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Shock models with dependence and asymmetric linkages

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Abstract

This paper introduces a new class of copulas and shows its relevance for the applications. In particular, a stochastic interpretation in terms of a system of dependence components affected by a global shock is given. As a main feature of the model, the global shock has an opposite effect on the different components of the system. Copulas generated by this mechanism are characterized in the bivariate case and their main properties are illustrated. Connections with concepts like semilinear copulas and conic aggregation functions are also highlighted. Moreover, a high dimensional extension is presented.

Keywords: Copula, Dependence concepts, Marshall-Olkin distribution, Shock models, Tail dependence.
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1. Introduction

Dependence concepts play a crucial role in multivariate statistical literature since it was recognized that the independence assumption cannot describe conveniently the behavior of a stochastic system. Since then, different attempts have been made in order to provide more flexible methods to describe the variety of dependence-types that may occur in practice. Copula models have become popular in different applications in view of their ability to describe the relationships among random variables in a flexible way. To this end, several families of copulas have been introduced, motivated by special needs from the scientific practice (see, for instance, [17, 20, 33]).

Consider, for instance, the case when one wants to build a stochastic model for describing the dependence among two (or more) lifetimes, i.e. positive random variables. In engineering

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applications, joint models of lifetimes may serve to estimate the expected lifetime of a system composed by several components. In a related situation like portfolio credit risk, instead, the lifetimes may have the interpretation of time-to-default of firms, or generally financial entities, while a stochastic model may estimate the price/risk of a relate derivative contract (e.g. CDO). In both cases, it is of interest to estimate the probability of the occurrence of a joint default, which means, in the case of a bivariate random vector (X, Y), the probability of the event $\{X = Y\}$, or more generally $\{f(X) = g(Y)\}$ for some measurable functions f and g. Obviously, if one requires the event $\{f(X) = g(Y)\}$ to have non-zero probability, then the copula for (X, Y) must have a singular component, as described in [10, 28].

The generation of convenient statistical distribution for modeling such situations originated from the seminal paper by Marshall-Olkin [30]; see [2] for an up-to-date overview. In [12], a general framework was introduced for such constructions, which is briefly recalled here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. For $d \geqslant 2$, consider a system composed by d components whose behavior is described by the continuous random variables (=r.v.'s) $X_1, \dots, X_d$ such that each X_i is distributed according to a continuous distribution function F_i, $X_i \sim F_i$. The r.v. X_i can be interpreted as a shock that effects only the i-th component of the system, i.e. the idiosyncratic shock. Let $\mathcal{S} \neq \emptyset$ be a collection of subsets $S \subseteq \{1, 2, \dots, d\}$ with at least 2 elements. For each $S \in \mathcal{S}$ consider the random variable Z_S with probability distribution function G_S. These r.v.'s Z_S can be interpreted as an (exogenous) shock that may affect the stochastic behavior of all the system components with index $i \in S$, i.e. the systemic shock. Furthermore, we assume a given dependence among the introduced random vectors $\mathbf{X}$ and $\mathbf{Z}$. To this end, according to Sklar's theorem [35], we assume that there exists a copula C such that

$$(\mathbf{X}, \mathbf{Z}) \sim C((F_i)_{i=1,\dots,d}, (G_S)_{S \in \mathcal{S}}).$$

The copula C hence describes how the shocks $\mathbf{X}$ and $\mathbf{Z}$ are related. Finally, for $i = 1, \dots, d$, assume the existence of a *linking function* Ψ_i that expresses how the effects produced by the shock X_i and all the shocks Z_S with $i \in S$ are combined together and act on the i-th component. Given the previous framework, the d-dimensional stochastic model $\mathbf{Y} = (Y_1, \dots, Y_d)$ can be constructed by setting, for $i = 1, \dots, d$,

$$Y_i = \psi_i(X_i, Z_{S:i \in S}).$$

Interestingly, under suitable assumptions on the d.f. s F_i's and G_S's, the d.f. of $\mathbf{Y}$ is given by a copula (for more details, see [11]). Several families of copulas can be interpreted by using the previous stochastic mechanism.

- If $(\mathbf{X}, \mathbf{Z})$ is a vector of independent components, then the resulting copula is of Marshall-Olkin type, according to different generalizations provided by [29, 30, 31] and by [4, 6, 15, 16] for the exchangeable case.

- If $\mathbf{X}$ and $\mathbf{Z}$ are independent vectors, but there is some dependence among the components of $\mathbf{X}$, then the resulting copula has been described in [11, 12] in the case only one exogenous shock affects the system.

- Furthermore, if a specific dependence is assumed between $\mathbf{X}$ and $\mathbf{Z}$, then several constructions have been provided in [32] (see also [8]) and [3].

Actually, in all previously cited examples, the linking function ψ_i does not change with i, i.e. the exogenous shocks affect all the components of the system in the same way. One attempt to weaken this assumption has been provided in [34]. Here, the main idea is to consider three independent r.v.'s X_1, X_2 and Z that are used to construct the bivariate vector (Y_1, Y_2) such that $Y_1 = \max\{X_1, Z\}$ and $Y_2 = \min\{X_2, Z\}$. The resulting copula has been characterized in [34, Theorem 9], and is also called *maxmin copula*, since it considers maximum and minimum as linking functions.

Various interpretations of this model can be found, since it is possible in many practical situations that the common exogenous shock will produce different effects on different system components. For instance, we may think of X_1 and X_2 as r.v.'s representing the respective wealth of two groups of people, and the exogenous shock Z is interpreted as an event that is beneficial to the first group and detrimental to the second one. Analogously, X_1 and X_2 can be thoughts as a short and a long investment, respectively, while Z is beneficial only to one of this type of investment.

One of the main goals of this paper is to extend the latter model in order to allow the two underlying system variables X_1 and X_2 to be dependent. In particular, the dependence is assumed to be governed by a general copula, while the third variable Z is assumed to be independent of both X_1 and X_2. Specifically, we prove that joint d.f. arising from the previously described stochastic mechanism is actually a copula (section 2) and we illustrate some of its main features (section 3). Finally, we discuss possible multivariate generalizations (section 4).

As a matter of fact, the obtained class of copulas may have some properties that are appealing in various contexts related to fuzzy set theory and multicriteria decision making. First, it includes non-exchangeable copulas, which are used for instance as more general fuzzy connectives. See, for instance, [1, 14]. Then, its associated measure may have a singular component, a fact of potential use in various copula-based integrals. See, for instance, [25, 26]. Finally, the main idea of the class originated from a probabilistic extension of semilinear copulas, which have been generalized in various directions in the recent literature (see among others [21, 22]).

2. Maxmin copulas generated by dependent shocks

Here, we present and discuss the generalization of maxmin copulas in the two-dimensional case. For the definition of a copula and other basic notions from the theory of copulas we refer the reader to [17, 20, 33]. Given the endogenous shocks X_1 and X_2 and an exogenous shock Z, we assume that Z has opposite effects to the two components X_1 and X_2 of the system, i.e. it has a beneficial effect on one component (as interpreted by the linking function max) and detrimental effect on the other one (as interpreted by the linking function min). Specifically, we will consider the distribution of

$$(Y_1, Y_2) = (\max\{X_1, Z\}, \min\{X_2, Z\}), \tag{2.1}$$

which can be easily described in view of the following preliminary result.

Lemma 2.1. *Let (X_1, X_2, Z) be distributed according to $C(F_1, F_2, G)$ and (Y_1, Y_2) be defined as in* (2.1). *Then the distribution function H of (Y_1, Y_2) is equal to*

$$\begin{aligned} H(y_1, y_2) = C(F_1(y_1), 1, \min\{G(y_1), G(y_2)\}) \\ + \max\{C(F_1(y_1), F_2(y_2), G(y_1)) - C(F_1(y_1), F_2(y_2), G(y_2)), 0\} \end{aligned} \tag{2.2}$$

for all $y_1, y_2 \in \mathbb{R}$. Additionally, we have that $F_{Y_1} \leq \min\{F_1, G\}$ and $F_{Y_2} \geq \max\{F_2, G\}$.

Proof. A short computation reveals that $H(y_1, y_2)$ is equal to

$$\begin{aligned} &P(\max\{X_1, Z\} \leq y_1, \min\{X_2, Z\} \leq y_2, Z \leq y_2) + P(\max\{X_1, Z\} \leq y_1, \min\{X_2, Z\} \leq y_2, Z > y_2) \\ &= P(X_1 \leq y_1, Z \leq \min\{y_1, y_2\}) + P(X_1 \leq y_1, X_2 \leq y_2, y_2 < Z \leq y_1) \\ &= C(F_1(y_1), 1, G(\min\{y_1, y_2\})) + \max\{C(F_1(y_1), F_2(y_2), G(y_1)) - C(F_1(y_1), F_2(y_2), G(y_1)), 0\}. \end{aligned}$$

Since distribution functions are increasing, we have that $G(\min\{y_1, y_2\}) = \min\{G(y_1), G(y_2)\}$ and therefore we have proven (2.2). The last part of the lemma follows from $F_{Y_1}(y_1) = P(X_1 \leq y_1, Z \leq y_1)$ and $1 - F_{Y_2}(y_2) = P(X_2 > y_2, Z > y_2)$. ■

In order to get more insight into this model, let us now state some definitions and notations. Throughout the paper, denote by $\mathbb{I}$ the interval $[0, 1]$, id the identity function on $[0, 1]$ and set $\mathrm{im}F := \{F(x) | x \in \mathbb{R}\}$ for a real function F. For a distribution function F the *quasi-inverse* of F is the function $F^{-1} : \mathbb{I} \longrightarrow [-\infty, \infty]$, defined by $F^{-1}(u) = \inf\{x \in \mathbb{R} | F(x) \geq u\}$, where the infimum of an empty set is equal to infinity. Observe that $F^{-1}(u) = \infty$ if and only if $u = 1 \notin \mathrm{im}F$, and $F^{-1}(u) = -\infty$ if and only if $u = 0$. Let us also fix the notation $f(x-)$, respectively $f(x+)$, for the left, respectively right, limit of the function f at x, if it exists. For $u \notin \mathrm{im}F \cup \{0, 1\}$ denote $\overline{u} = F(F^{-1}(u))$ and $\underline{u} = F(F^{-1}(u)-)$. Note that $\overline{u} \in \mathrm{im}F$, and that either $\underline{u} \in \mathrm{im}F$ or $\underline{u} \notin \mathrm{im}F$ but $\underline{u} - \varepsilon \in \mathrm{im}F$ for every $\varepsilon > 0$. Although introducing these notions needs only elementary calculus, Figure 1 may be helpful to many readers.

