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Invited review

A review of recent advances in time-dependent vehicle routing

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ABSTRACT

In late 2015 three of the co-authors of this paper published the first review on time-dependent routing problems. Since then, there have been several important algorithmic developments in the field. These include travel time prediction methods, real-time re-optimization by operating directly on the road graph, efficient exploration of solution neighborhoods, dynamic discretization discovery and *Machine Learning*-inspired methods. The aim of this survey is to present such research lines, together with indications on their further developments.

1. Introduction

Since the appearance of the first review on time-dependent routing problems (Gendreau et al. (2015)), the interest in the field has grown enormously. In this paper we provide a review of the most recent developments that have had a major impact on the literature.

Time-dependent (TD) routing amounts to designing *best* routes in a graph in which arc traversal times may vary over the planning horizon (Ichoua et al. (2003)). The problem can be formulated on either an undirected, directed or mixed graph, but here we consider the directed version unless specified differently. Let $G = (V, A)$ be a directed graph representing a *road network*, where V is a set of either vehicle depots, customer locations or road intersections, and $A \subseteq V \times V$ is an arc set. With each arc $(i, j) \in A$ is associated a function $\tau_{ij}(t)$ representing the amount of time that a vehicle, leaving origin $i \in V$ at time instant t, takes to reach destination $j \in V$ along $(i, j) \in A$. We assume that the travel time functions satisfy the *first-in-first-out* (FIFO) property, i.e., the arrival time is a strictly monotonic function of the starting time.

In classical (time-invariant) *Vehicle Routing Problems* (VRPs), information about point-to-point routing on the road network may be determined in advance. Therefore, the majority of the time-invariant VRP literature is based on an abstraction of the road network, referred to as a *customer-based* graph $G' = (V', A')$ (Huang et al. (2017)), where vertices $i \in V' \subseteq V$ represent "points of interest" (POIs), i.e., customers, depots, etc. Then the routing decisions on the *customer-based* graph and the path selection at road network level can be decoupled. On the contrary, in a TD setting each arc $(i, j) \in A'$ of the *customer-based* graph is associated to a time-dependent cost function $\tau'_{ij}(t)$ representing the time taken by a quickest path in the road network starting from $i \in V'$ at time instant t to reach destination $j \in V'$. In general, this quickest path may change over time, i.e., different start times t may lead to different quickest paths in the road network. More formally, if P_{ij} is the set of paths linking vertices $i, j \in V'$ in G' and $\tau^p_{ij}(t)$ is the traversal time of path $p \in P_{ij}$ starting at time t, then

$$\tau'_{ij}(t) = \min_{p \in P_{ij}} \tau^p_{ij}(t). \tag{1}$$

Part of the complexity of time-dependent VRPs (TDVRPs) is due to the fact that, for any given vehicle route $p_k = (v_0, v_1, \dots, v_k)$, on either the road graph G or a customer-based graph G', the time needed to reach v_k for a prescribed start time t, denoted as $z(p_k, t)$, must be computed recursively as:

$$z(p_k, t) = z(p_{k-1}, t) + \tau_{v_{k-1}, v_k}(t + z(p_{k-1}, t)) \quad k = 1, \dots, \tag{2}$$

with the initialization $z(p_0, t) = 0$.

1.1. Travel time models

In this paper, we focus on travel time variations induced by exogenous events like traffic congestion, weather conditions or mobile obstacles, excluding endogenous actions taken by drivers, e.g., adapting speed to balance fuel consumption and travel time (Franceschetti et al. (2013)). In particular, we address time-dependent problems related to the planning of ground vehicles (trucks, vans, ambulances, …) that are often affected by road traffic congestion. A typical traffic pattern is

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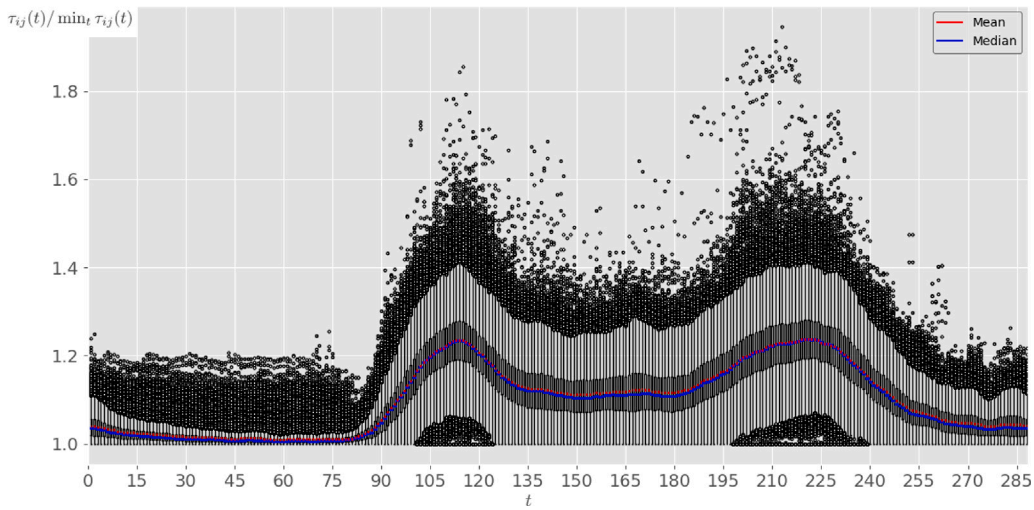


Fig. 1. Boxplots of normalized τ values across a typical day of operations for 100 origin–destination pairs in the Greater London (UK) area (a time slot t corresponds to a 5-min interval).

shown in Fig. 1 where the normalized travel times $\tau_{ij}(t)/\min_t \tau_{ij}(t)$ are depicted between 100 origin–destination pairs located in the Greater London (UK) area. The 24-h time horizon is partitioned in 288 time slots, each corresponding to a 5-min interval. For each time slot t, a boxplot shows the travel times between the 100 origin–destination pairs at time t: the gray segments describe the *boxes*, the black ones the *whiskers* while the upper and lower dots represent the *outliers*; moreover, the red line represents the average travel time function, while the blue one describes the median.

Most of the current time-dependent vehicle routing literature relies on the travel time model proposed by Ichoua et al. (2003) (IGP model in the following). The planning horizon $[0, T]$ is partitioned into a finite number H of time slots $[T_h, T_{h+1}]$ $(h = 0, \dots, H-1)$, where $T_0 = 0$ and $T_H = T$. Given a road network, vehicle speed is assumed to be equal to a constant level v_{ijh} on arc (i, j) during time period h. Algorithm 1 reports the pseudocode of the IGP procedure for computing the time $\tau_{ij}(t)$ required to traverse arc (i, j), of length L_{ij}, with a constant stepwise speed $v_{ij}(t) = v_{ijh}$, $t \in [T_h, T_{h+1}]$, $h = 0, \dots, H-1$, if the vehicle leaves node i at time t. The main idea of the IGP model is that the speed changes when the boundary between two consecutive time periods is crossed, resulting in a piecewise linear travel time functions $\tau_{ij}(t)$, which satisfies the FIFO property. The algorithm is initialized with the time slot p which the start time t belongs to. At steps 2 and 5, d is set equal to the arc length not yet traversed at the current time instant t. At each iteration, t' represents the arrival time of a vehicle traversing a distance d at constant speed v_{ijh} (step 8) with $h \geq p$. The algorithm terminates when t' belongs to the current time slot $[T_h, T_{h+1}]$. Fig. 2 gives an example of the travel speed function and its associated travel time function for an arc length 3 with a start time equal to 0.5. The corresponding travel time $\tau(0.5) = 2$ is computed by Algorithm 1 in 2 iterations. The initial value of d is set equal to the arc length 3. It is worth noting that the length traversed by a vehicle in a time interval $[t_1, t_2]$ is equal to the integral of its speed function between t_1 and t_2. Since the speed function is constant stepwise, it follows that:

$$L_{ij} = (t - T_1) \times v_{ij0} + (T_1 - T_2) \times v_{ij1} + (t' - T_2) \times v_{ij2}.$$

During the first iteration, d is decreased by the length traversed in 0.5 time units at speed $v_{ij0} = 1$, i.e., $d = d - (t - T_1) \times v_{ij0} = 2.5$. During the second iteration, d is decreased by the length traversed in 1 time unit at speed $v_{ij1} = 2$, i.e., $d = d - (T_1 - T_2) \times v_{ij1} = 0.5$. At the end of the second iteration the while loop stops since $t' = T_2 + d/v_{ij2} < T_3$. Then the algorithm terminates and returns $\tau_{ij}(0.5) = t' - t$.

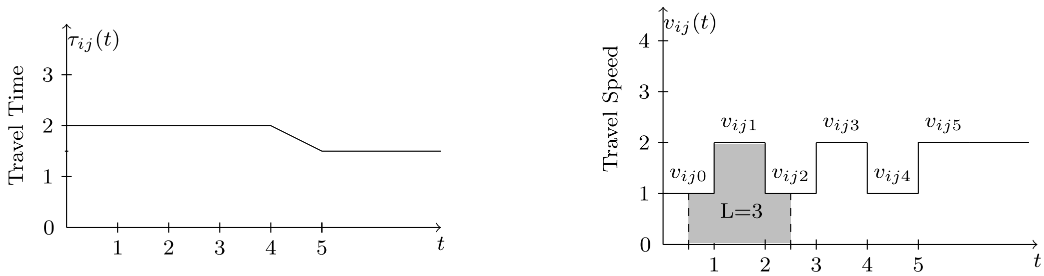
Ghiani and Guerriero (2014) exploit some properties of the IGP model. Firstly, they prove that any continuous piecewise linear travel time model, satisfying the FIFO property, can be generated by an appropriate IGP model. They also show that the model parameters can be obtained by solving a system of linear equations for each arc. Then, such parameters are proved to be nonnegative, which allows them to be interpreted as speeds. Finally, Ghiani and Guerriero (2014) prove that the speed and travel time functions v_{ijh} and $\tau'_{ij}(t)$ on an associated customer-based graph are piecewise constant and piecewise linear, respectively.

