Soft Comput (2014) 18:1463–1470
DOI 10.1007/s00500-014-1265-1

FOUNDATIONS

A note on standard completeness for some extensions of uninorm logic

Paolo Baldi

Published online: 9 April 2014

Abstract We prove standard completeness for uninorm logic extended with *knotted* axioms. This is done following a proof-theoretical approach, based on the elimination of the density rule in suitable hypersequent calculi.

Keywords Hypersequent · Density elimination · Standard completeness · Fuzzy logic · Knotted axioms · Uninorm logic

1 Introduction

In mathematical fuzzy logic, the intended or *standard* semantics is based on algebraic structures over the real interval [0, 1], see (Hajek 1998). Showing that a logic is *standard complete*, i.e., complete with respect to the standard semantics, is therefore of crucial importance to the field. Two main approaches have been developed to tackle the problem: *algebraic* and *proof-theoretical*. The former was introduced for proving standard completeness for monoidal t-norm logic MTL (Jenei and Montagna 2002). The method has since been extended to prove standard completeness for many axiomatic extensions of MTL, see e.g., (Cintula et al. 2009; Esteva et al. 2002; Horcik 2011). However, no algebraic proof of standard completeness has been found for uninorm logic UL (Metcalfe and Montagna 2007) (MTL without weakening/integrality). Only for a handful of its axiomatic extensions has standard completeness been proved algebraically (Wang 2012; Gabbay and Metcalfe 2007). In particular, Wang (2012) shows, algebraically, standard completeness for any logic extending UL with both the *n-mingle axiom* $\alpha^n \to \alpha^{n-1}$ and the *n-contraction* axiom $\alpha^{n-1} \to \alpha^n$, for given $n > 2$. An alternative, proof-theoretic approach for proving standard completeness was introduced in Metcalfe and Montagna (2007). This method has been used to establish standard completeness for UL and a few axiomatic extensions, as well as many axiomatic extensions of MTL (Baldi et al. 2012; Metcalfe and Montagna 2007; Ciabattoni and Metcalfe 2008). The main idea behind the proof-theoretic approach is that the addition of a special rule, called *density*, to any axiomatic extension of UL, makes the logic *rational complete*, i.e., complete with respect to algebras over the rationals in [0, 1], see (Ciabattoni and Metcalfe 2007, 2008; Metcalfe and Montagna 2007). Thus, showing the admissibility (or *elimination*) of the density rule entails rational completeness for the original logic. More precisely, given a logic L, the proof-theoretical approach to standard completeness can be summarized as follows:

(a) Define a cut-free proof system HL for L extended with the density rule.
(b) Prove that the density rule is eliminable in HL (density elimination), i.e., that it does not enlarge the set of provable formulas. This shows rational completeness for L.
(c) Standard completeness may then be obtained from rational completeness by means of the Dedekind–MacNeille completion.

In this paper, we follow the above steps to prove standard completeness for axiomatic extensions of UL with any set of *knotted axioms* $\alpha^k \to \alpha^n$, for $n, k > 1$. Knotted axioms were

Communicated by A. Ciabattoni.

Work supported by the project FWF START Y544-N23.

P. Baldi (✉)
Vienna University of Technology, Vienna, Austria
e-mail: baldi@logic.at

Table 1 Hypersequent calculus HUL for UL

$\dfrac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \alpha, \Delta \Rightarrow \Psi}{G \mid \Gamma, \Delta \Rightarrow \Psi}\,(cut)$	$\dfrac{}{G \mid \alpha \Rightarrow \alpha}\,(init)$	$\dfrac{}{G \mid \Gamma, \bot \Rightarrow \Psi}\,(\bot)$
$\dfrac{}{G \mid \Gamma \Rightarrow \top}\,(\top)$	$\dfrac{G \mid \Gamma \Rightarrow \Psi}{G \mid t, \Gamma \Rightarrow \Psi}\,(tl)$	$\dfrac{}{\Rightarrow t}\,(tr)$
$\dfrac{}{f \Rightarrow}\,(fl)$	$\dfrac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow f}\,(fr)$	
$\dfrac{G \mid \alpha, \beta, \Gamma \Rightarrow \Psi}{G \mid \alpha \cdot \beta, \Gamma \Rightarrow \Psi}\,(\cdot\, l)$	$\dfrac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Delta \Rightarrow \beta}{G \mid \Gamma, \Delta \Rightarrow \alpha \cdot \beta}\,(\cdot\, r)$	
$\dfrac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \beta, \Delta \Rightarrow \Psi}{G \mid \Gamma, \alpha \to \beta, \Delta \Rightarrow \Psi}\,(\to l)$	$\dfrac{G \mid \alpha, \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \to \beta}\,(\to r)$	
$\dfrac{G \mid \alpha_1, \Gamma \Rightarrow \Psi}{G \mid \alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Psi}\,(\wedge l)$	$\dfrac{G \mid \alpha_2, \Gamma \Rightarrow \Psi}{G \mid \alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Psi}\,(\wedge l)$	$\dfrac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \wedge \beta}\,(\wedge r)$
$\dfrac{G \mid \alpha, \Gamma \Rightarrow \Psi \quad G \mid \beta, \Gamma \Rightarrow \Psi}{G \mid \alpha \vee \beta, \Gamma \Rightarrow \Psi}\,(\vee l)$	$\dfrac{G \mid \Gamma \Rightarrow \alpha_1}{G \mid \Gamma \Rightarrow \alpha_1 \vee \alpha_2}\,(\vee r)$	$\dfrac{G \mid \Gamma \Rightarrow \alpha_2}{G \mid \Gamma \Rightarrow \alpha_1 \vee \alpha_2}\,(\vee r)$
$\dfrac{G \mid \Gamma \Rightarrow \Psi}{G \mid \Gamma \Rightarrow \Psi \mid \Gamma \Rightarrow \Psi}\,(ec)$	$\dfrac{G}{G \mid \Gamma \Rightarrow \Psi}\,(ew)$	
	$\dfrac{G \mid \Gamma_1, \Delta_1 \Rightarrow \Psi_1 \quad G \mid \Gamma_2, \Delta_2 \Rightarrow \Psi_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Psi_1 \mid \Delta_1, \Delta_2 \Rightarrow \Psi_2}\,(com)$	