Following the procedure described in [34], we define the functions $\phi : \mathbb{I} \longrightarrow \mathbb{I}$ and $\psi : \mathbb{I} \longrightarrow \mathbb{I}$ in order to derive a convenient expression for the copula of (2.1). The important part of the definition of ϕ, respectively ψ, is the one on $\mathrm{im}F_{Y_1}$, respectively $\mathrm{im}F_{Y_2}$. On the remaining part of the domain, we define ϕ and ψ so that they satisfy certain technical conditions which will be needed in the future and are not important for understanding. Define

$$\phi(u) = \begin{cases} 0, & \text{if } u = 0, \\ F_1(F_{Y_1}^{-1}(u)), & \text{if } u \in \mathrm{im}F_{Y_1} \backslash \{0, 1\}, \\ 1, & \text{if } u = 1, \\ \frac{\phi(\overline{u}) - \phi(\underline{u}-)}{\overline{u} - \underline{u}}(u - \underline{u}) + \phi(\underline{u}-), & \text{if } u \notin \mathrm{im}F_{Y_1} \cup \{0, 1\}, \end{cases}$$

where we obey the following convention: if $\underline{u} \in \mathrm{im}F_{Y_1}$, then $\phi(\underline{u}-)$ means $\phi(\underline{u})$; and if $\underline{u} = 0$, then $\phi(\underline{u}-)$ means 0. Observe that for $u \notin \mathrm{im}F_{Y_1} \cup \{0, 1\}$ the function ϕ is defined simply

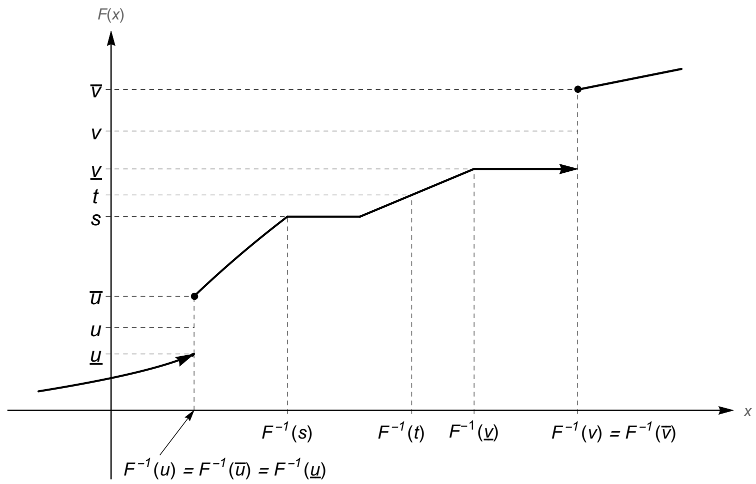

Figure 1: Graphical presentation of a distribution function, its quasi-inverse, and their properties.

as linear interpolation between $(\underline{u}, \phi(\underline{u}-))$ and $(\overline{u}, \phi(\overline{u}))$. Define

$$\psi(v) = \begin{cases} 0, & \text{if } v = 0, \\ F_2(F_{Y_2}^{-1}(v)), & \text{if } v \in \mathrm{im} F_{Y_2} \backslash \{0,1\}, \\ 1, & \text{if } v = 1, \\ \frac{\psi(\overline{v}) - \psi(\underline{v}-)}{\overline{v} - \underline{v}}(v - \underline{v}) + \psi(\underline{v}-), & \text{if } v \notin \mathrm{im} F_{Y_2} \cup \{0,1\}, \end{cases}$$

where we obey similar convention as with ϕ. A straightforward proof gives $F_1(y_1) = \phi(F_{Y_1}(y_1))$ for all $y_1 \in \mathbb{R}$ with $F_{Y_1}(y_1) > 0$, and $F_2(y_2) = \psi(F_{Y_2}(y_2))$ for all $y_2 \in \mathbb{R}$ with $F_{Y_2}(y_2) < 1$ (for details see the proof of Theorem 9 in [34]). Using this, Eq. (2.2) becomes

$$\begin{aligned} H(y_1, y_2) &= C(\phi(F_{Y_1}(y_1)), 1, \min\{G(y_1), G(y_2)\}) \\ &\quad + \max\{C(\phi(F_{Y_1}(y_1)), \psi(F_{Y_2}(y_2)), G(y_1)) - C(\phi(F_{Y_1}(y_1)), \psi(F_{Y_2}(y_2)), G(y_2)), 0\}. \end{aligned}$$

Now, contrarily to the case presented in [34], it is not possible to recover $G(y_i)$ in terms of ϕ, ψ, F_{Y_1} and F_{Y_2}, since we allow dependence between shocks X_1, X_2 and Z, i.e. we can express the distribution functions F_{Y_1} and F_{Y_2} only as

$$\begin{aligned} F_{Y_1}(y_1) &= H(y_1, \infty) = C(F_1(y_1), 1, G(y_1)) + \max\{C(F_1(y_1), 1, G(y_1)) - F_1(y_1), 0\} \\ &= C(F_1(y_1), 1, G(y_1)) = C(\phi(F_{Y_1}(y_1)), 1, G(y_1)), \\ F_{Y_2}(y_2) &= H(\infty, y_2) = G(y_2) + F_2(y_2) - C(1, F_2(y_2), G(y_2)) \\ &= G(y_2) + \psi(F_{Y_2}(y_2)) - C(1, \psi(F_{Y_2}(y_2)), G(y_2)). \end{aligned}$$

Therefore, in order to get a closed form expression, we consider the special case of the model (2.1) when the exogenous shock Z is independent of endogenous shocks (X_1, X_2), i.e. we assume that the d.f. of (X_1, X_2, Z) can be expressed as $C(F_1, F_2) \cdot G$. In such a case, the distribution functions of (Y_1, Y_2), Y_1 and Y_2 are equal, respectively, to

$$\begin{aligned}
H(y_1, y_2) &= F_1(y_1)\min\{G(y_1), G(y_2)\} + C(F_1(y_1), F_2(y_2))\max\{(G(y_1) - G(y_2)), 0\},\\
F_{Y_1}(y_1) &= F_1(y_1)G(y_1),\\
F_{Y_2}(y_2) &= G(y_2) + F_2(y_2) - F_2(y_2)G(y_2) = 1 - (1 - F_2(y_2))(1 - G(y_2)).
\end{aligned}$$

Now, if we define ϕ and ψ as before, $G(y_1)$ and $G(y_2)$ can be expressed as follows:

$$\begin{aligned}
G(y_1) &= \frac{F_{Y_1}(y_1)}{\phi(F_{Y_1}(y_1))},\\
G(y_2) &= \frac{F_{Y_2}(y_2) - \psi(F_{Y_2}(y_2))}{1 - \psi(F_{Y_2}(y_2))}.
\end{aligned}$$

Thus, we can deduce the copula associated with the random pair (Y_1, Y_2) derived from (2.1), where Z is independent of (X_1, X_2), in terms of the copula C of (X_1, X_2), and the two functions ϕ and ψ.

To this end, let $\mathscr{F}_1$ be the class of increasing functions $\phi : \mathbb{I} \longrightarrow \mathbb{I}$ such that $\phi(0) = 0$, $\phi(1) = 1$ and the function $\phi^* := \mathrm{id}/\phi$ is increasing on $(0, 1]$. Furthermore, let $\mathscr{F}_2$ be the class of increasing functions $\psi : \mathbb{I} \longrightarrow \mathbb{I}$ such that $\psi(0) = 0$, $\psi(1) = 1$ and the function

$$\psi_*(v) := \begin{cases} \frac{v-\psi(v)}{1-\psi(v)}, & \text{if } v \in [0, 1),\\ 1, & \text{if } v = 1, \end{cases}$$

is increasing. The functions belonging to the classes $\mathscr{F}_1$ and $\mathscr{F}_2$ satisfy the following properties (we refer the reader to [34] for proofs and some other related properties).

Proposition 2.2. *For all $\phi \in \mathscr{F}_1$ and $\psi \in \mathscr{F}_2$ it holds that*

(i) $\psi \le \mathrm{id} \le \phi$ *and* $\psi_* \le \phi^*$,

(ii) if there exists $u \in (0, 1]$ such that $\phi(u) = u$, then ϕ is equal to the identity function on the interval $[u, 1]$,

(iii) if there exists $v \in [0, 1)$ such that $\psi(v) = v$, then ψ is equal to the identity function on the interval $[0, v]$,

(iv) ϕ is continuous on the interval $(0, 1]$ and ψ is continuous on the interval $[0, 1)$.