Algorithm 1 Travel time calculation procedure in the IGP model

1: $t_0 \leftarrow t$
2: $h \leftarrow p : T_p \leq t \leq T_{p+1}$
3: $d \leftarrow L_{ij}$
4: $t' \leftarrow t + d/v_{ijp}$
5: **while** $t' > T_{h+1}$ **do**
6: $\quad d \leftarrow d - v_{ijh}(T_{h+1} - t)$
7: $\quad t \leftarrow T_{h+1}$
8: $\quad h \leftarrow h + 1$
9: $\quad t' \leftarrow t + d/v_{ijh}$
10: **end while**
11: **return** $t' - t_0$

1.2. Data sources

Although time-dependent vehicle routing problems have received increasing attention from the scientific community in recent years, the majority of researchers still rely on *synthetic* travel time functions. This can be explained as follows: traffic data (observations over time of vehicle GPS latitude and longitude, in addition to instantaneous speed) can be collected mainly from smart phone applications and probe vehicles. As far as the first technique is concerned, data are gathered by big Information Technology companies (such as Google, Apple, Microsoft, …) over many countries and utilized to provide valuable web services such as point-to-point routing services based on real-time traffic conditions. As such, companies have no incentive to share *data with fine granularity* since that provides them with an unparalleled competitive advantage. As for the second approach, data are collected by transportation companies over a given market (e.g., in Gmira et al. (2020) data are collected by a company providing home deliveries of appliances and furniture in the city of Montreal). Obviously, in this kind of business, data remain confidential. An example of available traffic data is given by Uber Movement (Uber, 2022), which shares anonymized data from over ten billion trips (located in 52 metropolitan



(a) Travel time function

(b) Speed function

Fig. 2. Example from Calogiuri et al. (2021): a continuous piecewise linear arc travel time function (a) and the associated constant stepwise speed function (b).

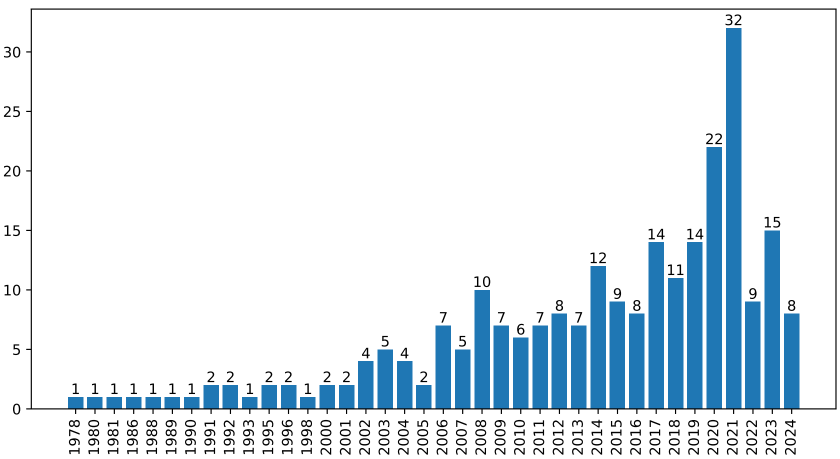


Fig. 3. Number of publications by year.

areas in North and South America, Europe, Africa, Asia and Oceania). However, such data are aggregated in five macro-periods: AM peak (7 AM–10 AM), Midday (10 AM–4 PM), PM peak (4 PM–7 PM), Evening (7 PM–12 AM), Early Morning (12 AM–7 AM). This level of granularity is not enough to take full advantage of traffic conditions in time-dependent vehicle routing algorithms. In conclusion, most of the scientific literature still relies on *synthetic* travel time functions.

1.3. Paper outline

The review covers 49 papers, the majority of which were published in international journals, with a specific focus on those published from 2015 onwards. Papers published prior to 2015 that are deemed particularly noteworthy due to their relevance to the topic have been included in this literature review. Fig. 3 illustrates the distribution of papers by year of publication. The majority of papers were published after 2008, coinciding with a surge in technological advances that stimulated increased interest in this field. According to Gendreau et al. (2015), until 2015, time-dependent travel times had largely been overlooked in various vehicle routing problems, such as point-to-point quickest path on a time-dependent continental-sized network, time-dependent general routing, and time-dependent arc routing. As shown in Fig. 3, there has been a noticeable increase in contributions post-2015, with a significant concentration observed between 2017 and 2024. Unlike the 2015 review (Gendreau et al. (2015)), this paper does not provide an exhaustive description of each and every contribution in the time-dependent vehicle routing field. Instead, we focus on describing five main sub-areas: travel time prediction methods, real-time re-optimization strategies directly on the road graph, efficient exploration of neighborhoods, dynamic discretization discovery for problems with narrow time windows, and *Machine Learning*-inspired methods. These areas have seen remarkable advancements in recent years, with further improvements expected in the future.

The remainder of the paper is organized as follows. In Sections 2 and 3, we present the main application areas and a taxonomy of time-dependent routing problems. In Section 4 we review the literature related to travel time prediction, computation and approximation methods. Then, in Section 5 we survey the literature on time-dependent routing with path flexibility. Section 6 introduces the dynamic discretization discovery framework, followed by efficient neighborhood evaluations in Section 7 and by Machine Learning-based methods in Section 8. Finally, conclusions are reported in Section 9.

2. Main application areas

Time-dependent routing problems naturally occur in a variety of applications, such as travel scheduling in public transportation systems, route planning on road networks, vehicle routing problems as well as some planning problems in robotics and military contexts. We will now provide an overview of the main application areas.

Travel planning in *public transit networks*, i.e., finding the most efficient connection in a bus, train, ferry or multimodal transportation network (Google Transit, 2014), is time-dependent and discontinuous in nature: see Fig. 4 for a typical travel time function on a transit line.

In this context, travel times are not guaranteed to respect the FIFO property: for instance, a traveler may choose between waiting for the next bus or reaching the destination on foot (Garcia et al., 2013). Hence, routing problems on transit networks are often modeled on a time-expanded graph (Bast et al., 2010), where each event in the timetable (e.g., a vehicle departure or arrival at a stop) is modeled by a vertex. The extremes of an arc represent departure and arrival events of a direct connection, a transfer connection or waiting at a stop. In this paper, we focus on travel time in road networks. The reader interested

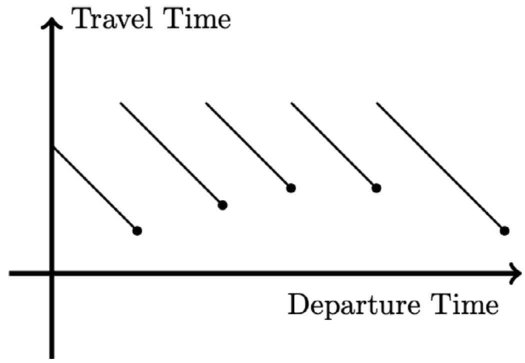


Fig. 4. Example from Gendreau et al. (2015) of a piecewise linear travel time function on a transit line.

in routing in public transportation networks is referred to Delling et al. (2013) and Dibbelt et al. (2013).

As aforementioned, travel time functions on *road networks* are usually assumed to be continuous and computed according to the speed model proposed by Ichoua et al. (2003) (Fig. 2). Such a time-dependent model plays a central role in both *route planning in road networks* and fleet management in *vehicle routing problems*. *Route planning in road networks*, which involves the computation of the quickest route between a source and a target on road networks, is the basic ingredient of a plethora of travel information services, e.g., Google, Baidu, Yandex, Bing, Apple or HERE Maps (Bast et al., 2016). *Route planning in road networks* is also used as a subroutine in VRPs arising in several application domains such as distribution planning, mail delivery, garbage collection, salt gritting. Even with the advancement of sophisticated algorithms for *route planning in road networks* (Batz et al., 2013), generating fast travel-time queries on large-scale road networks (usually within a fraction of a millisecond) still presents substantial methodological challenges.

Another peculiar class of time-dependent VRPs concerns the route planning of vehicles such as an aircraft, a ship or a submarine. Here the decisions are taken in a two or three-dimensional space and evaluated in terms of the minimization of travel time, fuel consumption or a combination of them (Perakis and Papadakis (1989), Norstad et al. (2011)). In recent years, this problem has received attention thanks to technological advancements in the collection of detailed real-time data about air and underwater currents, the state of the surface of the ocean, etc.

Another source of time-dependency is due to points of interest whose position change during the planning horizon. In these applications, the vehicles have to reach a set of moving targets which is the case, e.g., of resupplying moving units, such as patrolling boats (Helvig et al., 2003), or intercepting mobile ground units with an aircraft (Jiang et al., 2005). Travel time dependency can be also induced by the presence of moving obstacles, as customary in *robot motion planning* (Sutner and Maass (1988), Latombe (1990), Fujimura (1995)). Finally, time-dependent travel times may arise in routing problems where a fleet of *automated guided vehicles* (AGVs) is in charge of moving loads along a (physical) network whose arcs can be traversed by only one vehicle at a time, which is the case of manufacturing systems, port terminals and automated warehouses. Here, the time-dependency is related to the need of planning conflict-free routes. In particular, the decision includes not only routing but also scheduling the vehicles, which are allowed to wait at nodes of the network to avoid collisions (Adamo et al. (2018), Adamo et al. (2023b)).

3. Taxonomy

Time-dependent routing problems may be classified in *point-to-point* and *multiple-point* problems. Other criteria distinguish whether vehicles must service nodes, arcs or both, which is the case of *node*, *arc* and *general routing*, respectively. In the following, we provide a short description for each class of problems and provide some relevant recent references. The reader interested in an updated full list of papers on time-dependent routing can refer to the www.tdrouting.com web page.

