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Extreme biconic copulas: characterization, properties and extensions to aggregation functions

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Abstract

Motivated by optimization problems arising in the study of copula-based stochastic models, we study the extreme points (in the Krein-Milman sense) of the class of extreme biconic copulas and provide their characterization. Moreover, we show how this characterization may help in shedding light upon new aspects of biconic copulas, including their link with extreme-value copulas. The study is embedded into the more general setting of aggregation functions (especially, semi–copulas and quasi–copulas), where the determination of extreme points has also interest.

Keywords: Copula, Extreme Point, Stochastic Measures, Uncertainty quantification.

1. Introduction

In several problems arising in the analysis of multivariate systems, the main task is to select a convenient (stochastic) mathematical model for describing the behavior of the system, according to the available information. The model is hence used in order to derive some system quantities, which are of interest by practitioners and/or decision makers. For instance, in risk analysis, one may wonder how to calculate some risk measures like quantiles — also known as Value-at-Risk in a financial setting, return period in environmental sciences, etc. — by using the probability distribution of the whole system.

However, the issue of selecting the *right* model has to compare with the model uncertainty (and uncertainty quantification), since the system quantities can just be calculated under partial knowledge of the model, which is due, e.g., to scarce observations, corrupted

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or missing data and, in general, complexity of the whole system and its inter-connections. Model uncertainty has become particularly popular in (financial) risk management, e.g. in the estimation of optimal bounds for the risk of a given portfolio of losses when only the marginal distributions are known (see, e.g., [15, 30, 31]).

In view of the celebrated Sklar's theorem (see [34, 37]), most of these optimization problems do not depend on the behavior of the system components (i.e. on the marginal distributions), but on their dependence structure, as captured by the copula of the system. We recall that a copula is a (probability) distribution function on $[0,1]^d$, for $d \geq 2$, with univariate uniform margins (see, for instance, [13, 26]), whose class is denoted by $\mathcal{C}$. Specifically, in many cases, the interest is to study a special functional $\psi\colon \mathcal{C} \to \mathbb{R}$ (whose output is related to the "risk" of a system) and, specifically, to determine $\sup \psi(C)$ and $\inf \psi(C)$, where the supremum (respectively, infimum) is considered with respect to the whole $\mathcal{C}$ (or one of its subsets), which may help to quantify the uncertainty of our model (under possible different dependence structures).

Mathematically, since the class of copulas is convex and compact (under L^∞ norm), as stressed in [4], the search for a supremum (respectively, infimum) of a functional in $\mathcal{C}$ may be ease by considering the supremum (respectively, infimum) of the same functional in the class of extreme copulas. Here, we remind that a copula is extreme if it cannot be expressed as a (non-trivial) convex combination of two distinct copulas. Now, although extreme copulas are of potential interest in various optimization problems, even in the bivariate case, only a few examples of extreme copulas are available (e.g., shuffles of Min [25, 35], hairpin copulas [11]), and an "operational" characterization of extreme copulas is still out of reach (see, e.g., [1]).

Here, we focus on the subclass $\mathcal{C}_B$ of biconic copulas, which are constructed starting from the values that the copula assumes on its diagonal section (see [21] and also [8, 16]). Since the class of biconic copulas is convex and compact (this result will be proved later), first, we characterize its extreme points; then, we show how these copulas are also extreme points in $\mathcal{C}$.

In order to reach our main results, we embed this study in the general setting of aggregation functions [3, 19] and biconic aggregation functions [21]. In particular, we consider special subclasses of aggregation functions like semi–copulas [12, 14] and quasi–copulas [6, 18, 33]. Since biconic semi–copulas and quasi–copulas form, in fact, convex sets, it is also of general interest to investigate their extreme points.

The paper is organized as follows. After some preliminaries concerning biconic aggregation functions, we state our main problem about extreme points (Section 2). Section 3 investigates the extreme points of biconic semi–copulas and quasi–copulas. Section 4 is focused on the study of biconic copulas, while Section 5 underlines the main consequences of the characterization of extreme biconic copulas. In particular, we show how biconic copulas are related to the class of extreme-value copulas, which arise in the study of maxima of independent and identically distributed random vectors. Section 6 is devoted to the conclusions.

2. Preliminaries and problem statement

Since our results will also use concepts of geometric nature, in the text $\langle (a,b),(c,d)\rangle$ will denote the segment which joins the points (a,b) and (c,d) in $\mathbb{R}^2$, and $\langle (a,b),(c,d)\rangle_0$ will denote the same segment deprived of its extreme points. Moreover, for basic definitions of bivariate aggregation functions and copulas, we refer to [13, 19].

Here, we denote by $\mathcal{A}$, $\mathcal{S}$, $\mathcal{Q}$ and $\mathcal{C}$ the class of binary aggregation functions, semi–copulas, quasi–copulas and copulas, respectively. We recall that $\mathcal{C} \subset \mathcal{Q} \subset \mathcal{S} \subset \mathcal{A}$.

The *diagonal section* of a function $F\colon [0,1]^2 \to [0,1]$ is the function $\delta_F\colon [0,1] \to [0,1]$ defined by $\delta_F(t) = F(t,t)$ for every $t \in [0,1]$. In particular, the diagonal section δ of a copula satisfies the following conditions:

(D1) $\delta(0) = 0$ and $\delta(1) = 1$;

(D2) δ is increasing;

(D3) $\delta(t) \le t$ for all $t \in [0,1]$;

(D4) δ is *2-Lipschitz continuous*, i.e. $|\delta(t') - \delta(t)| \le 2|t' - t|$ for all $t, t' \in [0,1]$.

As a consequence of these properties, we have that δ is absolutely continuous and $\delta'(t) \in [0,2]$ for every t in $[0,1]$, if it exists. We denote by $\mathcal{D}_{\mathcal{C}}$ the sets of all functions $\delta\colon [0,1] \to [0,1]$ that satisfies (D1), (D2), (D3) and (D4). As known (see, for instance, [13, 26]), a function $\delta\colon [0,1] \to [0,1]$ is the diagonal section of a (quasi–)copula if, and only if, $\delta \in \mathcal{D}_{\mathcal{C}}$.

Analogously, a function $\delta\colon [0,1] \to [0,1]$ is the diagonal section of a binary aggregation function (respectively, semi-copula) if, and only if, it satisfies (D1) and (D2) (respectively, (D1), (D2) and (D3)). See, for instance, [12, 21]. We denote by $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{S}}$, respectively, the sets of all diagonal sections of binary aggregation functions and semi–copulas. As known, $\mathcal{D}_{\mathcal{C}} \subset \mathcal{D}_{\mathcal{S}} \subset \mathcal{D}_{\mathcal{A}}$.

Various methods have been presented in the literature to construct aggregation functions starting from the values assumed by their diagonal sections (see [17] and references therein). Here, we focus on one of these constructions, called *biconic aggregation functions* as investigated in [21] (see also [22]) and recalled here.

Let $\delta \in \mathcal{D}_{\mathcal{A}}$ and $\alpha, \beta \in [0,1]$. Let $A_\delta^{\alpha,\beta}$ be the function defined by

$$A_\delta^{\alpha,\beta}(x,y) = \begin{cases} \alpha(x-y) + (1+y-x)\delta\left(\dfrac{y}{1+y-x}\right), & \text{if } y \le x, \\[2ex] \beta(y-x) + (1+x-y)\delta\left(\dfrac{x}{1+x-y}\right), & \text{otherwise}, \end{cases} \tag{1}$$

with the convention $\delta\left(\frac{0}{0}\right) := 0$. This function is called *biconic function* (*with a given diagonal section*) *of type* (α,β)—for the sake of simplicity, we will only write *biconic function.* The elements of $\mathcal{D}_{\mathcal{A}}$ for which the function given by (1) is an aggregation function, a semi-copula, a quasi-copula or a copula, respectively, are characterized in [8, 21] and reported here.

Theorem 2.1. *Let $\delta \in \mathcal{D}_{\mathcal{A}}$ and $\alpha, \beta \in [0,1]$ such that $\alpha \leq \beta$. Then the function $A_\delta^{\alpha,\beta}$ given by* (1) *is an aggregation function if, and only if, the function $\frac{\delta(t)-\alpha}{t}$ is increasing for every $t \in]0,1]$ and $\frac{\delta(t)-\beta}{1-t}$ is increasing for every $t \in [0,1[$.*

Furthermore, if $\alpha = \beta = 0$, it holds:

(i) *For every $\delta \in \mathcal{D}_{\mathcal{S}}$, $A_\delta^{0,0}$ is a semi–copula if, and only if, the function $\frac{\delta(t)}{t}$ is increasing for all $t \in]0,1]$.*

(ii) *For every $\delta \in \mathcal{D}_{\mathcal{C}}$, $A_\delta^{0,0}$ is a quasi–copula if, and only if, the function $\frac{\delta(t)}{t}$ is increasing for all $t \in]0,1]$ and the function $\frac{t-\delta(t)}{1-t}$ is increasing for all $t \in [0,1[$.*

(iii) *For every $\delta \in \mathcal{D}_{\mathcal{C}}$, $A_\delta^{0,0}$ is a copula if, and only if, δ is convex.*

Observe that, for $\alpha = \beta = 0$, the function $A_\delta^{\alpha,\beta}$ given by (1) can be written as

$$A_\delta^{0,0}(x,y) = \delta\left(\frac{x \wedge y}{1 + (x \wedge y) - (x \vee y)}\right)[1 + (x \wedge y) - (x \vee y)], \quad (x,y) \in [0,1]^2. \tag{2}$$

Remark 2.1. Notice that the increasingness of the function $\frac{t-\delta(t)}{1-t}$ for all $t \in [0,1[$ in Theorem 2.1(ii) is equivalent to the increasingness of the function $\frac{1-\delta(t)}{1-t}$ for all $t \in [0,1[$, a fact that will be used in the sequel.

