1)\,q^7 + \cdots \end{aligned} \tag{C.5}$$

and

$$\begin{aligned}\mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{F}}(\eta;q) = {}& 1 + 2\,s_1\,q + (4\,s_2+2)\,q^2 + (7\,s_3+4\,s_1)\,q^3 + (12\,s_4+6\,s_2+8)\,q^4 \\ &+ (19\,s_5+10\,s_3+12\,s_1)\,q^5 + (30\,s_6+14\,s_4+20\,s_2+18)\,q^6 \\ &+ (45\,s_7+22\,s_5+28\,s_3+30\,s_1)\,q^7 + \cdots . \end{aligned} \tag{C.6}$$

We obtain

$$\mathrm{I}^{\mathrm{U}(2)}_{\mathrm{F}} - \mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{F}} = -\frac{1}{\eta^2}(1+\eta^2+\eta^4)\,q^2 - \frac{1}{\eta^3}(3+4\eta^2+4\eta^4+3\eta^6)\,q^3 + \cdots , \tag{C.7}$$

$$\mathrm{I}^{\mathrm{U}(3)}_{\mathrm{F}} - \mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{F}} = -\frac{1}{\eta^3}(1+\eta^2+\eta^4+\eta^6)\,q^3 - \frac{1}{\eta^4}(3+4\eta^2+5\eta^4+4\eta^6+3\eta^8)\,q^4 + \cdots , \tag{C.8}$$

$$\begin{aligned}\mathrm{I}^{\mathrm{U}(4)}_{\mathrm{F}} - \mathrm{I}^{\mathrm{U}(\infty)}_{\mathrm{F}} = {}& -\frac{1}{\eta^4}(1+\eta^2+\eta^4+\eta^6+\eta^8)\,q^4 \\ &- \frac{1}{\eta^5}(3+4\eta^2+5\eta^4+5\eta^6+4\eta^8+3\eta^{10})\,q^5 + \cdots , \end{aligned} \tag{C.9}$$

$$\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(5)} - \mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)} = -\frac{1}{\eta^5}(1+\eta^2+\eta^4+\eta^6+\eta^8+\eta^{10})\, q^5 - \frac{1}{\eta^6}(3+4\eta^2+5\eta^4+5\eta^6+5\eta^8+4\eta^{10}+3\eta^{12})\, q^6 + \cdots\,, \tag{C.10}$$

$$\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(6)} - \mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)} = -\frac{1}{\eta^6}(1+\eta^2+\eta^4+\eta^6+\eta^8+\eta^{10}+\eta^{12})\, q^6 - \frac{1}{\eta^7}(3+4\eta^2+5\eta^4+5\eta^6+5\eta^8+5\eta^{10}+4\eta^{12}+3\eta^{14})\, q^7 + \cdots\,. \tag{C.11}$$

The pattern is

$$\mathrm{I}_{\mathrm{F}}^{\mathrm{U}(N)} - \mathrm{I}_{\mathrm{F}}^{\mathrm{U}(\infty)} = \frac{\eta}{1-\eta^2}(\eta^{N+1}-\eta^{-N-1})\, q^N + \frac{\eta}{1-\eta^2}(3\eta^{N+2}+\eta^N+\eta^{N-2}-\eta^{2-N}-\eta^{-N}-3\eta^{-2-N})\, q^{N+1}+\cdots+\mathcal{O}(q^{2N}). \tag{C.12}$$

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