3.1. Time-dependent point-to-point route planning

Route planning on road networks is a key ingredient of travel information services, used by millions of users on a daily basis. The core component is the *Time-Dependent Quickest Path Problem* (TDQPP), which involves the computation of a path $p = (s = v_0, v_2, \dots, v_k = d)$ between two vertices s and d, whose duration $z(p,t)$, defined by (2), is minimum. As its time-independent (*classical*) counterpart, the TDQPP is polynomially solvable in FIFO networks (Kaufman & Smith, 1993) even with the presence of traffic lights (Ahuja et al., 2002). The TDQPP has been extensively studied in the past decade. To achieve fast running times on continental road networks with millions of vertices, additional arcs (*shortcuts*) are introduced in a preprocessing phase (Strasser et al., 2021).

3.2. The time-dependent traveling salesman problem and its variants

The *Time-dependent Traveling Salesman Problem* (TDTSP) aims to determine a least-cost Hamiltonian circuit on the customer-based graph G', starting from the depot node at time $t = 0$. Malandraki and Dial (1996) is the first contribution to address the TDTSP and propose an approximate dynamic programming algorithm, where travel time functions are constant step-wise and, therefore, do not satisfy the FIFO property. In the last decade several contributions have proposed exact solutions for the TDTSP, based on FIFO travel time models. Among others, we cite the branch-and-cut approach by Cordeau et al. (2014), the branch-and-bound algorithm by Arigliano et al. (2018) and the constraint programming approach by Melgarejo et al. (2015). The proposed approaches are able to solve realistic instances with up to 50 POIs. Several contributions studied the TDTSP with Time Windows (TDTSPTW). Two different objective functions are considered: *makespan* and *duration*. In the former case, the vehicle has to return to the depot as soon as possible after the start of the planning horizon. This problem has received attention in the literature, see for example Arigliano et al. (2019), Montero et al. (2017) and Albiach et al. (2008). Vu et al. (2020) has been the first to devise a solution approach for minimizing the *duration* of the tour, which is the case when the vehicle has to return to the depot as quickly as possible after the departure from the depot. Vu et al. (2020) formulate the TDTSPTW as an integer programming problem defined on (partially) time expanded networks. Then, they devise a solution approach relying on a *Dynamic Discretization Discovery* framework, which is thoroughly discussed in Section 6. It is worth noting that exact solution approaches do not scale well for both TDTSP and TDTSPTW with wide time windows (Adamo et al. (2023a) and Pralet (2023)). A number of contributions have dealt with generalizations of the TDTSP. This is the case of the *Moving-Target TSP*, where POIs are targets moving at constant speed to be intercepted in minimum time by a pursuer (Wang & Wang, 2023). We also mention the *TDTSP with Profits*, usually presented as the time-dependent orienteering problem (Khodadadian et al., 2022). Finally, we observe that there are contributions dealing with a scheduling problem referred to as the TDTSP, aiming to determine a sequence of jobs on a single machine, where the processing times are position-dependent.

3.3. The time-dependent vehicle routing problem

The TDTSP is generalized by *Time-dependent Vehicle Routing Problem* (TDVRP), in which a homogeneous or heterogeneous fleet of vehicles has to visit a set of customers subject to capacity constraints. Malandraki and Daskin (1992) is the first to introduce a *Mixed Integer*

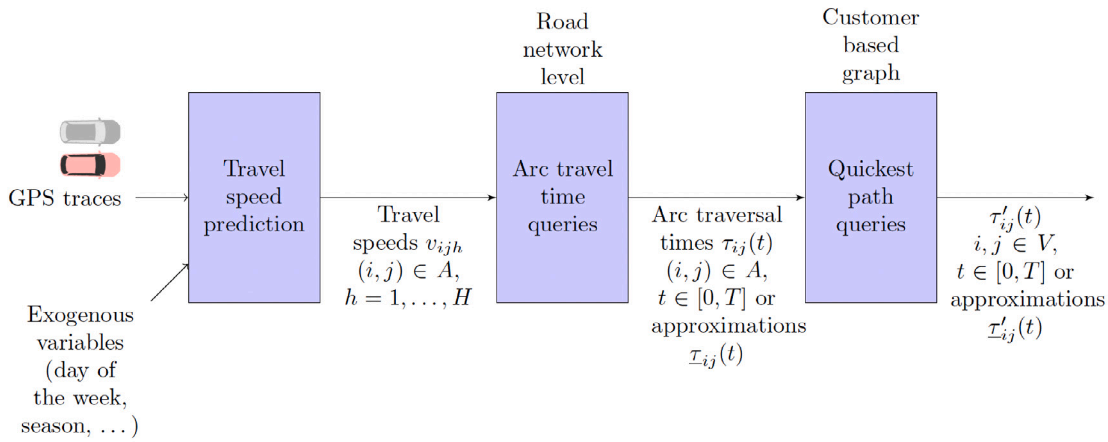


Fig. 5. From GPS vehicle traces to quickest path queries.

Linear Programming (MILP) formulation for the TDVRP (without time windows) on a customer-based graph. The time horizon is partitioned into intervals and the travel time on each arc is modeled as a stepwise function. Unfortunately, this travel time model does not satisfy the FIFO property. This limit is overcome by the IGP model proposed by Ichoua et al. (2003), in a paper dealing with the TDVRP with soft time windows. Other contributions focusing on this problem are Jung and Haghani (2001) and Haghani and Jung (2005), where genetic algorithms are developed with the objective of minimizing lateness, routing costs and fixed costs for the vehicles. Donati et al. (2008) deal with the TDVRP with hard time windows. A hierarchy of two artificial ant colonies and a local search method are proposed and tested on instances derived from Solomon's VRPTW instances and a real network in Italy. Figliozzi (2012) proposes a fast iterative route construction and improvement method to solve the TDVRP with hard and soft time windows. Taş et al. (2014) address the TDVRP with stochastic travel times and soft time windows. The authors devise a tabu search and an adaptive large neighborhood search to optimize both efficiency and reliability. It is only in 2013 that Dabia et al. (2013) propose an exact method to solve a TDVRP with time windows aiming to minimize the total duration of the routes computed by the IGP model. Experiments are conducted on randomly generated instances of different sizes. Computational results show that instances with twenty-five vertices can be consistently solved to the optimum. To the best of our knowledge, no other contributions have devised an exact method tailored for this problem. In the last decade there have been several contributions on TDVRP dealing with the integration of routing decisions with path selection on the road network. This research stream is thoroughly reviewed in Section 5.

3.4. Time-dependent arc routing

In *Arc Routing Problems* (ARPs), a homogeneous or heterogeneous fleet of vehicles has to traverse a predefined set of arcs and edges of a graph. Even though most related applications occur in urban contexts, the ARP literature considering time-dependent travel times is quite limited (Corberán et al., 2021). Sun et al. (2011) and Wang and Wen (2002) study the time-dependent *Chinese Postman Problem*, which aims to determine a least-duration circuit that visits each arc of the graph at least once. Calogiuri et al. (2019) propose a branch-and-bound algorithm to solve to optimality the time-dependent variant of the *Rural Postman Problem* (RPP), where the postman is not required to cover every arc in the network, but only a subset of them. The objective is to determine a least-duration tour that includes all the required arcs and edges. Xin et al. (2022) propose a time–space network model for the time-dependent RPP and a heuristic genetic algorithm to solve it. In the *Capacitated ARP* (CARP), a demand is associated with each required edge or arc in a RPP. A routes is feasible when the total serviced demand does not exceed the vehicle capacity. The CARP aims to determine a set of feasible routes that satisfies the total demand and minimizes the total cost. The time dependent variant of the CARP is studied in Tagmouti et al. (2007, 2010, 2011) and Vidal et al. (2021). We finally mention ARPs with profits, where the number of required links is too large to be serviced all in a day. In these cases each requested link has a profit. Black et al. (2013) deal with the time-dependent prize-collecting ARP, where the objective is to maximize the total profit collected minus the total travel cost.

3.5. Time-dependent general routing

In *Time-dependent General Routing Problems* (TDGRPs), a homogeneous or heterogeneous fleet of vehicles must visit certain required vertices and must traverse certain required arcs and edges on a time-dependent graph. The main applications of the TDGRP are waste collection and newspaper delivery. Ahabchane et al. (2020) present a multi-vehicle mixed integer programming formulation where time-dependent demands are considered for snow removal problems. Adamo et al. (2021) give some properties as well as an exact algorithm for solving the single-vehicle general routing problem with time-dependent traversal times.

4. Predicting, computing and approximating travel times in road networks and customer-based graphs

Traffic conditions are highly non-linear and dynamic, changing over time and space. Causes of variability in travel times include hourly, daily, weekly or seasonal variations from the average traffic volumes, random events like accidents, and weather conditions (Malandraki & Daskin, 1992). The former are periodic in nature and can be modeled by deterministic time-dependent travel times while the latter can be cast as random variables.

Traffic data can be exploited to enhance vehicle routes in two ways: by considering the historical traffic patterns while building a-priori routes; by re-routing vehicles in real-time (if operationally feasible) as soon as fresh traffic data become available. Starting from raw data (most often, GPS vehicle traces), forecasting methods aim to predict travel time functions over road networks and customer-based graphs. This is done through the three-stage process depicted in Fig. 5: travel speed predictions; arc travel time queries, quickest path queries.

As explained in Section 1, the planning horizon $[0, T]$ is usually partitioned into a finite number H of time slots $[T_h, T_{h+1}]$ $(h = 0, \ldots, H-1)$, where $T_0 = 0$ and $T_H = T$ and vehicle speed is assumed to be equal to a constant level v_{ijh} on any arc (i,j) of the road graph during time

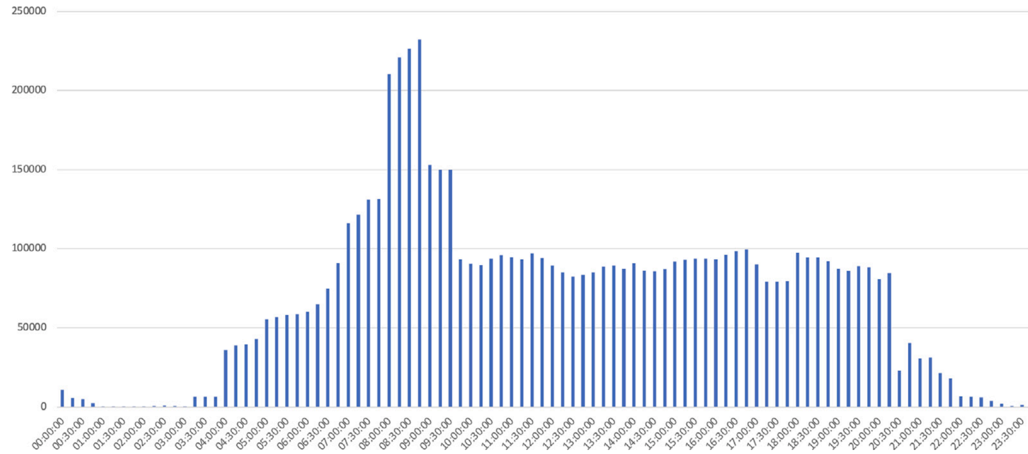


Fig. 6. Number of observations scattered over a typical day in the Gmira et al. (2020) case study.

period h. The resulting travel time functions $\tau_{ij}(t)$ are piecewise linear and satisfy the FIFO property.