Remark 2.3. Notice that in [34], the function ϕ^* (respectively ψ_*) is defined in a different way, i.e. it corresponds to $1/\phi^*$ (respectively $1/\psi_*$) in the present context. We have changed the notation for the sake of simplicity, since the new notation can give directly $G(y_1) = (\phi^* \circ F_{Y_1})(y_1)$ and $G(y_2) = (\psi_* \circ F_{Y_2})(y_2)$.

In the next proposition, we give sufficient conditions on ϕ and ψ that guarantee $\phi \in \mathscr{F}_1$ and $\psi \in \mathscr{F}_2$, respectively.

Proposition 2.4. *Let $\phi, \psi : \mathbb{I} \longrightarrow \mathbb{I}$ be increasing functions satisfying $\phi(0) = \psi(0) = 0$ and $\phi(1) = \psi(1) = 1$.*

(i) If ϕ is a concave function, then $\phi \in \mathscr{F}_1$.

(ii) If ψ is a convex function, then $\psi \in \mathscr{F}_2$.

Proof. The claim (i) follows easily from the fact that, if ϕ is concave, then it is anti-star-shaped (see, e.g., [12]). For the sake of completeness we present also the following straightforward proof. Since ϕ is a concave function, we have that $\phi(u) \geq \phi(a) + (u - a)\,(\phi(b) - \phi(a))/(b - a)$ for all $u \in [a, b]$ and arbitrary $0 \leq a < b \leq 1$. Take $u_1 < u_2$ and set $a = 0$, $b = u_2$ and $u = u_1$. Since $\phi(0) = 0$, we get $\phi^*(u_1) \leq \phi^*(u_2)$, i.e. ϕ^* is increasing and by that $\phi \in \mathscr{F}_1$.

Claim (ii): If ψ is a convex function, it holds that $\psi(v) \leq \psi(a) + (v - a)\,(\psi(b) - \psi(a))/(b - a)$ for all $v \in [a, b]$ and arbitrary $0 \leq a < b \leq 1$. Now take $v_1 < v_2$ and set $a = v_1$ $b = 1$ and $v = v_2$. Using $\psi(1) = 1$ we derive $\psi_*(v_1) \leq \psi_*(v_2)$. So, ψ_* is increasing and therefore $\psi \in \mathscr{F}_2$. ∎

Remark 2.5. Notice that functions from classes $\mathscr{F}_1$ (respectively, $\mathscr{F}_2$) need not be concave (respectively, convex). Consider, for instance, the following piecewise linear functions

$$\phi(u) = \begin{cases} \frac{b}{a}\,u, & \text{if } u \in [0, a], \\ b, & \text{if } u \in (a, b], \\ u, & \text{if } u \in (b, 1]; \end{cases} \qquad \psi(v) = \begin{cases} v, & \text{if } v \in [0, c), \\ c, & \text{if } v \in [c, d), \\ \frac{1-c}{1-d}\,v - \frac{d-c}{1-d}, & \text{if } v \in [d, 1], \end{cases}$$

with parameters $0 \leq a \leq b \leq 1$ and $0 \leq c \leq d \leq 1$, respectively.

Now, let $\mathscr{C}$ be the set of all bivariate copulas. For all $\phi \in \mathscr{F}_1$, $\psi \in \mathscr{F}_2$ and $C \in \mathscr{C}$ define

$$\begin{aligned} T(\phi, \psi, C)(u, v) &:= \phi(u)\min\{\phi^*(u), \psi_*(v)\} + C(\phi(u), \psi(v))\max\{\phi^*(u) - \psi_*(v), 0\} \\ &= \phi(u)\big(\phi^*(u) - \max\{\phi^*(u) - \psi_*(v), 0\}\big) + C(\phi(u), \psi(v))\max\{\phi^*(u) - \psi_*(v), 0\} \\ &= \max\{\phi^*(u) - \psi_*(v), 0\}\big(C(\phi(u), \psi(v)) - \phi(u)\big) + u. \end{aligned} \tag{2.3}$$

In order to prove that (2.3) defines a *bona fide* copula, we will use the characterization of bivariate copulas provided in [13].

To this end, we recall some definitions and results which will be essential in the proof of the next theorem. Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let x be a point in $[a, b)$. The limits

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x + h) - f(x)}{h}, \qquad D_+ f(x) = \liminf_{h \to 0^+} \frac{f(x + h) - f(x)}{h},$$

are called, respectively, *rightside upper* and *lower Dini derivatives* of f at x. Note that the rightside Dini derivatives take values in $[-\infty, +\infty]$.

Lemma 2.6. *A function $C : \mathbb{I}^2 \to \mathbb{I}$ is a copula if and only if C satisfies the following conditions:*

(C1) $C(u, 0) = C(0, v) = 0$ *for all* $u, v \in \mathbb{I}$, *i.e.* C *is grounded;*

(C2) $C(u, 1) = u$ *and* $C(1, v) = v$ *for all* $u, v \in \mathbb{I}$*;*

(C3) C *is continuous;*

(C4) *there exists a countable set* $\mathcal{Z} \subset \mathbb{I}$ *such that for every* $u \in \mathbb{I} \setminus \mathcal{Z}$ *the following conditions hold:*

(i) $D^+C_v(u)$ *is finite for every* $v \in \mathbb{I}$,

(ii) $D^+C_v(u) \leqslant D^+C_{v'}(u)$ *whenever* $0 \leqslant v < v' \leqslant 1$.

Here, $C_v(u)$ *denotes the function* $u \mapsto C(u, v)$ *for a fixed* $v \in \mathbb{I}$.

We can now prove the main result of this section.

Theorem 2.7. *For all* $\phi \in \mathscr{F}_1$, $\psi \in \mathscr{F}_2$ *and* $C \in \mathscr{C}$, *the function* $T(\phi, \psi, C)$ *given by* (2.3) *is a copula.*

Proof. The proof is based on the characterization provided in Lemma 2.6. Conditions (C1) and (C2) can be easily verified. In order to show that Condition (C3) holds true we only need to prove continuity of T at the points $u = 0$ and $v = 1$ by Proposition 2.2 (part (iv)). First, consider the point $u = 0$ and assume that ϕ^* has a limit 0 at this point. Then, the first term of (2.3) is a product of a function whose limit is zero and a bounded function; hence, it is continuous at $u = 0$. In case that 0 is not the limit of ϕ^* at $u = 0$, finite limit at $u = 0$ still exists since ϕ^* is increasing. In turn, ϕ is continuous at $u = 0$ and, hence, the first term of (2.3) is a product of a bounded function and a function whose limit is zero; so it is continuous at $u = 0$. The case $v = 1$ goes similarly.

It remains to prove Condition (C4). For a fixed $v \in \mathbb{I}$ the differential quotient

$$D(u, h; v) := \frac{1}{h} \left[T(\phi, \psi, C)(u + h, v) - T(\phi, \psi, C)(u, v)\right]$$

can be written as $1 + D_1(u, h; v) + D_2(u, h; v)$ where

$$D_1(u, h; v) := \frac{1}{h}\big(\max\{\phi^*(u{+}h){-}\psi_*(v), 0\}{-}\max\{\phi^*(u){-}\psi_*(v), 0\}\big)\big(C(\phi(u{+}h), \psi(v)){-}\phi(u{+}h)\big) \tag{2.4}$$

and

$$D_2(u, h; v) := \big(\max\{\phi^*(u){-}\psi_*(v), 0\}\big)\frac{1}{h}\big(C(\phi(u{+}h), \psi(v)){-}C(\phi(u), \psi(v)){-}\phi(u{+}h){+}\phi(u)\big). \tag{2.5}$$

We first consider the case $\phi^*(u) < \psi_*(v)$. This implies that $\phi^*(u+h) < \psi_*(v)$ for a sufficiently small $h > 0$, so that $D_1(u,h;v) = D_2(u,h;v) = 0$ and $D(u,h;v) = 1$. On the other hand, if $\phi^*(u) \geqslant \psi_*(v)$, then $\phi^*(u+h) \geqslant \psi_*(v)$ since ϕ^* is increasing. Therefore, (2.4) becomes

$$D_1(u,h;v) = \frac{\phi^*(u+h) - \phi^*(u)}{h}\big(C(\phi(u+h),\psi(v)) - \phi(u+h)\big). \tag{2.6}$$

Let us now turn to the proof of the two statements of Condition (C4) and let us start by item (ii). It will turn out that inequality in (ii) actually holds for all $u \in \mathbb{I}$. Since C is a copula, it follows that

$$\frac{C(z+k,w) - C(z,w)}{k}$$

is a positive increasing function of w, so that it attains its maximum at $w = 1$ which is equal to 1. After introducing the notation $w = \psi(v)$, $z = \phi(u)$ and $k = \phi(u+h) - \phi(u)$ we can rewrite Eq. (2.5) into

$$D_2(u,h;v) = \max\{\phi^*(u) - \psi_*(v), 0\}\frac{\phi(u+h) - \phi(u)}{h}\left(\frac{C(z+k,w) - C(z,w)}{k} - 1\right). \tag{2.7}$$

Thus D_2 (regarded as a function of v when all the other variables are held fixed) is a product of a positive decreasing function, a positive constant, and a negative increasing function. So, it is a negative increasing function with maximum 0 at $v = 1$. Similarly, from Eq. (2.6), D_1 can be seen as a product of a positive constant and a negative increasing function. Recall that $D(u,h;v) = 1 + D_1(u,h;v) + D_2(u,h;v)$, so that it is an increasing function of v.