The functions in Theorem 2.1 are called *biconic aggregation functions, biconic semi-copulas, biconic quasi-copulas* and *biconic copulas* (*with a given diagonal section*), and they are denoted by $A_\delta^{\alpha,\beta}, S_\delta, Q_\delta$ and C_δ, respectively, for an appropriate δ.

For $\alpha, \beta \in [0,1]$, let $\mathcal{A}_B^{\alpha,\beta}$ denote the set of all biconic aggregation functions A such that $A(1,0) = \alpha$ and $A(0,1) = \beta$. We also denote by $\mathcal{S}_B$, $\mathcal{Q}_B$ and $\mathcal{C}_B$ the sets of all biconic semi-copulas, biconic quasi-copulas and biconic copulas, respectively. Moreover, $\mathcal{D}_{\mathcal{S}_B}$, $\mathcal{D}_{\mathcal{Q}_B}$ and $\mathcal{D}_{\mathcal{C}_B}$ denote the sets of the diagonal sections of all biconic semi-copulas, quasi-copulas and copulas, respectively, whose properties are characterized in Theorem 2.1.

Now, we are ready to state our main research question that is related to the extreme points of a given convex set of aggregation functions. The importance of extreme points is well recognized by the following two results from Functional Analysis (see [29], for instance).

Theorem 2.2 (Krein-Milman)**.** *Let S be a non-empty compact convex subset of a locally convex Hausdorff topological vector space. Then S is the closure of the convex hull of the set of extreme points of S.*

Given a set S, we will denote by $\mathrm{Ext}(S)$ the set of its extreme points.

Theorem 2.3 (Choquet)**.** *For a compact convex subset K of a normed space V, given $k \in K$, there exists a probability measure ν supported on $\mathrm{Ext}(K)$ such that, for any affine function f on K, we have*

$$f(k) = \int_K f(e)\, d\nu(e).$$

Now, for the sake of presentation, we illustrate our main goal on the class $\mathcal{C}_B$, but similar considerations hold for $\mathcal{A}_B^{\alpha,\beta}$, $\mathcal{S}_B$ and $\mathcal{Q}_B$.

The class $\mathcal{C}_B$ is a convex set, which can be proved to be also compact under L^∞ norm (see Proposition 6.1 in Appendix). As such, in view of Krein–Milman Theorem 2.2, $\mathcal{C}_B$ is the closure of the convex hull of its extreme points. Roughly speaking, convex combinations of the extreme points of $\mathcal{C}_B$ can be used to approximate any element of $\mathcal{C}_B$ (and, hence, to derive useful information about suitable functionals of $\mathcal{C}_B$). Thus, it could be of great interest to determine extreme copulas in $\mathcal{C}_B$.

As a matter of fact, $\mathcal{C}_B$ is also in one-to-one correspondence to the set $\mathcal{D}_{\mathcal{C}_B}$, which is the set of the generators of the whole class. On the other hand, $\mathcal{D}_{\mathcal{C}_B}$ is also a convex set, which is a compact subset of all continuous functions in $[0,1]$ with L^∞ norm.

Thus, one may wonder whether a suitable strategy to find an extreme copula in $\mathcal{C}_B$ is to study the extreme points in $\mathcal{D}_{\mathcal{C}_B}$, since this last problem may be easier to handle due to the reduction in the dimensionality of the involved functions.

In general, it is not possible to find any special relationship between extreme copulas and extreme diagonal sections, as the following example shows (for the characterization of the extreme elements of $\mathcal{D}_{\mathcal{C}}$, see Theorem 6.1 in the Appendix).

Example 2.1. Four cases are provided:

(a) Extreme copula with extreme diagonal. The copula $W(x,y) = \max\{0, x+y-1\}$ is an extreme copula in $\mathcal{C}$ and its diagonal section $\delta_W(t) = \max\{0, 2t-1\}$ is also an extreme point of $\mathcal{D}_{\mathcal{C}}$.

(b) Extreme copula with non-extreme diagonal. For every $\delta \in \mathcal{D}_{\mathcal{C}}$, consider the Fredricks–Nelsen copula

$$C_\delta^{\mathbf{FN}}(x,y) = \min\left(x, y, \frac{\delta(x)+\delta(y)}{2}\right).$$

If $\delta(t) = t^2$ for all $t \in [0,1]$, then we have that $C_\delta^{\mathbf{FN}}$ is an extreme copula (see [27, 32]), but δ is not an extreme point of $\mathcal{D}_{\mathcal{C}}$.

(c) Non-Extreme copula with extreme diagonal. Consider the diagonal δ_W and the copula C whose mass is spread uniformly on $([0,1/2] \times [1/2,1]) \cup ([1/2,1] \times [0,1/2])$. Then δ_W is extreme in $\mathcal{D}_{\mathcal{C}}$, but the copula C is not extreme in $\mathcal{C}$.

(d) Non-extreme copula with non-extreme diagonal. Finally, the independent copula $\Pi(x,y) = xy$ is not extreme and, at the same time, its diagonal section is not extreme in $\mathcal{D}_{\mathcal{C}}$.

However, in the class of biconic copulas (or biconic aggregation functions, in general), a relationship between extreme copulas and extreme diagonals does exist, as the following result shows.

Let $\mathcal{F}_B$ denote one element of the set $\mathcal{A}_B^{\alpha,\beta} \cup \mathcal{S}_B \cup \mathcal{Q}_B \cup \mathcal{C}_B$. Here, we consider on $\mathcal{F}_B$ the pointwise topology. Notice that, as known, in $\mathcal{Q}_B$ (and, hence, in $\mathcal{C}_B$) the pointwise topology is equivalent to the topology induced by L^∞ norm.

Let $\mathcal{D}_{\mathcal{F}_B}$ denote the set of diagonal sections of all the elements of $\mathcal{F}_B$. Note that this set is convex and compact (in the topology of pointwise convergence).

Theorem 2.4. *Let $\delta \in \mathcal{D}_{\mathcal{F}_B}$, and let $A_\delta^{\alpha,\beta} \in \mathcal{F}_B$ be the function given by (1). Then $A_\delta^{\alpha,\beta} \in \mathrm{Ext}(\mathcal{F}_B)$ if, and only if, $\delta \in \mathrm{Ext}(\mathcal{D}_{\mathcal{F}_B})$.*

Proof. Let $\delta, \delta_1, \delta_2 \in \mathcal{D}_{\mathcal{F}_B}$ such that there exists $\gamma \in]0,1[$ for which $\delta(x) = \gamma\delta_1(x) + (1-\gamma)\delta_2(x)$ for all $x \in [0,1]$. On the other hand, if $A_\delta^{\alpha,\beta}, A_{\delta_1}^{\alpha,\beta}, A_{\delta_2}^{\alpha,\beta} \in \mathcal{F}_{BC}$ are given by (1), then we have $A^{\alpha,\beta}_{\gamma\delta_1+(1-\gamma)\delta_2}(x,y) = A_\delta^{\alpha,\beta}(x,y) = \gamma A_{\delta_1}^{\alpha,\beta}(x,y) + (1-\gamma)A_{\delta_2}^{\alpha,\beta}(x,y)$ for all $(x,y) \in [0,1]^2$. Thus, δ is an extreme point of $\mathcal{D}_{\mathcal{F}_{BC}}$ if, and only if, $A_\delta^{\alpha,\beta}$ is an extreme point of $\mathcal{F}_B$. □

As a consequence of Theorem 2.4, the study of the set of extreme points of $\mathcal{F}_B$ is equivalent to study of the set of extreme points of $\mathcal{D}_{\mathcal{F}_B}$. In the following sections, we will use this equivalence to determine the extreme points of $\mathcal{S}_B$, $\mathcal{Q}_B$ and $\mathcal{C}_B$.

3. Extreme biconic semi-copulas and quasi–copulas

In this section, we study the extreme points of the set $\mathcal{S}_B$ and $\mathcal{Q}_B$ by considering the extreme points of $\mathcal{D}_{\mathcal{S}_B}$ and $\mathcal{D}_{\mathcal{Q}_B}$, respectively.