first introduced in Hori et al. (1994) and provide a generalization of both n-mingle and n-contraction. Thus, we generalize the result in (Wang 2012) and simplify its proof, which contains many hard-to-check case distinctions. The paper is organized as follows: Sect. 2 contains a Gentzen-style calculus for any axiomatic extension of UL with knotted axioms (step (a)). The calculus is based on *hypersequents*, a natural generalization of Gentzen sequents. Following the approach in (Ciabattoni and Metcalfe 2008), in Sect. 3 we prove density elimination (step (b)). This shows the rational completeness of our logics. Standard completeness can then be obtained by *Dedekind–MacNeille completion* (step (c)). Results in (Ciabattoni et al. 2011) ensure indeed that the knotted axioms are preserved under this construction.

2 Proof theory for axiomatic extensions of UL with knotted axioms

We present hypersequent calculi for axiomatic extensions of UL with knotted axioms. We start recalling uninorm logic UL, first introduced in (Metcalfe and Montagna 2007). UL is based on a propositional language with propositional variables, the constants $\top, \bot, t, f$ and the connectives $\cdot, \wedge, \vee$ and $\to$. Propositional formulas are generated in the usual way. An Hilbert system for UL is obtained by adding the prelinearity axiom $((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$ to MAILL (the multiplicative–additive fragment of intuitionistic linear logic, see e.g., Galatos et al. 2007). A Gentzen-style calculus for UL was introduced in (Metcalfe and Montagna 2007) as well and is based on *hypersequents*.

Definition 1 (Avron 1987) A *hypersequent* is a finite multiset $S_1 \mid \ldots \mid S_n$ where for $i = 1 \ldots n$, each S_i is a single-conclusioned sequent, called a *component* of the hypersequent; i.e., S_i is an object of the kind $\Gamma_i \Rightarrow \Psi_i$, where Γ_i is a multiset of formulas and Ψ_i is either empty or a single formula.

The symbol "$\mid$" in a hypersequent is intended to denote a disjunction of sequents. The formula-interpretation $\mathbb{I}$ of a hypersequent $H = \Gamma_1 \Rightarrow \Psi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Psi_n$ is defined as follows, see, e.g., (Avron 1991; Ciabattoni et al. 2008; Metcalfe et al. 2008):

- $\mathbb{I}(\Gamma_1 \Rightarrow \Psi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Psi_n) = \mathbb{I}(\Gamma_1 \Rightarrow \Psi_1) \vee \cdots \vee \mathbb{I}(\Gamma_n \Rightarrow \Psi_n)$

where the interpretation $\mathbb{I}$ of a sequent $\Gamma \Rightarrow \Psi$ is the usual one, i.e.,

- $\mathbb{I}(\Gamma \Rightarrow \Psi) = \odot\Gamma \to \beta$, if Ψ is a formula β
- $\mathbb{I}(\Gamma \Rightarrow) = \odot\Gamma \to f$, otherwise

$\odot\Gamma$ stands for the multiplicative conjunction $\cdot$ of all the formulas in Γ, and it is t when Γ is empty.

The hypersequent calculus HUL for UL is given in Table 1. HUL is obtained by adding to the (hypersequent version of the) calculus for MAILL, the rules (ec), (ew) and the communication rule (com), which is equivalent to the prelinearity axiom. The context G appearing in all rules in Table 1 stands for an arbitrary hypersequent. With a slight abuse of notation, in the following we will denote multisets and formulas with the same symbol as for the corresponding metavariables. The notation Γ^k stands for k comma-separated occurrence $\Gamma, \ldots, \Gamma$ of a multiset Γ. By α^k, we will denote both the multiplicative conjunction $\alpha \cdots \alpha$ of k occurrences of the formula α, and k comma-separated occurrences $\alpha, \ldots, \alpha$. The meaning will be clear from the context. We will denote repeated applications of a rule (r) by $(\overline{r})$. Derivability in a logic L (Hilbert style) and in a hypersequent calculus HL is defined as usual and is denoted by $\vdash_L$ and $\vdash_{HL}$, respectively. The calculus HUL is sound and complete for UL (see Metcalfe and Montagna 2007), i.e., for any hypersequent H we have that $\vdash_{HUL} H$ if and only $\vdash_{UL} \mathbb{I}(H)$. Notice that, although the rule (cut) is useful for proving the completeness of HUL, it does not extend the set of provable hypersequents. Indeed, as shown in (Metcalfe and Montagna 2007), HUL admits *cut-elimination*, i.e., any derivation containing applications of the rule (cut) can be converted into one which does not contain any application of (cut) (a cut-free derivation). All the rule applications in a cut-free derivation enjoy the *subformula property*, i.e., all formulas occurring in the premises are subformulas of the formulas in the conclusion. This is an essential property, that will be used for our proof of density elimination. We present below hypersequent calculi which admit cut-elimination and are sound and complete for extensions of UL with *knotted* axioms.

Lemma 1 *Let L be the logic UL* $+ \alpha^k \to \alpha^n$ *and HL the hypersequent calculus HUL* $+ (knot_{k,n})$*, where for* $n, k \geq 1$ $(knot_{k,n})$ *is the rule:*

$$\frac{G \mid \Pi, \Gamma_1^n \Rightarrow \Psi \quad \ldots \quad G \mid \Pi, \Gamma_k^n \Rightarrow \Psi}{G \mid \Pi, \Gamma_1, \ldots \Gamma_k \Rightarrow \Psi} (knot_{k,n})$$

HL is sound and complete for L.