Regarding forecasts, most Intelligent Transportation Systems currently in use generate only short-term traffic prediction, a topic on which a huge literature has been published in the last decades (see, e.g., Alam et al. (2016)). On the other hand, vehicle routing tools based on a time-dependent framework need the forecast of travel time functions extending over larger time intervals (e.g., from 6 am to 8 pm on a given day). Such predictions typically rely on historical travel speed values and a set of exogenous variables.

In the following subsections, we focus on traveling information at road-network level from two points of view: accurate predictions of travel speeds; reducing time and space complexity of travel time queries on customer-based graphs. We also deal with the efficient evaluation of a route duration through approximated travel time profiles.

4.1. Speed prediction

All methods start from GPS traces (Fig. 5) transmitted by mobile devices installed on-board. Each GPS observation includes an identifier, a latitude–longitude pair, an instantaneous speed, a mobile identifier, a driver identifier, a date and a time stamp. In order to estimate reliable travel time functions in a typical application, at least one year of GPS data are needed. Observations are usually scattered irregularly over a day, as shown in Fig. 6 for the Gmira et al. (2020) case study, in which an arc speed pattern is made of 96 average speed values taken over time intervals of 15 min on a given day.

Speed prediction is only the final phase in a process made up of (at least) three main macro-steps: data preparation and representation; size reduction and clustering; prediction (possibly with further data processing). The literature addressing these phases is not unified and leads to a number of possible algorithmic choices, where different *Machine Learning* and *Data Mining* methods are applied. Thus, the best approach has to be determined for each specific context.

In the *data preparation and representation* phase, GPS points are mapped to the underlying road network to generate daily speed patterns for each individual arc. For instance, in Gmira et al. (2020) the road network is made up of 233,914 arcs and the data are collected over 515 days, with a total of $233,914 \times 515 = 120,506,910$ speed patterns. Then data are cleaned, e.g., by deleting GPS points with abnormal speeds, by aggregating GPS points associated with parking stops for deliveries, etc. Finally, a map-matching algorithm has to be applied to the remaining GPS points. This amounts to assigning each GPS observation to an arc of the road network which is a difficult problem, especially in dense urban road networks. See Hashemi and Karimi (2016) for a description of such an algorithm.

In the *size reduction and clustering* phase, an elimination procedure must be first applied to get rid of speed patterns or time intervals with too few data. For instance, in Gmira et al. (2020) a speed pattern is automatically eliminated when the fraction of real average speed values over the 96 time intervals was less than 5%, i.e., speed patterns with only 4 average speeds or less are removed. This process ends up with a total of 6,667,459 speed patterns associated to only 3485 arcs (delivery routes traversed only a few streets of the Montreal area). Then arcs with similar speed patterns may be clustered before predicting travel speed values. In Gmira et al. (2020) the 96 time intervals of 15 min are aggregated into 24 time intervals of one hour in order to compute each average speed value with an adequate number of observations. Finally, intervals with no or very few observations over the entire network (e.g., night hours) have to be discarded. For instance, in the above-mentioned paper only 1-h time intervals starting from 7:00 a.m. to 7:00 p.m. are kept in every speed pattern. After this stage a speed pattern is modeled as an array of 13 average speed values, one for each 1-h time interval between 7:00 a.m. and 7:00 p.m.. First, an average speed pattern is calculated for each arc over all its corresponding speed patterns (in Gmira et al. (2020) this leads to 3485 average speed patterns. Second, a clustering approach is used. Gmira et al. (2020) utilize the *K-means algorithm* (MacQueen (1967), Jain and Dubes (1988)) to get a first classification of the arcs, followed by *affinity propagation* Frey and Dueck (2007) to get the final classes; this is devised because the number of average speed patterns is too large for a direct application of the affinity propagation algorithm.

Regarding *prediction*, methods can be classified (Van Hinsbergen et al. (2007)) in: naive (i.e., without any model assumption); parametric; non-parametric; a combination of the last two (the so-called hybrid methods). *Naive methods* do not use any model, are easy to implement and require few data. However, they lack accuracy and are usually used as a baseline to compare more advanced methods. *Parametric methods* rely on a predefined functional form linking the variable to be predicted (travel time, traffic flow at a given time in the future, …) and a number of independent variables. The most basic parametric models are linear regression, *Autoregressive Moving Average* (ARMA) models, and its integrated version (ARIMA), the *Kalman filter* and its extension to non-linear systems, named *Extended Kalman Filter*. The latter is particularly suited for traffic prediction because of the nonlinear and dynamic relationship between state variables in this application domain. Finally, *non-parametric methods* (also known as *data-driven* methods) use data to determine both the model structure and its parameters. They are usually very accurate but require a lot of data. Popular non-parametric models include *support vector machines* (SVMs), *artificial neural networks* (ANNs) and *non-parametric regression*. SVMs can be trained to guarantee global convergence and deal well with noise in the data. ANNs are among the best for travel time forecasting and include *Multi-layer Perceptrons* (MLPs), *Recurrent Neural Networks* (RNNs) and *Long Short-Term Memory Networks* (LSTMNs). The latter two architectures are widely used for their proven ability to capture the relationship among sequential time-dependent data. Finally, non-parametric regression approaches estimate a regression function without relying on any explicit form. They use a search function (e.g., a *k-Nearest Neighbor* approach) to find the most similar observations in the database; then the selected observations are averaged to provide a prediction.

4.2. Computing travel times

When travel time information is provided at the road network level, the computation of quickest paths among customers and facilities is computationally demanding and can become a major obstacle to the application of routing optimization algorithms. Two types of queries are recurrent in the solution process of time-dependent vehicle routing:

- travel time queries: queries that evaluate the arrival time values $\Phi_{ij}(t)$ at node j when *traveling on road arc* (i,j) starting at time t, calculated as:

  $$\Phi_{ij}(t) = \{x | \int_t^x v_{ij}(t')dt' = d_{ij}\},$$

  where $v_{ij}(t)$ is the (piecewise constant) speed function of arc (i,j);
- quickest path queries: queries that evaluate the earliest arrival time $\Gamma_{ij}(t)$ at node j when traveling from node i *along a path* with starting time t; of course, $\Gamma_{ij}(t) = t + \tau_{ij}(t)$.

4.2.1. Travel time queries

Most of the literature use the iterative algorithm proposed by Ichoua et al. (2003), requiring $O(H)$ time. To avoid this overhead, Vidal et al. (2021) propose to express (continuous) functions $\Phi_{ij}(t)$ as closed-form piecewise-linear (PL) functions. They also demonstrate that each PL arrival time function can be stored as a simple array of linear function pieces, giving indexed access in $O(1)$ time if the index of the piece is known and, by binary search, in $O(\log H)$ time otherwise.

4.2.2. Quickest path queries

As observed in Section 1, most VRPs assume that the optimization problem is defined on a customer-based graph $G' = (V', A')$ for which a travel time matrix $\tau'_{ij}(t)$ has to be computed efficiently. Computing such a matrix starting from the road-network level (functions $\tau_{ij}(t)$) can be computationally demanding. Indeed, on a customer-based graph the travel time matrix depends on an additional continuous parameter, the start time, and the quickest paths may change over time. As observed by Ichoua et al. (2003), when travel speed functions are piecewise constant, the arrival time functions $\Gamma_{ij}(t)$ are continuous piecewise linear. A naive approach to quickest path queries is to run a quickest path algorithm (see, e.g., Delling and Wagner (2009), Rahunathan et al. (2021)) any time a start time is specified or, alternatively, precompute approximate quickest paths based on some time discretization. To circumvent the drawbacks of such methods, Vidal et al. (2021) propose a continuous pre-processing approach, during which they compute closed-form representations of the arrival time $\Gamma_{ij}(t)$ function at each destination j for all departure times t at origin i. This effectively avoids the computational overhead of approaches based on iterative travel time queries as well as the memory overhead and inaccuracy of approaches based on time discretization. It is worth noting that a breakpoint of PL functions $\Gamma_{ij}(t)$ can be of two types: a *primitive breakpoint*, i.e., a breakpoint of a road arc travel time function; a *minimization breakpoint*, i.e., a time instant when the quickest path from $i \in V$ to $j \in V$ changes. Fig. 7 illustrates this concept (Gmira et al. (2021b)): there are 4 alternative paths between an origin–destination pair $i \in V$ and $j \in V$ and t_1, t_2 and t_3 constitute the minimization breakpoints ($a_\cdot$ and $b_\cdot$ represent the earliest and latest arrival times at a node $\cdot$).