Theorem 3.1. *Let $\delta \in \mathcal{D}_{\mathcal{S}_B}$. Then $\delta \in \mathrm{Ext}(\mathcal{D}_{\mathcal{S}_B})$ if, and only if, there exists $x_0 \in [0,1]$ such that $\delta \in \{\delta_{x_0}, \delta^*_{x_0}\}$, where*

$$\delta_{x_0}(x) = \begin{cases} 0, & \text{if } x \in [0,x_0[, \\ x, & \text{if } x \in [x_0,1], \end{cases} \quad \text{and} \quad \delta^*_{x_0}(x) = \begin{cases} 0, & \text{if } x \in [0,x_0], \\ x, & \text{if } x \in]x_0,1]. \end{cases} \tag{3}$$

Proof. If $\delta \in \mathcal{D}_{\mathcal{S}_B}$ is such that $\frac{\delta(x)}{x} \in \{0,1\}$ for every $x \in]0,1]$, we prove that $\delta \in \mathrm{Ext}(\mathcal{D}_{\mathcal{S}_B})$. On one hand, it is clear $\frac{\delta(x)}{x} \geq 0$ for every $x \in]0,1]$. If $\delta(x) = 0$ for every $x \in [0,1[$ and there exist $\delta_1, \delta_2 \in \mathcal{D}_{\mathcal{S}_{BC}}$ such that $\delta(x) = \gamma\delta_1(x) + (1-\gamma)\delta_2(x)$ for all $x \in [0,1]$ and for some $\gamma \in]0,1[$, then $\delta_1(x) = \delta_2(x) = 0$—note that if there exists $x' \in [0,1]$ such that $\delta_1(x') > 0$ (similarly, if $\delta_2(x') > 0$), then $0 = \gamma\delta_1(x') + (1-\gamma)\delta_2(x') > (1-\gamma)\delta_2(x')$, i.e. $\delta_2(x') < 0$, which is absurd. On the other hand, we have $\frac{\delta(x)}{x} \leq 1$ for every $x \in [0,1]$.
If $\delta(x) = x$ and $\delta(x) = \gamma\delta_1(x) + (1-\gamma)\delta_2(x)$, then $\delta_1(x) = \delta_2(x) = x$ for every x; otherwise, if there exists $x' \in [0,1]$ such that $\delta_1(x') < x'$ (similarly, if $\delta_2(x') < x'$), then $x' = \gamma\delta_1(x') + (1-\gamma)\delta_2(x') < \gamma x' + (1-\gamma)\delta_2(x')$, i.e. $\delta_2(x') > x'$, which is absurd.

Now, let δ be in $\mathcal{D}_{\mathcal{S}_{BC}}$ such that $\delta(x)$ and $\frac{\delta(x)}{x}$ are increasing for every $x \in]0,1]$ and $\delta(0) = 1 - \delta(1) = 0$, with $\frac{\delta(x)}{x} \notin \{0,1\}$ for some $x_0 \in]0,1[$. We show that δ can be expressed as convex combination of two different diagonal sections. We distinguish two cases:

(a) $\frac{\delta(x)}{x}$ is not continuous at the point x_0. We have to consider two subcases, depending on the value of $\delta(x_0^-)$.

(a.1) $\delta(x_0^-) > 0$.

(a.1.1) $\frac{\delta(x_0^-)}{x_0} < \frac{\delta(x_0)}{x_0}$. For $x \in [0, x_0[$ and $\varepsilon > 0$ sufficiently small, we define $\delta_1(x) = (1-\varepsilon)\delta(x)$ and $\delta_2(x) = (1+\varepsilon)\delta(x)$, with $(1+\varepsilon)\frac{\delta(x_0^-)}{x_0} < \frac{\delta(x_0)}{x_0}$. For $x \in [x_0, 1]$, we define $\delta_1(x) = \delta_2(x) = \delta(x)$. Then we have $\frac{\delta_1(x)+\delta_2(x)}{2} = \delta(x)$ for all $x \in [0, 1]$—note that, for $i = 1, 2$, the functions $\delta_i(x)$ and $\frac{\delta_i(x)}{x}$ are increasing for every x, and $\delta_i(0) = 1 - \delta_i(1) = 0$.

(a.1.2) $\frac{\delta(x_0^-)}{x_0} = \frac{\delta(x_0)}{x_0} < \frac{\delta(x_0^+)}{x_0}$. The reasoning is similar to case (a.1.1). by taking the intervals $[0, x_0]$ and $]x_0, 1]$.

(a.2) $\delta(x_0^-) = 0$.

(a.2.1) $\delta(x_0^+) > \delta(x_0) > 0$. The reasoning is similar to that in (a.1.2), and we omit it.

(a.2.2) $\delta(x_0^+) < x_0$. The reasoning is similar to (a.1.1) (or (a.1.2)) by taking $\delta_1(x) = (1 + \varepsilon)\delta(x) - \varepsilon x$ and $\delta_2(x) = (1 - \varepsilon)\delta(x) + \varepsilon x$.

(b) $\frac{\delta(x)}{x}$ is continuous. Let $x_0 = \sup\{x \in [0, 1] : \delta(x) = 0\}$ and $x_1 = \inf\{x \in [0, 1] : \delta(x) = x\}$. Let $x_2 \in]x_0, x_1[$ such that $\delta(x_2) = \frac{x_2}{2}$. We define the functions

$$f(x) = \begin{cases} \dfrac{\alpha\delta(x)}{x}, & \text{if } x \in]x_0, x_2[, \\ \alpha\left(1 - \dfrac{\delta(x)}{x}\right), & \text{if } x \in]x_2, x_1[, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha \in [0, 1[$, $\delta_1(x) = \delta(x)+xf(x)$ and $\delta_2(x) = \delta(x)-xf(x)$, then $\frac{\delta_1(x)+\delta_2(x)}{2} = \delta(x)$ for all $x \in [0, 1]$. Note that, for $i = 1, 2$, the functions $\delta_i(x)$ and $\frac{\delta_i(x)}{x}$ are increasing for every x, and $\delta_i(0) = 1 - \delta_i(1) = 0$.

The proof is hence complete. □

In view of Theorem 2.4, it follows that $S_\delta \in \text{Ext}(\mathcal{S}_B)$ if, and only if, there exists $x_0 \in [0, 1]$ such that its diagonal section $\delta \in \{\delta_{x_0}, \delta^*_{x_0}\}$. Moreover, as a consequence of Theorem 2.3 the following characterization of the extreme diagonals of the biconic semi-copulas can be given.

Corollary 3.1. *Let F_1 be a distribution function on $[0, 1]$ and let F_2 be a left-continuous distribution function on $]0, 1[$. Then $\delta \in \mathcal{D}_{\mathcal{S}_B}$ if, and only if, $\delta(x) = x(\alpha F_1(x)+(1-\alpha)F_2(x))$ for all $x \in [0, 1]$, with $\alpha \in [0, 1]$.*

Proof. For $x_0 \in]0, 1]$ define the function δ_{x_0} as in (3) and, for $x_0 \in]0, 1[$ define the function $\delta^*_{x_0}$ as in (3). For $x_0 = 0$, set $\delta_0(x) = x$ for every $x \in [0, 1]$.

Consider the function $F\colon (\text{Ext}(\mathcal{D}_{\mathcal{S}_B}), \|\cdot\|_\infty) \to]-1, 1]$, such that $F(\delta_{x_0}) = x_0$ and $F(\delta^*_{x_0}) = -x_0$. It can be easily seen that such an F is a homeomorphism.

It is clear $\delta_{x_0}(x) = x\Delta_{x_0}(x)$ and $\delta^*_{x_0}(x) = x\Delta^*_{x_0}(x)$, where Δ_{x_0} is the Dirac distribution and $\Delta^*_{x_0}$ is the left-continuous Dirac distribution. From Theorem 2.3, for every $\delta \in \mathcal{D}_{\mathcal{S}_B}$, there exists a measure μ_δ in $(]-1,1], \mathcal{B}(]-1,1]))$ such that

$$\delta(x) = \int_{]-1,0[} \delta^*_{-t}(x)\, d\mu_\delta(t) + \int_{[0,1]} \delta_t(x)\, d\mu_\delta(t) = x \int_{]-1,0[} \Delta^*_{-t}(x)\, d\mu_\delta(t) + x \int_{[0,1]} \Delta_t(x)\, d\mu_\delta(t).$$

For every $x \in [0,1]$, define

$$\int_{[0,1]} \Delta_t(x)\, d\mu_\delta(t) = \mu_\delta([0,x]) := G_1(x)$$

and

$$\int_{]-1,0[} \Delta^*_{-t}(x)\, d\mu_\delta(t) = \int_{]0,1[} \Delta^*_t(x)\, d\mu'_\delta(t) = \mu'_\delta(]0,x[) := G_2(x),$$

where μ'_δ is the translation of μ_δ through the function $h\colon]-1,0[\to]0,1[$ given by $h(x) = -x$. Thus

$$G_1(x) = G_1(1) \cdot \frac{G_1(x)}{G_1(1)} = G_1(1) \cdot F_1(x) = \alpha \cdot F_1(x),$$

where F_1 is a distribution function, and, similarly, $G_2(x) = (1-\alpha)F_2(x)$, where F_2 is a left-continuous distribution function; whence $\delta(x) = x(\alpha F_1(x) + (1-\alpha)F_2(x))$.

Conversely, we note that, if $\delta(x) = x(\alpha F_1(x) + (1-\alpha)F_2(x))$ for all $x \in [0,1]$, with $\alpha \in [0,1]$, it is easy to check that $\frac{\delta(x)}{x}$ is increasing for all $x \in]0,1]$, which completes the proof. □

Note that the set $\mathrm{Ext}(\mathcal{D}_{\mathcal{S}_B})$ is closed. Moreover, the set $\mathrm{Ext}(\mathcal{S}_B)$ is also closed. Clearly, this set is not dense in $\mathcal{S}_B$ (actually, it is nowhere-dense in $\mathcal{S}_B$). This is not the case of $\mathrm{Ext}(\mathcal{Q}_B)$, as the next result shows.

First, let us construct a possible diagonal section $\delta \in \mathcal{D}_{\mathcal{Q}_B}$.