Proof Soundness and completeness of HUL with respect to UL have been shown in (Metcalfe and Montagna 2007). Thus, we just need to deal with the knotted axioms. Showing that $\alpha^k \to \alpha^n$ is derivable in HL is easy (see e.g., the derivation of $\alpha^{n-1} \to \alpha^n$ in Theorem 6 in Ciabattoni et al. 2002). For the other direction, we reason by induction on the length of the derivation of a hypersequent H in HL. Assume that the last applied rule is $(knot_{k,n})$. We limit ourselves to a simple application of the kind:

$$\frac{\Gamma_1^n \Rightarrow \beta \quad \ldots \quad \Gamma_k^n \Rightarrow \beta}{\Gamma_1, \ldots \Gamma_k \Rightarrow \beta} (knot_{k,n})$$

Our aim is to show:

$$(*) \quad \alpha_1^n \to \beta, \ldots, \alpha_k^n \to \beta \vdash_L \alpha_1, \ldots, \alpha_k \to \beta$$

where $\alpha_i = \odot\Gamma_i$, for $i = \{1, \ldots, k\}$. The proof can be extended to a more general application of $(knot_{k,n})$, containing a hypersequent context G. Indeed, from $(*)$ one easily gets $\mathbb{I}(G) \vee (\alpha_1^n \to \beta), \ldots, \mathbb{I}(G) \vee (\alpha_k^n \to \beta) \vdash_L \mathbb{I}(G) \vee (\alpha_1, \ldots, \alpha_k \to \beta)$. For proving $(*)$, notice that we have $(\alpha_1^n \to \beta), \ldots, (\alpha_k^n \to \beta) \vdash_L (\alpha_1^n \vee \cdots \vee \alpha_k^n) \to \beta$. Moreover, by the axiom schema $\alpha^k \to \alpha^n$ and basic properties of L, we get $\vdash_L (\alpha_1^k \vee \cdots \vee \alpha_k^k) \to (\alpha_1^n \vee \cdots \vee \alpha_k^n)$. Hence, we obtain $(\alpha_1^n \to \beta), \ldots, (\alpha_k^n \to \beta) \vdash_L (\alpha_1^k \vee \cdots \vee \alpha_k^k) \to \beta$. Finally, to complete our proof of $(*)$, it is sufficient to show that L derives $\alpha_1, \ldots, \alpha_k \to (\alpha_1^k \vee \cdots \vee \alpha_k^k)$, which is already the case for UL (see the proof in the Appendix). □

Lemma 2 *Let* R *be any set of rules of the kind* $(knot_{k,n})$*, with* $n, k \geq 1$*. The calculus HUL extended with the rules in* R *admits cut-elimination.*

Proof It is shown in (Metcalfe and Montagna 2007) that HUL admits cut-elimination. We can easily check that any rule $(knot_{k,n})$ satisfies the syntactic conditions in (Ciabattoni et al. 2008) for preserving cut-elimination (i.e., in the terminology of Ciabattoni et al. 2008, $(knot_{k,n})$ is a *completed* rule). Therefore, the calculus HUL + R admits cut-elimination as well, as a consequence of Corollary 8.6 in (Ciabattoni et al. 2008). □

3 Density elimination

We now consider the extension of our hypersequent calculi with the *density rule*. This rule, first introduced Hilbert style in Takeuti and Titani (1984), in a hypersequent calculus has the following form (Metcalfe and Montagna 2007):

$$\frac{G \mid \Lambda \Rightarrow p \mid \Sigma, p \Rightarrow \Psi}{G \mid \Lambda, \Sigma \Rightarrow \Psi} (D)$$

where p is a propositional variable satisfying the *eigenvariable* condition, i.e., it should not appear in the lower hypersequent. We show density elimination for any hypersequent calculus extending HUL with (D) and any knotted rule $(knot_{k,n})$, with $n, k > 1$. This means that each derivation containing an application of (D) can be transformed into a derivation of the same end-hypersequent which does not contain (D). We first recall the idea behind the density elimination proof in (Ciabattoni and Metcalfe 2008). Assume we have a derivation d ending in an application of density:

$$\frac{\vdots \atop G \mid \Lambda \Rightarrow p \mid \Sigma, p \Rightarrow \Psi}{G \mid \Lambda, \Sigma \Rightarrow \Psi} (D)$$

As our calculi admit cut-elimination, we can safely assume d to be cut-free. We remove the application of (D) by substituting occurrences of p in d in an "asymmetric" way. More precisely, each occurrence of p on the left-hand side of each sequent is replaced by Λ, and each occurrence on the right-hand side by a Σ on the left and a Ψ on the right. The application of (D) would then be simply replaced by (ec). However, this procedure does not lead to a valid derivation, in general. Indeed, d might contain, for instance, an axiom $p \Rightarrow p$, and the application of the asymmetric substitution would return $\Lambda, \Sigma \Rightarrow \Psi$, that is not an axiom anymore. The idea is then to deal in a different way with sequents of kind $\Pi, p^k \Rightarrow p$, replacing each of them by $\Pi, \Lambda^{k-1} \Rightarrow t$. Thus, for instance, the axiom $p \Rightarrow p$ would be turned into the axiom $\Rightarrow t$. This method is applied in (Ciabattoni and Metcalfe 2008) to extensions of HUL with *balanced* rules, i.e., rules for which the number of occurrences of metavariables in the premises and in the conclusion is the same. This is not the case of the knotted rules, for which the asymmetric substitution might be problematic. For instance, consider the following application of the rule $(knot_{2,3})$ (also known as 3-contraction):

$$\dfrac{\Pi, p^3 \Rightarrow p \quad \Pi, p^3 \Rightarrow p}{\Pi, p^2 \Rightarrow p}\,(knot_{2,3})$$

The substitution sketched above would give us

$$\dfrac{\Pi, \Lambda^2 \Rightarrow t \quad \Pi, \Lambda^2 \Rightarrow t}{\Pi, \Lambda \Rightarrow t}$$

which is clearly not an application of $(knot_{2,3})$ anymore. We show below how to overcome this problem for any knotted rule with $n, k > 1$. The key observation is that the knotted rules allow us to use a restricted form of contraction and weakening on the left, when the right-hand side of a sequent is equal to t.