Choose $0 \leqslant v < v' \leqslant 1$ and observe that

$$\sup_{0<h<\varepsilon} D(u,h;v) \leqslant \sup_{0<h<\varepsilon} D(u,h;v'),$$

so that

$$\limsup_{h\downarrow 0} D(u,h;v) = \lim_{\varepsilon\downarrow 0}\sup_{0<h<\varepsilon} D(u,h;v) \leqslant \lim_{\varepsilon\downarrow 0}\sup_{0<h<\varepsilon} D(u,h;v') = \limsup_{h\downarrow 0} D(u,h;v').$$

Note that the quantities in the inequality above may not be finite. Nevertheless, the inequality holds for all $u \in \mathbb{I}$. It remains to show item (i) of Condition (C4) and, in view of the remark above, we only need to consider the case that $\phi^*(u) \geqslant \psi_*(v)$, which implies that $D_1(u,h;v)$ can be expressed as (2.6). Let $\mathcal{Z}_0$ denotes the set of all $z \in \mathbb{I}$ for which the upper Dini derivative of $C_w(z)$ is not finite at least for some $w \in \mathbb{I}$. Let $\mathcal{Z}_1$ be the set of all $u \in \mathbb{I}$ such that $\phi(u) \in \mathcal{Z}_0$. Moreover, let $\mathcal{Z}_2$ and $\mathcal{Z}_3$ be sets of all points of $\mathbb{I}$ for which the upper Dini derivative of ϕ, respectively ϕ^*, is not finite. Consequently, the limit superior of $D_1(u,h;v)$ is finite for all $u \in \mathbb{I} \setminus \mathcal{Z}_3$ by Eq. (2.6). The limit superior of $D_2(u,h;v)$ is finite for all $u \in \mathbb{I} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$ by equations (2.5) ($D_2(u,h;v) = 0$ if there exists $h > 0$ such that $k = \phi(u+h) - \phi(u) = 0$) and (2.7). Finally, item (i) of Condition (C4) is fulfilled with $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3$.

It remains to prove that the set $\mathcal{Z}$ is countable. Since C is a copula, $\mathcal{Z}_0$ is countable by Lemma 2.6 which implies that $\mathcal{Z}_1$ is countable. In order to finish the proof of this theorem we only need to show that the sets $\mathcal{Z}_2$ and $\mathcal{Z}_3$ are also countable. To this end introduce $H(u,v) = \min\{\phi(u)v, u\}$ for $u, v \in \mathbb{I}$, and observe that this is a copula by, say, the fact that it is the maxmin copula (page 3) and using results of [34]. So, by Lemma 2.2 of [13] there exists a countable set $\mathcal{Z}_H$ such that for every $u \in \mathbb{I} \setminus \mathcal{Z}_H$ it holds that Dini derivatives $D^+H_v(u)$ and $D_+H_v(u)$ are finite for all $v \in \mathbb{I}$. Let $\mathcal{Z}_\phi$ be a possible set such that $D^+\phi(u)$ and $D_+\phi(u)$ are finite for all $u \in \mathbb{I} \setminus \mathcal{Z}_\phi$. It is clear that

$$D^+H_v(u) = \begin{cases} 1, & \text{if } \phi^*(u) \leqslant v; \\ vD^+\phi(u), & \text{otherwise.} \end{cases}$$

A similar expression can be obtained for $D_+H_v(u)$. Thus, $\mathcal{Z}_\phi = \mathcal{Z}_H$ is countable and consequently $\mathcal{Z}_2 \subseteq \mathcal{Z}_\phi$ is countable. Next, observe that $\phi \cdot \phi^* = \mathrm{id}$ and therefore

$$\frac{(\phi \cdot \phi^*)(u+h) - (\phi \cdot \phi^*)(u)}{h} = 1.$$

After a short straightforward computation, it can be seen that

$$\phi(u+h)\frac{\phi^*(u+h) - \phi^*(u)}{h} = 1 - \phi^*(u)\frac{\phi(u+h) - \phi(u)}{h}.$$

We take limit superior on both sides of this latter expression to conclude that

$$\phi(u)D^+\phi^*(u) = 1 - \phi^*(u)D_+\phi(u),$$

so that $\mathcal{Z}_3 \subseteq \mathcal{Z}_\phi$ proving that $\mathcal{Z}_3$ is countable and consequently $\mathcal{Z}$ is countable. ■

Example 2.8. In model (2.1) assume that X_1, X_2 and Z are identically distributed with a common distribution function G. In this case, functions ϕ and ψ have to satisfy $\phi \circ F_{Y_1} = G$ and $\psi \circ F_{Y_2} = G$. Since $F_{Y_1} = G^2$ and $F_{Y_2} = 2G - G^2$, we define $\phi(u) = \sqrt{u}$ for $u \in \mathbb{I}$, and $\psi(v) = 1 - \sqrt{1-v}$ for $v \in \mathbb{I}$. In Figures 2 and 3 we present scatterplots of copulas $T(\phi, \psi, C)$ of this type. Specifically, when C is the independence copula Π, we get maxmin copula from [34]. From these figures, some features of the obtained copulas can be easily visualized, namely the asymmetry, the presence of singular component, and also the support of the copula measure (which often has an obvious non-empty complement in the unit square). All these aspects will be discussed in Section 3.

3. Properties and examples of maxmin copulas with dependent shocks

Here we discuss some properties of the proposed general class of copulas.

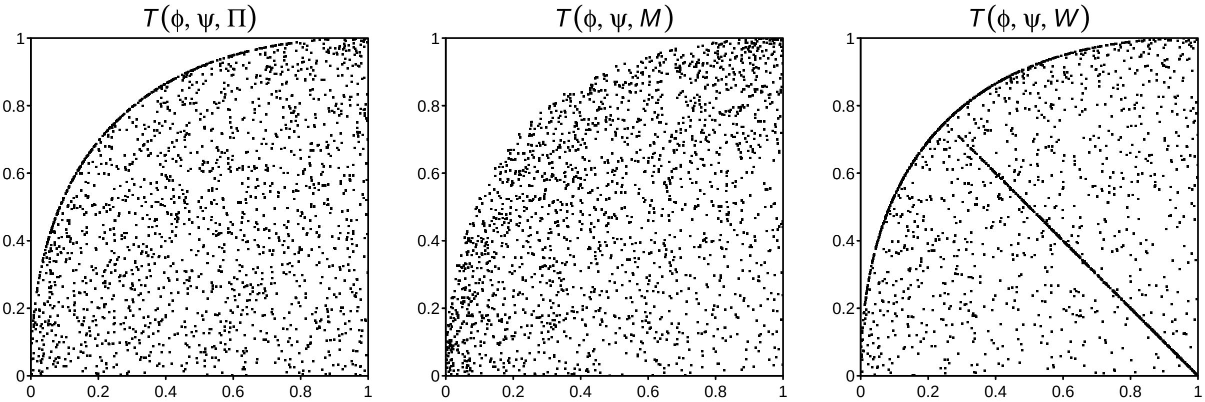

Figure 2: Scatterplots of 2000 points generated from the copula $T(\phi, \psi, C)$ of Example 2.8, where C is equal to (from left to right) the independence copula Π, the comonotonicity copula M and the countermonotonicity copula W.

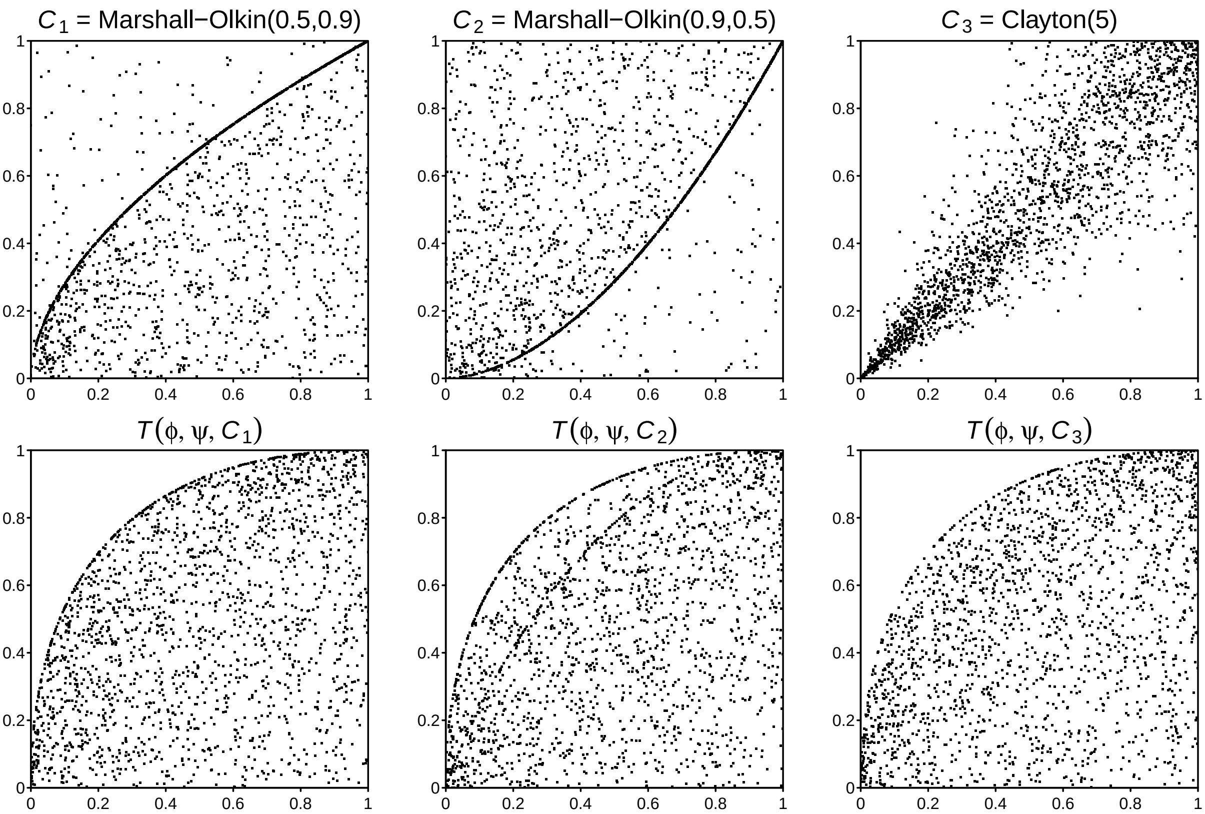

Figure 3: Scatterplots of 2000 points generated from the copula C (first row) and $T(\phi, \psi, C)$ of Example 2.8 (second row), where C is equal to (from left to right) Marshall-Olkin copula with parameters $(\alpha, \beta) = (0.5, 0.9)$, $(\alpha, \beta) = (0.9, 0.5)$, and Clayton copula with parameter $\theta = 5$.