Let $n \in \mathbb{N}$. Define the sequence $(m_k)_{k=0,1,\ldots,n-1}$

$$0 = m_0 < m_1 < \cdots < m_{n-1} < 1 \tag{4}$$

and the sequence $(m'_k)_{k=0,1,\ldots,n-1}$

$$1 < m'_0 < m'_1 < \cdots < m'_{n-1} = 2. \tag{5}$$

Consider the sequence

$$0 = x_0 < x_1 < \cdots < x_{2n-1} < x_{2n} = 1$$

such that, for $k = 0,1,\ldots,n-1$, x_{2k+1} is the first coordinate of the intersection point between the line of equation $y = m_k x$ and the line of equation $y = m'_k(x-1)+1$, that is

$$x_{2k+1} = \frac{m'_k - 1}{m'_k - m_k}.$$

Moreover, for $k = 1, \dots, n-1$, let x_{2k} be the first coordinate of the intersection point between the line of equation $y = m_k x$ and the line of equation $y = m'_{k-1}(x-1)+1$, that is

$$x_{2k} = \frac{m'_k - 1}{m'_k - m_k}.$$

Then, define the function $\delta\colon [0,1] \to [0,1]$ given by

$$\delta(x) = \begin{cases} m_k x, & \text{if } x \in [x_{2k}, x_{2k+1}] \text{ for some } k = 0, 1, \dots, n-1 \\ m'_k(x-1)+1, & \text{if } x \in [x_{2k+1}, x_{2k+2}] \text{ for some } k = 0, 1, \dots, n-1. \end{cases} \tag{6}$$

An example of such a function is depicted in Figure 1. It can be checked that such a δ belongs to $\mathcal{D}_{\mathcal{Q}_B}$ since it verifies the condition (ii) of Theorem 2.1.

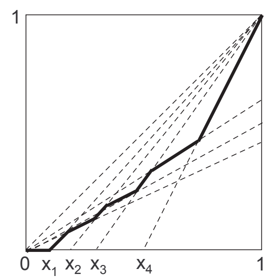

Figure 1: Construction of the diagonal section δ (in bold) defined in (6).

Theorem 3.2. *For every $n \in \mathbb{N}$ and for all sequences $(m_k)_{k=0,1,\dots,n-1}$ and $(m'_k)_{k=0,1,\dots,n-1}$ defined as in* (4) *and* (5)*, respectively, the function δ given by* (6) *is an extreme point of $\mathcal{D}_{\mathcal{Q}_B}$.*

Proof. Let δ be the function defined by (6). Assume there exist two diagonals δ_1 and δ_2 such that $\delta(x) = \gamma\delta_1(x) + (1-\gamma)\delta_2(x)$ for all $x \in [0,1]$ and for any $\gamma \in]0,1[$. We distinguish some cases:

(a) If $x \in [0, x_1]$ then we have $\delta(x) = 0$. If $\delta_1(x^*) > 0$ for some $x^* \in [0, x_1]$ (similarly, if $\delta_2(x^*) > 0$), then $\delta_2(x^*) < 0$, which is absurd.

(b) In the first interval, $[0, x_1]$, we have $\delta_1(x) = \delta_2(x) = \delta(x)$. Suppose that the first interval in which these equalities do not hold is $[x_{2k}, x_{2k+1}]$, for some $k \in \{1, 2, \dots, n-1\}$, and consider $x \in [x_{2k}, x_{2k+1}]$. If $\delta(x) = \gamma\delta_1(x) + (1-\gamma)\delta_2(x)$ for every x, with $\delta_1(\bar{x}) < m_k\bar{x}$ for some $\bar{x} \in [x_{2k}, x_{2k+1}]$ (similarly, if $\delta_2(\bar{x}) < m_k\bar{x}$), then we have $\frac{\delta_1(\bar{x})}{\bar{x}} < m_k = \frac{\delta_1(x_{2k})}{x_{2k}}$, which contradicts condition (ii) of Theorem 2.1.

(c) Suppose the first interval in which the equalities $\delta_1(x) = \delta_2(x) = \delta(x)$ are not satisfied is $[x_{2k+1}, x_{2k+2}]$, for some $k \in \{0, 1, \dots, n-1\}$, and let $x \in [x_{2k+1}, x_{2k+2}]$. Let $\delta(x) =$

$\gamma\delta_1(x)+(1-\gamma)\delta_2(x)$ for every x, with $\delta_1(\bar{x}) > m'_k(\bar{x}-1)+1$ for some $\bar{x} \in [x_{2k+1}, x_{2k+2}]$ (similarly, if $\delta_2(\bar{x}) > m'_k(\bar{x}-1)+1$), then we have $\frac{\delta_1(\bar{x})-1}{\bar{x}-1} < m'_k = \frac{\delta_1(x_{2k+1})-1}{x_{2k+1}-1}$, which contradicts condition (ii) of Theorem 2.1.

Therefore, $\delta(x) = \delta_1(x) = \delta_2(x)$ for all $x \in [0,1]$, i.e. $\delta \in \mathrm{Ext}(\mathcal{D}_{\mathcal{Q}_B})$. □

As a consequence of Theorems 2.4 and 3.2, we have the following result.

Corollary 3.2. *The set* $\mathrm{Ext}(\mathcal{Q}_B)$ *is dense in* $\mathcal{Q}_B$ *(under* L^∞*-norm).*

Proof. Let $\delta^* \in \mathcal{D}_{\mathcal{Q}_B}$. Our objective is to approximate δ^* by a diagonal δ of type (6) that belongs to $\mathrm{Ext}(\mathcal{D}_{\mathcal{Q}_B})$. To this end, let P be the partition of $[0,1]$ given by

$$0 = x_0 < x_1 < x_2 = \frac{1}{2n} < x_3 < x_4 = \frac{2}{2n} < \cdots < x_{4n} = \frac{2n}{2n} = 1.$$

To construct a diagonal section $\delta \in \mathrm{Ext}(\mathcal{D}_{\mathcal{Q}_B})$ proceed in the following manner:

- If $x \in [0, x_1]$, then define $\delta(x) = 0$;
- If $x \in [x_1, 1/(2n)]$, then define the graph of δ as the segment

$$\langle (x_1, 0), (1/(2n), \delta^*(1/(2n))) \rangle$$

  with slope $\frac{1-\delta^*(1/(2n))}{1-1/(2n)}$, so that $x_1 = \frac{\delta^*(1/(2n))-1/(2n)}{\delta^*(1/(2n))-1}$.

Moreover, for every $k \in \{1, 2, \ldots, 2n-1\}$,

- If $x \in [k/(2n), x_{2k+1}]$, then define $\delta(x) = m_k x$ with $m_k = \frac{2n\delta^*(k/(2n))}{k}$.
- If $x \in [x_{2k+1}, (k+1)/(2n)]$, then define the graph of δ as the segment

$$\langle (x_{2k+1}, 2n\delta^*(k/(2n))x_{2k+1}/k), ((k+1)/(2n), \delta^*((k+1)/(2n))) \rangle$$

  with slope $\frac{1-\delta^*((k+1)/(2n))}{1-(k+1)/(2n)}$, so that

$$x_{2k+1} = \frac{k(2n\delta^*((k+1)/(2n)) - k - 1)}{2n\delta^*(k/(2n))(2n-k-1) - 2kn(1-\delta^*((k+1)/(2n)))}.$$

As a byproduct of the previous procedure, all the points of P are hence determined. The function δ so defined can be proved to belong to $\mathrm{Ext}(\mathcal{D}_{\mathcal{Q}_B})$ by using Theorem 3.2.

Finally, note

$$|\delta^*(x) - \delta(x)| \le \max_{k=0,1,\ldots,n-1} |\delta^*(x_{2k}) - \delta^*(x_{2k+2})| \le 2 \cdot \frac{2}{2n} = \frac{2}{n},$$

which tends to 0 as $n \to \infty$. This completes the proof. □

4. Extreme biconic copulas

Now, we focus on the class of biconic copulas. As above, we will characterize the extreme points of $\mathcal{C}_B$ by investigating the class of its diagonal sections.

Theorem 4.1. *The extreme points of $\mathcal{D}_{\mathcal{C}_B}$ are given by the diagonal sections defined, for every $t \in [0,1]$, by*

$$\delta_{\alpha,1}(t) = 0 \vee \left(\frac{t-\alpha}{1-\alpha}\right), \tag{7}$$

$$\delta_{0,\beta}(t) = \frac{(2\beta-1)t}{\beta} \vee (2t-1)\,, \tag{8}$$

$$\delta_{\alpha,\beta}(t) = \left(0 \vee \frac{(2\beta-1)(t-\alpha)}{\beta-\alpha}\right) \vee (2t-1), \tag{9}$$

$$\delta_{1/2,1/2}(t) = 0 \vee (2t-1), \tag{10}$$

for every $\alpha \in [0,1/2[$ and $\beta \in]1/2,1]$.