Lemma 3 *Let $n, k > 1$. The following rules are derivable in the calculus $HUL + (knot_{k,n})$:*

$$\dfrac{G \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}\,(w_t) \qquad \dfrac{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Gamma_1 \Rightarrow t}\,(c_t)$$

Proof First, notice that in HUL, for any $m > 1$, the rule

$$\dfrac{G \mid \Pi^m \Rightarrow t}{G \mid \Pi \Rightarrow t}\,(*_m)$$

is derivable. We reason by induction on m. First, we show that $(*_2)$ is derivable, as follows:

$$\dfrac{G \mid \Pi \Rightarrow t \quad \dfrac{G \mid \Pi \Rightarrow t}{G \mid \Pi, t \Rightarrow t}\,(tl)}{G \mid \Pi, \Pi \Rightarrow t}\,(cut)$$

Assuming that $(*_{m-1})$ is derivable, we derive $(*_m)$ as follows

$$\dfrac{G \mid \Pi \Rightarrow t \quad \dfrac{\dfrac{G \mid \Pi \Rightarrow t}{G \mid \Pi^{m-1} \Rightarrow t}\,(*_{m-1})}{G \mid \Pi^{m-1}, t \Rightarrow t}\,(tl)}{G \mid \Pi^m \Rightarrow t}\,(cut)$$

Similarly, we can prove that for any $m > 1$ the rule

$$\dfrac{G \mid \Pi^m \Rightarrow t}{G \mid \Pi \Rightarrow t}\,(*^m)$$

is derivable. The base case $(*^2)$ can be derived as follows:

$$\dfrac{\dfrac{G \mid \Pi, \Pi \Rightarrow t \quad \dfrac{\Rightarrow t}{G \mid \ \Rightarrow t}\,(ew)}{G \mid \Pi \Rightarrow t \mid \Pi \Rightarrow t}\,(com)}{G \mid \Pi \Rightarrow t}\,(ec)$$

Assuming $(*^{m-1})$ is derivable, we get:

$$\dfrac{\dfrac{\dfrac{G \mid \Pi^m \Rightarrow t \quad \dfrac{\Rightarrow t}{G \mid \ \Rightarrow t}\,(ew)}{G \mid \Pi^{m-1} \Rightarrow t \mid \Pi \Rightarrow t}\,(com)}{G \mid \Pi \Rightarrow t \mid \Pi \Rightarrow t}\,(*^{m-1})}{G \mid \Pi \Rightarrow t}\,(ec)$$

Using $(*_2)$ and $(*^2)$, we can easily show that the two rules (c_t) and (w_t) are interderivable. Indeed, if we have (w_t), we can derive (c_t) as follows:

$$\dfrac{\dfrac{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}\,(w_t)}{G \mid \Pi, \Gamma_1 \Rightarrow t}\,(*^2)$$

And analogously:

$$\dfrac{\dfrac{G \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}\,(*_2)}{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}\,(c_t)$$

In what follows, it is therefore enough to prove that either (c_t) or (w_t) is derivable. In particular, we show that (c_t) is derivable in case the knotted rule $(knot_{k,n})$ has $n > k$, and that (w_t) is derivable, otherwise.

1. Assume $n > k$. Suppose that we are given a derivation of $G|\Pi, \Gamma_1, \Gamma_1 \Rightarrow t$. Consider the following application of $(knot_{k,n})$:

$$\dfrac{\dfrac{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi^n, \Gamma_1^n, \Gamma_1^n \Rightarrow t}\,(*_n) \quad \ldots \quad \dfrac{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi^n, \Gamma_1^n, \Gamma_1^n \Rightarrow t}\,(*_n)}{G \mid \Pi^n, \Gamma_1^k, \Gamma_1^k \Rightarrow t}\,(knot_{k,n})$$

If $2k > n$, then apply $(knot_{k,n})$ with k identical premises $G_1|\Pi^n, \Gamma_1^{2k-n}, \Gamma_1^n \Rightarrow t$ to obtain $G_1|\Pi^n, \Gamma_1^{2k-n+k}$. Apply the rule $(knot_{k,n})$ once more using this sequent as the premises. Repeat in this way until we get $G_1|\Pi^n, \Gamma_1^l \Rightarrow t$, for some $l \leq n$. The proof that (c_t) is derivable is then completed as follows

$$\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\vdots}{G \mid \Pi^n, \Gamma_1^l \Rightarrow t} \quad G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi^n, \Gamma_1^{l+1} \Rightarrow t \mid \Pi, \Gamma_1 \Rightarrow t}\,(com) \quad \dfrac{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t \mid \Pi, \Gamma_1 \Rightarrow t}\,(ew)}{G \mid \Pi^n, \Gamma_1^{l+2} \Rightarrow t \mid \Pi, \Gamma_1 \Rightarrow t \mid \Pi, \Gamma_1 \Rightarrow t}\,(com)}}{}$$

$$\vdots$$

$$\dfrac{\dfrac{G \mid \Pi^n, \Gamma_1^n \Rightarrow t \mid \Pi, \Gamma_1 \Rightarrow t \mid \ldots \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Gamma_1 \Rightarrow t \mid \ldots \mid \Pi, \Gamma_1 \Rightarrow t}\,(*^n)}{G \mid \Pi, \Gamma_1 \Rightarrow t}\,(\overline{ec})$$

2. Consider now the case where $n < k$. Suppose that we are given a derivation of $G|\Pi, \Gamma_1 \Rightarrow t$. We prove that (w_t) is derivable in our calculus. First, consider the following application of $(knot_{k,n})$:

$$\dfrac{\dfrac{G \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi^n, \Gamma_1^n \Rightarrow t}(*_n) \quad \dots \quad \dfrac{G \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi^n, \Gamma_1^n \Rightarrow t}(*_n)}{G \mid \Pi^n, \Gamma_1^k \Rightarrow t}(knot_{k,n})$$

We iterate similar application of $(knot_{k,n})$, increasing the occurrences of Γ_1 by $(k-n)$, until we get $G \mid \Pi^n, \Gamma_1^l \Rightarrow t$, for some $l \geq 2n$.