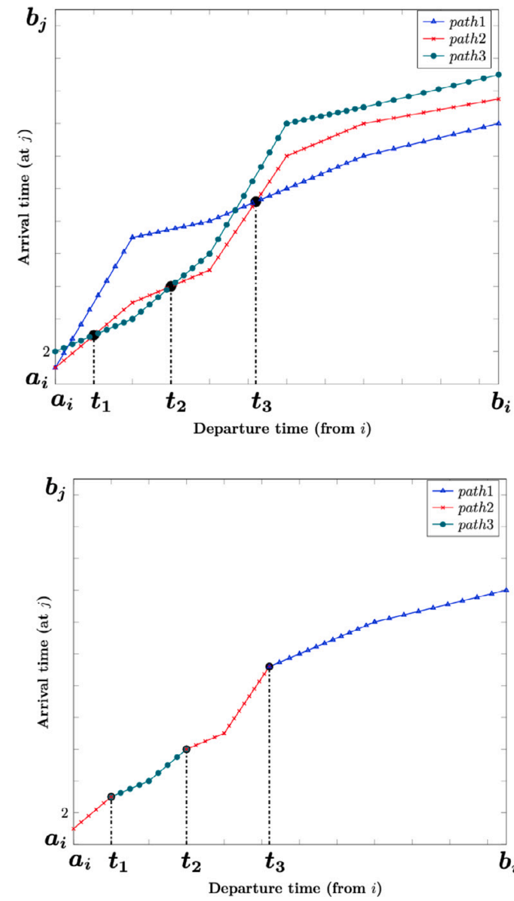


Fig. 7. Example from Gmira et al. (2021b): (a) the three alternative paths between points i and j; (b) the resulting arrival time function with the three minimization breakpoints t_1, t_2 and t_3.

The worst case analysis by Foschini et al. (2014) predicts an exponential growth of the number of PL function pieces, when the traveling time Φ of the road network is encoded on a customer-based graph as a travel time profile Γ. Nevertheless, as observed by Vidal et al. (2021) in their computational experiments, the number of PL pieces associated to $\Gamma_{ij}(t)$ remains sufficiently small to be efficiently computed and stored in a few seconds. Indeed, the increase in the number of PL function pieces is mainly due to the number of minimization breakpoints. However, in most cases, given an origin–destination pair, there are few quickest path changes occurring on a road-network in a single day.

4.3. Approximating travel times for efficient evaluation of route durations

In routing optimization algorithms, approximated travel times can be useful in a number of ways. Vidal et al. (2021) approximate the time-dependent quickest path duration with its value at the best starting time, which is obtained as a by-product of the Φ preprocessing. Gmira et al. (2021b) evaluate the impact of a CROSS exchange by an approximate computation of a penalty function of the delays induced by a local move.

Calogiuri et al. (2019), Adamo et al. (2021) and Adamo et al. (2020) define a lower approximation $\underline{\tau}()$ on $\tau(t)$ by decomposing the speed values v_{ijh} as (Cordeau et al. (2014)):

$$v_{ijh} = u^0_{ij} b^0_h \delta^0_{ijh}, \quad (3)$$

where:

- u^0_{ij} is the maximum travel speed across arc $(i,j) \in A$, i.e., $u^0_{ij} = \max_{h=0,\dots,H-1} v_{ijh}$;

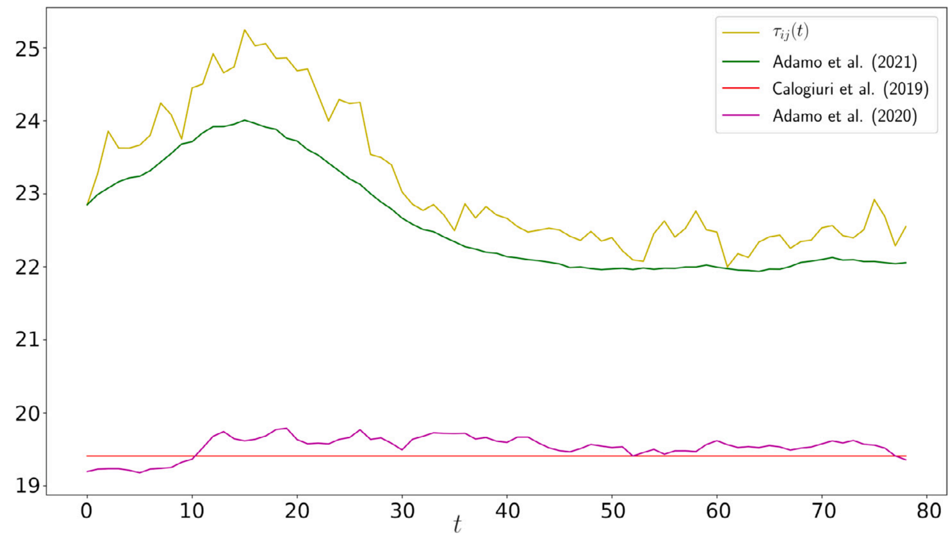


Fig. 8. Comparing the $\underline{\tau}$ functions determined by Adamo et al. (2021) with those of Calogiuri et al. (2019) and Adamo et al. (2020) (a time slot t corresponds to a 5-min interval).

- b_h^0 belongs to $]0,1]$ and is the best (i.e., lightest) congestion factor during $h-th$ interval, i.e., $b_h^0 = \max_{(i,j)\in A} v_{ijh}/u_{ij}^0$;
- δ_{ijh}^0 belongs to $]0,1]$ and represents the degradation of the congestion factor of arc (i,j) in the $h-th$ interval with respect to the less-congested arc in the same period.

Before delving into these approximations, we describe some properties of the Cordeau et al. (2014) decomposition. Let

$$\Delta^0 = \min_{(i,j)\in A, h=0,\dots,H} \delta_{ijh}^0$$

be the heaviest degradation of the congestion factor of any arc $(i,j)\in A$ over $[0,T]$. If $\Delta^0 = 1$, then all arcs $(i,j)\in A$ share a common congestion factor b_h^0 during intervals $[T_h, T_{h+1}]$ $(h=0,\dots,H-1)$. In such a case, the graph satisfies the *path ranking invariance* property, i.e., the quickest path between any two vertices does not change over time (while, of course, its traversal time may vary). Moreover, the optimal tour of the *Time-Dependent Traveling Salesman Problem* (TD-TSP) is the same as the optimal tour of the classical *Traveling Salesman Problem* (TSP) w.r.t. (time-invariant) optimistic u_{ij}^0 travel times (of course, the objective function values are different). On the other hand, if $\Delta^0 < 1$ this TSP solution corresponds to a lower bound on the optimal solution value of the associated TD-TSP. With reference to the *Time-Dependent Rural Postman Problem*, Calogiuri et al. (2019) define a lower approximation $\underline{\tau}$ as the optimal solution value of a *Rural Postman Problem* defined w.r.t. the most favorable congestion factor during each interval i.e., $\underline{v}'_{ijh} := f_h u_{ij}$ $(h=1,\dots,H)$. The main drawback of this approach is that, if a subset of the arcs is time-invariant, then traffic congestion factors b_h are all equal to 1. In this case, the lower approximation $\underline{\tau}$ determined by Calogiuri et al. (2019) is constant and equal to L_{ij}/u_{ij} (see Fig. 8).

As already observed, the condition $\Delta = 1$ is a sufficient (but not necessary) condition for strict path ranking invariance. Indeed, the path ranking invariance is basically due to a low variability of δ values, which can also occur when the Δ value is low. Fig. 9 shows the δ^0 factors associated to the Greater London application described in Section 1. The instance exhibits a Δ value remarkably lower than 1, but also significant portions of the time horizon during which the mean, the median and the inter-quartile range are almost constant. Based on this remark, Adamo et al. (2020) enhanced the speed decomposition of Cordeau et al. (2014), by preliminary observing that factorization (3) still holds if parameters b_h and δ_{ijh} $(h=0,\dots,H-1)$ are computed on the basis of a maximum speed u_{ij} *greater* than u_{ij}^0:

$$u_{ij} \ge u_{ij}^0 \qquad (i,j)\in A. \tag{4}$$

This amounts to adding an additional time slot $h=H$ (in which the vehicle has already returned to the depot) with speed $u_{ij} = v_{ijH} \ge v_{ijh}$ $(h=0,\dots,H-1)$. Let $\mathbf{u}$ be the vector of u_{ij} associated to arcs $(i,j)\in A$. Then, the travel speeds can be expressed as

$$v_{ijh} = u_{ij} b_h(\mathbf{u}) \delta_{ijh}(\mathbf{u}), \tag{5}$$

where:

- $b_h(\mathbf{u}) \in [0,1]$ is the best congestion factor during interval $[T_h, T_{h+1}]$ w.r.t. $\mathbf{u}$, i.e.,

$$b_h(\mathbf{u}) = \max_{(i,j)\in A} \frac{v_{ijh}}{u_{ij}}; \tag{6}$$

- $\delta_{ijh}(\mathbf{u}) = \dfrac{v_{ijh}}{b_h(\mathbf{u}) u_{ij}}$ belongs to $[0,1]$ and represents the degradation of the congestion factor of arc (i,j) in interval $[T_h, T_{h+1}]$ w.r.t. the least congested arc in $[T_h, T_{h+1}]$.

With each vector $\mathbf{u}$, we denote with $\underline{z}(\mathbf{u})$ the traversal time of the optimal (time-invariant) TSP solution w.r.t. L_{ij}/u_{ij}. Hence, a lower bound $LB(\mathbf{u})$ is the traversal time, w.r.t. the $\underline{\tau}_{ij}(t)$ functions, of the (time-invariant) TSP solution. In addition, an upper bound $UB(\mathbf{u})$ is provided by the traversal time of this tour w.r.t. the $\tau_{ij}(t)$ functions.

By increasing the u_{ij} variables, $\underline{z}(\mathbf{u})$ decreases (or remains the same) since some of the L_{ij}/u_{ij} arc costs decrease. At the same time, the traffic factors b_h decrease (or remain the same). Hence, the effects of the two terms in $\phi()$ are conflicting. As a result, ϕ may increase, decrease or remain unchanged. Adamo et al. (2020) study the associated optimization problem and propose an efficient heuristic to determine "good" $\mathbf{u}$ values.