3.1. Exchangeability

One of the properties that are shared by numerous constructions of copulas is the *exchangeability* (or *symmetry*). We say that a bivariate copula C is *exchangeable* if $C(u, v) =$

$C(v,u)$ for all $(u,v) \in \mathbb{I}^2$. In some situations exchangeability is a desirable property; consider, for instance, the case of an engineering system composed by two identical components. However, in many other cases this assumption is somehow too restrictive, as discussed for instance in [19].

In the case of copulas of type (2.3), in general, it is hard to expect exchangeability, in view of the very stochastic interpretation of the family. However, notice that, if we take $\phi = \psi$, then $\phi = \psi = \mathrm{id}$ as a consequence of the first statement in Proposition 2.2, we get $T(\mathrm{id}, \mathrm{id}, C) = C$.

3.2. Dependence properties

In the next theorem we investigate which dependence properties of a copula C are preserved under the application of transformation (2.3). Consider a random vector (Y_1, Y_2) with continuous marginal distribution functions F_{Y_1} and F_{Y_2}, and associated copula C_{Y_1,Y_2}. Random vector (Y_1, Y_2) (or copula C_{Y_1,Y_2}) is *positively quadrant dependent*, PQD, if $C_{Y_1,Y_2}(u,v) \geq \Pi(u,v)$ for all $u, v \in \mathbb{I}$. The *lower tail dependence coefficient* λ_L of random vector (Y_1, Y_2) (or copula C_{Y_1,Y_2}) is the following limit (if it exists):

$$\lambda_L(C_{Y_1,Y_2}) = \lim_{t \downarrow 0} P(Y_2 \leq F_{Y_2}^{-1}(t) | Y_1 \leq F_{Y_1}^{-1}(t)) = \lim_{t \downarrow 0} \frac{C_{Y_1,Y_2}(t,t)}{t}.$$

For further explanation of the property PQD and tail dependence coefficients we refer the reader to [33, Chapter 5].

Theorem 3.1. *Let C be an arbitrary copula, $\phi \in \mathscr{F}_1$, $\psi \in \mathscr{F}_2$, and (Y_1, Y_2) a random vector with continuous marginal distribution functions and associated copula $T(\phi, \psi, C)$. Then it holds that:*

(i) if C is PQD, then $T(\phi, \psi, C)$ is PQD,

(ii) if ϕ is continuous at 0, then

$$\lambda_L(T(\phi, \psi, C)) = \lim_{t \downarrow 0} \frac{C(\phi(t), \psi(t))}{\phi(t)},$$

otherwise we have $\lambda_L(T(\phi, \psi, C)) = \phi(0+)(1 - \psi'(0+))$.

Proof.

(i) If $\phi^*(u) \leq \psi_*(v)$, then $T(\phi, \psi, C)(u,v) = u \geq uv$ and the thesis follows. So, we consider the case $\phi^*(u) > \psi_*(v)$ and prove that $(T(\phi, \psi, C) - \Pi)(u,v) \geq 0$. Since $C(u,v) \geq uv$, we have that

$$\begin{aligned}
(T(\phi, \psi, C) - \Pi)(u,v) &= u - (\phi^*(u) - \psi_*(v))\big(\phi(u) - C(\phi(u), \psi(v))\big) - uv \\
&\geq u(1-v) - (\phi^*(u) - \psi_*(v))\phi(u)(1 - \psi(v)) \\
&= \phi(u)\big(\phi^*(u)(1-v) - (\phi^*(u) - \psi_*(v))(1 - \psi(v))\big) \\
&= \phi(u)\big(-\phi^*(u)(v - \psi(v)) + v - \psi(v)\big) \\
&= \phi(u)(v - \psi(v))(1 - \phi^*(u)) \\
&\geq 0.
\end{aligned}$$

(ii) By item (i) of Proposition 2.2 we have

$$\begin{aligned} T(\phi,\psi,C)(t,t) &= (\phi^*(t)-\psi_*(t))(C(\phi(t),\psi(t))-\phi(t))+t \\ &= C(\phi(t),\psi(t))(\phi^*(t)-\psi_*(t))+\phi(t)\psi_*(t). \end{aligned}$$

So, we get

$$\frac{T(\phi,\psi,C)(t,t)}{t} = \frac{C(\phi(t),\psi(t))}{\phi(t)}\left(1-\frac{\psi_*(t)}{\phi^*(t)}\right)+\frac{\psi_*(t)}{\phi^*(t)}.$$

In order to compute the lower tail dependence coefficient we take the limit of the above expression as t approaches 0. Using $\psi(0+)=0$ and L'Hôpital's rule on the limit of the function $(t-\psi(t))/t$ we compute that the limit of $\psi_*(t)/\phi^*(t)$ is equal to $\phi(0+)(1-\psi'(0+))$. This implies

$$\lambda_L(T(\phi,\psi,C)) = \big(1-\phi(0+)(1-\psi'(0+))\big)\lim_{t\downarrow 0}\frac{C(\phi(t),\psi(t))}{\phi(t)}+\phi(0+)(1-\psi'(0+)),$$

from which (ii) follows. To conclude, let us emphasize that $\psi'(0+)$ exists and is between zero and one.

■

If we keep applying recursively the transformation (2.3), many different copulas are obtained with a variety of forms of dependence.

For a copula C, and fixed functions ϕ and ψ, denote $T^2C := T(\phi,\psi,T(\phi,\psi,C))$ and $T^nC := T(\phi,\psi,T^{n-1}C)$.

Example 3.2. Given $\alpha,\beta\in(0,1)$ define $\phi_\alpha(u)=u^{1-\alpha}$, $u\in\mathbb{I}$, and $\psi_\beta(v)=1-(1-v)^{1-\beta}$, $v\in\mathbb{I}$. For all $\alpha,\beta\in(0,1)$ it holds that $\phi_\alpha\in\mathscr{F}_1$ and $\psi_\beta\in\mathscr{F}_2$ by Proposition 2.4.

Let $\alpha=\beta=0.5$, i.e. functions ϕ and ψ are as in Example 2.8. In Figures 4 and 5 we present scatterplots of copula T^nC, where C is equal to Clayton copula with parameter $\theta=-0.7$ and the countermonotonicity copula W. In both cases, the transformation is applied to symmetric copula, but produces asymmetric copulas. Moreover, in both cases the support of the copula measure (and of its singular component) changes.

Expression (2.3) makes it difficult to compute Spearman's rho and Kendall's tau of $T(\phi,\psi,C)$ even for simple choice of functions ϕ and ψ, and copula C. We refer the reader to [33, Chapter 5] for definitions and explanation of Spearman's rho and Kendall's tau. In the paper [34], Spearman's rho and Kendall's tau for $C=\Pi$, $\phi=\phi_\alpha$ and $\psi=\psi_\beta$ are given for all $\alpha,\beta\in(0,1)$. In order to check how ρ and τ change with multiple applications of transformation T, we simulate 10^4 points from a given copula C, recursively apply transformation T for functions $\phi=\phi_\alpha$ and $\psi=\psi_\beta$ with fixed α and β, and compute the approximated values of ρ and τ on each step. In Tables 1, 2 and 3 we present these calculations, where the copula C is equal to Π, M and W, respectively, showing possible changes in the strength of dependence. In the case of Π, respectively W, one can notice that the recursive application of the transformation plays in favor of an increase in the strength of positive dependence, respectively changes the negative dependence into a positive one. In the case of M, however, the values of the measures of association decrease, but still remain positive.

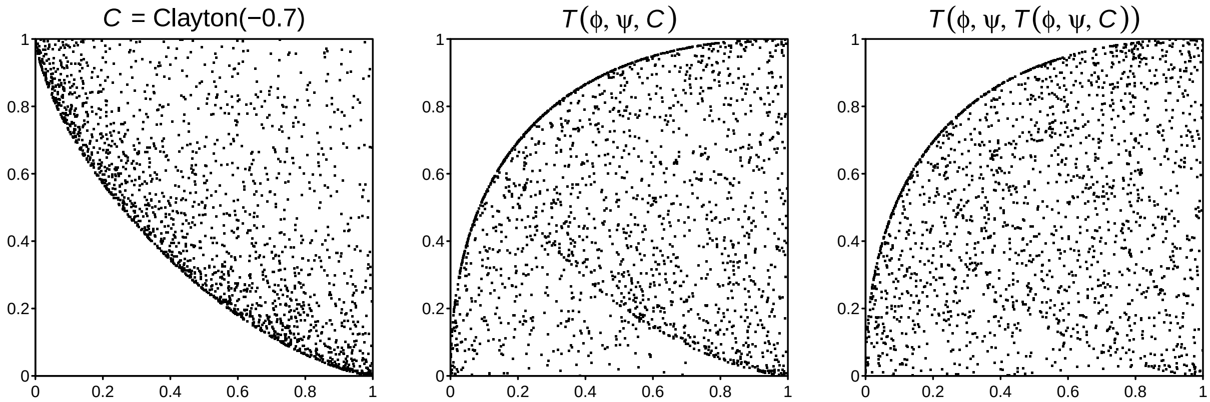

Figure 4: Scatterplots of 2000 points generated from the copulas C, TC, and T^2C (from left to right) of Example 3.2, where C is equal to Clayton copula with parameter $\theta = -0.7$, $\phi = \phi_{0.5}$ and $\psi = \psi_{0.5}$.