The proof that (w_t) is derivable is then completed as follows:

$$\dfrac{\dfrac{\dfrac{\vdots \atop G \mid \Pi^n, \Gamma_1^l \Rightarrow t \quad G \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi^n, \Gamma_1^{l-1} \Rightarrow t \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}(com) \qquad \dfrac{G \mid \Pi, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Gamma_1 \Rightarrow t \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}(ew)}{G \mid \Pi^n, \Gamma_1^{l-2} \Rightarrow t \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}(com)}{\vdots}$$

$$\dfrac{\dfrac{G \mid \Pi^n, \Gamma_1^n, \Gamma_1^n \Rightarrow t \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t \mid \dots \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t \mid \dots \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}(*^n)}{G \mid \Pi, \Gamma_1, \Gamma_1 \Rightarrow t}(\overline{ec})$$

□

We are now ready for the result on density elimination. The proof follows closely that in (Ciabattoni and Metcalfe 2008).

Theorem 1 *Let R be any set of rules of the kind $(knot_{k,n})$, with $n, k > 1$. The calculus HUL extended with (D) and the rules in R admits density-elimination.*

Proof It proceeds by induction on the length of the derivations. Consider a derivation d ending in a topmost application of the density rule

$$\dfrac{\vdots \atop G \mid \Lambda \Rightarrow p \mid \Sigma, p \Rightarrow \Psi}{G \mid \Lambda, \Sigma \Rightarrow \Psi}(D)$$

By Lemma 2, we can safely assume d to be cut-free. Let H be a hypersequent $H = S_1 \mid \dots \mid S_n$. We let $H^* = G \mid \Lambda, \Sigma \Rightarrow \Psi \mid S'_1 \mid \dots \mid S'_n$ where, for each component S_i, the sequent S'_i is defined as follows:

(a) If $S_i = \Pi, p^k \Rightarrow p$ (with $p \notin \Pi$), $k > 0$, $S'_i = \Pi, \Lambda^{k-1} \Rightarrow t$.
(b) If $S_i = \Pi \Rightarrow p$ (with $p \notin \Pi$), then $S'_i = \Pi, \Sigma \Rightarrow \Psi$
(c) If $S_i = \Pi, p^k \Rightarrow \Psi_1$ (with $p \notin \Pi, \Psi_1$), $k > 0$, then $S'_i = \Pi, \Lambda^k \Rightarrow \Psi_1$
(d) Otherwise, $S'_i = S_i$

Henceforth, we call a sequent $\Pi, p^k \Rightarrow p$ a *pp-component*.

We prove the following:

Claim: *For each hypersequent* H in d *one can find a* (D)-*free derivation of* H^*.

The result on density elimination follows from this claim. Just let H be $G \mid \Lambda \Rightarrow p \mid \Sigma, p \Rightarrow \Psi$. We get that $G \mid G \mid \Lambda, \Sigma \Rightarrow \Psi \mid \Lambda, \Sigma \Rightarrow \Psi$ is (D)-free derivable (note that $G^* = G$ by the eigenvariable condition on p). The desired (D)-free proof of $G \mid \Lambda, \Sigma \Rightarrow \Psi$ follows then by multiple applications of (ec).

For proving the claim, we reason by induction on the length of the derivation of a hypersequent H in d. Notice that, if H is an axiom of kind $p \Rightarrow p$, then H^* is $G \mid \Lambda, \Sigma \Rightarrow \Psi \mid \; \Rightarrow t$, which is derivable by applying (ew) to the axiom $\Rightarrow t$. The claim is easily provable also in case the last applied rule is (ew), (ec).

The cases for logical rules and communication (com) are as in (Ciabattoni and Metcalfe 2008). Assume now that the last applied rule is a rule $(knot_{k,n})$ in R. We distinguish three cases, according to the presence of *pp*-components in the premises:

(i) None of the premises contains a *pp*-component. This implies that neither the conclusion does: the claim hence holds, by simply using the induction hypothesis and applying the knotted rule again.

(ii) All of the premises contain *pp*-components, i.e., we have a rule application as the following

$$\dfrac{G_1 \mid \Pi, \Gamma_1^n, p^{n_1} \Rightarrow p \quad \dots \quad G_1 \mid \Pi, \Gamma_k^n, p^{n_k} \Rightarrow p}{G_1 \mid \Pi, \Gamma_1, \dots, \Gamma_k, p^l \Rightarrow p}(knot_{k,n})$$

where each $n_i \geq 1$ is the number of p appearing in the left hand side of the ith premise, for $i = \{1, \dots, k\}$, and $l \geq k$ is the number of p appearing in the left-hand side of the conclusion. By the induction hypothesis, we have density-free derivations of $G_1^* \mid \Pi, \Gamma_1^n, \Lambda^{n_1 - 1} \Rightarrow t, \dots, G_1^* \mid \Pi, \Gamma_k^n, \Lambda^{n_k - 1} \Rightarrow t$. Consider the following derivation:

$$\dfrac{G_1^* \mid \Pi, \Gamma_1^n, \Lambda^{n_1-1} \Rightarrow t \quad \dfrac{G_1^* \mid \Pi, \Gamma_2^n, \Lambda^{n_2-1} \Rightarrow t}{G_1^* \mid \Pi, \Gamma_2^n, t, \Lambda^{n_2-1} \Rightarrow t}(tl)}{G_1^* \mid \Pi^2, \Gamma_1^n, \Gamma_2^n, \Lambda^{n_1-1}, \Lambda^{n_2-1} \Rightarrow t}(cut)$$

Starting from the end-hypersequent above, we can iterate similar applications of (cut) with each of the $G_1^* \mid \Pi, \Gamma_3^n \Rightarrow t, \dots, G_1^* \mid \Pi, \Gamma_k^n \Rightarrow t$ until we get:

$$G_1^* \mid \Pi^k, \Gamma_1^n, \Gamma_2^n, \dots, \Gamma_k^n, \Lambda^{n_1 + \dots + n_k - k} \Rightarrow t$$

The desired hypersequent $G_1^* \mid \Pi, \Gamma_1, \dots, \Gamma_k, \Lambda^{l-1} \Rightarrow t$ is then obtained by suitable repeated applications of (c_t) and (w_t) to the hypersequent above.