As shown in the example reported in Fig. 8, Adamo et al. (2021) get a lower approximation $\underline{\tau}$ that fits the original τ better than Adamo et al. (2020). Indeed, the fitting procedure by Adamo et al. (2020) aims to minimize the deviation between the (original) speed values v_{ijh} and the most favorable speed value $\underline{v}_{ijh}$, during some (not necessarily consecutive) periods. Fig. 8 compares the approximations provided by Calogiuri et al. (2019), Adamo et al. (2020) and Adamo et al. (2021). It is worth noting that the latter approach guarantees that there always exists at least one time instant in which the lower approximation $\underline{\tau}$ equals the original travel time function τ.

5. Time-dependent routing with path flexibility

As observed in Section 1, most studies are based on an abstraction of the road network, referred to as *customer-based* graph, where vertices

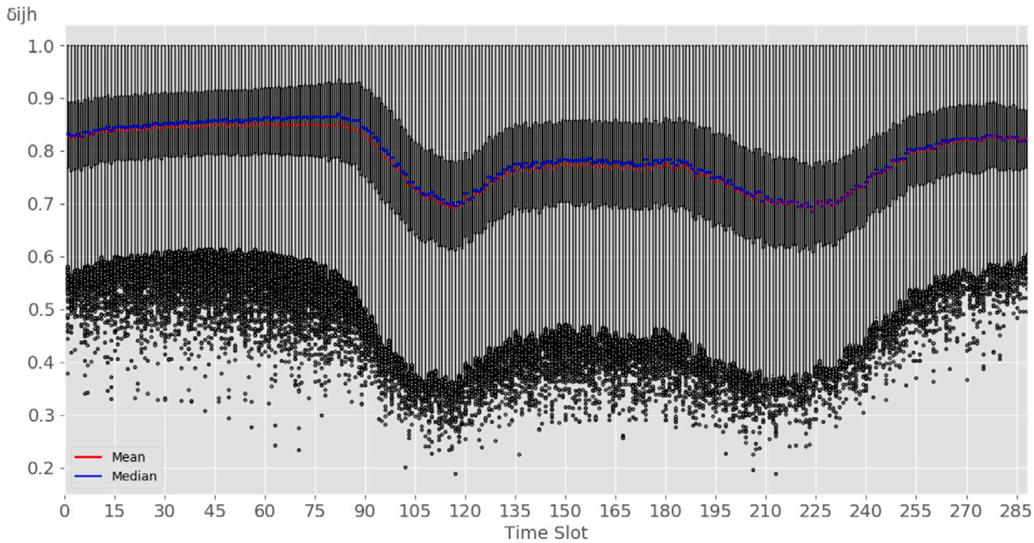


Fig. 9. Boxplots of the δ^0 factors in a typical day of operations in the Greater London (UK) application described in Section 1.

represent "points of interest" (POIs), i.e., customers, depots, etc., and arcs correspond to quickest paths between POIs. In general, these quickest paths may change over time, i.e., different start times may lead to different quickest paths. If the VRP depends on parameters (transportation cost, ...) proportional to arc traversal times, this approach allows to pre-compute such quickest paths and decouple the problem into: (1) the path selection at road network level; (2) the routing decisions on the *customer-based* graph. However, as observed in Vidal et al. (2020), in urban areas road networks have a complex structure, with traffic congestion affecting several road attributes other than driving time, i.e., tolls, transportation mode, attractiveness, energy consumption and emissions. This may lead to complex time-dependent cost functions, such that each arc start traversal time might not correspond to a single *min-cost* path, but several non-dominated *best paths* have to be taken into account. This aspect is especially relevant in time-dependent vehicle routing problems, where the objective function (e.g., distance) and constraints (e.g., time windows) are defined in terms of different attributes (Ben Ticha et al. (2021)). Further issues arise in time-dependent green vehicle routing problems, where the minimization of emissions depends not only on vehicle speed but also on vehicle load. Therefore, in order to determine the time-dependent emission-minimizing path, it is required to know the vehicle load, which cannot be determined in advance since it depends on the points-of-interest sequence (Ehmke et al. (2016b)). Similar considerations apply to urban distribution with battery-powered vehicles, where the cost function models time-dependent energy consumption. Even more complicated is the situation when routing decisions are integrated with time-dependent decisions to be taken en-route at the road network level, such as travel speed for each road segment, driver breaks at intermediate nodes, recharging stops for electric vehicles, etc. (Vidal et al. (2020)). Accounting for such *alternative* point-to-point paths gives rise to time-dependent routing problems with *path flexibility*, where routing decisions are integrated with path selection on the road network. Research on this topic is fairly recent and scattered. Two alternative approaches stand out in literature. The first approach represents the road network as a multigraph, containing multiple alternative paths between every pair of POIs. In the second approach, the problem is directly solved on the original road network. In the following we review the literature on VRP with *path flexibility* (PF) according to these two criteria.

5.1. *Path flexibility on a customer-based multigraph*

Time-dependent path flexibility comes into play when multiple objectives are relevant, requiring the selection of paths constituting a good trade-off among two or more criteria. This may require the construction of a multigraph on which multi-criteria cheapest path problems are solved. Several contributions pre-compute, as a first step, a (non-exhaustive) set of such paths between each pair of nodes. Then, when constructing a solution, each path is evaluated according to the cost function actually optimized.

In Wang et al. (2014), the authors consider the TDVRP with Time Windows, where path flexibility is modeled by connecting each pair of nodes with two arcs. The first arc represents the best path when the traffic is low and is characterized by a time-dependent travel time function defined according to the Ichoua et al. (2003) model. The second arc represents the quickest path during the peak hours and is assigned a constant travel time. Thus, one arc dominates the other one, depending on the departure time. The authors propose a heuristic algorithm based on *Particle Swarm Optimization*. The goal is the minimization of the total route cost, consisting of the fixed transportation cost, a time-dependent cost and distance cost.

In Alinaghian and Naderipour (2016), the authors apply *Particle Swarm Optimization* to solve the *Time-Dependent Pollution Routing Problem* (TDPRP), a variant of the TDVRP aiming to minimize environmental pollution and the total cost. They model path flexibility with two alternative paths between each pair of customers, corresponding respectively to low-traffic and high-traffic. Emissions are determined by taking into account vehicle speed, vehicle load and road gradient. The proposed approach is used to deal with a dairy distribution problem in Esfahan (Iran).

In Qian and Eglese (2016), the authors propose a Tabu Search-based column generation algorithm to solve the TDVRP with Time Windows. The goal is the minimization of greenhouse gas (GHG) emissions. When new routes (columns) are generated, the path between two consecutive customers is determined by NHA algorithm, a heuristic approach proposed in Qian and Eglese (2014). The idea is to first select a (non-exhaustive) set of candidate paths on the road network and then determine speed/stopping at customers so that the GHG-minimizing path is selected. To solve the problem, the authors propose two solution approaches: a dynamic programming algorithm and a heuristic adaptive search method. The latter first selects a set of candidate paths and then defines the travel speed for each road segment on the selected paths.

The authors in Setak et al. (2015) investigate a TDVRP defined on a multigraph where multiple arcs represent alternative time-dependent paths. The goal is to minimize a transportation cost including energy consumption, total travel time, vehicle acquisition cost and tolls. A

Tabu Search metaheuristic is proposed to solve the problem with neighborhoods based on swapping two customers and reversing the sequence of in-between customers. The computational campaign is carried out on instances with two (alternative) arcs connecting each pair of nodes.

In Huang et al. (2017), the authors formulate the TDVRP where path flexibility is evaluated under both deterministic and stochastic traffic conditions. In this section we review only the deterministic approach, while the stochastic version is discussed in Section 5.2 since it is road-network based. The (*deterministic*) quickest paths between each pair of customers are determined in advance for a given (discrete) set of departure times. Besides the time-dependent quickest paths, the set of alternative paths also includes the distance-minimizing path, with the aim of (partially) taking into account the effect of load-dependency. The authors also deal with *time flexibility*, meant as flexible departure time at the depot or post service waiting time at customers in order to save future cost by avoiding traffic. The objective is to minimize the total cost consisting of fuel cost and vehicle depreciation cost. A MILP model, in which each alternative path is associated with a decision variable, is proposed. The authors generate instances based on the road network of Beijing. The time horizon is discretized. For each (discretized) time point, an alternative path is generated by computing the time-dependent quickest path between arc endpoints. Different levels of time discretization are considered and a multigraph with 20 customers is generated for each of them. The maximum number of alternative arcs is equal to nine, obtained with the finest discretization level (10 min). Instances are solved using IBM ILOG Cplex. Results show that the benefit of a finer discretization is very small. Moreover the role of time flexibility is not relevant compared to path flexibility.

To the best of our knowledge, *the computation of an exhaustive set of all eligible alternative paths* has been addressed for the first time as an open research issue in Ehmke et al. (2016b). The authors study a time-dependent VRP with the objective of minimizing carbon emissions. They demonstrate the following property: when the min-cost path is the same for an empty and a full-loaded vehicle, it will remain the same for any intermediate load. This result allows expected time-dependent fuel consumption for all those paths shown to be load-independent to be precomputed. As discussed in the following section, they then exploit this result to partially reduce the computational burden of a solution approach, based on a road-network.

In Raeesi and Zografos (2019), the authors introduce the *Time-Dependent Steiner Pollution Routing Problem* with time windows, flexible departure times, multi-trips and heterogeneous fleet. The problem is multi-objective since it aims to minimize three objective functions related to vehicle hiring cost, total amount of fuel consumed, and total makespan of the routes. The authors advance the result of Ehmke et al. (2016b). In particular, they propose a *Path Elimination Procedure* (PEP) that reduces the number of eligible paths, by identifying and discarding proven redundant paths in a pre-processing step. The authors propose a MILP formulation based on the PEP.

5.2. Path flexibility on the road network

To overcome the issues inherent in the pre-computation of the complete set of alternative paths, some contributions propose to define explicitly their TDVRP at the road-network level.

Ehmke et al. (2016b) deal with a time-dependent VRP aiming to minimize carbon emissions. The authors propose a Tabu Search heuristic, where local search moves are evaluated by computing optimal point-to-point paths on-the-fly for given starting time and load. To this aim, they use a heuristic algorithm proposed in Ehmke et al. (2016a). Best paths computed on-the-fly are stored to avoid wasting time in repeated computations. As afore discussed, the authors exploit the above mentioned property to precompute a set of load-independent paths.