Proof. First, we prove that the diagonal section given by (7) is extreme for every $\alpha \in [0,1/2]$ (note that the diagonal section given by (10) can be included here). To this end, we will verify that $\delta'_{\alpha,1}$ — for convenience, we consider the right-hand derivative — cannot be written as the convex sum of two increasing functions $f_1, f_2\colon [0,1] \to [0,2]$ such that $\int_0^1 f_1(t)\,dt = \int_0^1 f_2(t)\,dt = 1$. Since f_1 and f_2 are non-negative, for all $t \in [0,1]$ and every $\gamma \in]0,1[$, we have that $\gamma f_1(t) + (1-\gamma) f_2(t) = 0$ implies $f_1(t) = f_2(t) = 0$, i.e. the functions f_1 and f_2 are equal to 0 on the same interval where $\delta'_{\alpha,1}$ is equal to 0. Now, let $t_0 = \sup\{t \in [0,1] : \delta'_{\alpha,1}(t) = 0\}$. Assume $f_1(t_0) \geq \delta'_{\alpha,1}(t_0)$. If $f_1(t_0) > \delta'_{\alpha,1}(t_0)$, since f_1 is increasing, we have $f_1(t) \geq f_1(t_0) > \delta'_{\alpha,1}(t_0) = \delta'_{\alpha,1}(t)$ for every $t \in [t_0,1]$; thus

$$\int_{t_0}^1 f_1(t)\,dt > \int_{t_0}^1 \delta'_{\alpha,1}(t) = \int_{t_0}^1 \frac{1}{1-\alpha}\,dt = \frac{1-t_0}{1-\alpha}.$$

However, since $t_0 = \alpha$, $\int_{t_0}^1 f_1(t)\,dt > 1$, which is absurd; therefore $f_1(t_0) = \delta'_1(t_0)$, and this implies $f_2(t_0) = \delta'_1(t_0)$. Similarly, we obtain the same result by assuming $f_2(t_0) \geq \delta'_{\alpha,1}(t_0)$. Furthermore, since, for $i = 1,2$, $f_i(t) \geq \delta'_{\alpha,1}(t)$ for all $t \in [t_0,1]$, it follows that $\gamma f_1(t) + (1-\gamma) f_2(t) = \delta'_{\alpha,1}(t)$ implies $f_1(t) = f_2(t) = \delta'_{\alpha,1}(t)$ for all $t \in [t_0,1]$, which ends up showing that $\delta'_{\alpha,1}$ cannot be written as a convex sum of f_1 and f_2 with $f_i \neq \delta'_{\alpha,1}$, for $i = 1,2$, and hence $\delta_{\alpha,1} \in \mathrm{Ext}(\mathcal{D}_{\mathcal{C}_B})$.

A similar reasoning leads us to determine that the diagonals given by (8) and (9) are extreme points of $\mathcal{D}_{\mathcal{C}_B}$.

It only remains to prove that there are no other extreme diagonals in $\mathcal{D}_{\mathcal{C}_B}$ In this respect, consider a convex diagonal δ different from those given by (7)–(10). Our goal is to find two increasing functions $f_1, f_2\colon [0,1] \longrightarrow [0,2]$ such that $\int_0^1 f_1(t)\,dt = \int_0^1 f_2(t)\,dt = 1$ and $\delta'(t) = \frac{f_1(t)+f_2(t)}{2}$ for all $t \in [0,1]$. Suppose, without loss of generality, that δ' is right-continuous. Given the graph of δ', we complete it with vertical segments, if necessary. Let

$x_0 \in [0,1]$ be a point of continuity of δ' such that $\delta'(x_0) < \delta'(x_0+h)$ for all $h > 0$. Let $x_1 \in]x_0, 1]$ and $h_1 \in [0,1]$ such that $x_0 + h_1 \leq x_1$. We define the functions

$$f_1(t) = \begin{cases} 2\delta'(t) - \delta'(x_0), & \text{if } x_0 \leq t \leq x_0 + h_1, \\ 2\delta'(x_0+h_1) - \delta'(x_0), & \text{if } x_0 + h_1 < t \leq x_1, \end{cases} \tag{11}$$

and

$$f_2(t) = \begin{cases} \delta'(x_0), & \text{if } x_0 \leq t \leq x_0 + h_1, \\ 2\delta'(t) + \delta'(x_0) - 2\delta'(x_0+h_1), & \text{if } x_0 + h_1 < t \leq x_1, \end{cases} \tag{12}$$

Note that $\frac{f_1(t)+f_2(t)}{2} = \delta'(t)$ for all $t \in [x_0, x_1]$, and

$$\int_{x_0}^{x_1} (f_1(t) - \delta'(t))\, dt = \int_{x_0}^{x_1} (\delta'(t) - f_2(t))\, dt$$
$$= 2\delta(x_0+h_1) - \delta(x_0) - \delta(x_1) + 2\delta'(x_0+h_1)(x_1 - x_0 - h_1) - \delta'(x_0)(x_1 - x_0),$$

i.e. the area between f_1 and δ' is the same as the area between δ' and f_2, and, in addition, both these areas depend continuously on h_1.

Next, we consider two cases:

(a) Suppose there exist two points x_0, x_0' in $[0,1]$ in which the function δ' is continuous and $\delta'(x_0) < \delta'(x_0+h)$ and $\delta'(x_0') < \delta'(x_0'+h)$ for all $h > 0$. Let $x_1 \in]x_0, 1]$, $x_1' \in]x_0', 1]$, and let h_1, h_1 be in $[0,1]$ such that $x_0 + h_1 \leq x_1$ and $x_0' + h_1' \leq x_1'$. For the interval $[x_0, x_1]$ we define the functions f_1 and f_2 as in (11) and (12), respectively. For the interval $[x_0', x_1']$ we define

$$f_1(t) = \begin{cases} \delta'(x_0'), & \text{if } x_0' \leq t \leq x_0' + h_1', \\ 2\delta'(t) + \delta'(x_0') - 2\delta'(x_0'+h_1'), & \text{if } x_0' + h_1' \leq t \leq x_1', \end{cases}$$

and

$$f_2(t) = \begin{cases} 2\delta'(t) - \delta'(x_0'), & \text{if } x_0' \leq t \leq x_0' + h_1', \\ 2\delta'(x_0'+h_1') - \delta'(x_0'), & \text{if } x_0' + h_1' \leq t \leq x_1'. \end{cases}$$

Moreover, $f_1(t) = f_2(t) = \delta'(t)$ for all $t \in [0,1] \backslash ([x_0, x_1] \cup [x_0', x_1'])$, and, in addition, h_1 and h_1' are taken so that $\int_0^1 f_i(t)\, dt = 1$ for $i = 1, 2$.

(b) Suppose there do not exist two points x_0 and x_0' as in the previous case. Then, there exist, at least, two maximal intervals I and I' with extremes x_0, x_1 for I and x_0', x_1' for I' in which δ' is constant, with $\delta'(t) \neq \{0, 2\}$ for every $t \in I \cup I'$. We define $f_1(t) = \delta'(t) - h_1$ and $f_2 = \delta'(t) + h_1$ for $t \in I$, $f_1(t) = \delta'(t) + h_1'$ and $f_2 = \delta'(t) - h_1'$ for $t \in I'$, and $f_1(t) = f_2(t) = \delta'(t)$ for $t \in [0,1] \backslash (I \cup I')$, where h_1 and h_1' are two appropriate numbers for which $\int_0^1 f_1(t)\, dt = \int_0^1 f_2(t) = 1$ and $\delta'(t) = \frac{f_1(t)+f_2(t)}{2}$ for all $t \in [0,1]$.

In all cases we have $\delta'(t) = \frac{f_1(t)+f_2(t)}{2}$ for $t \in [0,2]$ except, maybe, in two points, whence

$$\delta(t) = \frac{\int_0^1 f_1(t)\,dt + \int_0^1 f_2(t)\,dt}{2}.$$

Since, for $i = 1, 2$, $\int_0^x f_i(t)\,dt$ is increasing and $\int_0^1 f_i(t)\,dt = 1$, it follows that $\delta \notin \mathrm{Ext}(\mathcal{D}_{\mathcal{C}_B})$, which concludes the proof. □

Thus, the extreme points of $\mathcal{D}_{\mathcal{C}_B}$ belong to the two-parameter family $(\delta_{\alpha,\beta})$ for the values α and β indicated in (7)–(10). The diagonal sections of extreme biconic copulas from (7)–(10) are depicted in Figure 2. The supports of their corresponding biconic copulas obtained via (2) are represented in Figure 3.

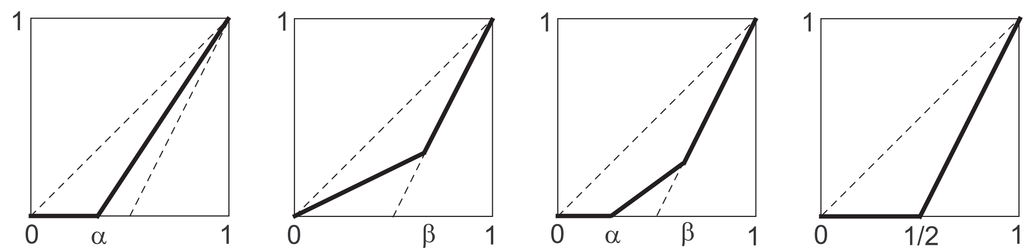

Figure 2: The diagonal sections defined in (7), (8), (9) and (10) (in bold line), from left to right, respectively (see Theorem 4.1).

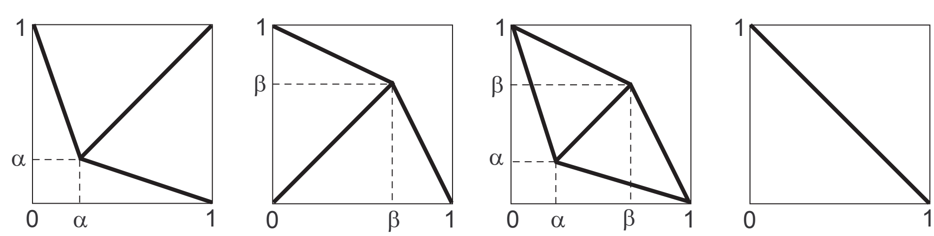

Figure 3: The support (in bold line) of the extreme biconic copulas with diagoanl sections (7), (8), (9) and (10), from left to right, respectively.