(iii) Only some premises (say m, with $m < k$) are *pp*-component, while others are not, namely w.l.o.g. we have a rule application as the following:

$$\dfrac{\begin{array}{c} G_1 \mid \Pi, \Gamma_1^n, p^{n_1} \Rightarrow p \dots G_1 \mid \Pi, \Gamma_m^n, p^{n_m} \Rightarrow p \\ G_1 \mid \Pi, \Gamma_{m+1}^n \Rightarrow p \dots G_1 \mid \Pi, \Gamma_k^n \Rightarrow p \end{array}}{G_1 \mid \Pi, \Gamma_1, \dots, \Gamma_k, p^l \Rightarrow p}(knot_{k,n})$$

By the induction hypothesis, we have density-free derivations of $G_1^* \mid \Pi, \Gamma_1^n, \Lambda^{n_1-1} \Rightarrow t, \dots, G_1^* \mid \Pi, \Gamma_m^n, \Lambda^{n_m-1} \Rightarrow$

t and $G_1^* \mid \Pi, \Gamma_{m+1}^n, \Sigma \Rightarrow \Psi, \ldots, G_1^* \mid \Pi, \Gamma_k^n, \Sigma \Rightarrow \Psi$. Recall that our aim is to obtain a derivation of

$$G_1^* \mid \Pi, \Gamma_1, \ldots, \Gamma_k, \Lambda^{l-1} \Rightarrow t$$

As in the previous case, we do repeated cuts, but only on the m premises containing pp-components, thus obtaining:

$$G_1^* \mid \Pi^m, \Gamma_1^n, \ldots, \Gamma_m^n, \Lambda^{n_1+\cdots+n_m-m} \Rightarrow t$$

We repeatedly apply (c_t) or (w_t) to the previous hypersequent, to get:

$$G_1^* \mid \Pi^m, \Gamma_1^n, \ldots, \Gamma_m^n, \Lambda^{(l-1)+(k-m)} \Rightarrow t$$

We "remove" then extra occurrences of Λ from this hypersequent, using applications of (com) as the following:

$$\dfrac{\dfrac{G_1^* \mid \Pi^m, \Gamma_1^n, \ldots, \Gamma_m^n, \Lambda^{(l-1)+(k-m)} \Rightarrow t \quad G_1^* \mid \Pi, \Gamma_{m+1}^n, \Sigma \Rightarrow \Psi}{G_1^* \mid \Pi^{m+1}, \Gamma_1^n, \ldots, \Gamma_{m+1}^n, \Lambda^{(l-1)+(k-m-1)} \Rightarrow t \mid \Lambda, \Sigma \Rightarrow \Psi}(com)}{G_1^* \mid \Pi^{m+1}, \Gamma_1^n, \ldots, \Gamma_{m+1}^n, \Lambda^{(l-1)+(k-m-1)} \Rightarrow t}(ec)$$

Similarly, by an application of (com) to the conclusion above and the premise $G_1^* \mid \Pi, \Gamma_{m+2}^n, \Sigma \Rightarrow \Psi$, we get $G_1^* \mid \Pi^{m+2}, \Gamma_1^n, \ldots, \Gamma_{m+2}^n, \Lambda^{(l-1)+(k-m-2)} \Rightarrow t$. We can iterate applications of (com) of this kind for all the $(k-m)$ premises of $(knot_{k,n})$ which do not contain pp-components, until we finally get:

$$\dfrac{\vdots \atop G_1^* \mid \Pi^k, \Gamma_1^n, \ldots, \Gamma_k^n, \Lambda^{l-1} \Rightarrow t}{G_1^* \mid \Pi, \Gamma_1, \ldots, \Gamma_k, \Lambda^{l-1} \Rightarrow t}(\overline{c_t})$$

This concludes the proof of the main claim, thus showing density elimination for our calculus. □

Standard completeness for our logics can finally be obtained as a consequence of Theorem 1 and of known results in (Ciabattoni et al. 2011; Metcalfe 2011; Metcalfe and Montagna 2007), which we summarize below.

Let us first recall the algebraic structures providing a semantics for our logics.

Definition 2 An UL-algebra is a structure $(A, \wedge, \vee, \cdot, \rightarrow, \top, \bot, t, f)$ where

- $(A, \wedge, \vee, \top, \bot)$ is a bounded lattice, with $\top$ and $\bot$ the maximum and minimum, respectively.
- $(A, \cdot, t)$ is a monoid, $f \in A$.
- $x \cdot z \leq y \Leftrightarrow z \leq x \rightarrow y$ for any $x, y, z \in A$ (*Residuation*).
- $t \leq ((x \rightarrow y) \wedge t) \vee ((y \rightarrow x) \wedge t)$ for any $x, y \in A$ (*Prelinearity*).

Following common practice, given any logic L which extends UL with an axiom α, we call L-*algebras* the class of UL-algebras satisfying the corresponding algebraic equation $t \leq \alpha$. Henceforth, we let L be a logic extending UL with a set of knotted axioms $\alpha^k \rightarrow \alpha^n$, with $k, n > 1$.

First we recall that, by density elimination, we can prove the completeness of the logic L with respect to the class of linearly, densely ordered L-algebras (dense L-chains, for short). This establishes rational completeness, i.e., the completeness of L with respect to L-algebras with lattice reduct $\mathbb{Q} \cap [0, 1]$. In what follows, we assume that the reader is familiar with the usual semantic notion of consequence relation $\models_{\mathcal{K}}$ with respect to a class of algebras $\mathcal{K}$, see e.g., (Galatos et al. 2007; Horcik 2011).