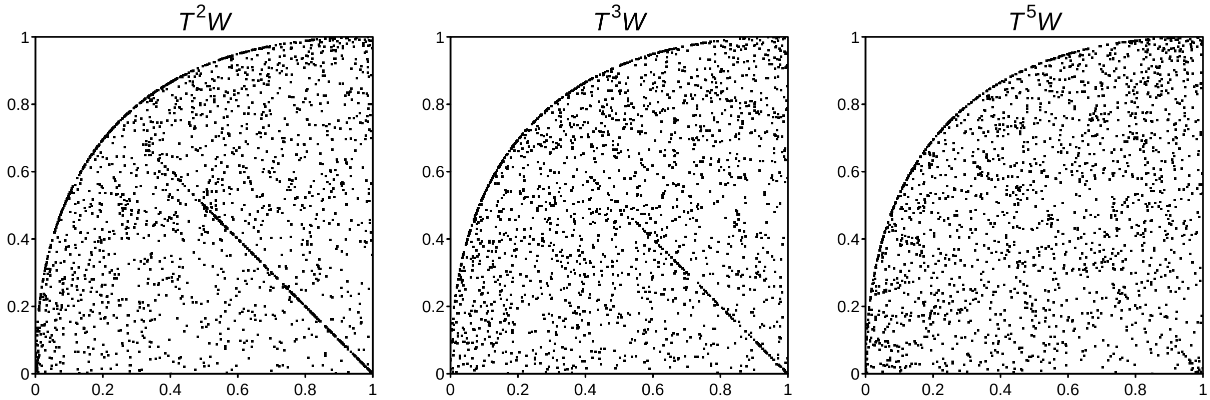

Figure 5: Scatterplots of 2000 points generated from the copulas T^2W, T^3W, and T^5W (from left to right) of Example 3.2, where $\phi = \phi_{0.5}$ and $\psi = \psi_{0.5}$.

3.3. Support and singular components

As clarified in Example 2.8, copulas generated by shock models are, in general, not absolutely continuous and have a support that is strictly contained in $\mathbb{I}^2$. We refer the reader to [33, Section 2.4] and [9] for basic notions on support of a copula, and singular and absolutely continuous components of a copula.

In fact, if ϕ and ψ are strictly increasing, then the support of the copula given by (2.3) lies in the set $\{(u, v) \in \mathbb{I}^2 \,|\, \phi^*(u) \geq \psi_*(v)\}$, i.e. below the curve $\{(u, v) \in \mathbb{I}^2 \,|\, \phi^*(u) = \psi_*(v)\}$. If, in addition, the functions ϕ^* and ψ_* are strictly increasing and C is absolutely continuous, then this curve is the support of the singular part of $T(\phi, \psi, C)$. Notice that the presence of singular components of a copula along different curves is also relevant to detect various paths of tail dependence, as recently stressed in [18].

Now, if we consider the flipping transformation [5] of $T(\phi, \psi, C)$ given by $u - T(\phi, \psi, C)(u, 1-v)$, the resulting copula has a zero set below the curve

$$\Gamma_{\phi,\psi} := \{(u, v) \in \mathbb{I} \,|\, \phi^*(u) = \psi_*(1 - v)\}.$$

α	β	$\rho(\Pi)$	$\rho(T^1\Pi)$	$\rho(T^2\Pi)$	$\rho(T^3\Pi)$	$\rho(T^4\Pi)$	$\tau(\Pi)$	$\tau(T^1\Pi)$	$\tau(T^2\Pi)$	$\tau(T^3\Pi)$	$\tau(T^4\Pi)$
0.1	0.1	-0.008	-0.011	-0.004	0.006	0.018	-0.006	-0.007	-0.003	0.004	0.012
0.1	0.5	-0.005	0.051	0.059	0.088	0.076	-0.003	0.034	0.040	0.059	0.051
0.1	0.9	-0.012	0.102	0.144	0.137	0.120	-0.008	0.071	0.098	0.094	0.083
0.5	0.1	0.017	0.057	0.080	0.077	0.081	0.011	0.038	0.054	0.051	0.054
0.5	0.5	0.005	0.295	0.323	0.330	0.333	0.004	0.211	0.228	0.233	0.235
0.5	0.9	0.002	0.552	0.543	0.558	0.558	0.001	0.445	0.440	0.447	0.449
0.9	0.1	0.022	0.129	0.116	0.133	0.143	0.015	0.089	0.080	0.091	0.098
0.9	0.5	0.004	0.540	0.560	0.550	0.549	0.003	0.434	0.449	0.441	0.441
0.9	0.9	0.021	0.852	0.863	0.866	0.871	0.014	0.794	0.805	0.806	0.815

Table 1: Each row presents approximated values of Spearman's rho and Kendall's tau of the copula $T^k\Pi$ for $k \in \{0, 1, 2, 3, 4\}$, where functions $\phi = \phi_\alpha$ and $\psi = \psi_\beta$ are fixed with parameters α and β specified at the first two columns (see also Example 3.2).

α	β	$\rho(M)$	$\rho(T^1M)$	$\rho(T^2M)$	$\rho(T^3M)$	$\rho(T^4M)$	$\tau(M)$	$\tau(T^1M)$	$\tau(T^2M)$	$\tau(T^3M)$	$\tau(T^4M)$
0.1	0.1	1.000	0.860	0.734	0.617	0.510	1.000	0.800	0.641	0.510	0.400
0.1	0.5	1.000	0.554	0.301	0.178	0.121	1.000	0.442	0.212	0.120	0.081
0.1	0.9	1.000	0.207	0.140	0.138	0.134	1.000	0.144	0.096	0.095	0.092
0.5	0.1	1.000	0.556	0.301	0.167	0.128	1.000	0.445	0.212	0.114	0.086
0.5	0.5	1.000	0.480	0.369	0.337	0.339	1.000	0.344	0.261	0.238	0.240
0.5	0.9	1.000	0.564	0.558	0.556	0.549	1.000	0.451	0.448	0.445	0.445
0.9	0.1	1.000	0.209	0.137	0.154	0.133	1.000	0.145	0.094	0.105	0.091
0.9	0.5	1.000	0.561	0.548	0.555	0.563	1.000	0.444	0.437	0.446	0.448
0.9	0.9	1.000	0.871	0.877	0.856	0.862	1.000	0.814	0.819	0.798	0.802

Table 2: Each row presents approximated values of Spearman's rho and Kendall's tau of the copula T^kM for $k \in \{0, 1, 2, 3, 4\}$, where functions $\phi = \phi_\alpha$ and $\psi = \psi_\beta$ are fixed with parameters α and β specified at the first two columns (see also Example 3.2).

The obtained copula is hence given by

$$\begin{aligned} T_2(\phi,\psi,C)(u,v) &= u - T(\phi,\psi,C)(u,1-v) \\ &= \max\{\phi^*(u) - \psi_*(1-v), 0\}\big(\phi(u) - C(\phi(u),\psi(1-v))\big). \end{aligned}$$

Thus, the introduced copulas of type (2.3) can provide a useful class of copulas when the zero set is known; a question connected with the study of conic aggregation functions (see, for instance, [23, 24]).

Example 3.3. For a function $\phi \in \mathscr{F}_1$ set $\psi_\phi(t) = 1 - \phi(1-t)$. Then $\psi_\phi \in \mathscr{F}_2$ and

$$T_3(\phi,C)(u,v) = T_2(\phi,\psi_\phi,C)(u,v) = W(\phi^*(u),\phi^*(v))\big(\phi(u) - C(\phi(u), 1-\phi(v))\big),$$

where W is the countermonotonicity copula. This construction can be seen as a transformation $T_3 : \mathscr{F}_1 \times \mathscr{C} \longrightarrow \mathscr{C}$. Observe that $\mathscr{F}_1$, equipped with the function composition, is a semigroup with identity. However, the transformation T_3 restricted to $\mathscr{F}_1$ is not a semigroup homomorphism or anti-homomorphism.

Let us explore the properties of the zero set of the copula in this case. Assume that the functions ϕ and ϕ^* are strictly increasing and that $\Gamma = \{(u,\gamma(u)) \,|\, u \in \mathbb{I}\}$ for some function

α	β	$\rho(W)$	$\rho(T^1W)$	$\rho(T^2W)$	$\rho(T^3W)$	$\rho(T^4W)$	$\tau(W)$	$\tau(T^1W)$	$\tau(T^2W)$	$\tau(T^3W)$	$\tau(T^4W)$
0.1	0.1	-1.000	-0.853	-0.724	-0.624	-0.528	-1.000	-0.798	-0.644	-0.533	-0.437
0.1	0.5	-1.000	-0.490	-0.222	-0.071	0.007	-1.000	-0.402	-0.169	-0.053	0.003
0.1	0.9	-1.000	-0.019	0.144	0.118	0.112	-1.000	-0.010	0.099	0.082	0.077
0.5	0.1	-1.000	-0.520	-0.249	-0.084	-0.003	-1.000	-0.421	-0.186	-0.060	-0.003
0.5	0.5	-1.000	0.018	0.194	0.265	0.302	-1.000	0.027	0.148	0.194	0.218
0.5	0.9	-1.000	0.470	0.529	0.539	0.536	-1.000	0.403	0.430	0.433	0.431
0.9	0.1	-1.000	0.005	0.139	0.129	0.131	-1.000	0.005	0.096	0.089	0.090
0.9	0.5	-1.000	0.466	0.538	0.562	0.563	-1.000	0.403	0.434	0.451	0.451
0.9	0.9	-1.000	0.828	0.867	0.865	0.866	-1.000	0.801	0.812	0.806	0.810

Table 3: Each row presents approximated values of Spearman's rho and Kendall's tau of the copula T^kW for $k \in \{0,1,2,3,4\}$, where functions $\phi = \phi_\alpha$ and $\psi = \psi_\beta$ are fixed with parameters α and β specified at the first two columns (see also Example 3.2).