As a consequence of Theorems 2.4 and 4.1, the extreme biconic copulas are given in the following result.

Corollary 4.1. *Let $C_\delta \in \mathcal{C}_B$. Then $C_\delta \in \mathrm{Ext}(\mathcal{C}_B)$ if, and only if, δ is equal to one of the expressions provided in* (7), (8), (9) *and* (10).

Now, in general, if R and T be two convex sets such that $R \subset T$, then it is known that $r \in \mathrm{Ext}(R)$ does not imply $r \in \mathrm{Ext}(T)$. This is not the case for $\mathcal{C}_B \subset \mathcal{C}$, as the next result will show. First, we need to make two preliminary observations.

If $C \in \mathcal{C}$ is such that $C \in \mathrm{Ext}(\mathcal{C})$, then it is easy to check that the copula C^* given by

$$C^*(x, y) = y - C(1 - x, y), \ (x, y) \in [0, 1]^2, \tag{13}$$

is also an extreme point of $\mathcal{C}$.

Moreover, we also need the following lemma, which is adapted from Proposition 2 in [32].

Lemma 4.1. *Let L and H be two increasing homeomorphisms such that $L(x) < x < H(x)$ on $]0, 1[$. Let C be a copula such that, for every $x, y \in [0, 1]$, the measure μ_C over $J = [0, x] \times [0, 1] \cup [0, 1] \times [0, y]$ is concentrated on the graphs of L and H. Then the probability mass distribution on J is uniquely determined, and, in particular,*

$$\mu_C([0, x] \times [0, 1] \cap Graph(L)) = L(x) - H^{-1}L(x) + LH^{-1}L(x) - H^{-1}LH^{-1}L(x) + \cdots$$

We are now in position to prove our result.

Theorem 4.2. $\mathrm{Ext}(\mathcal{C}_B) \subset \mathrm{Ext}(\mathcal{C})$.

Proof. For the diagonal given by (10), the copula is W, and the result is trivial.

Let $0 < \alpha < 1/2 < \beta < 1$. For the diagonal section δ given by (7)—and similarly for (8)— the related biconic copula C_δ spreads the probability mass uniformly on the three segments $\langle(0, 1), (\alpha, \alpha)\rangle$, $\langle(\alpha, \alpha), (1, 0)\rangle$ and $\langle(\alpha, \alpha), (1, 1)\rangle$. Now, C_δ can be considered as a patchwork copula (see [10]) that applies either the copula W or the copula M to each of the rectangles $[0, \alpha] \times [\alpha, 1]$, $[\alpha, 1] \times [\alpha, 1]$ and $[\alpha, 1] \times [0, \alpha]$. In such rectangles, the probability mass of the copula can be assigned in a unique way (notice that W and M are extreme copulas), and hence the related patchwork copula C_δ is extreme.

Now, let C_δ be the biconic copula obtained from the diagonal section given by (9), and let C^* be the copula given by (13), whose support is depicted in Figure 4. Let H (respectively, L) be the function whose graph consists of the segments $\langle(0, 0), (1-\beta, \beta)\rangle$ and $\langle(1 - \beta, \beta), (1, 1)\rangle$ (respectively, $\langle(0, 0), (1 - \alpha, \alpha)\rangle$ and $\langle(1 - \alpha, \alpha), (1, 1)\rangle$). The probability mass that is spread on the segment $\langle(0, 0), (1 - \alpha, \alpha)\rangle$ is uniquely determined (see [32]), and from Lemma 4.1 the probability mass on the segment $\langle(0, 0), (x, \frac{\alpha x}{1-\alpha})\rangle$, $x \in [0, 1 - \alpha]$, is given by

$$L(x) - (H^{-1} \circ L)(x) + (L \circ H^{-1} \circ L)(x) - (H^{-1} \circ L \circ H^{-1} \circ L)(x) + \cdots$$

Thus, the probability mass on the segment $\langle(0, 0), (1 - \beta, \beta)\rangle$ is also uniquely determined. Using the survival copula of C^* (see, e.g., [26]), the mass on the segments $\langle(1-\beta, \beta), (1, 1)\rangle$ and $\langle(1 - \alpha, \alpha), (1, 1)\rangle$ is uniquely determined as well, and, consequently, on the segment $\langle(1-\alpha, \alpha), (1-\beta, \beta)\rangle$. Therefore $C^* \in \mathrm{Ext}(\mathcal{C})$ and, thus, $C \in \mathrm{Ext}(\mathcal{C})$, which concludes the proof. □

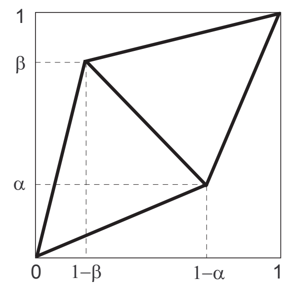

Figure 4: The support (in bold) of the copula C^* in the proof of Theorem 4.2.

5. Consequences of the characterization of extreme biconic copulas

Here, we illustrate some consequence of the results obtained in Section 4.

5.1. Measure-theoretic properties of biconic copulas

We recall that the measure-theoretic properties of biconic copulas have been studied in [16], where the following result has been obtained.

Theorem 5.1. *Let $\delta \in \mathcal{D}_\mathcal{C}$. Then the biconic copula C_δ given by* (2) *has its probability mass distributed on the main diagonal of the unit square and, eventually, in the sets $K_D(\delta)$, $K_A(\delta)$, and $K_S(\delta)$, where:*

$$K_D(\delta) = \{\langle (b_i, b_i), (0,1)\rangle_0 \cup \langle (b_i, b_i), (1,0)\rangle_0 \colon \nexists\, \delta'(b_i)\};$$
$$K_A(\delta) = \{\langle (b, b), (0,1)\rangle_0 \cup \langle (b, b), (1,0)\rangle_0 \colon \delta''(b) > 0\};$$
$$K_S(\delta) = \{\langle (b, b), (1,0)\rangle_0 \cup \langle (b, b), (0,1)\rangle_0 \colon b \in T\},\ \text{with } T = \{x \in [0,1] \colon \nexists\, \delta''(x)\}.$$

Biconic copulas that assign probability mass in the set $K_D(\delta)$ (respectively, $K_A(\delta)$ and $K_S(\delta)$) are said to have *discrete spectrum* (respectively, *absolutely continuous spectrum*, *singular spectrum*).

As a consequence of Theorem 4.1 and Krein-Milman Theorem 2.2, we can provide a property of biconic copulas with discrete spectrum.

Corollary 5.1. *Biconic copulas with only discrete spectrum are dense in $\mathcal{C}_B$.*

Proof. All the extreme biconic copulas obtained in Corollary 4.1 have discrete spectrum, as can be also observed from Figure 3. Moreover, the convex sum of biconic copulas with discrete spectrum is a biconic copula with discrete spectrum, as can be easily checked. Since, from Theorem 2.2, the convex sum of extreme biconic copulas is dense in $\mathcal{C}_B$, the result follows. □

Moreover, the following result can be also given.

Corollary 5.2. *Let $\delta_{\alpha,\beta}$ be the diagonal section given by* (9)*. Then every copula $C \in \mathcal{C}_B$ can be expressed, for every $(x, y) \in [0,1]^2$, in the form*

$$C(x,y) = \int_E C_{\delta_{\alpha,\beta}}(x,y)\, d\nu_C(\alpha,\beta),$$

where ν_C is a measure concentrated on $E = ([0,1/2[\times]1/2,1]) \cup \{(1/2,1/2)\}$.

Proof. Consider the bijection $\Psi\colon E \to \mathrm{Ext}(\mathcal{D}_{\mathcal{C}_B})$, where $\Psi(\alpha,\beta) = \delta_{\alpha,\beta}$. Then Ψ is a homeomorphism. Thus, by applying Choquet Theorem 2.3, the result follows. □

We apply Corollary 5.2 to provide examples of biconic copulas with some interesting properties.

Example 5.1. Given a measure ν concentrated on $[0,1/2]$, then

$$C(u,v) = \int_{[0,1/2]} C_{\delta_{x,1-x}}(u,v)\, d\nu(x)$$

is a copula. Taking different types of measures ν, it is possible to establish the following types of copulas whose supports are $[0,1]^2$.

(a) Defining a numbering $n\colon \mathbb{Z}^+ \longrightarrow \mathbb{Q}\cap]0,1/2[$ and the measure ν with $\nu(n(i)) = \frac{1}{2^{n(i)}}$, we can generate a copula $C \in \mathcal{C}_B$ with discrete spectrum and full support.

(b) Taking a singular measure ν with support on $[0,1/2]$, we generate a copula $C \in \mathcal{C}_B$ with singular spectrum and full support.

5.2. Biconic copulas and their associated Markov operators

Thanks to Corollary 5.2, we are also able to prove that biconic copulas induce Markov operators that satisfy the Feller property (see, for instance, [2]).

To prove this, we need some preliminary considerations.

Let $(\Omega, \mathcal{B}, \mu)$ be a measurable space, and let $\mathcal{L}^\infty(\Omega,\mathcal{B},\mu)$ — $\mathcal{L}^\infty$, for short — be the set of the equivalence classes of *essentially bounded measurable functions*, i.e. the set of the equivalence classes of all measurable functions $f\colon \Omega \longrightarrow \mathbb{C}$ such that there exists an $M \in \mathbb{R}$, with $M > 0$, for which it holds $|f(x)| \le M$ μ-almost everywhere on Ω. A function $T\colon \mathcal{L}^\infty \longrightarrow \mathcal{L}^\infty$ satisfies the *Feller property* if, $f \in \mathcal{L}^\infty$ and f continuous implies that $T(f)$ is continuous.