In Huang et al. (2017), the authors observe that vehicle routes obtained under a deterministic assumption are good approximations of solutions under stochastic traffic conditions. For this reason, besides a two-stage stochastic MILP, they devise a route-path approximation method encoding a two-stage stochastic strategy. During the first stage, sequences of customers are determined considering expected traffic conditions. In the second stage, information on travel times is known and therefore path selection is made directly on the road network according to real-time traffic conditions. Results show that path flexibility is a natural recourse action for coping with different traffic conditions.

In Ben Ticha et al. (2021), a branch-and-price algorithm is directly applied at the road network level to solve the TDVRP with Time Windows, where the objective function is the minimization of the total traveled distance. The authors adapt a branch-and-price algorithm previously developed for the same problem with constant travel times (Ben Ticha et al. (2019)). The pricing problem is reduced to a *Time-Dependent Shortest Path Problem with Resource Constraints*, addressed with a time-dependent labeling algorithm, where a bi-directional search strategy aims to limit the length of the subpaths defined by the labels. The branching rule is designed to account for road arcs traversed multiple times. Computational experiments show that the solution cost on the road network provides average improvements of 1.7% and 7.3% over the min-distance customer-based graph and min-time customer-based graph, respectively.

Gmira et al. (2021b) proposed a Tabu Search heuristic for the TDVRPTW on road network graphs. A major result of the paper is a CROSS exchange operator, which allows neighborhood solutions to be evaluated in constant time with a special structure named *dominant shortest path structure* (see Fig. 7). Computational results show that the proposed solution approach allows instances with up to 200 nodes and 580 arcs to be handled in very reasonable computing times.

As stated in Ben Ticha et al. (2018), a promising research area is represented by dynamic time-dependent routing problems with real-time traffic information provided at road-network level. We are aware of only one contribution on this subject. In Gmira et al. (2021a), the authors consider the dynamic version of the problem studied in Gmira et al. (2021b). In particular, they devise a reactive procedure for accounting for real-time changes in road travel speeds. For each route affected by the speed update, the new arrival and departure times are computed for each customer along the planned route. If the route is not feasible, the procedure attempts to repair feasibility by computing a new minimal travel time path on the road network. If the route remains infeasible, the procedure modifies the sequence of customers by applying a local search based on Or-opt exchanges. The obtained route is kept, by handling (possible) infeasibility as a penalty for lateness at customers and at the depot. The results obtained on a real road network show the benefits of the proposed procedure over a non-reactive procedure.

6. Dynamic discretization discovery

Time-dependent vehicle routing problems fall in the broader area of time-dependent optimization problems, i.e., problems in which activities and resources have to be scheduled over time. A first approach is to use compact models in which nodes $i \in V$ (or $i \in V'$) are associated to continuous arrival and departure times (say, $t_i^a \in [0, H]$, $t_i^d \in [0, H]$, respectively). Such formulations are nonlinear in nature and their linearizations provide weak relaxations (i.e., poor dual bounds) (Ghiani and Guerriero (2019)). An alternative approach, especially successful with time-window constraints, is based on a discretization of time. In such *Integer Programming* (IP) formulations, the planning horizon $[0, H]$ is replaced by a discrete time horizon $D_t = \{t_1, \dots, t_K\}$, where $t_k \in [0, H]$ for $k = 1, \dots, K$ with $t_1 < t_2 < \dots < t_K$. Each vertex i of the road (or customer-based) graph is time-expanded, i.e., associated to a number of (i, k) nodes ($k = 1, \dots, K$), each representing the visit of vertex i at time t_k. Moreover time-expanded arcs $a = ((i, k), (j, k'))$ are used to model the transition of a vehicle leaving i at time t_k and reaching j at time $t_{k'}$. In time-dependent VRPs, $t_{k'}$ may correspond exactly to the arrival time $\Gamma_{ij}(t_k)$, i.e., $t_k + \tau_{ij}(t_k)$ or can be an upper or lower approximation of the arrival time. The *time-indexed* (TI) model is then cast as a shortest path from a dummy source node to a dummy sink node. Here, binary decision variables x_a are associated to the selection of time-expanded arcs a. Depending on whether lower or upper approximations are used, a dual or a primal bound to the optimal objective function value is obtained. To limit the computational effort, coarse discretizations are often used (Albiach et al. (2008), Soler et al. (2009)). Consequently, the approach may generate poor solutions or may be unable to find a feasible solution even if one exists. On the other hand, solving fine discretizations to optimality may be time and memory-consuming. For TD vehicle routing problem, the existence of a finite number of solutions guarantees that the possible arrival/departure times are finite as well. Therefore, there exists a TI model providing the optimal solution of the original continuous problem. Unfortunately, finding such a *complete* time-discretization is computationally challenging. To overcome this drawback, Boland and Savelsbergh (2019) proposed a novel paradigm, dubbed *Dynamic Discretization Discovery* (DDD) allowing the solution of a complete TI model to be constructed without fully generating it. The solution approach is initialized with a formulation on an "easy to solve" partially time-expanded network, obtained by *aggregating* the time instants of the (*original*) time discretization. In such a partially time-expanded network, there may be multiple arcs between some pairs of (aggregated) nodes. In time-window constrained problems, earliest and latest departure times, as well as travel times, are used to select which arcs are included into the partially expanded network.

More formally, let e_i and l_i be the earliest and latest departure times at customer $i \in V'$. Moreover, with a little abuse of notation, let $\tau_{ij}^{-1}(t)$ denote the time needed to traverse arc (i,j) as a function of the arrival time t at node j. For a single route, a *complete* TI model includes all the arcs $((i,t),(j,\max\{e_j, t+\tau_{ij}(t)\}))$, for $i,j \in V'$ modeling travel from node i to j departing at time instants $t = e_i, \dots, \min\{l_i, l_j - \tau_{ij}^{-1}(l_j)\}$. On the contrary, in a partially time-expanded network only a subset $\mathcal{T}_i$ of such time instants is included. Let $\mathcal{V}$ and $\mathcal{A}$ be the vertices and arcs of such an aggregation, respectively. The timed arcs have the form $((i,t),(j,\bar{t}))$, where $(i,j) \in A'$, $t \in \mathcal{T}_i$, and $\bar{t} \in \mathcal{T}_j$. It is worth noting that arc $((i,t),(j,\bar{t}))$ does not have to satisfy $\bar{t} = t + \tau_{ij}(t)$, even if the vehicle does not wait at j. This feature provides a mechanism to control both the size of the time-expanded network and the approximation properties of the resulting IP model.

In order to obtain a relaxation, Vu et al. (2020) show that three conditions guarantee a lower bound:

(i) $(i, e_i) \in \mathcal{T}_i$ for all $i \in V'$;
(ii) for all $(i,t) \in \mathcal{V}$ and $j \in V'$ with $t + \tau_{ij}(t) \le l_j$, there exists a time instant $\bar{t}$ with $((i,t),(j,\bar{t})) \in \mathcal{A}$;
(iii) every timed arc is *short*, i.e., $\bar{t} \le \max\{e_j, t + \tau_{ij}(t)\}$.

It can be proved that the best such lower bound is obtained by selecting $\bar{t} = \max\{t' : t' \le \max\{e_j, t+\tau_{ij}(t)\}, (j,t') \in \mathcal{V}\}$ with no other arc from (i,t) to j in $\mathcal{A}$. On the other hand, an upper bound from the IP can be obtained, provided the IP is feasible, if all arcs $((i,t),(j,\bar{t})) \in \mathcal{A}$ are "long", i.e., $\bar{t} \ge \max\{e_j, t + \tau_{ij}(t)\}$.

A dynamic discretization discovery algorithm iteratively refines an initial partially time-expanded model by *disaggregating* time-indexed nodes until the optimality gap is closed. More precisely, the algorithm is made up of the following steps:

Step 1 A partially time-expanded model providing a lower bound on the optimal value of the complete TI model is solved;
Step 2 The corresponding optimal solution is checked for feasibility and/or optimality for the complete IP model;
Step 3 If this control fails, the algorithm identifies time points that, added to the partial discretization, make the current optimal solution no longer feasible for the new lower-bound IP model; then Step 1 is executed again.

This approach has been used in a number of contributions, including He et al. (2022) for solving the time-dependent quickest path problem, Vu et al. (2020) for the time-dependent TSP with Time Windows, Vu et al. (2022) for the time-dependent minimum tour duration problem and the time-dependent delivery man problem. Very recently, in Gnegel and Fügenschuh (2023) the authors propose a branch-and-refine paradigm, where *the iterative refinement of the time-expanded model* is embedded in a branch-and-bound tree instead of restarting whenever the optimal solution of the partially time-expanded model is found to be infeasible.

7. Efficient neighborhood evaluation

In spite of a number of recent advancements, exact algorithms can currently solve only TDVRP instances of moderate size (see, e.g., Adamo et al. (2021) and Dabia et al. (2013)). Hence, heuristics are needed to tackle large scale instances arising in real world applications in the attempt to obtain "good" solutions in a "reasonable" amount of time. The majority of such algorithms (Vidal et al. (2013)) are based on some sort of Neighborhood Search (NS), which attempts to enhance an initial feasible solution $s^{(0)}$. At iteration k, the NS procedure searches a neighborhood $N(s^{(k)})$ made up of solutions "close" to the current solution $s^{(k)}$. Neighbors are obtained through a transformation called a "move". If an improving solution $s^{(k+1)}$ is found, a new iteration is performed. Otherwise, the NS stops and a local optimum is obtained. To escape local optima, the NS is often embedded into a metaheuristic scheme. In time-dependent vehicle routing, the most time-consuming part of such algorithms is the assessment of the feasibility and the objective function value for all the moves in a neighborhood, an aspect which is particularly relevant when time windows and time constraints are imposed.