$\gamma : \mathbb{I} \longrightarrow \mathbb{I}$. We will derive some properties of γ if $\Gamma = \Gamma_{\phi,\psi_\phi} = \{(u,v) \in \mathbb{I} \mid \phi^*(u)+\phi^*(v) = 1\}$. From the symmetry of the copula, it follows that $\gamma \circ \gamma = \mathrm{id}$. Since $\gamma(u) = (\phi^*)^{-1}(1-\phi^*(u))$, γ is a strictly decreasing function.

We observe that $T_3(\phi, C)$ actually belongs to constructions considered in [7, 27]. In particular, Corollary 3 in [7] states that, if C_1 and C_2 are bivariate copulas, and $f, g \in \mathscr{F}_1$, then

$$(C_1 \diamond C_2)(u,v) \coloneqq C_1(f^*(u), g^*(v))C_2(f(u), g(v))$$

is a copula. So, $T_3(\phi, C) = W \diamond \tilde{C}$ for $\tilde{C}(u,v) = u - C(u, 1-v)$ where operation $\diamond$ is used with respect to $f = g = \phi$.

Curiously, if we set $W^2 \diamond \tilde{C} = W \diamond (W \diamond \tilde{C})$, $W^k \diamond C = W \diamond (W^{k-1} \diamond \tilde{C})$ for $k \geq 3$, the following recursive formula holds

$$(W^n \diamond \tilde{C})(u,v) = \prod_{k=1}^{n} W\left(\frac{\phi^{k-1}(u)}{\phi^k(u)}, \frac{\phi^{k-1}(v)}{\phi^k(v)}\right) C(\phi^n(u), \psi^n(v)).$$

where ϕ^k denotes the k-fold composition of ϕ with itself.

4. Extensions to higher dimensions

In this section we propose an extension of maxmin copulas to higher dimensions. We start with the description of the generating mechanism and, to this end, we consider n r.v.'s $X_1, X_2, ..., X_n$ divided into two groups, one consisting of p r.v.'s and the other one of the remaining q r.v.'s, where $n = p + q$. The dependence of these r.v.'s is described by a copula C. Furthermore, there is a r.v. Z, independent of all of them, that may effect the behavior of the system in two different ways. To interpret this we introduce p random variables $Y_i = \max\{X_i, Z\}$ for $i = 1, 2 \ldots, p$ and q random variables $Y_i = \min\{X_i, Z\}$ for $i = p+1, p+2, \ldots, n$. Having our application in mind we can think of the first p components of the vector $\mathbf{Y}$ as default times of the companies with recovery option (this means that we assume these companies to have a spare component which absorbs the first shock that arrives and consequently they default only on the second one), and of the remaining q components to represent the respective default times of companies defaulting on the first shock.

Our aim is to give a closed expression for the distribution function of the r.v. $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$ via the high-dimensional extension of our copula. Unfortunately, the expression given in Eq. (4.4) is quite involved. In order to provide a more readable formula we introduce vector notation for the r.v.'s as well as for their joint and/or marginal distribution functions (d.f.'s) and finally also for the distortion functions. Furthermore, we need to introduce notation for subvectors written as subscripts of the vectors in order to express actions on them and relations between them. The following few paragraphs serve to develop Eq. (4.4) together with the sketch of the proof of Theorem 4.1.

First we introduce vector notation for the underlying r.v.'s $\mathbf{X} = (X_1, X_2, ..., X_n)$. We will understand relations between (random) vectors $\leqslant$ componentwise. Similarly, operations on vectors (like max and min) will be understood to be done componentwise and, in case of random vectors, they are assumed to hold almost surely. We will denote $\mathbf{X}_{p+1:n} = (X_{p+1}, X_{p+2}, ..., X_n)$ or, more generally, $\mathbf{X}_K = (X_k)_{k \in K \subseteq \{1,2,...,n\}}$.

Using these notations, the vector $\mathbf{Y}$ of Eq. (2.1) may be extended to higher dimensions as $\mathbf{Y}_{1:p} = \max\{\mathbf{X}, Z\}_{1:p}$ and $\mathbf{Y}_{p+1:n} = \min\{\mathbf{X}, Z\}_{p+1:n}$. This means that for indices $1 \leqslant i \leqslant p$ we have that $Y_i = \max\{X_i, Z\}$, while for indices $p+1 \leqslant i \leqslant n$ we have that $Y_i = \min\{X_i, Z\}$ which corresponds to the definition loosely given above.

Next let us explain the main idea of our proof. For every $\mathbf{y} \in \mathbb{R}^n$ the joint probability distribution of $\mathbf{Y}$ can be rewritten as

$$\begin{aligned} H(\mathbf{y}) &= \mathbb{P}\left(\max\left\{\mathbf{X}, Z\right\}_{1:p} \leqslant \mathbf{y}_{1:p}, \min\left\{\mathbf{X}, Z\right\}_{p+1:n} \leqslant \mathbf{y}_{p+1:n}\right) \\ &= \mathbb{P}\left(\mathbf{X}_{1:p} \leqslant \mathbf{y}_{1:p}, Z \leqslant \min\{\mathbf{y}_{1:p}\}, \min\left\{\mathbf{X}, Z\right\}_{p+1:n} \leqslant \mathbf{y}_{p+1:n}\right). \qquad (4.1) \end{aligned}$$

Now we try to get components of the subvector $\mathbf{X}_{p+1:n}$, roughly speaking, out of the minimum condition above which will be achieved one by one. On each step we may use the identity

$$\mathbb{P}\left(\min\left\{X_i, Z\right\} \leqslant y_i\right) = \mathbb{P}\left(Z \leq y_i\right) + \mathbb{P}\left(Z > y_i, X_i \leqslant y_i\right) \qquad (4.2)$$

for $i = p+1, \ldots, n$. In order to see how it works apply this rule for $i = p+1$ on Eq. (4.1) to get

$$\begin{aligned} H(\mathbf{y}) = \mathbb{P}&\left(\mathbf{X}_{1:p} \leqslant \mathbf{y}_{1:p}, Z \leqslant \min\{\mathbf{y}_{1:p}\}, Z \leqslant y_{p+1}, \min\left\{\mathbf{X}, Z\right\}_{p+2:n} \leqslant \mathbf{y}_{p+2:n}\right) \\ &+ \mathbb{P}\left(\mathbf{X}_{1:p} \leqslant \mathbf{y}_{1:p}, Z \leqslant \min\{\mathbf{y}_{1:p}\}, Z > y_{p+1}, X_{p+1} \leqslant y_{p+1}, \min\left\{\mathbf{X}, Z\right\}_{p+2:n} \leqslant \mathbf{y}_{p+2:n}\right). \end{aligned}$$

Note that, in the first of the two summands of the latter expression, the r.v. X_{p+1} does not appear (we shall say, just during the development of our formula, that in an expression like that *index* $p+1$ *is of the first kind*), while in the second summand the r.v. X_{p+1} appears in relation of the form $X_{p+1} \leqslant y_{p+1}$ (in which case we shall say that *index* $p+1$ *is of the second kind*). We continue by deciding on the type of index $p+2$ yielding two new summands from each of existing ones by applying (4.2). On this step we get 4 summands and we proceed in a similar way through all the indices of $S := \{p+1 : n\}$. At the end of this procedure we are faced with a disjoint set of 2^q events. Let $K \subseteq S$ denote the set of indices of the first type, while $K^c = S \setminus K$ represents the opposite event, i.e. the set of indices of the second type. Since all the events K are disjoint, they yield a partition of our universe. So, the rightmost probability of (4.1) can be written as a sum of probabilities each under the additional assumption that exactly the indices of K are of the first type.