Moreover, every copula C induces a doubly stochastic (Markov) operator $T_C\colon \mathcal{L}^\infty \longrightarrow \mathcal{L}^\infty$ given by $T_C(f)(x) = \int_{[0,1]} f\, d\mu_{C,x}$, where $\mu_{C,x}$ is the measure given in the disintegration of μ_C (see, e.g., [13]).

In general, the Markov operator associated with a copula need not satisfy the Feller property. Consider, for instance, the ordinal sum of two copies of the independence copula with respect to the partition $([0,1/2],[0,1/2])$. However, in the class of biconic copula the following result holds.

Theorem 5.2. *Let $C \in \mathcal{C}_B$. Then the Markov operators associated with C satisfies the Feller property.*

Proof. We prove that every copula $C \in \mathrm{Ext}(\mathcal{C}_{BC})$ has the Feller property. Consider the diagonal $\delta_{\alpha,1}$ given by (7) — for the rest of the diagonals in Theorem 4.1 the proof is similar, and we can omit it—and let C be the copula with diagonal $\delta_{\alpha,1}$. We have

$$\mu_{C,x} = \begin{cases} \Delta_{\frac{\alpha-1}{\alpha}x+1}, & \text{if } x \in [0,\alpha], \\ \dfrac{\alpha}{1-\alpha}\Delta_{\frac{\alpha}{1-\alpha}(x-1)} + \dfrac{1-2\alpha}{1-\alpha}\Delta_x, & \text{if } x \in]\alpha, 1], \end{cases} \tag{14}$$

such that if $y \to x$ then $\mu_{C,y} \overset{w}{\to} \mu_{C,x}$, where $\overset{w}{\to}$ denotes the weak convergence (see, e.g., [5]. Furthermore, if $f \in \mathcal{L}^\infty$ is continuous, then $T_C(f)(y) = \int_{[0,1]} f\, d\mu_{C,y}$ converges to $\int_{[0,1]} f\, d\mu_{C,x} = T_C(f)(x)$, as y tends to x, and $T_C(f)$ is continuous.

From Corollary 5.2, if $C \in \mathcal{C}_{BC}$, then C can be expressed as

$$C(x, y) = \int_E C_{\delta_{\alpha,\beta}}(x, y)\, d\nu_C(\alpha, \beta).$$

If R is a rectangle in $[0,1]^2$, then

$$\mu_C(R) = \int_E \mu_{C_{\delta_{\alpha,\beta}}}(R)\, d\nu_C(\alpha, \beta).$$

Thus, by standard arguments from measure theory, it follows that

$$\int_E \mu_{C_{\delta_{\alpha,\beta}}}(B)\, d\nu_C(\alpha, \beta) = \mu_C(B)$$

for every $B \in \mathcal{B}([0,1]^2)$. Let $B_x = \{x \in [0,1] : \exists y \in [0,1] \text{ satisfying } (x,y) \in B\}$. It is clear that

$$\mu_{C_x}(B) = \int_E \mu_{C_{\delta_{\alpha,\beta},x}}(B)\, d\nu_C(\alpha, \beta)$$

because, using Fubini's theorem, we have

$$\begin{aligned} \int_0^1 \int_E \mu_{C_{\delta_{\alpha,\beta},x}}(B_x)\, d\nu_C(\alpha, \beta) d\lambda(x) &= \int_E \int_0^1 \mu_{C_{\delta_{\alpha,\beta},x}}(B_x)\, d\lambda(x) d\nu_C(\alpha, \beta) \\ &= \int_E \mu_{C_{\delta_{\alpha,\beta}}}(B)\, d\nu_C(\alpha, \beta) = \mu_C(B). \end{aligned}$$

Therefore, if f is a simple function, then

$$\int_0^1 f\, d\mu_{C_x} = \int_E \int_0^1 f\, d\mu_{C_{\delta_{\alpha,\beta},x}} d\nu_C(\alpha, \beta). \tag{15}$$

Applying the Lebesgue dominated convergence theorem, we have that Equation (15) holds for every $f \in \mathcal{L}^\infty$; in particular, if f is continuous. For every continuous function f, the function $g(\alpha, \beta, x) = \int_0^1 f\, d\mu_{C_{\alpha,\beta},x}$ is continuous in x. Moreover, $|g(\alpha, \beta, x)| \le \|f\|_\infty$ for all $((\alpha, \beta), x) \in E \times [0,1]$. Again, applying the Lebesgue dominated convergence theorem we have that $T_C(f)(x) = \int_E \int_0^1 f\, d\mu_{C_{\delta_{\alpha,\beta}}} d\nu_C(\alpha, \beta)$ is continuous, i.e. the Feller property is satisfied. ◻

5.3. Biconic copulas and extreme-value copulas

In this section we find a relationship between certain types of power stable aggregation functions and the corresponding biconic aggregation functions, with a special attention to biconic (quasi-)copulas.

We recall that a binary aggregation function $A\colon [0,1]^2 \longrightarrow [0,1]$ is called *power stable* whenever, for any constant $r > 0$ and all $(x,y) \in [0,1]^2$ it holds $A(x^r, y^r) = (A(x,y))^r$. Power stable aggregation functions are studied in [23, 24], where the following results are given.

Theorem 5.3. *A continuous function $A\colon [0,1]^2 \longrightarrow [0,1]$ is a power stable aggregation function if, and only if, there is a continuous non-zero function $d\colon [0,1] \longrightarrow [0,\infty[$ such that*

(i) the function $\frac{d(t)}{t}$ is decreasing on $]0,1]$, and either $d(0) > 0$ or $\lim_{t\to 0^+} \frac{d(t)}{t} < \infty$,

(ii) the function $\frac{d(t)}{1-t}$ is increasing on $[0,1[$, and either $d(1) > 0$ or $\lim_{t\to 1^-} \frac{d(t)}{1-t} < \infty$,

and, for all $(x,y) \in]0,1[^2$, it holds

$$A(x,y) = (xy)^{d\left(\frac{\log(x)}{\log(xy)}\right)}. \tag{16}$$

Every function d as in Theorem 5.3 is called *dependence function*.

Theorem 5.4. *A continuous function $Q\colon [0,1]^2 \longrightarrow [0,1]$ is a power stable quasi–copula if, and only if, it can be written in the form (16) for a dependence function d such that $d(0) = d(1) = 1$, $d(\alpha x) \le 1 - \alpha + \alpha\, d(x)$ and $d(\alpha x + 1 - \alpha) \le 1 - \alpha + \alpha\, d(x)$ for all $\alpha, x \in [0,1]$.*

As known (see, e.g., [20]), power-stable copulas are also known under the name *extreme-value* copulas (EVC's, for short), which are usually represented in the form

$$C_P(x,y) = (xy)^{P\left(\frac{\log(x)}{\log(xy)}\right)} \tag{17}$$

for all $(x,y) \in]0,1]^2 \setminus \{(1,1)\}$, where $P\colon [0,1] \to [1/2,1]$ is a convex function fulfilling $t \vee (1-t) \le P(t) \le 1$ for all $t \in [0,1]$. The function P is known as *Pickands dependence function* [28]. Here, the set of all Pickands dependence functions is denoted by $\mathcal{P}$. Notice that the set of the EVC's, denoted by $\mathcal{C}_{EV}$, is log-convex, since $C_1^{\alpha} \cdot C_2^{1-\alpha} \in \mathcal{C}_{EV}$ for every $C_1, C_2 \in \mathcal{C}_{EV}$ and for all $\alpha \in [0,1]$.

Now, consider the transformation $T\colon [0,1]^2 \to \mathbb{R}^2$ given by

$$T(x,y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (x, x+y-1), \tag{18}$$

which transforms the triangle of vertices $(0,1)$, $(1/2,1/2)$ and $(1,1)$ into the triangle of vertices $(0,0)$, $(1/2,0)$ and $(1,1)$. Replacing y by $d(x)$ in (18), we have $T(x,d(x)) = (x, x + d(x) - 1)$. Let $\delta^d(x) = x + d(x) - 1$ for every $x \in [0,1]$ (and similarly, $\delta^P(x) = x + P(x) - 1$). Then the following result, whose proof is straightforward and can be omitted, holds.

Theorem 5.5. *The following statements hold:*

(a) *For any dependence function d such that $d(0) = d(1) = 1$, $d(\alpha x) \leq 1 - \alpha + \alpha\, d(x)$, $d(\alpha x + 1 - \alpha) \leq 1 - \alpha + \alpha\, d(x)$ and $x \vee (1 - x) \leq d(x)$ for all $\alpha, x \in [0, 1]$, the function δ^d is a diagonal such that $\frac{\delta^d(x)}{x}$ is increasing for all $x \in]0, 1]$ and $\frac{\delta^d(x)-1}{x-1}$ is increasing for all $x \in [0, 1[$, i.e. $\delta^d \in \mathcal{D}_{\mathcal{Q}_B}$.*

(b) *For any Pickands dependence function $P \in \mathcal{P}$, the function δ^P is a diagonal section in $\mathcal{D}_{\mathcal{C}_B}$, i.e. it is convex.*

Furthermore, both the transformations in (a) *and* (b) *are homeomorphisms (with L^∞ norm) and transform extreme points into extreme points.*

From Theorems 2.1 and 5.5, by taking $\delta = \delta^P$ (respectively, $\delta = \delta^d$) in (2), we have that, for every $(x, y) \in [0, 1]^2$, the function defined by

$$D_P(x, y) = P\left(\frac{x \wedge y}{1 + (x \wedge y) - (x \vee y)}\right)[1 + (x \wedge y) - (x \vee y)] + (x \vee y) - 1 \qquad (19)$$

is a biconic copula (respectively, biconic quasi-copula).