Two kinds of speed-up techniques have been proposed: pre-checks and pre-calculations. Pre-checks are fast computations to prove the infeasibility or dominance of a move without performing the complete time-consuming exact feasibility or objective function calculations. Examples include checking time window violations by using bounds on travel times, and evaluating the objective function using bounds on the exact value. On the other hand, pre-calculations try to speed up the evaluation of neighbors $s' \in N(s^{(k)})$ by storing partial calculation results, common to all the neighbors of the current solution $s^{(k)}$. As such, these data need only to be updated at each new iteration of the NS procedure.

This research line initially considered time-invariant VRPs and has been recently extended to include some TD VRPs. Savelsbergh (1992) shows how to implement an edge-improvement exchange method for the VRP-TW in constant time instead of the $O(n)$ time of the straightforward approach (n being the number of customers). Campbell and Savelsbergh (2004) show how to efficiently incorporate complicating constraints (such as time windows and shift time limits) into an insertion algorithm for the VRP, resulting into a $O(n^3)$ or $O(n^3 log(n))$ time complexity instead of the $O(n^4)$ complexity of a naive approach. Kindervater et al. (1993) use precalculated values related to time windows and capacity constraints of a route stored in memory as global data. Moves are then evaluated in a so-called lexicographic order by using such global variables. Vidal et al. (2015) review and generalize this approach in a "reoptimization by concatenation" method for many timing problems. Using this method, move evaluations can be done in constant time for the time-dependent VRPTW (without duration constraints). Hashimoto et al. (2008) deal with the time-dependent VRPTW and show how to identify the best solution in a (slightly modified) 2-opt neighborhood by utilizing prior information in a dynamic programming algorithm. Visser and Spliet (2020) show that the earliest and latest arrival time global variables introduced by Campbell and Savelsbergh (2004) can be generalized to the time-dependent version of the VRPTW. In particular, they present a procedure, dubbed F/B method, which stores forward and backward ready time functions in memory, which reduces the move evaluation complexity from $O(n^2)$ to $O(n)$. This turns out to be particularity useful for evaluating advanced neighborhoods such as k-exchange. Empirical results on instances with 1000 customers showed a speed-up of 8.89 times for a construction heuristic and 3.94 times for an exchange neighborhood improvement procedure. Moreover, Visser and Spliet (2020) extend their method to the case of multiple time windows.

8. *Machine learning* -based methods

Leveraging *Machine Learning* (ML) to solve *Combinatorial Optimization* (CO) problems has been recently the focus of a promising research line, with many potential applications in logistics, transportation, scheduling, and resource allocation. Studies at the intersection of CO and ML can be classified into two primary directions: *ML-augmented CO* and *End-to-End CO learning*. The former focuses on the utilization of ML to assist optimization algorithms to make good decisions. For instance, computationally heavy tasks, such as constraint or objective function evaluation, can be replaced by fast approximation of input/output patterns provided by ML algorithms, resulting in more efficient optimization algorithms. Other examples are the choice of a branching variable in a branch-and-bound algorithm and the selection of the next neighborhood to be explored in a metaheuristic. On the other hand, *End-to-End CO learning* aims at predicting solutions to optimization problems. For a methodological overview of *ML-augmented CO* see Bengio et al. (2021) while for a survey of *End-to-End CO learning* see Kotary et al. (2021). As far as Vehicle Routing is concerned, automated learning mechanisms can be used to guide the optimization process, by exploiting the information gained when solving instances with similar features (e.g., with the same traffic and demand patterns) in previous days, weeks, etc, as it is customary in distribution management. The resulting speed-up can be particularly useful in TD routing problems where a major role is played by dynamic variables such as traffic congestion and road closures. Despite its potential, research in this field is still at a very early stage: indeed, at the time of writing, the authors are aware of only four contributions on ML-based methods for solving the TDTSP. In Ghiani et al. (2020), the authors illustrate how to integrate supervised and unsupervised ML techniques in a simple constructive heuristic for the TDTSP. They use a multilayer perceptron regressor to predict the arrival time at each customer zone based on the demand rate and spatial distribution. The predictions are then used to guide a fast heuristic. The customer zones are preliminarily defined through an unsupervised technique that considers traffic patterns. Computational results show that the proposed solution approach is promising in a real-time context.

More recently, Adamo et al. (2023a) develop a time-invariant approximation of the TDTSP that is then used to devise fast upper bounds. The approximation is based on the solution of a (typically very large) linear program. A multilayer perceptron regressor is used to predict the arrival times at each customer zone, which allows to reduce heuristically the size of the linear program. Computational results show that this approach outperforms (Ghiani et al., 2020) in those (non-real-time) settings where it can be reasonable to determine high quality TDTSP solutions in half a minute.

In Wu et al. (2021), the TDTSP with time windows is modeled as a Markov decision process and deep reinforcement learning is used to train an agent to make a decision at each location. Extensive experiments indicate that the proposed method can capture the real-time traffic changes and yield high-quality solutions within a very short time, compared with a greedy heuristic and a simulated annealing algorithm.

Hansknecht et al. (2018) deal with the problem of selecting a branching rule for the TDTSP in the presence of several fractional variables with the objective to yield smaller branch-and-bound trees. They propose to learn a branching rule based on several generic MIP features (such as fractionality, pseudocosts, etc.) as well as some features peculiar to the TDTSP. They use the *LambdaRank algorithm* (see, e.g., Burges et al. (2006)), a technique where the various (branching) alternatives are ranked by solving pairwise classification or regression problems.

9. Main challenges and research opportunities

The literature review presented in this article confirms that *time-dependent vehicle routing* is a dynamic and evolving research field. In the following, we discuss a number of related challenges and research opportunities. In particular, we focus on problems arising in distribution management (and, in particular, last-mile distribution) albeit many of the following considerations also apply to fleet management in other settings (such as emergency services, garbage collection, street cleaning, etc.). For ease of presentation, we identify three major areas: research opportunities arising from the rapid advancement of information and communication technologies (ICTs), those deriving from the increase in size of the distribution problems posed by the e-commerce explosion and those motivated by the adoption of new business practices in last-mile delivery.

9.1. Opportunities arising from the rapid advancement of information and communication technologies

As shown in Section 4, the data coming from sensors, probe vehicles and smart phone applications can be exploited to estimate historical traffic patterns and then used to build a-priori routes. However, it is often the case that some unexpected events may affect travel times significantly. A remarkable challenge is to develop a real-time system that, based on both *structured* data (those produced by sensors) and *unstructured* data (data sources in a *natural language* such as incident reports, weather reports,…), may update speed predictions in the next few hours as soon as fresh data become available. In turn, these forecasts might be used to re-route vehicles in real-time (if operationally feasible) as in Gmira et al. (2021a). To this purpose Gmira et al. (2020) suggest, as a future research direction, to extend the LSTM neural network model they use for the historical pattern prediction. However, this modification might not be straightforward, especially because of the unstructured data whose processing requires specialized techniques to extract significant features from text (see Jurafsky (2000)). A further interesting challenge motivated by the availability of "big data" is the exploitation of "tacit" knowledge in time-dependent VRPs. As highlighted by the *Amazon Last Mile Routing Research Challenge* held in 2021 (MIT, 2021), the definition of vehicle routes may be influenced by a number of factors (driver preferences, parking availability, etc.) that cannot be included explicitly into an optimization model. Current ICTs allow plenty of data to be collected as drivers follow their routes, including how drivers with different profiles react to traffic variability, ease of finding a parking spot, etc. Then ML models could be used in principle to extract knowledge from such data, e.g. by performing a prediction of the changes made by a driver with given features. In this context, it would be interesting to develop time-dependent VRP algorithms generating routes for specific drivers, along the lines of Cook et al. (2022).

9.2. Opportunities arising from the increase in size of the distribution problems posed by the e-commerce explosion

From a methodological point of view, there is a lack of contributions in solving very large-scale time-dependent routing problems. Current heuristics are evaluated on benchmark instances with up 1000 points of interest (Visser and Spliet (2020)). As pointed out in Arnold et al. (2019), emerging application contexts motivate the development of solution methods that can handle thousands of points of interest. To this aim it would be interesting to extend to the time-dependent case the results in Bramel and Simchi-Levi (1995, 1996) where the authors present a characterization of the asymptotic optimal solution of the (time-invariant) capacitated VRP with and without time windows for general distributions of service times, time windows, customer loads and locations. This characterization led to the development of asymptotically optimal heuristics based on formulating the problems as capacitated location problems. It would also be interesting to investigate how the increase in space and time complexity can be mitigated by the latest results provided by Visser and Spliet (2020) for neighborhood evaluation combined with travel time approximation by Adamo et al. (2023a), as well as the closed-form travel time functions by Vidal et al. (2021).

9.3. Opportunities arising from the adoption of new business practices in last-mile delivery

A promising research line is motivated by companies and governing authorities that are stimulating the adoption of new transportation paradigms in which a *primary* vehicle can transfer a portion of its load to *secondary* vehicles, such as bikes or scooters or ground drones, which are less sensitive to traffic congestion. Such delivery configurations make use of transhipment points (as in Baldacci et al. (2017)) and constitute two-echelon systems (see Cuda et al. (2015) for a review) in which each echelon refers to one level of the distribution network. These features introduce new operational constraints to time-dependent VRPs due to the heterogeneous travel time functions and possibly to synchronization issues (Drexl (2012)). In this framework, a crucial issue is to define the location of the transhipment points (which is a location-routing problem, see Schiffer et al. (2019)) and the number and type of the vehicles (which is a fleet sizing and mix problem, see Masmoudi et al. (2022) for a recent state of the art). Here the rationale is to identify areas of the service territory (usually densely populated areas) where it is worth using vehicles less sensitive to traffic while designing routes. This is a quite new area except for the contributions by Schmidt et al. (2019, 2023) that however use rather simplistic travel time models that do not satisfy the FIFO property.

CRediT authorship contribution statement

Tommaso Adamo: Conceptualization, Writing – original draft, Writing – review & editing. **Michel Gendreau:** Conceptualization, Writing – original draft, Writing – review & editing. **Gianpaolo Ghiani:** Conceptualization, Writing – original draft, Writing – review & editing. **Emanuela Guerriero:** Conceptualization, Writing – original draft, Writing – review & editing.

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