Now is the time to give the actual development of the formula. For a fixed K we will denote by $C(F_{\mathbf{X}}(\mathbf{y})_{\{1:p\}\cup K^c})$ the value of copula C at the vector point $\mathbf{u}$ in which u_j is set to $F_{X_j}(y_j)$ for all $j \in \{1:p\} \cup K^c$, while $u_j = 1$ otherwise. The final formula becomes

$$H(\mathbf{y}) = \sum_{K \subseteq S} \mathbb{P}\left(\mathbf{X}_{\{1:p\}\cup K^c} \leqslant \mathbf{y}_{\{1:p\}\cup K^c}, Z \leqslant \min\left\{\mathbf{y}_{\{1:p\}\cup K}\right\}, Z > \max\left\{\mathbf{y}_{K^c}\right\}\right)$$

which can be expressed in terms d.f.'s as

$$H(\mathbf{y}) = \sum_{K \subseteq S} C(F_{\mathbf{X}}(\mathbf{y})_{\{1:p\}\cup K^c}) \max\left\{0, F_Z\left(\min\left\{\mathbf{y}_{\{1:p\}\cup K}\right\}\right) - F_Z\left(\max\left\{\mathbf{y}_{K^c}\right\}\right)\right\}. \tag{4.3}$$

In order to get the corresponding copula we have to introduce appropriate distortion functions. To this end, we will use similar approach as in Section 2. First, we observe that $F_{\mathbf{Y}}(\mathbf{y})_{1:p} = F_{\mathbf{X}}(\mathbf{y})_{1:p} F_Z(\mathbf{y})_{1:p}$ and $1 - F_{\mathbf{Y}}(\mathbf{y})_{p+1:n} = (1 - F_{\mathbf{X}}(\mathbf{y})_{p+1:n})(1 - F_Z(\mathbf{y})_{p+1:n})$. We then define $\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_n)$, so that $F_{\mathbf{X}} = \mathbf{\Phi}(F_{\mathbf{Y}})$ on all indices $\{1:n\}$. Furthermore, we introduce

$$\mathbf{\Phi}^*_{1:p}(u) = \frac{u}{\mathbf{\Phi}_{1:p}(u)} \quad \text{and} \quad \mathbf{\Phi}^*_{p+1:n}(u) = \begin{cases} \dfrac{u - \mathbf{\Phi}_{p+1:n}(u)}{1 - \mathbf{\Phi}_{p+1:n}(u)}, & \text{for } u < 1; \\ 1, & \text{for } u = 1. \end{cases}$$

Obviously, it holds that $F_Z = \mathbf{\Phi}^*(F_{\mathbf{Y}})$ on all indices $1, \ldots, n$. Using these functions and the fact that F_Z is an increasing function, we can translate formula (4.3) for distribution functions into the desired copula of our model, which becomes

$$T(\mathbf{\Phi}, C)(\mathbf{u}) = \sum_{K \subseteq S} C(\mathbf{\Phi}(\mathbf{u})_{\{1:p\}\cup K^c}) \max\left\{0, \min\left\{\mathbf{\Phi}^*(\mathbf{u})_{\{1:p\}\cup K}\right\} - \max\left\{\mathbf{\Phi}^*(\mathbf{u})_{K^c}\right\}\right\}. \tag{4.4}$$

Here again, similarly as above, in the copula C the values of all entries with missing indices are set equal to 1 (thus, they are actually the related lower-dimensional margins of C). Let us also recall that $S = \{p+1 : n\}$ and $K^c = S \backslash K$ for $K \subseteq S$. Summarizing, the following result can be formulated.

Theorem 4.1. *Under the previous assumptions, the function given by* (4.4) *is a copula.*

In order to further clarify the notation used in the definition of $T(\mathbf{\Phi}, C)$, let us write (4.4) without the vector notation in case that $n = 3$ and $p = 1$ (cf. Example 4.2):

$$\begin{aligned} T(\phi_1, \phi_2, \phi_3, C)(u_1, u_2, u_3) = {} & C(\phi_1(u_1), \phi_2(u_2), \phi_3(u_3)) \max\{0, \phi_1^*(u_1) - \max\{\phi_2^*(u_2), \phi_3^*(u_3)\}\} \\ & + C(\phi_1(u_1), 1, \phi_3(u_3)) \max\{0, \min\{\phi_1^*(u_1), \phi_2^*(u_2)\} - \phi_3^*(u_3)\} \\ & + C(\phi_1(u_1), \phi_2(u_2), 1) \max\{0, \min\{\phi_1^*(u_1), \phi_3^*(u_3)\} - \phi_2^*(u_2)\} \\ & + \phi_1(u_1) \min\{\phi_1^*(u_1), \phi_2^*(u_2), \phi_3^*(u_3)\}. \end{aligned}$$

Example 4.2. In Figure 6, respectively Figure 7, we present scatterplots of multivariate maxmin copulas for the dimension $n = 3$ and $p = 1$, respectively $p = 2$, where $C = \Pi$, $\phi_i(u) = \sqrt{u}$ for $i \leq p$ and $\phi_i(u) = 1 - \sqrt{1-u}$ for $i > p$. Note that the cases $p = 0$ and $p = 3$ reduce to the Marshall-Olkin copulas. Related to the graph presented in Figure 6 we also show its bivariate marginal distributions; Figure 8, respectively Figure 9, shows scatterplot with respect to axes Y_1 and Y_2, respectively axes Y_2 and Y_3. The marginal distribution with respect to axes Y_1 and Y_3 is the same as the one with respect to axes Y_1 and Y_2.

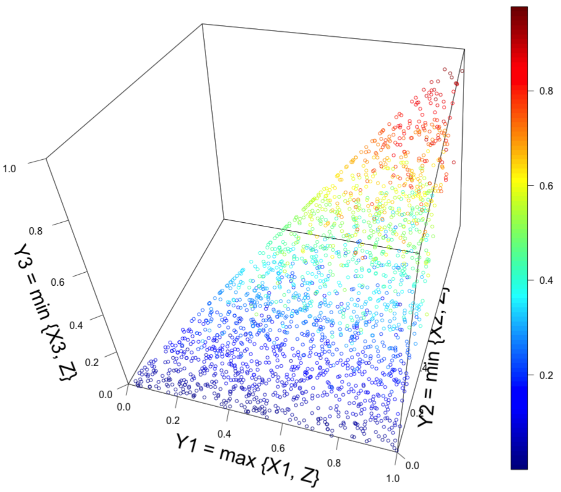

Figure 6: Scatterplot of 2000 points generated from a copula $T(\mathbf{\Phi}, C)$ of type (4.4), where $C = \Pi$, $\mathbf{\Phi}$ as in Example 4.2 and $p = 1$.

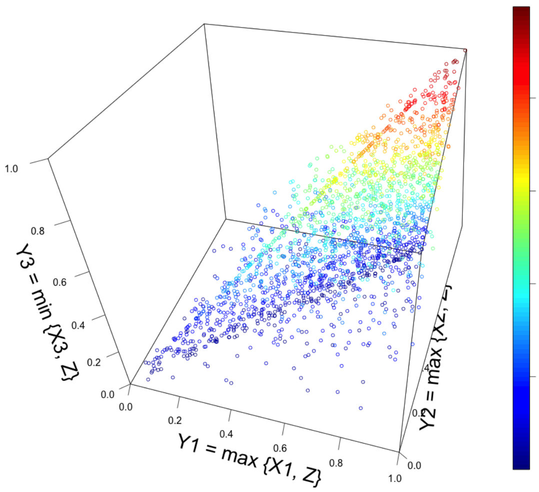

Figure 7: Scatterplot of 2000 points generated from a copula $T(\mathbf{\Phi}, C)$ of type (4.4), where $C = \Pi$, $\mathbf{\Phi}$ as in Example 4.2 and $p = 2$.

5. Conclusions

The present work explores a possible specification in the general framework of copula models that can be interpreted in terms of shock models (also called Marshall-Olkin mechanism). The main idea of the proposed model is that a given dependence structure is modified by the presence of an exogenous shock that has opposite effects on the involved variables. In a stochastic setting, the proposed methodology can be used, for instance, to show how a given dependence may react to a "stress scenario" (or to a multiple application of the same stress scenario).

At a more theoretical level, the copula of Eq. (2.3) can be seen as a way to transform a given copula C into another copula by means of auxiliary functions. As such, it generates copulas that may exhibit some desirable properties like asymmetries, singularities and non-trivial tail behavior. Interestingly, Eq. (2.3) is related to various semilinear-type families of copula that have been recently introduced.

As a take-home message for the readers with interest in aggregation functions and multi-criteria decision making, the present work suggests that it may be convenient to consider

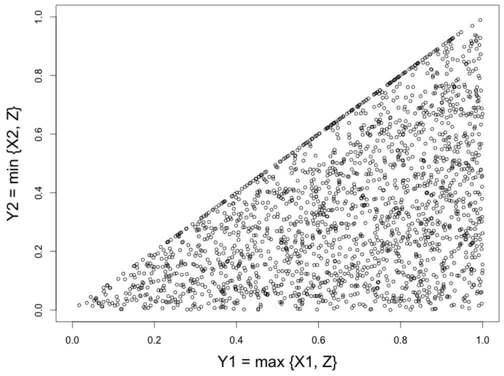

Figure 8: Bivariate marginal scatterplot of 2000 points generated from a copula $T(\mathbf{\Phi}, C)$ of type (4.4), showing axes Y_1 and Y_2, where $C = \Pi$, $\mathbf{\Phi}$ as in Example 4.2 and $p = 1$.

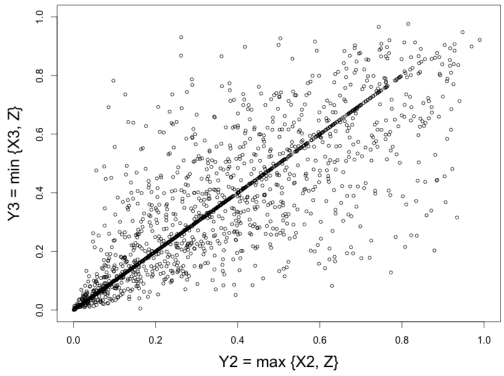

Figure 9: Bivariate marginal scatterplot of 2000 points generated from a copula $T(\mathbf{\Phi}, C)$ of type (4.4), showing axes Y_2 and Y_3, where $C = \Pi$, $\mathbf{\Phi}$ as in Example 4.2 and $p = 1$.

transformation methods for aggregation functions that act differently on the involved variables, especially when some input values need to have a "prominent role" in the aggregation process. In this respect, we think that the present methodology can be also modified in such a way that the transformation acts "with a different degree" on the various inputs (and not just with a positive/negative effect). Future investigations will be devoted to this latter extension.

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