As a consequence of Theorem 5.5, the following results hold:

- Extreme biconic quasi-copulas are in a one-to-one correspondence with extreme power stable quasi-copulas.
- Biconic copulas of type (19) are in a one-to-one correspondence with EVC's. This bijection allows us to describe the set $\mathrm{Ext}(\mathcal{P})$ and provide a proof alternative to the methods presented in [36].

Example 5.2. The dependence function $d(x) = \min\{\max\{1 - x, \frac{1+x}{2}\}, \max\{x, \frac{2-x}{2}\}\}$ defined for all $x \in [0, 1]$ (see Example 1 in [23]) generates a power stable quasi-copula Q, but not a copula. We note that Q cannot be expressed in the form $Q_1^\alpha \cdot Q_2^{1-\alpha}$ for every $\alpha \in]0, 1[$, where Q_1 and Q_2 are two power stable quasi-copulas different from Q.

Unlike quasi-copulas and copulas, the set $\mathcal{S}_B$ and the set of of power-stable semi–copulas are not in a one-to-one correspondence, as the following example shows.

Example 5.3. Consider the function

$$d(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1/2, \\ 1, & \text{if } 1/2 < x \leq 1. \end{cases}$$

Then the corresponding function A given by (16) is not a semi-copula. However, for the function δ^d it holds that $\frac{\delta^d(x)}{x}$ is increasing for all $x \in]0, 1]$, and thus, the function D_{δ^d}, given by (19), is a biconic semi-copula.

6. Conclusions

We have studied the extreme points of biconic semi–copulas, biconic quasi–copulas and biconic copulas, and the relationships between these functions and the corresponding power stable functions. In particular, we have proved that an extreme point of a biconic copula is characterized by the corresponding extreme convex diagonal and shown that every extreme point of a biconic copula is also an extreme point of the set of all copulas.

Although we have determined a dense set in the class of all extreme biconic quasi–copulas, it remain an open problem the full characterization of the elements of this class.

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Appendix

Here we prove that the class of biconic copulas is compact (it is trivial to prove that it is convex).

Proposition 6.1. *The class of biconic copulas is compact.*

Proof. If $(C_n)_n$ is a sequence of biconic copulas that converges to C, as $n \to +\infty$, then Equation (2) shows that C is a biconic copula whose diagonal section δ is the pointwise limit of the sequence of the diagonal sections δ_{C_n}; thus $\mathcal{C}_B$ is a closed set of $\mathcal{C}$. Moreover, since $\mathcal{C}_B$ is a closed set in $\mathcal{C}$, which is compact with the topology of the pointwise convergence (see [9]), then $\mathcal{C}_B$ is compact too. □

Now we characterize the extreme points of the set of all diagonal sections of a copula. This result appeared in [7, Lemma 2.1], where no proof is given, and is reported here for the sake of completeness.

Theorem 6.1. *The set $\mathcal{D}_{\mathcal{C}}$ is convex and compact with respect to L^∞ norm. Its extreme points are the diagonal sections δ such that*

$$\lambda\left(\{x \in [0,1] \colon \delta(x) = x\} \cup \{x \in [0,1] \colon \delta'(x) = 0\} \cup \{x \in [0,1] \colon \delta'(x) = 2\}\right) = 1, \tag{20}$$

where λ denotes the Lebesgue measure on $[0,1]$.

Proof. If $\delta(x) = x$ for all $x \in [0,1]$, the result is trivial.

Suppose that Equation (20) holds. Suppose $\delta(x) < x$ for all $x \in]0,1[$, and let δ_1 and δ_2 be two diagonals such that $\delta(x) = \frac{\delta_1(x)+\delta_2(x)}{2}$ for every $x \in [0,1]$. Then we have $\delta'(x) = \frac{\delta_1'(x)+\delta_2'(x)}{2}$ as long as the three derivatives exist. Let $K = \{x \in]0,1[\colon \exists \delta'(x), \delta_1'(x), \delta_2'(x)\}$. Since δ, δ_1 and δ_2 are increasing, they are differentiable almost everywhere, whence we have $\lambda(K) = 1$. If $\delta'(x) = 0$ for $x \in K$ then $\delta_1'(x) = \delta_2'(x) = 0$ (similarly, if $\delta'(x) = 2$ for $x \in K$ then $\delta_1'(x) = \delta_2'(x) = 2$), i.e. $\delta'(x) = \delta_1'(x) = \delta_2'(x)$ in a set of measure 1. Since diagonal sections are absolutely continuous, we have $\delta_1(x) = \delta_2(x) = \delta(x)$ for all $x \in]0,1[$, and hence δ is an extreme point.

If there exists $x_0 \in]0,1[$ such that $\delta(x_0) < x_0$, then there exist $x_1, x_2 \in [0,1]$, with $x_0 \in]x_1, x_2[$, such that $\delta(x_1) = x_1$ and $\delta(x_2) = x_2$ and for which $\delta(x) < x$ for all $x \in]x_1, x_2[$. Consider the diagonal

$$\delta^*(t) = \frac{\delta(x_1 + (x_2 - x_1)t) - x_1}{x_2 - x_1} \tag{21}$$

for all $t \in [0,1]$. Note that δ^* is a diagonal such that $(\delta^*)' \in \{0,2\}$ in a set of measure 1 and $\delta^*(t) < t$ for all $t \in]0,1[$, so that δ^* is extreme, and hence, if there exist two diagonals δ_1, δ_2 such that $\delta(x) = \frac{\delta_1(x)+\delta_2(x)}{2}$ for all $x \in]x_1, x_2[$, then $\delta_1(x) = \delta_2(x) = \delta(x)$ for every $x \in]x_1, x_2[$.

Finally, if $Z = \{x \in [0,1] : \delta(x) = x\}$, then it is easy to check $\delta_1(x) = \delta_2(x) = x$ for every $x \in Z$, and we conclude that δ is extreme.

Conversely, suppose $\delta \in \mathrm{Ext}(\mathcal{D}_\mathcal{C})$ and suppose that there exists $x_0 \in [0,1]$ for which $\delta(x_0) < x_0$. Let $x_1, x_2 \in [0,1]$ such that $\delta(x_1) = x_1$, $\delta(x_2) = x_2$ and $\delta(x) < x$ for every $x \in]x_1, x_2[$. If $\lambda(\{x \in]x_1, x_2[\colon \delta'(x) = 0\} \cup \{x \in]x_1, x_2[\colon \delta'(x) = 2\}) \neq x_2 - x_1$, then $\lambda(\{x \in]x_1, x_2[\colon (\delta^*)'(x) = 0\} \cup \{x \in]x_1, x_2[\colon (\delta^*)'(x) = 2\}) \neq 1$, where δ^* is the diagonal section given by (21). Let $\Gamma = \{x \in]x_1, x_2[\colon \delta'(x) \notin \{0,2\}\}$. It is clear $\lambda(\Gamma) > 0$, and there exist $a, b \in]0,2[$ such that $N = \Gamma \cap [a,b]$ and $\lambda(N) > 0$. Let $\varepsilon = \frac{1}{2}\min\{a, 1-b\}$. Using arguments of continuity of the function $\int_{[0,x]} \chi_N(x)\,dx$, where χ_N is the characteristic function of N, we partition N into two sets, N_1 and N_2, such that $N = N_1 \cup N_2$, $\lambda(N_1) = \lambda(N_2) = \frac{1}{2}\lambda(N)$ and $(x,y) \in N_1 \times N_2$ implying $x < y$. We define the functions

$$g_1(x) = \begin{cases} (\delta^*)'(x), & \text{if } x \notin N, \\ (\delta^*)'(x) - \varepsilon, & \text{if } x \in N_1, \\ (\delta^*)'(x) + \varepsilon, & \text{if } x \in N_2, \end{cases} \qquad g_2(x) = \begin{cases} (\delta^*)'(x), & \text{if } x \notin N, \\ (\delta^*)'(x) + \varepsilon, & \text{if } x \in N_1, \\ (\delta^*)'(x) - \varepsilon, & \text{if } x \in N_2. \end{cases}$$

and $\delta_i^*(x) = \int g_i(x)\,dx$ for $i = 1,2$. Since $\delta^*(x) = \frac{\delta_1^*(x) + \delta_2^*(x)}{2}$ for every x, consider the diagonal section

$$\delta_i(x) = \begin{cases} \delta(x), & \text{if } x \notin]x_1, x_2[, \\ (x_2 - x_1)\delta_i^*(x) + x_1, & \text{if } x \in]x_1, x_2[, \end{cases}$$

for $i = 1,2$. Thus, it holds that $\delta(x) = \frac{\delta_1(x) + \delta_2(x)}{2}$ for every $x \in [0,1]$, which contradicts the fact that δ is extreme. Therefore, $\lambda\left(\{x \in [0,1] : \delta'(x) = 0\} \cup \{x \in [0,1] : \delta'(x) = 2\}\right) = 1 - \lambda\left(\{x \in [0,1] : \delta(x) = x\}\right)$, which completes the proof. $\square$