Theorem 2 *Let $T \cup \{\gamma\}$ be any set of formulas in the language of L. $T \vdash_L \gamma$ if and only if $\{t \leq \beta \mid \beta \in T\} \models_{\mathcal{K}} t \leq \gamma$, where $\mathcal{K}$ is the class of dense L-chains.*

Proof By Theorem 1, the hypersequent calculus HL corresponding to L admits density elimination. By Proposition 5.37 in (Metcalfe et al. 2008), density elimination for HL is equivalent to the admissibility of the density rule in the Hilbert-style calculus L. The claim then follows by Theorem 3.64 in (Metcalfe et al. 2008). □

We can now obtain the standard completeness for L, i.e., the equivalence between the consequence relations $\vdash_L$ and $\models_{\mathcal{K}}$, when $\mathcal{K}$ is the class of L-algebras with lattice reduct [0, 1]. In terms of universal algebra, this means that L-algebras are generated as quasivarieties by their members in [0, 1].

Theorem 3 (Standard completeness) *Let $T \cup \{\gamma\}$ be any set of formulas in the language of L. $T \vdash_L \gamma$ if and only if $\{t \leq \beta \mid \beta \in T\} \models_{\mathcal{K}} t \leq \gamma$, where $\mathcal{K}$ is the class of L-algebras with lattice reduct* [0, 1].

Proof By Theorem 2, the logic L is complete with respect to L-algebras with lattice reduct $\mathbb{Q} \cap [0, 1]$ (dense, countable L-chains). To prove standard completeness, it is then enough to show that any L-algebra A with lattice reduct $\mathbb{Q} \cap [0, 1]$ can be embedded into an L-algebra with lattice reduct [0, 1], see e.g., (Cintula et al. 2009). Consider the embedding of A into its Dedekind–Macneille completion $DM(A)$, see e.g., (Galatos et al. 2007; Horcik 2011). The lattice reduct of $DM(A)$ is clearly [0, 1]. We just need to show that $DM(A)$ is an L-algebra. As A is a UL-algebra, $DM(A)$ is a UL-algebra as well, by Theorem 27 in (Metcalfe and Montagna 2007). Furthermore, Theorem 2.7 in (Ciabattoni et al. 2011) shows the preservation under DM-completion for a large class of algebraic equations, which includes $t \leq \alpha^k \rightarrow \alpha^n$, for any $n, k \geq 1$. Hence, $DM(A)$ is an L-algebra with lattice reduct [0, 1]. This completes the proof. □

Concluding remark

Our standard completeness proof does not apply to the knotted axioms $\alpha \to \alpha^n$ and $\alpha^k \to \alpha$. It can be easily shown that, in presence of prelinearity, these axioms are equivalent to $\alpha \to \alpha^2$ and $\alpha^2 \to \alpha$, respectively. For the corresponding (hyper)sequent rules 2-mingle (mgl) and 2-contraction (c) (Metcalfe et al. 2008), even though Lemma 3 trivially holds, our proof of density elimination does not go through. Indeed, it is not clear how to handle, for instance, an application of (c), as below left. Our substitution would make it into the rule below right, which is not a correct application of (c) and cannot be handled just using Lemma 3 to change the number of the occurrences of Λ, as done in the proof of Theorem 1.

$$\dfrac{\Pi, p, p \Rightarrow p}{\Pi, p \Rightarrow p}\,(c) \qquad \dfrac{\Pi, \Lambda \Rightarrow t}{\Pi \Rightarrow t}$$

Note that density elimination for HUL extended with both (mgl) and (c) has actually been shown in (Metcalfe and Montagna 2007). However, this proof cannot be easily adapted to show density elimination for HUL extended either with (c) or with (mgl).

Acknowledgments I would like to thank the anonymous referees for their comments, which helped me to substantially improve the final form of the paper, and David Cerna, for linguistic advice. Furthermore, I would like to thank my supervisor Agata Ciabattoni for her guidance and advice throughout the whole preparation of this paper.

Appendix: Proof of the derivability of $\alpha_1, \ldots, \alpha_k \to \alpha_1^k \vee \cdots \vee \alpha_k^k$ in UL

First, we show that HUL derives the hypersequent $\alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \mid \ldots \mid \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_k^k$. We proceed by induction on k. For $k = 2$, we have

$$\dfrac{\dfrac{\alpha_1 \Rightarrow \alpha_1 \quad \alpha_1 \Rightarrow \alpha_1}{\alpha_1, \alpha_1 \Rightarrow \alpha_1^2}\,(\cdot r) \qquad \dfrac{\alpha_2 \Rightarrow \alpha_2 \quad \alpha_2 \Rightarrow \alpha_2}{\alpha_2, \alpha_2 \Rightarrow \alpha_2^2}\,(\cdot r)}{\alpha_1, \alpha_2 \Rightarrow \alpha_1^2 \mid \alpha_1, \alpha_2 \Rightarrow \alpha_2^2}\,(com)$$

For the induction step, we assume to have a derivation d of $\alpha_1, \ldots, \alpha_{k-1} \Rightarrow \alpha_1^{k-1} \mid \ldots \mid \alpha_1, \ldots, \alpha_{k-1} \Rightarrow \alpha_{k-1}^{k-1}$ in HUL. First, we show that for any α_i with $i = \{1, \ldots, k-1\}$ we have a derivation d_i in HUL of the hypersequent $\alpha_k, \alpha_i^{k-1} \Rightarrow \alpha_k^k \mid \alpha_k \Rightarrow \alpha_i$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha_k \Rightarrow \alpha_k \quad \alpha_i \Rightarrow \alpha_i}{\alpha_i \Rightarrow \alpha_k \mid \alpha_k \Rightarrow \alpha_i}\,(com) \qquad \dfrac{\alpha_k \Rightarrow \alpha_k \quad \alpha_i \Rightarrow \alpha_i}{\alpha_i \Rightarrow \alpha_k \mid \alpha_k \Rightarrow \alpha_i}\,(com)}{\alpha_i, \alpha_i \Rightarrow \alpha_k^2 \mid \alpha_k \Rightarrow \alpha_i}\,(\cdot r)}{\vdots}}{\dfrac{\alpha_i^{k-1} \Rightarrow \alpha_k^{k-1} \mid \alpha_k \Rightarrow \alpha_i \qquad \alpha_k \Rightarrow \alpha_k}{\alpha_k, \alpha_i^{k-1} \Rightarrow \alpha_k^k \mid \alpha_k \Rightarrow \alpha_i}\,(\cdot r)}$$

Consider now the following derivation. For space reasons, we abbreviate the hypersequent $\alpha_1, \ldots, \alpha_{k-1} \Rightarrow \alpha_2^{k-1} \mid \ldots \mid \alpha_1, \ldots, \alpha_{k-1} \Rightarrow \alpha_{k-1}^{k-1}$ with H.

$$\dfrac{\begin{matrix}\vdots\, d \\ \alpha_1, \ldots, \alpha_{k-1} \Rightarrow \alpha_1^{k-1} \mid H\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, d_1 \\ \alpha_k, \alpha_1^{k-1} \Rightarrow \alpha_k^k \mid \alpha_k \Rightarrow \alpha_1\end{matrix} \quad \begin{matrix}\vdots\, d \\ \alpha_1, \ldots, \alpha_{k-1} \Rightarrow \alpha_1^{k-1} \mid H\end{matrix}}{\alpha_k, \alpha_1^{k-1} \Rightarrow \alpha_k^k \mid \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \mid H}\,(\cdot r)}{\alpha_1, \ldots, \alpha_k \Rightarrow \alpha_k^k \mid \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \mid H}\,(cut)$$

Starting from the end-hypersequent above and repeating similar derivations for any of the d_i, we eventually obtain the desired derivation of

$$\alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \mid \ldots \mid \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_k^k$$

From this, we get:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \mid \ldots \mid \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_k^k\end{matrix}}{\alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \vee \cdots \vee \alpha_k^k \mid \ldots \mid \alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \vee \cdots \vee \alpha_k^k}\,(\overline{\vee r})}{\alpha_1, \ldots, \alpha_k \Rightarrow \alpha_1^k \vee \cdots \vee \alpha_k^k}\,(\overline{ec})$$

Finally, by the completeness of HUL with respect to UL (see Metcalfe and Montagna 2007), we have that $\vdash_{UL} (\alpha_1, \ldots, \alpha_k) \to (\alpha_1^k \vee \cdots \vee \alpha_k^k)$. This completes the proof.

References

Avron A (1987) A constructive analysis of RM. J Symb Log 52(4):939–951

Avron A (1991) Hypersequents, logical consequence and intermediate logics for concurrency. Ann Math Artif Intell 4:225–248

Baldi P, Ciabattoni A, Spendier L (2012) Standard completeness for extensions of MTL: an automated approach. In: Ong L, de Queiroz R (eds) Workshop on logic, language, information and computation (WoLLIC 2012). LNCS 7456. Springer, Heidelberg, pp 154–167

Ciabattoni A, Esteva F, Godo L (2002) T-norm based logics with n-contraction. Neural Netw World 12(5):441–452

Ciabattoni A, Galatos N, Terui K (2008) From axioms to analytic rules in nonclassical logics. In: IEEE symposium on logic in computer science (LICS 2008). IEEE, pp 229–240

Ciabattoni A, Galatos N, Terui K (2011) Macneille completions of FL-algebras. Algebra universalis 66(4):405–420

Ciabattoni A, Metcalfe G (2007) Density elimination and rational completeness for first-order logics. In: Symposium on logical foundations of computer science (LFCS 2007), LNCS 4514. pp 132–146

Ciabattoni A, Metcalfe G (2008) Density elimination. Theor Comput Sci 403(2–3):328–346

Cintula P, Esteva F, Gispert J, Godo L, Montagna F, Noguera C (2009) Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies. Annals Pure Appl Logic 160(1):53–81

Esteva F, Gispert J, Godo L, Montagna F (2002) On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. Studia Logica 71(2):199–226

Gabbay DM, Metcalfe G (2007) Fuzzy logics based on [0, 1)-continuous uninorms. Arch Math Log 46(5–6):425–449

Galatos N, Jipsen P, Kowalski T, Ono H (2007) Residuated lattices: an algebraic glimpse at substructural logics: studies in logics and the foundations of mathematics. Elesevier, Amsterdam
Hájek P (1998) Metamathematics of fuzzy logic. Kluwer, Dordrecht
Horčík R (2011) Algebraic semantics: semilinear FL-algebras. In: Cintula P, Hájek P, Noguera C (eds) Handbook of mathematical fuzzy logic-volume 1, volume 37 of studies in logic. Mathematical logic and foundations. College Publications, London, pp 103–207
Hori R, Ono H, Schellinx H (1994) Extending intuitionistic linear logic with knotted structural rules. Notre Dame J Form Logic 35(2):219–242
Jenei S, Montagna F (2002) A proof of standard completeness for Esteva and Godo's logic MTL. Studia Logica 70(2):183–192
Metcalfe G (2011) Proof theory for mathematical fuzzy logic. In: Cintula P, Hájek P, Noguera C (eds) Handbook of mathematical fuzzy logic-volume 1, volume 37of studies in logic. Mathematical logic and foundations. College Publications, London, pp 103–207
Metcalfe G, Montagna F (2007) Substructural fuzzy logics. J Symb Log 72(3):834–864
Metcalfe G, Olivetti N, Gabbay D (2008) Proof theory for fuzzy logics, volume 36 of Springer series in applied logic. Springer, New York
Takeuti G, Titani S (1984) Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. J Symb Log 49(3):851–866
Wang SM (2012) Uninorm logic with the n-potency axiom. Fuzzy Sets Syst 205:116–126